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PAINLEVÉ PROPERTY OF THE HÉNON-HEILES HAMILTONIANS

by

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Abstract. — Time independent Hamiltonians of the physical type

$$H = (P_1^2 + P_2^2)/2 + V(Q_1, Q_2)$$

pass the Painlevé test for only seven potentials V, known as the Hénon-Heiles Hamiltonians, each depending on a finite number of free constants. Proving the Painlevé property was not yet achieved for generic values of the free constants. We integrate each missing case by building a birational transformation to some fourth order first degree ordinary differential equation in the classification (Cosgrove, 2000) of such polynomial equations which possess the Painlevé property. The properties common to each Hamiltonian are:

(i) the general solution is meromorphic and expressed with hyperelliptic functions of genus two,

(ii) the Hamiltonian is complete (the addition of any time independent term would ruin the Painlevé property).

***Résumé* (Propriété de Painlevé des hamiltoniens de Hénon-Heiles).** — Les hamiltoniens, indépendants du temps, de la forme

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satisfont au test de Painlevé pour seulement sept potentiels V ; ceux-ci sont connus sous le nom de hamiltoniens de Hénon-Heiles et ils dépendent d'un nombre fini de constantes libres. La propriété de Painlevé restait à établir pour des valeurs génériques des constantes libres. Nous traitons chacun des cas en suspens en construisant une transformation birationnelle vers une équation différentielle ordinaire d'ordre quatre qui figure dans la liste exhaustive (Cosgrove, 2000) de telles équations polynomiales possédant la propriété de Painlevé. Les propriétés communes à ces hamiltoniens sont :

(i) la solution générale est méromorphe et peut être exprimée en termes de fonctions hyperelliptiques de genre deux,

(ii) le hamiltonien est complet au sens où l'addition de tout terme indépendant du temps ferait perdre la propriété de Painlevé.

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1. Introduction

Let us consider the most general two-degree of freedom, classical, time-independent Hamiltonian of the physical type (i.e, the sum of a kinetic energy and a potential energy),

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{1}$$

and let us require that the general solution $q_1^{n_1}, q_2^{n_2}$, with n_1, n_2 integers to be determined, be single valued functions of the complex time t, i.e., what is called the *Painlevé property* of these equations.

A necessary condition is that the Hamilton equations of motion, when written in these variables $q_1^{n_1}, q_2^{n_2}$, pass the Painlevé test ([**12**]). This selects seven potentials V (three "cubic" and four "quartic") depending on a finite number of arbitrary constants, which are known as the Hénon-Heiles Hamiltonians ([**24**]). In order to prove the sufficiency of these conditions, one must then perform the explicit integration and check the singlevaluedness of the general solution. We present here a review on this subject.

The paper is organized as follows:

In section 2, we enumerate the seven cases isolated by the Painlevé test, together with the second constant of the motion K in involution with the Hamiltonian. In section 3, we recall the separating variables in the four cases where they are known. In section 4, we display confluences from quartic cases to all the cubic cases, thus restricting the problem to the consideration of the quartic cases only. In section 5, due to the lack of knowledge of the separating variables in the three remaining cases, we state the equivalence of the equations of motion and the conservation of energy with some fourth order first degree ordinary differential equations (ODEs). In section 6, since these fourth order equations do not belong to any set of already classified equations, we build a birational transformation between each quartic case and some fourth order ODE belonging to a classification of Cosgrove ([**17**]), thus proving the Painlevé property for the quartic cases.

To summarize, the results are twofold:

1. each case is integrated by solving a Jacobi inversion problem involving a hyperelliptic curve of genus two, which proves the meromorphy of the general solution,
2. each case is *complete* in the sense of Painlevé, i.e, it is impossible to add any time-independent term to the Hamiltonian without ruining the Painlevé property.

2. The seven Hénon-Heiles Hamiltonians

By application of the Painlevé test, one isolates two classes of potentials $V(q_1, q_2)$, called "cubic" and "quartic" for simplification.

1. In the cubic case HH3 ([**10, 13, 21**]),

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1 q_1^2 + \omega_2 q_2^2) + \alpha q_1 q_2^2 - \frac{1}{3}\beta q_1^3 + \frac{1}{2}\gamma q_2^{-2}, \quad \alpha \neq 0, \tag{2}$$

in which the constants $\alpha, \beta, \omega_1, \omega_2$ and γ can only take the three sets of values,

$$(\text{SK}): \quad \beta/\alpha = -1, \omega_1 = \omega_2, \tag{3}$$

$$(\text{KdV5}): \quad \beta/\alpha = -6, \tag{4}$$

$$(\text{KK}): \quad \beta/\alpha = -16, \omega_1 = 16\omega_2. \tag{5}$$

2. In the quartic case HH4 ([**23, 32**]),

$$\begin{aligned} H = \frac{1}{2}(P_1^2 + P_2^2 + \Omega_1 Q_1^2 + \Omega_2 Q_2^2) + CQ_1^4 \\ + BQ_1^2 Q_2^2 + AQ_2^4 + \frac{1}{2}\left(\frac{\alpha}{Q_1^2} + \frac{\beta}{Q_2^2}\right) + \gamma Q_1, \quad B \neq 0, \end{aligned} \tag{6}$$

in which the constants $A, B, C, \alpha, \beta, \gamma, \Omega_1$ and Ω_2 can only take the four values (the notation $A : B : C = p : q : r$ stands for $A/p = B/q = C/r =$ arbitrary),

$$\left\{ \begin{array}{ll} A : B : C = 1 : 2 : 1, & \gamma = 0, \\ A : B : C = 1 : 6 : 1, & \gamma = 0, \ \Omega_1 = \Omega_2, \\ A : B : C = 1 : 6 : 8, & \alpha = 0, \ \Omega_1 = 4\Omega_2, \\ A : B : C = 1 : 12 : 16, & \gamma = 0, \ \Omega_1 = 4\Omega_2. \end{array} \right. \tag{7}$$

For each of the seven cases so isolated there exists a second constant of the motion K ([**7, 18, 25**]) ([**6, 7, 26**]) in involution with the Hamiltonian,

(SK) $$K = \big(3p_1 p_2 + \alpha q_2 (3q_1^2 + q_2^2) + 3\omega_2 q_1 q_2\big)^2 + 3\gamma(3p_1^2 q_2^{-2} + 4\alpha q_1 + 2\omega_2),$$

(KdV5) $$\begin{aligned} K = 4\alpha p_2 (q_2 p_1 - q_1 p_2) + (4\omega_2 - \omega_1)(p_2^2 + \omega_2 q_2^2 + \gamma q_2^{-2}) \\ + \alpha^2 q_2^2 (4q_1^2 + q_2^2) + 4\alpha q_1 (\omega_2 q_2^2 - \gamma q_2^{-2}), \end{aligned}$$

(KK) $$\begin{aligned} K = (3p_2^2 + 3\omega_2 q_2^2 + 3\gamma q_2^{-2})^2 + 12\alpha p_2 q_2^2 (3q_1 p_2 - q_2 p_1) \\ - 2\alpha^2 q_2^4 (6q_1^2 + q_2^2) + 12\alpha q_1 (-\omega_2 q_2^4 + \gamma) - 12\omega_2 \gamma, \end{aligned}$$

(1 : 2 : 1) $$\left\{ \begin{aligned} & K = (Q_2 P_1 - Q_1 P_2)^2 + Q_2^2 \frac{\alpha}{Q_1^2} + Q_1^2 \frac{\beta}{Q_2^2} \\ & \qquad - \frac{\Omega_1 - \Omega_2}{2}\left(P_1^2 - P_2^2 + Q_1^4 - Q_2^4 + \Omega_1 Q_1^2 - \Omega_2 Q_2^2 + \frac{\alpha}{Q_1^2} - \frac{\beta}{Q_2^2}\right), \\ & A = \frac{1}{2}, \end{aligned} \right.$$

$$
(1:6:1) \quad \begin{cases} K = \left(P_1P_2 + Q_1Q_2\Big(-\dfrac{Q_1^2+Q_2^2}{8} + \Omega_1\Big)\right)^2 \\ \qquad\qquad - P_2^2\dfrac{\kappa_1^2}{Q_1^2} - P_1^2\dfrac{\kappa_2^2}{Q_2^2} + \dfrac{1}{4}\left(\kappa_1^2Q_2^2 + \kappa_2^2Q_1^2\right) + \dfrac{\kappa_1^2\kappa_2^2}{Q_1^2Q_2^2}, \\ \alpha = -\kappa_1^2, \quad \beta = -\kappa_2^2, \quad A = -\dfrac{1}{32}, \end{cases}
$$

$$
(1:6:8) \quad \begin{cases} K = \left(P_2^2 - \dfrac{Q_2^2}{16}(2Q_2^2 + 4Q_1^2 + \Omega_2) + \dfrac{\beta}{Q_2^2}\right)^2 \\ \qquad - \dfrac{1}{4}Q_2^2(Q_2P_1 - 2Q_1P_2)^2 + \gamma\Big(- 2\gamma Q_2^2 - 4Q_2P_1P_2 \\ \qquad\qquad + \dfrac{1}{2}Q_1Q_2^4 + Q_1^3Q_2^2 + 4Q_1P_2^2 - 4\Omega_2Q_1Q_2^2 + 4Q_1\dfrac{\beta}{Q_2^2}\Big), \\ A = -\dfrac{1}{16}, \end{cases}
$$

$$
(1:12:16) \quad \begin{cases} K = \Big(8(Q_2P_1 - Q_1P_2)P_2 - Q_1Q_2^4 - 2Q_1^3Q_2^2 \\ \qquad\qquad + 2\Omega_1Q_1Q_2^2 - 8Q_1\dfrac{\beta}{Q_2^2}\Big)^2 + \dfrac{32\alpha}{5}\left(Q_2^4 + 10\dfrac{Q_2^2P_2^2}{Q_1^2}\right), \\ A = -\dfrac{1}{32}., \end{cases}
$$

Remark. — *Performing the reduction $q_1 = 0, p_1 = 0$ in the three HH3 Hamiltonians (2) yields $H = p^2/2 + (1/2)\omega q^2 + (1/2)\gamma q^{-2}$, for which q^2 obeys a linearizable Briot-Bouquet ODE. Similarly, the reduction $Q_1 = 1, P_1 = 0$ in the four HH4 Hamiltonians (6) yields $H = P^2/2 + (1/2)\omega Q^2 + AQ^4 + (1/2)\beta Q^{-2}$, for which Q^2 obeys the Weierstrass elliptic equation.*

These seven Hénon-Heiles Hamiltonians can be studied from various points of view such as: separation of variables ([**37**]), Painlevé property, algebraic complete integrability ([**3**]). For the interrelations between these various approaches, the reader can refer to the plain introduction in Ref. [**1**]. In the present work, we only deal with proving the Painlevé property (PP).

In order to prove or disprove the PP, it is sufficient to obtain an (explicit) canonical transformation to new canonical variables (the so-called *separating variables*) which separate the *Hamilton-Jacobi equation* for the action $S(q_1, q_2)$ ([**5**, chap. 10]), which for two degrees of freedom is

$$
H(q_1, q_2, p_1, p_2) - E = 0, \ p_1 = \frac{\partial S}{\partial q_1}, \ p_2 = \frac{\partial S}{\partial q_2}. \tag{8}
$$

Indeed, if such separating variables are obtained, depending on the genus g of the hyperellitic curve $r^2 = P(s)$ involved in the associated Jacobi inversion problem,

$$\frac{\mathrm{d}s_1}{\sqrt{P(s_1)}} + \frac{\mathrm{d}s_2}{\sqrt{P(s_2)}} = 0, \qquad \frac{s_1\mathrm{d}s_1}{\sqrt{P(s_2)}} + \frac{s_2\mathrm{d}s_2}{\sqrt{P(s_2)}} = \mathrm{d}t, \tag{9}$$

the elementary symmetric functions s_1+s_2 and s_1s_2 are either meromorphic functions of time ($g \le 2$), or multivalued ($g > 3$).

3. The four cases with known separating variables

Two of the seven cases (KdV5, 1:2:1) have a second invariant K equal to a second degree polynomial in the momenta, therefore there exists a classical method ([**38, 39**]) to obtain the canonical transformation $(q_1, q_2, p_1, p_2) \to (s_1, s_2, r_1, r_2)$ with the separating variables (s_1, s_2) obeying the canonical system (9). For the KdV5 case, one obtains ([**4, 18, 45**])

$$\begin{cases} q_1 = -(s_1+s_2+\omega_1-4\omega_2)/(4\alpha), \quad q_2^2 = -s_1s_2/(4\alpha^2), \\ p_1 = -4\alpha\dfrac{s_1r_1-s_2r_2}{s_1-s_2}, \qquad p_2^2 = -16\alpha^2\dfrac{s_1s_2(r_1-r_2)^2}{(s_1-s_2)^2}, \\ H = \dfrac{f(s_1,r_1)-f(s_2,r_2)}{s_1-s_2}, \\ f(s,r) = -\dfrac{s^2(s+\omega_1-4\omega_2)^2(s-4\omega_2)-64\alpha^4\gamma}{32\alpha^2 s} + 8\alpha^2r^2s, \\ f(s_j,r_j) - Es_j + \dfrac{K}{2} = 0,\ j=1,2, \\ P(s) = s^2(s+\omega_1-4\omega_2)^2(s-4\omega_2) + 32\alpha^2Es^2 - 16\alpha^2Ks - 64\alpha^4\gamma. \end{cases} \tag{10}$$

For 1:2:1, one obtains

$$\begin{cases} q_j^2 = (-1)^j\dfrac{(s_1+\omega_j)(s_2+\omega_j)}{\omega_1-\omega_2}, \quad j=1,2, \\ p_j = 2q_j\dfrac{\omega_{3-j}(r_2-r_1)-s_1r_1+s_2r_2}{s_1-s_2}, \quad j=1,2, \\ H = \dfrac{f(s_1,r_1)-f(s_2,r_2)}{s_1-s_2}, \\ f(s,r) = 2(s+\omega_1)(s+\omega_2)r^2 - \dfrac{s^3}{2} - \dfrac{\omega_1+\omega_2}{2}s^2 \\ \qquad -\dfrac{\omega_1\omega_2}{2}s + \dfrac{\omega_2-\omega_1}{2}\Big(\dfrac{\alpha}{s+\omega_1} - \dfrac{\beta}{s+\omega_2}\Big), \\ f(s_j,r_j) = -\Big(s_j + E\dfrac{\omega_1+\omega_2}{2}\Big) - \dfrac{\alpha+\beta}{2} - \dfrac{K}{2}, \quad j=1,2, \\ P(s) = s(s+\omega_1)^2(s+\omega_2)^2 - \alpha(s+\omega_2)^2 - \beta(s+\omega_1)^2 \\ \qquad -(s+\omega_1)(s+\omega_2)\left[E(2s+\omega_1+\omega_2)-K\right]. \end{cases} \tag{11}$$

The two cubic cases SK and KK,

$$H_{\text{SK}} = \frac{1}{2}(P_1^2 + P_2^2) + \frac{\Omega_1}{2}(Q_1^2 + Q_2^2) + \frac{1}{2}Q_1 Q_2^2 + \frac{1}{6}Q_1^3 + \frac{\lambda^2}{8}Q_2^{-2}, \tag{12}$$

$$H_{\text{KK}} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega_2}{2}(16q_1^2 + q_2^2) + \frac{1}{4}q_1 q_2^2 + \frac{4}{3}q_1^3 + \frac{\lambda^2}{2}q_2^{-2}, \tag{13}$$

are equivalent under a birational canonical transformation ([**8, 36**]). Therefore, the separating variables (s_1, s_2) are common to these two cases.

In the nongeneric case $\lambda = 0$, the separating variables have been built ([**33**]) by a method ([**2, 40**]) based on the local representation of the general solution $q_1(t), q_2(t)$ by a Laurent series of $t - t_0$ near a movable singularity t_0. The algebraic curves defined by the values of the two invariants H, K in terms of the arbitrary coefficients of the Laurent series are then geometrically interpreted, with, in principle, the separating variables as the final output. However, some technical difficulty prevents this method to handle the generic case $\lambda \neq 0$.

The generic case can nevertheless be separated ([**42**]) and the result is

$$\left\{\begin{array}{l}
q_1 = -6\left(\dfrac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2}\right)^2 - \dfrac{\tilde{Q}_1 + \tilde{Q}_2}{2}, \qquad q_2^2 = 24\dfrac{f(\tilde{Q}_1, \tilde{P}_1) - f(\tilde{Q}_2, \tilde{P}_2)}{\tilde{Q}_1 - \tilde{Q}_2}, \\[2ex]
p_1 = -4\tilde{Q}_1\dfrac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2} - 2\dfrac{\tilde{Q}_1\tilde{P}_2 - \tilde{Q}_2\tilde{P}_1}{\tilde{Q}_1 - \tilde{Q}_2}, \quad p_2 = \tilde{Q}_2\dfrac{\tilde{P}_1 - \tilde{P}_2}{\tilde{Q}_1 - \tilde{Q}_2}, \\[2ex]
H = f(\tilde{Q}_1, \tilde{P}_1) + f(\tilde{Q}_2, \tilde{P}_2) + \dfrac{\lambda^2}{24}\dfrac{\tilde{Q}_1 - \tilde{Q}_2}{f(\tilde{Q}_1, \tilde{P}_1) - f(\tilde{Q}_2, \tilde{P}_2)}, \\[2ex]
f(q, p) = p^2 + \dfrac{1}{12}q^3 - 4\omega_2^2 q, \\[2ex]
\left(f(\tilde{Q}_j, \tilde{P}_j) - \dfrac{E}{2}\right)^2 + \dfrac{\lambda^2}{24}\tilde{Q}_j + K = 0, \quad j = 1, 2, \\[2ex]
\tilde{Q}_1 = s_1^2 - \dfrac{3K}{\lambda^2}, \ \tilde{Q}_2 = s_2^2 - \dfrac{3K}{\lambda^2}, \ \tilde{P}_1 = \dfrac{r_1}{2s_1}, \ \tilde{P}_2 = \dfrac{r_2}{2s_2}, \\[2ex]
P(s) = -\dfrac{1}{3}\left(s^2 - 3\dfrac{K}{\lambda^2}\right)^3 + \Omega_1^2\left(s^2 - 3\dfrac{K}{\lambda^2}\right) + \dfrac{\lambda}{\sqrt{3}}s + 2E.
\end{array}\right. \tag{14}$$

It is remarkable that the canonical transformation

$$(q_1, q_2, p_1, p_2) \longrightarrow \left(\frac{\tilde{Q}_1 + \tilde{Q}_2}{2} + \Omega_1, \frac{\tilde{Q}_1 - \tilde{Q}_2}{2}, \tilde{P}_1 + \tilde{P}_2, \tilde{P}_1 - \tilde{P}_2\right) \tag{15}$$

coincides with the canonical transformation between the SK variables and the KK variables in the particular case $\lambda = 0$.

In the three remaining cases, the quartic 1:6:1, 1:6:8, 1:12:16, the separating variables are only known in nongeneric cases ([**41, 43**]), and the associated *particular* solutions are single valued. In order to decide about the Painlevé property, which only involves the *general* solution, one must therefore integrate by different means.

4. Confluences from the quartic cases to the cubic ones

A possible way to integrate would be to take advantage of some confluence from an integrated case to a not yet integrated case. For instance, the property of single valuedness of the general solution of the second Painlevé equation P2 implies, from the classical confluence from P2 to P1, the same property for P1.

The confluence from the quartic 1:6:8 case to the cubic KK case found in Ref. [**35**] is not an isolated feature ([**41**]), and in fact all the cubic cases can be obtained by a confluence of at least one quartic case. Just like between the six Painlevé equations, one of the parameters in the Hamiltonian is lost in the process. Consider, for instance, the quartic 1:12:16 and the cubic KK cases,

$$(16)\quad \left\{\begin{array}{rcl} h_{1:12:16}(t) &=& \frac{1}{2}(p_1^2+p_2^2)+\frac{\omega}{8}(4q_1^2+q_2^2) \\ && \quad -\frac{n}{32}(16q_1^4+12q_1^2q_2^2+q_2^4)+\frac{1}{2}\Big(\frac{\alpha}{q_1^2}+\frac{\beta}{q_2^2}\Big), \\ H_{\rm KK}(T) &=& \frac{1}{2}(P_1^2+P_2^2)+\frac{\Omega}{2}(16Q_1^2+Q_2^2)+N\Big(Q_1Q_2^2+\frac{16}{3}Q_1^3\Big)+\frac{B}{2Q_2^2}, \end{array}\right.$$

The confluence in this case is

$$1{:}12{:}16 \to {\rm KK}\left\{\begin{array}{l} t=\varepsilon T,\ q_1=\varepsilon^{-1}+Q_1,\ q_2=Q_2,\ n=-\frac{4}{3}\varepsilon^{-1}N, \\ \alpha=\varepsilon^{-7}\Big(-\frac{4}{3}N+4\Omega\varepsilon\Big),\ \beta=\varepsilon^{-2}B, \\ \omega=\varepsilon^{-3}(-4N+4\Omega\varepsilon),\ h=\varepsilon^{-5}(-2N+4\Omega\varepsilon+H\varepsilon^3),\ \varepsilon\to 0, \end{array}\right.$$

and the two quartic parameters (α,ω) coalesce to the single cubic parameter Ω.

We have checked that all the generic cubic cases can be obtained by confluence from at least one quartic case, as indicated in the following list:

$$(17)\quad \left\{\begin{array}{l} \text{HH4 1:2:1} \to \text{HH3 KdV5}, \\ \text{HH4 1:6:8} \to \text{HH3 KK}, \\ \text{HH4 1:6:8} \to \text{HH3 KdV5}, \\ \text{HH4 1:12:16} \to \text{HH3 KK}, \\ \text{HH4 1:12:16} \to \text{HH3 SK}. \end{array}\right.$$

Since these confluences are not invertible and always go from quartic to cubic, they are unfortunately of no help to integrate the missing cases, which are all quartic. In section 6, we present another class of transformations, these one invertible, between some of the seven cases, which indeed helps to integrate the missing cases.

5. Equivalent fourth order ODEs

The Painlevé school has "classified" (i.e, enumerated the integrable equations and integrated them) several types of ODEs (e.g., second order first degree, third order first degree of the polynomial type, etc), but no four-dimensional first order differential system such as the Hamilton equations

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = p_j,\ \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial V}{\partial q_j},\ j=1,2 \tag{18}$$

has ever been classified. However, some types of fourth order ODEs have been classified, in particular the polynomial class ([**9, 11, 16, 17**])

$$u'''' = P(u''', u'', u', u, x), \tag{19}$$

in which P is polynomial in u''', u'', u', u and analytic in x. Therefore, if one succeeds, by elimination of either q_1 or q_2 (or another combination) between the two Hamilton equations and the equation $H = E$ expressing the conservation of the energy, to build a fourth order ODE in the class (19), and if this ODE is *equivalent* to the original system, then the question is settled.

In the cubic case, the two Hamilton equations

$$q_1'' + \omega_1 q_1 - \beta q_1^2 + \alpha q_2^2 = 0, \tag{20}$$

$$q_2'' + \omega_2 q_2 + 2\alpha q_1 q_2 - \gamma q_2^{-3} = 0, \tag{21}$$

together with $H - E = 0$, see (2), are indeed equivalent ([**21**]) to the single fourth order first degree ODE for $q_1(t)$,

$$\begin{aligned} &q_1'''' + (8\alpha - 2\beta) q_1 q_1'' - 2(\alpha+\beta) q_1'^2 - \frac{20}{3}\alpha\beta q_1^3 \\ &\quad + (\omega_1 + 4\omega_2) q_1'' + (6\alpha\omega_1 - 4\beta\omega_2) q_1^2 + 4\omega_1\omega_2 q_1 + 4\alpha E = 0. \end{aligned} \tag{22}$$

The equivalence results from the conservation of the number of parameters between the system (20)–(21) and the single equation (22), since the coefficient γ of the non-polynomial term q_2^{-2} has been replaced by the constant value E of the Hamiltonian H. The results of the classification ([**17**]) enumerate as expected only three Painlevé-integrable such equations and they provide their general solution (for the first time in the SK and KK cases).

In the quartic case, the similar fourth order equation is built by eliminating Q_2 and $Q_1'''^2$ between the two Hamilton equations,

$$Q_1'' + \Omega_1 Q_1 + 4CQ_1^3 + 2BQ_1Q_2^2 - \alpha Q_1^{-3} + \gamma = 0, \tag{23}$$

$$Q_2'' + \Omega_2 Q_2 + 4AQ_2^3 + 2BQ_2Q_1^2 - \beta Q_2^{-3} = 0, \tag{24}$$

and the Hamiltonian (6), which results in

$$
\begin{aligned}
(25)\qquad & -Q_1'''' + 2\frac{Q_1' Q_1'''}{Q_1} + \left(1 + 6\frac{A}{B}\right)\frac{Q_1''^2}{Q_1} - 2\frac{{Q_1'}^2 Q_1''}{Q_1^2} \\
& + 8\left(6\frac{AC}{B} - B - C\right) Q_1^2 Q_1'' + 4(B - 2C) Q_1 {Q_1'}^2 + 24C\left(4\frac{AC}{B} - B\right) Q_1^5 \\
& + \left[12\frac{A}{B}\omega_1 - 4\omega_2 + \left(1 + 12\frac{A}{B}\right)\frac{\gamma}{Q_1} - 4\left(1 + 3\frac{A}{B}\right)\frac{\alpha}{Q_1^4}\right] Q_1'' \\
& + 6\frac{A}{B}\frac{\alpha^2}{Q_1^7} + 20\frac{\alpha}{Q_1^5}{Q_1'}^2 - 12\frac{A}{B}\frac{\gamma\alpha}{Q_1^4} + 4\left(3\frac{A}{B}\omega_1 - \omega_2\right)\left(\gamma - \frac{\alpha}{Q_1^3}\right) - 2\gamma\frac{{Q_1'}^2}{Q_1^2} \\
& + 6\left(\frac{A}{B}\gamma^2 + 2B\alpha - 8\frac{AC}{B}\alpha\right)\frac{1}{Q_1} + \left(6\frac{A}{B}\omega_1^2 - 4\omega_1\omega_2 - 8BE\right) Q_1 \\
& + 48\frac{AC}{B}\gamma Q_1^2 + 4\left(12\frac{AC}{B} - B - 4C\right)\omega_1 Q_1^3 = 0.
\end{aligned}
$$

The equivalence with the Hamilton equations results from the dependence on E but not on β. However, this type of fourth order first degree ODEs has not yet been classified, and this would be quite useful to do so, in order to check that no Painlevé-integrable case has been omitted when performing the Painlevé test on the coupled system made of the two Hamilton equations.

6. Birational transformations between the quartic cases and integrated equations

Between Hamiltonians with one degree of freedom such as $H = p^2/2 + aq^2 + bq^3 + q^4$ and $H = p^2/2 + Aq^2 + q^3$, there exist invertible transformations which allow one to carry out the solution from one case to the other. These are the well known homographies between the Jacobi and the Weierstrass elliptic functions. In the present case of two degrees of freedom, the simplest example of such a transformation is ([**15**, Eq. (7.14)])

$$
(26)\qquad \begin{cases}
Q_1^2 + Q_2^2 + \dfrac{\Omega_1 + \Omega_2}{5} = \alpha q_1 + \dfrac{\omega_1 + 4\omega_2}{20}, \\
(\Omega_1 - \Omega_2)(Q_1^2 - Q_2^2) = \dfrac{\alpha^2}{2} q_2^2 - \dfrac{4\omega_1 + 26\omega_2}{5}\alpha q_1 - \dfrac{(\omega_1 + 4\omega_2)^2}{100} + 2E, \\
\Omega_1 = \omega_1, \qquad \Omega_2 = 4\omega_2,
\end{cases}
$$

between the quartic 1:2:1 case $H(Q_j, P_j, \Omega_1, \Omega_2, A, B)$ and the cubic KdV5 case $H(q_j, p_j, \omega_1, \omega_2, \alpha, \gamma)$. Its action on the hyperelliptic curves is just a translation.

An attempt to find transformations between the other quartic cases and any cubic case which would be as simple as (26) has been unsuccessful for the moment.

However, it is possible to obtain a birational transformation ([**15**]) between every remaining quartic case (1:6:1, 1:6:8, 1:12:16) and some classified fourth order ODE of the type (19). Indeed, for each of the seven cases, the two Hamilton equations are equivalent ([**6, 21, 22**]) to the traveling wave reduction of a soliton system made either of a single PDE (HH3) or of two coupled PDEs (HH4), most of them appearing in lists established from group theory ([**19**]). Among the various soliton equations which are equivalent to them *via* a Bäcklund transformation, some of them admit a traveling wave reduction to a classified ODE. This property defines a path ([**31, 44**]) which starts from one of the three remaining HH4 cases, goes up to a soliton system of two coupled 1+1-dimensional PDEs admitting a reduction to the considered case, then goes *via* a Bäcklund transformation to another equivalent 1+1-dim PDE system, finally goes down by reduction to an already integrated ODE or system of ODEs.

6.1. Integration of the 1:6:1 and 1:6:8 cases with the a-F-VI equation

In this section, the integration is performed *via* a birational transformation to the autonomous F-VI equation (a-F-VI) in the classification of Cosgrove ([**17**]):

$$\text{a-F-VI}:\ y'''' = 18yy'' + 9y'^2 - 24y^3 + \alpha_{\mathrm{VI}}y^2 + \frac{\alpha_{\mathrm{VI}}^2}{9}y + \kappa_{\mathrm{VI}}t + \beta_{\mathrm{VI}},\ \kappa_{\mathrm{VI}} = 0. \tag{27}$$

The two considered Hamiltonians, with their second constant of the motion, are the following,

$$1:6:1 \left\{ \begin{aligned} H &= \frac{1}{2}(P_1^2 + P_2^2) + \frac{\Omega}{2}(Q_1^2 + Q_2^2) \\ &\qquad - \frac{1}{32}(Q_1^4 + 6Q_1^2Q_2^2 + Q_2^4) - \frac{1}{2}\Big(\frac{\kappa_1^2}{Q_1^2} + \frac{\kappa_2^2}{Q_2^2}\Big) = E, \\ K &= \bigg(P_1P_2 + Q_1Q_2\Big(-\frac{Q_1^2 + Q_2^2}{8} + \Omega\Big)\bigg)^2 \\ &\qquad - P_2^2\frac{\kappa_1^2}{Q_1^2} - P_1^2\frac{\kappa_2^2}{Q_2^2} + \frac{1}{4}\left(\kappa_1^2Q_2^2 + \kappa_2^2Q_1^2\right) + \frac{\kappa_1^2\kappa_2^2}{Q_1^2Q_2^2}, \end{aligned} \right. \tag{28}$$

and

$$1:6:8 \left\{ \begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega}{2}(4q_1^2 + q_2^2) \\ &\qquad - \frac{1}{16}(8q_1^4 + 6q_1^2q_2^2 + q_2^4) - \gamma q_1 + \frac{\beta}{2q_2^2} = E, \\ K &= \bigg(p_2^2 - \frac{q_2^2}{16}(2q_2^2 + 4q_1^2 + \omega) + \frac{\beta}{q_2^2}\bigg)^2 \\ &\qquad - \frac{1}{4}q_2^2(q_2p_1 - 2q_1p_2)^2 + \gamma\Big(-2\gamma q_2^2 - 4q_2p_1p_2 \\ &\qquad\quad + \frac{1}{2}q_1q_2^4 + q_1^3q_2^2 + 4q_1p_2^2 - 4\omega q_1q_2^2 + 4q_1\frac{\beta}{q_2^2}\Big). \end{aligned} \right. \tag{29}$$

There exists a canonical transformation ([**6**]) between these two cases, mapping the constants as follows:

$$E_{1:6:8} = E_{1:6:1},\quad K_{1:6:8} = K_{1:6:1},\quad \omega = \Omega,\quad \gamma = \frac{\kappa_1 + \kappa_2}{2},\quad \beta = -(\kappa_1 - \kappa_2)^2. \tag{30}$$

Therefore, one only needs to integrate either case.

The path to an integrated ODE comprises the following three segments.

The coordinate $q_1(t)$ of the 1:6:8 case can be identified ([**7, 6**]) to the component F of the traveling wave reduction $f(x,\tau)=F(x-c\tau), g(x,\tau)=G(x-c\tau)$ of a soliton system of two coupled KdV-like equations (c-KdV system) denoted c-KdV$_1$ ([**7, 6**])

$$(31)\quad \begin{cases} f_\tau + \left(f_{xx} + \frac{3}{2} f f_x - \frac{1}{2} f^3 + 3fg\right)_x = 0, \\ -2g_\tau + g_{xxx} + 6gg_x + 3fg_{xx} + 6gf_{xx} + 9f_x g_x - 3f^2 g_x \\ \qquad + \frac{3}{2} f_{xxxx} + \frac{3}{2} f f_{xxx} + 9 f_x f_{xx} - 3f^2 f_{xx} - 3 f f_x^2 = 0, \end{cases}$$

with the identification

$$(32)\quad \begin{cases} q_1 = F,\ q_2^2 = -2\left(F' + F^2 + 2G - 2\omega\right), \\ c = -\omega,\ K_1 = \gamma,\ K_2 = E, \end{cases}$$

in which K_1 and K_2 are two constants of integration.

There exists a Bäcklund transformation between this soliton system and another one of c-KdV type, denoted bi-SH system ([**19**]):

$$(33)\quad \begin{cases} -2u_\tau + \left(u_{xx} + u^2 + 6v\right)_x = 0, \\ v_\tau + v_{xxx} + uv_x = 0. \end{cases}$$

This Bäcklund transformation is defined by the Miura transformation ([**31**])

$$(34)\quad \begin{cases} u = \frac{3}{2}\left(2g - f_x - f^2\right), \\ v = \frac{3}{4}\left(2f_{xxx} + 4ff_{xx} + 8gf_x + 4fg_x + 3f_x^2 - 2f^2 f_x - f^4 + 4gf^2\right). \end{cases}$$

Finally, the traveling wave reduction

$$\begin{cases} u(x,\tau) = U(x-c\tau), \\ v(x,\tau) = V(x-c\tau) \end{cases}$$

can be identified ([**44**]) to the autonomous F-VI equation (a-F-VI) (27), whose general solution is meromorphic, expressed with genus two hyperelliptic functions ([**17**, Eq. (7.26)]). The identification is

$$(35)\quad \begin{cases} U = -6\left(y + \frac{c}{18}\right), \\ V = y'' - 6y^2 + \frac{4}{3}cy + \frac{16}{27}c^2 - \frac{K_A}{2}, \\ \alpha_{\mathrm{VI}} = -4c,\ \beta_{\mathrm{VI}} = K_B - 2cK_A + \frac{512}{243}c^3, \end{cases}$$

in which K_A, K_B are two constants of integration.

In order to perform the integration of both the 1:6:1 and the 1:6:8 cases, it is sufficient to express (F, G) rationally in terms of (U, V, U', V'). The result is

$$(36)\quad \begin{cases} F &= \dfrac{W'}{2W} + \dfrac{K_1}{24W}\left[-3U'^2 - 2(U-3c)\left(12V + (U+3c)^2\right) + 36K_B - 54K_1^2\right], \\ G &= \dfrac{U}{3} + \dfrac{1}{8W}\left[(2V+3K_2)\left(2V'' + K_1U' - 3K_1^2\right)\right. \\ &\qquad \left. - 2(U-3c)\left(2K_1V' + K_1^2(U+3c)\right)\right], \\ W &= \left(V + \dfrac{3}{2}K_2\right)^2 + \dfrac{3}{2}K_1^2(U-3c), \\ K_A &= K_2. \end{cases}$$

Making the product of the successive transformations (32), (36), (35), one obtains a meromorphic general solution for q_1, q_2^2:

$$(37)\quad \begin{cases} q_1 &= \dfrac{W'}{2W} + \dfrac{\gamma}{W}\left[9j - 3\left(y + \dfrac{4}{9}\omega\right)(h+E) - \dfrac{9}{4}\gamma^2\right], \\ q_2^2 &= -16\left(y - \dfrac{5}{9}\omega\right) + \dfrac{1}{W}\left[12\left(y' + \dfrac{\gamma}{2}\right)^2 - 48y^3 - 16\omega y^2\right. \\ &\qquad + \left(24E + \dfrac{128}{9}\omega^2\right)y + \dfrac{1280}{243}\omega^3 - \dfrac{40}{3}\omega E + \dfrac{3}{4}\beta \\ &\qquad \left. - 24\gamma\left(y - \dfrac{5}{9}\omega\right)h' - 144\gamma^2\left(y - \dfrac{5}{9}\omega\right)^2\right], \\ W &= (h+E)^2 - 9\gamma^2\left(y - \dfrac{5}{9}\omega\right), \\ \alpha_{\mathrm{VI}} &= 4\omega, \qquad \beta_{\mathrm{VI}} = \dfrac{3}{4}\gamma^2 + 2\omega E - \dfrac{3}{16}\beta - \dfrac{512}{243}\omega^3, \\ K_{1,\mathrm{VI}} &= \dfrac{3}{32}K - \dfrac{1}{2}E^2, \qquad K_{2,\mathrm{VI}} = \dfrac{3}{32}EK - \dfrac{1}{3}E^3 + \dfrac{9}{64}\beta\gamma^2, \\ K_1 &= \gamma, \quad K_2 = E, \quad K_A = E, \quad K_B = -\dfrac{3}{16}\beta + \dfrac{3}{4}\gamma^2, \end{cases}$$

in which h and j are the convenient auxiliary variables ([**17**, Eqs. (7.4)–(7.5)])

$$(38)\quad \begin{cases} y &= \dfrac{Q(s_1,s_2) + \sqrt{Q(s_1)Q(s_2)}}{2\left(\sqrt{s_1^2 - C_{\mathrm{VI}}} + \sqrt{s_2^2 - C_{\mathrm{VI}}}\right)^2} + \dfrac{5}{36}\alpha_{\mathrm{VI}}, \\ h &= -\dfrac{3}{4}E_{\mathrm{VI}}\dfrac{s_1s_2 + C_{\mathrm{VI}} + \sqrt{(s_1^2 - C_{\mathrm{VI}})(s_2^2 - C_{\mathrm{VI}})}}{s_1+s_2} - \dfrac{F_{\mathrm{VI}}}{2}, \\ j &= \dfrac{1}{6}(2h + F_{\mathrm{VI}})\left\{y + \dfrac{\alpha_{\mathrm{VI}}}{9} - \dfrac{E_{\mathrm{VI}}}{4(s_1+s_2)}\right\}. \end{cases}$$

In the above, the variables s_1, s_2 are defined by the hyperelliptic system ([**17**])

$$(39) \quad \begin{cases} (s_1 - s_2)s_1' = \sqrt{P(s_1)}, \qquad (s_2 - s_1)s_2' = \sqrt{P(s_2)}, \\ P(s) = (s^2 - C_{\mathrm{VI}})Q(s), \\ Q(s,t) = (s^2 - C_{\mathrm{VI}})(t^2 - C_{\mathrm{VI}}) - \dfrac{\alpha_{\mathrm{VI}}}{2}(s^2 + t^2 - 2C_{\mathrm{VI}}) + \dfrac{E_{\mathrm{VI}}}{2}(s+t) + F_{\mathrm{VI}}, \\ Q(s) = Q(s,s). \end{cases}$$

Despite their square roots, the symmetric expressions in (38) are nevertheless meromorphic ([**20, 30**]).

The completeness of both the 1:6:1 and 1:6:8 Hamiltonians results from the completeness of the a-F-VI ODE and the following counting. The 1:6:8 depends on the parameters $(\omega, \beta, \gamma, E, K)$, the a-F-VI ODE and its hyperelliptic system depend on the same number of parameters $(\alpha, \beta, C, E, F)_{\mathrm{VI}}$, and these two sets of five parameters are linked by exactly five algebraic relations ([**17**, Eqs. (7.9)-(7.12)]):

$$(40) \quad \begin{cases} \alpha_{\mathrm{VI}} = 4\omega, \\ \beta_{\mathrm{VI}} = \dfrac{3}{4}\gamma^2 + 2\omega E - \dfrac{3}{16}\beta - \dfrac{512}{243}\omega^3, \\ E_{\mathrm{VI}}^2 = -\dfrac{16}{3}\omega(F_{\mathrm{VI}} - 2E) - \beta + 4\gamma^2, \\ C_{\mathrm{VI}}E_{\mathrm{VI}}^2 = \dfrac{4}{3}(F_{\mathrm{VI}}^2 - 4E^2) + K, \\ (F_{\mathrm{VI}} - 2E)^2(F_{\mathrm{VI}} + 4E) + \frac{9K}{4}(F_{\mathrm{VI}} - 2E) - \frac{27}{4}\beta\gamma^2 = 0. \end{cases}$$

The algebraic nature (instead of rational like in the 1:2:1 case and the three cubic cases) of these dependence relations could explain the difficulty to separate the variables in the Hamilton-Jacobi equation. In the nongeneric case $\beta\gamma = 0$, i.e, $\kappa_1^2 = \kappa_2^2$, for which the separating variables are known ([**34**]), the coefficients $(\alpha, \beta, C, E^2, F)_{\mathrm{VI}}$ become rational functions of $(\omega, \beta, \gamma, E, K)$, see [**43**]. Since these separating variables have been obtained by the same method as in the cubic SK-KK case, it would be quite useful to remove the difficulty which remains in the method based on Laurent series, see Section 3.

6.2. Integration of the 1:12:16 case by a birational transformation. — This is the only case for which the integration, which can indeed be performed with the same results (meromorphy of the general solution, completeness of the Hamiltonian) is not satisfying. Indeed, the hyperelliptic system to which the 1:12:16 has been mapped by a birational transformation ([**15**]) is essentially different from the hyperelliptic system resulting from the separating variables ([**41**]) in the nongeneric case $\alpha\beta = 0$ for which they are known. Since the nongeneric subcase $\alpha = 0$ belongs to the

Stäckel class (two invariants quadratic in p_1, p_2), for which the separating variables are unambiguous, this indicates that some progress has still to be made.

The main remarkable feature of the 1:12:16 is the existence of a twin system to which it is mapped by a canonical transformation ([**6, 7**]) which only differs by numerical coefficients from the canonical transformation between the cubic SK and KK cases. The two systems are the following ones:

$$
(41)\quad 1:12:16 \left\{ \begin{array}{rcl} H &=& \dfrac{1}{2}(P_1^2+P_2^2)+\dfrac{\Omega}{8}(4Q_1^2+Q_2^2) \\ & & \quad -\dfrac{1}{32}(16Q_1^4+12Q_1^2Q_2^2+Q_2^4)-\dfrac{1}{2}\Big(\dfrac{\kappa_1^2}{Q_1^2}+\dfrac{4\kappa_2^2}{Q_2^2}\Big)=E, \\ K &=& \dfrac{1}{16}\Big(8(Q_2P_1-Q_1P_2)P_2-Q_1Q_2^4-2Q_1^3Q_2^2 \\ & & \quad +2\Omega Q_1Q_2^2+32Q_1\dfrac{\kappa_2^2}{Q_2^2}\Big)^2+\kappa_1^2\left(Q_2^4-4\dfrac{Q_2^2P_2^2}{Q_1^2}\right). \end{array} \right.
$$

and [this system is not the sum of a kinetic energy and a potential energy]

$$
(42)\quad 5:9:4 \left\{ \begin{array}{l} H=\dfrac{1}{2}\Big(p_1^2+\Big(p_2-\dfrac{3}{2}q_1q_2\Big)^2\Big)-\dfrac{1}{8}(4q_1^4+9q_1^2q_2^2+5q_2^4) \\ \qquad\qquad +\dfrac{\omega}{2}(q_1^2+q_2^2)-\kappa q_1+\dfrac{\zeta}{2q_2^2}=E, \\ K=\dfrac{1}{q_2^2}(2q_2^2p_1+2q_1^2q_2^2-2q_1q_2p_2-q_2^4-4\kappa q_1)^2\Big(2q_2^2p_1+2q_1^2q_2^2 \\ \quad +p_2^2-4q_1q_2p_2-2q_2^4+\Omega q_2^2+4\dfrac{\kappa^2}{q_2^2}+8\kappa q_1-4\kappa\dfrac{p_2}{q_2} \\ \quad +4(\zeta+4\kappa^2)\Big)\Big(\Big(-2q_1\dfrac{p_2}{q_2}+4q_1^2+q_2^2+4q_1\dfrac{\kappa}{q_2^2}\Big)p_1 \\ \quad -\dfrac{1}{q_2^4}(q_1^2q_2^2+q_2^4+2\kappa q_1)^2+2\dfrac{q_1^2}{q_2^2}(p_2-\dfrac{3}{2}q_1q_2)^2 \\ \quad +\dfrac{(q_1^2+q_2^2)^2}{2}+q_1^2\dfrac{\zeta}{q_2^4}\Big). \end{array} \right.
$$

The canonical transformation maps the constants as follows:

$$
(43)\quad E_{5:9:4}=E_{1:12:16},\quad K_{5:9:4}=K_{1:12:16},\quad \omega=\Omega,\quad \kappa=\frac{\kappa_1+\kappa_2}{2},\quad \zeta=-(\kappa_1-\kappa_2)^2.
$$

The path to an integrated ODE is quite similar to that described in detail in section 6.1, in particular it is also made of three segments ([**6, 31, 41**]). The result is the following ([**15**]):

$$
(44)\qquad Q_1, Q_2^2=\mathrm{rational}(y,y',y'',y'''),
$$

in which y obeys the F-IV equation in the classification of Cosgrove ([**17**]),

$$\text{(45)}\qquad \text{F-IV}\quad \begin{cases} y'''' = 30yy'' - 60y^3 + \alpha_{\text{IV}} y + \beta_{\text{IV}}, \\ y = \dfrac{1}{2}\left(s_1' + s_2' + s_1^2 + s_1 s_2 + s_2^2 + A\right), \\ (s_1 - s_2)s_1' = \sqrt{P(s_1)}, \qquad (s_2 - s_1)s_2' = \sqrt{P(s_2)}, \\ P(s) = (s^2 + A)^3 - \dfrac{\alpha_{\text{IV}}}{3}(s^2 + A) + Bs + \dfrac{\beta_{\text{IV}}}{3}, \\ K_{1,\text{IV}} = \left(\dfrac{3B}{4}\right)^2, \qquad K_{2,\text{IV}} = -\dfrac{9AB^2}{64} \end{cases}$$

with $(K_{1,\text{IV}}, K_{2,\text{IV}})$ two polynomial first integrals of F-IV. The general solution of this ODE is meromorphic, expressed with genus two hyperelliptic functions ([**17**]). This proves the PP for the 1:12:16.

In the two nongeneric cases $\kappa_1\kappa_2 = 0$ where the separating variables are known, the hyperelliptic curve is

$$\text{(46)}\qquad \kappa_1\kappa_2 = 0 : \; P(s) = s^6 - \omega s^3 + 2Es^2 + \frac{K}{20}s + \kappa_1^2 + \kappa_2^2 = 0,$$

and it does not coincide in this case with the hyperelliptic curve of F-IV. Therefore, F-IV (as well as its birationally equivalent ODE F-III) is not the good ODE to consider, and it should be quite instructive to directly integrate the fourth order equivalent ODE (26) in that case.

7. Conclusion and open problems

All the time independent two-degree-of-freedom Hamiltonians which possess the Painlevé property have a meromorphic general solution, expressed with hyperelliptic functions of genus two. Moreover, all such Hamiltonians are complete in the Painlevé sense, i.e, it is impossible to add any term to the Hamiltonian without ruining the Painlevé property.

As to the remaining open problems, depending on the center of interest, they are

1. from the point of view of Hamiltonian theory, one has to find the separating variables in the three missing quartic cases. This should be possible by the methods of Sklyanin and van Moerbeke and Vanhaecke;
2. from the point of view of the integration of differential equations, the problem remains to enumerate all the fourth order first degree differential equations in a given precise class which possess the Painlevé property.

Let us finally mention that the time dependent extension of these seven cases has been studied in Refs. [**27, 28**].

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