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Characteristic Algebras of Fully Discrete Hyperbolic Type Equations

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Abstract. The notion of the characteristic Lie algebra of the discrete hyperbolic type equation is introduced. An effective algorithm to compute the algebra for the equation given is suggested. Examples and further applications are discussed.

Key words: discrete equations; invariant; Lie algebra; exact solution; Liuoville type equation

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1 Introduction

It is well known that the characteristic Lie algebra introduced by A.B. Shabat in 1980, plays the crucial role in studying the hyperbolic type partial differential equations. For example, if the characteristic algebra of the equation is of finite dimension, then the equation is solved in quadratures, if the algebra is of finite growth then the equation is integrated by the inverse scattering method. More details and references can be found in [1]. Recently it has been observed by A.V. Zhiber that the characteristic algebra provides an effective tool for classifying the nonlinear hyperbolic equations. Years ago the characteristic algebra has been used to classify integrable systems of a special type [2]. However, the characteristic algebras have not yet been used to study the discrete equations, despite the fact that the discrete equations had become very popular the last decade (see, for instance, survey [3]).

In this paper we show that the characteristic algebra can be defined for any discrete equation of the hyperbolic type and it inherits most of the important properties of its continuous counterpart. However, it has essentially more complicated structure. The work was stimulated by [4], where the discrete field theory is studied and the question is posed which types of algebraic structures are associated with the finite field discrete 3D Toda chains.

2 Invariants and vector fields

Consider a discrete nonlinear equation of the form

$$t(u+1,v+1) = f\big(t(u,v), t(u+1,v), t(u,v+1)\big), \tag{1}$$

where $t = t(u,v)$ is an unknown function depending on the integers u, v, and f is a smooth function of all three arguments. The following notations are used to shorten formulae: $t_u = t(u+1,v)$, $t_v = t(u,v+1)$, and $t_{uv} = t(u+1,v+1)$. By using these notations one can rewrite the equation (1) as follows $t_{uv} = f(t, t_u, t_v)$. Actually, the equation (1) is a discrete analog of the partial differential equations. Particularly, the class of equations (1) contains difference schemes for the hyperbolic type PDEs on a quadrilateral grid.

The notations above are commonly accepted, but not very convenient to indicate the iterated shifts. Below we use also different ones. Introduce the shift operators D and $\bar{D}$, which act as follows $Df(u,v) = f(u+1,v)$ and $\bar{D}f(u,v) = f(u,v+1)$. For the iterated shifts we introduce the notations $f_j = D^j(f)$ and $\bar{f}_j = \bar{D}^j(f)$, so that $t(u+1,v) = t_1$, $t(u,v+1) = \bar{t}_1$, $t(u+2,v) = t_2$, $t(u,v+2) = \bar{t}_2$ and so on.

The equation (1) is supposed to be hyperbolic. It means that it can be rewritten in any of the forms: $t_u = g(t,t_v,t_{uv})$, $t_v = r(t,t_{uv},t_u)$, and $t = s(t_u,t_{uv},t_v)$ with some smooth functions g, r, and s.

A function $F = F(v,t,t_1,\bar{t}_1,\dots)$, depending on v and a finite number of the dynamical variables is called v-invariant, if it is a stationary "point" of the shift with respect to v so that (see also [5])

$$\bar{D}F(v,t,t_1,\bar{t}_1,\dots) = F(v,t,t_1,\bar{t}_1,\dots), \tag{2}$$

and really the function F solves the equation $F(v+1,t_v,f,f_1,\bar{t}_2,\dots) = F(v,t,t_1,\bar{t}_1,\dots)$. Examining carefully the last equation one can find that:

Lemma 1. *The v-invariant does not depend on the variables in the set $\{\bar{t}_j\}_{j=1}^{\infty}$.*

If any v-invariant is found, then each solution of the equation (1) can be represented as a solution of the following ordinary discrete equation $F(t,t_u,\dots,t_j) = c(u)$, where $c(u)$ is an arbitrary function of u.

Due to the Lemma 1 the equation (2) can be rewritten as

$$F(v+1,t_v,f,f_1,\dots) = F(v,t,t_1,t_2,\dots).$$

The left hand side of the equation contains t_v, while the right hand side does not. Hence the total derivative of $\bar{D}F$ with respect to t_v vanishes. In other words, the operator $X_1 = \bar{D}^{-1}\frac{d}{dt_v}\bar{D}$ annihilates the v-invariant F: $X_1F = 0$. In a similar way one can check that any operator of the form $X_j = \bar{D}^{-j}\frac{d}{dt_v}\bar{D}^j$, where $j \geq 1$, satisfies the equation $X_jF = 0$. Really, the right hand side of the equation $\bar{D}^jF(v,t,t_1,\bar{t}_1,\dots) = F(v,t,t_1,\bar{t}_1,\dots)$ (which immediately follows from (2)) does not depend on t_v and it implies the equation $X_jF = 0$. As a result, one gets an infinite set of equations for the function F. For each j the operator X_j is a vector field of the form

$$X = \sum_{j=0}^{\infty} x(j)\frac{\partial}{\partial t_j}. \tag{3}$$

Consider now the Lie algebra L_v of the vector fields generated by the operators X_j with the usual commutator of the vector fields $[X_i,X_j] = X_iX_j - X_jX_i$. We refer to this algebra as characteristic algebra of the equation (1).

Remark 1. Note that above we started to consider the discrete equation of the form (1) conjecturing that it admits nontrivial v-invariant. The definition of the algebra was motivated by the invariant. However the characteristic Lie algebra is still correctly defined for any equation of the form (1) even if it does not admit any invariant.

3 Algebraic criterion of existence of the invariants

Theorem 1. *The equation (1) admits a nontrivial v-invariant if and only if the algebra L_v is of finite dimension.*

Proof. Suppose that the equation (2) admits a non-constant solution. Then the following system of equations

$$X_j F(v, t, t_1, t_2, \dots, t_N) = 0, \qquad j \geq 1 \tag{4}$$

has a non-constant solution. It is possible only if the linear envelope of the vector-fields $\{X_j\}_{j=1}^{\infty}$ is of finite dimension.

It is worth mentioning an appropriate property of the vector fields above. If for a fixed j the operator X_j is linearly expressed through the operators $X_1, X_2, \dots, X_{j-1}$, then any operator X_k is a linear combination of these operators. Really, we are given the expression $X_j = a_1 X_1 + \cdots + a_{j-1} X_{j-1}$. Note that $X_{j+1} = \bar{D}^{-1} X_j \bar{D}$, hence $X_{j+1} = D^{-1}(a_1) X_1 + \cdots + D^{-1}(a_{j-1}) X_j$. Thus, in this case the characteristic algebra is generated by the first $j-1$ operators which are linearly independent. Due to the classical Jacoby theorem the system (4) has a non-constant solution only if dimension of the Lie algebra generated by the vector fields X_i is no greater than N. Thus, one part of the theorem is proved.

Suppose now that the dimension of the characteristic algebra is finite and equals, say, N, show that in this case the equation (1) admits a v-invariant. Evidently, there exists a function $G(t, t_1, \dots, t_N)$, which is not a constant and that $XG = 0$ for any X in L_v. Such a function is not unique, but any other solution is expressed as $h(G)$. Due to the construction the map $X \to \bar{D}^{-1} X \bar{D}$ leaves the algebra unchanged, hence $G_1 = \bar{D}G$ is also a solution of the same system $XG = 0$ and therefore $G_1 = h(G)$. In other words, one gets a discrete first order equation: $\bar{D}G = h(G)$, write its general solution in the following form: $C = F(v, G)$ where C does not depend on v. Evidently the function F found is just the v-invariant needed. ■

4 Computation of the characteristic algebra

In this section the explicit forms of the operators $\{X_j\}$ are given. We show that the operators are vector fields and give a convenient way to compute the coefficients of the expansion (3).

Start with the operator X_1. Directly by definition one gets $X_1 F(t, t_1, \dots) = \bar{D}^{-1} \frac{\partial}{\partial t_v} F(t_v, f, f_1, \dots)$. Computing the derivative by the chain rule one obtains

$$X_1 F(t, t_1, \dots) = \bar{D}^{-1} \left(\frac{\partial}{\partial t_v} + \frac{\partial f}{\partial t_v} \frac{\partial}{\partial f} + \frac{\partial f_1}{\partial t_v} \frac{\partial}{\partial f_1} + \cdots \right) F(t_v, f, f_1, \dots),$$

and finally

$$X_1 = \frac{\partial}{\partial t} + \bar{D}^{-1} \left(\frac{\partial f}{\partial t_v} \right) \frac{\partial}{\partial t_u} + \bar{D}^{-1} \left(\frac{\partial f_1}{\partial t_v} \right) \frac{\partial}{\partial t_2} + \cdots + \bar{D}^{-1} \left(\frac{\partial f_j}{\partial t_v} \right) \frac{\partial}{\partial t_{j+1}} + \cdots . \tag{5}$$

So the operator X_1 can be represented as $X_1 = \sum\limits_{i=0}^{\infty} x_i \frac{\partial}{\partial t_i}$ where the coefficients x_j are found as $x_j = \bar{D}^{-1} \left(\frac{\partial f_{j-1}}{\partial t_v} \right)$ for $j > 0$ and $x_0 = 1$. Actually the coefficients can be computed by the following more convenient formula

$$x_{j+1} = x_1 D(x_1) D^2(x_1) \cdots D^j(x_1), \tag{6}$$

or $x_{j+1} = x_j D^j(x_1)$. Really,

$$\begin{aligned} \bar{D} x_{j+1} &= \frac{\partial f_j}{\partial t_v} = \frac{\partial f_j}{\partial f_{j-1}} \frac{\partial f_{j-1}}{\partial f_{j-2}} \cdots \frac{\partial f}{\partial t_v} = D^j \left(\frac{\partial f}{\partial t_v} \right) \cdots D \left(\frac{\partial f}{\partial t_v} \right) \frac{\partial f}{\partial t_v} \\ &= \bar{D} \big(D^j(x_1) \cdots D(x_1) x_1 \big). \end{aligned}$$

To find X_2, use the following formula

$$X_2F = \bar{D}^{-1}X_1\bar{D}F(t,t_1,\dots) = \bar{D}^{-1}\left(\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial t_u} + x_2\frac{\partial}{\partial t_2} + \cdots\right)F(t_v, f, f_1, \dots).$$

After opening the parentheses and some transformation the right hand side of the last formula gets the form

$$X_2F(t,t_1,\dots) = \bar{D}^{-1}\left(X_1(f)\frac{\partial}{\partial f} + X_1(f_1)\frac{\partial}{\partial f_1} + X_1(f_2)\frac{\partial}{\partial f_2} + \cdots\right)F(t_v, f, f_1, \dots)$$

So the operator can be written as

$$X_2 = \bar{D}^{-1}\left(X_1(f)\right)\frac{\partial}{\partial t_1} + \bar{D}^{-1}\left(X_1(f_1)\right)\frac{\partial}{\partial t_2} + \bar{D}^{-1}\left(X_1(f_2)\right)\frac{\partial}{\partial t_3} + \cdots. \tag{7}$$

Continuing this way, one gets

$$X_j = \bar{D}^{-1}\left(X_{j-1}(f)\right)\frac{\partial}{\partial t_1} + \bar{D}^{-1}\left(X_{j-1}(f_1)\right)\frac{\partial}{\partial t_2} + \bar{D}^{-1}\left(X_{j-1}(f_2)\right)\frac{\partial}{\partial t_3} + \cdots. \tag{8}$$

One can derive an alternative way to compute the coefficients of the vector fields X_k above. Represent the operators as follows

$$X_k = \sum_{j=0}^{\infty} n_{kj}\frac{\partial}{\partial t_j}. \tag{9}$$

We will show that the coefficients n_{kj} of the operators satisfy the following linear equation

$$\bar{D}n_{k+1,j+1} = D^j\left(\frac{\partial f}{\partial t}\right)n_{k,j} + D^j\left(\frac{\partial f}{\partial t_1}\right)n_{k,j+1} + D^j\left(\frac{\partial f}{\partial t_v}\right)\bar{D}n_{k+1,j}, \tag{10}$$

closely connected with the direct linearization of the initial nonlinear equation (1). In order to derive this formula, apply the operator X_k to the iterated shift $f_j = f(t_j, t_{j+1}, f_{j-1})$ and use the chain rule

$$X_k(f_j) = D^j\left(\frac{\partial f}{\partial t}\right)\bar{D}^{-1}(X_{k-1}(f_{j-1})) + D^j\left(\frac{\partial f}{\partial t_1}\right)\bar{D}^{-1}X_{k-1}(f_j) + D^j\left(\frac{\partial f}{\partial t_v}\right)X_k(f_{j-1}). \tag{11}$$

Comparison of the two representations (8) and (9) of the operator X_k yields $X_k(f_j) = \bar{D}n_{k+1,j+1}$. By replacing in (11) X with n one gets the formula (10) required.

The characteristic algebra is invariant under the map $X \to D^{-1}XD$. First prove the formula $D^{-1}X_1D = D^{-1}(x_1)X_1$. To this end use the coordinate representation of the operator $X_1 = \sum_{i=0}^{\infty} x_i\frac{\partial}{\partial t_i}$ and the formula (6) for the coefficients. Then check that $D^{-1}X_2D = \rho X_2$. Really,

$$D^{-1}X_2D = D^{-1}\bar{D}^{-1}X_1\bar{D}D = \bar{D}^{-1}D^{-1}X_1D\bar{D} = \bar{D}^{-1}\big(D^{-1}(x_1)\big)X_2.$$

Obviously similar representations can be derived for any generator of the characteristic algebra.

The following statement turns out to be very useful for studying the characteristic algebra.

Lemma 2. *Suppose that the vector field*

$$X = \sum_{j=0}^{\infty} x_j\frac{\partial}{\partial t_j}$$

satisfies the equation

$$D(X) = X, \tag{12}$$

then $X \equiv 0$.

Proof. It follows from (12) that

$$\sum_{j=0}^{\infty} D\big(x_j(\bar{t}_1, \bar{t}_{-1}, t, t_1, t_2, \ldots, t_{k_j})\big)\frac{\partial}{\partial t_{j+1}} = \sum_{j=0}^{\infty} x_j(\bar{t}_1, \bar{t}_{-1}, t, t_1, t_2, \ldots, t_{k_j})\frac{\partial}{\partial t_j}.$$

Comparison of the coefficients before the operators $\frac{\partial}{\partial t_j}$ in both sides of this equation yields: $x_0 = 0$, $x_1 = 0$, $\ldots$ and so on. ∎

5 The commutativity property of the algebra

One of the unexpected properties of the characteristic Lie algebra is the commutativity of the operators X_j. Consider first an auxiliary statement.

Lemma 3. *The coefficients x_{ki} of the vector fields X_k, $k \geq 1$, $i \geq 0$ do not depend on the variable t_v.*

In other words, the coefficients of the expansions

$$X_k = \sum_{j=0}^{\infty} x_{kj}\frac{\partial}{\partial t_j}$$

satisfy the equation $\frac{d}{dt_v}x_{ki} = 0$.

Proof. For the operator X_1 one has $x_{1,j+1} = \bar{D}^{-1}\left(\frac{\partial f_j}{\partial t_v}\right)$. But the function f_j does not depend on $\bar{t}_2 = \bar{D}t_v$ and on $\bar{t}_3, \bar{t}_4, \ldots$ as well. Hence the coefficients do not depend on $\bar{t}_1 = t_v$. Similarly, the functions $X_1(f_j)$ do not depend on $\bar{t}_2$, so $x_{2,j+1} = \bar{D}^{-1}(X_1(f_j))$ do not contain t_v. One can complete the proof by using induction with respect to k. ∎

Now return to the main statement of the section.

Theorem 2. *For any positive integers i, j the equation holds $[X_i, X_j] = 0$.*

Proof. Remind that $X_j = \bar{D}^{-j}\frac{d}{dt_v}\bar{D}^j$, so that one can deduce

$$[X_i, X_j] = \left[\bar{D}^{-i}\frac{d}{dt_v}\bar{D}^i, \bar{D}^{-j}\frac{d}{dt_v}\bar{D}^j\right].$$

Suppose for the definiteness that $k = j - i \geq 1$. Then

$$[X_i, X_j] = \bar{D}^{-i}\left[\frac{d}{dt_v}, \bar{D}^{i-j}\frac{d}{dt_v}\bar{D}^{j-i}\right]\bar{D}^i = \bar{D}^{-i}\left[\frac{d}{dt_v}, X_k\right]\bar{D}^i = \bar{D}^{-i}\sum_{j=0}^{\infty}\frac{d}{dt_v}(x_{kj})\frac{\partial}{\partial t_j}\bar{D}^i.$$

But due to the Lemma 3 the last expression vanishes that proves the theorem. ∎

6 Characteristic algebra for the discrete Liouville equation

The well known Liouville equation $\frac{\partial^2 v}{\partial x \partial y} = e^v$ admits a discrete analog of the form [6]

$$t_{uv} = \frac{1}{t}(t_u - 1)(t_v - 1), \tag{13}$$

which can evidently be rewritten as

$$t_{u,-v} = \frac{1}{t-1} t_u t_{-v} + 1.$$

Specify the coefficients of the expansions (5)–(7) representing the vector fields X_1 and X_2 for the discrete Liouville equation (13). Find the coefficient $x_1 = \bar{D}^{-1}\left(\frac{\partial f}{\partial t_v}\right)$, remind that $f(t, t_u, t_v) = \frac{1}{t}(t_u - 1)(t_v - 1)$ and $g(t_{-v}, t, t_u) = \frac{1}{t-1} t_u t_{-v} + 1$,

$$x_1 = \bar{D}^{-1}\left(\frac{\partial f}{\partial t_v}\right) = \bar{D}^{-1}\left(\frac{t_u - 1}{t}\right) = \frac{g-1}{t_{-v}} = \frac{t_u}{t-1}.$$

Similarly

$$x_2 = \bar{D}^{-1}\left(\frac{\partial f_1}{\partial f}\frac{\partial f}{\partial t_v}\right) = \frac{t_2}{t_1 - 1}\frac{t_1}{t-1}.$$

It can easily be proved by induction that $x_j = \prod_{k=0}^{j-1} \frac{t_{k+1}}{t_k - 1}$ for $j \geq 1$, here $t_0 := t$. Remind also that $x_0 = 1$. So that the vector field is

$$X_1 = \frac{\partial}{\partial t} + \frac{t_1}{t-1}\frac{\partial}{\partial t_1} + \frac{t_1 t_2}{(t-1)(t_1-1)}\frac{\partial}{\partial t_2} + \frac{t_1 t_2 t_3}{(t-1)(t_1-1)(t_2-1)}\frac{\partial}{\partial t_3} + \cdots,$$

It is a more difficult problem to find the operator X_2, it can be proved that

$$X_2 = \frac{(t-1)t}{(t_{-v}-1)t_{-v}}(X_1 - X_{-1}),$$

where the operator X_{-1} is defined as follows $X_{-1} = \bar{D}\frac{d}{dt_{-1}}\bar{D}^{-1}$. For the coefficients $x_{-,j}$ of the expansion $X_{-1} = \sum_{j=0}^{\infty} x_{-,j}\frac{\partial}{\partial t_j}$ one can deduce the formula $x_{-,j} = \prod_{k=0}^{j-1}\frac{t_{k+1}-1}{t_k}$, so that the operator is represented as

$$X_{-1} = \frac{\partial}{\partial t} + \frac{t_1 - 1}{t}\frac{\partial}{\partial t_1} + \frac{(t_1-1)(t_2-1)}{t t_1}\frac{\partial}{\partial t_2} + \frac{(t_1-1)(t_2-1)(t_3-1)}{t t_1 t_2}\frac{\partial}{\partial t_3} + \cdots,$$
$$X_3 = \bar{D}^{-1} X_2 \bar{D} = \frac{(\bar{t}_{-1}-1)\bar{t}_{-1}}{(\bar{t}_{-2}-1)\bar{t}_{-2}} X_2.$$

Theorem 3. *Dimension of the characteristic Lie algebra of the Liouville equation* (13) *equals two.*

Proof. It is more easy to deal with the operators $Y_+ = (t-1)X_1$ and $Y_- = tX_{-1}$ rather than the operators X_1 and X_{-1}. In order to prove the theorem it is enough to check the formula

$$[Y_+, Y_-] - Y_+ + Y_- = 0. \tag{14}$$

Denote through X the left hand side of the equation (14) and compute $D(X)$ to apply the Lemma 2. It is shown straightforwardly that

$$D(Y_+) = \frac{t_1 - 1}{t_1}\left(Y_+ - (t-1)\frac{\partial}{\partial t}\right) \qquad \text{and} \qquad D(Y_-) = \frac{t_1}{t_1 - 1}\left(Y_- - t\frac{\partial}{\partial t}\right).$$

Compute now the shifted operator $D([Y_+,Y_-]) = [D(Y_+),D(Y_-)]$ which can be represented as follows

$$[D(Y_+),D(Y_-)] = \left[\frac{t_1-1}{t_1}\left(Y_+ - (t-1)\frac{\partial}{\partial t}\right), \frac{t_1}{t_1-1}\left(Y_- - t\frac{\partial}{\partial t}\right)\right] = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \left[\frac{t_1-1}{t_1}Y_+, \frac{t_1}{t_1-1}Y_-\right], \qquad A_2 = -\left[\frac{(t_1-1)(t-1)}{t_1}\frac{\partial}{\partial t}, \frac{t_1}{t_1-1}Y_-\right],$$
$$A_3 = -\left[\frac{t_1-1}{t_1}Y_+, \frac{tt_1}{t_1-1}\frac{\partial}{\partial t}\right], \qquad A_4 = \left[\frac{(t_1-1)(t-1)}{t_1}\frac{\partial}{\partial t}, \frac{tt_1}{t_1-1}\frac{\partial}{\partial t}\right].$$

Direct computations give

$$A_1 = [Y_+,Y_-] - \frac{1}{t_1-1}Y_- \frac{1}{t_1Y_+}, \qquad A_2 = \frac{1-t-t_1}{t_1}\frac{\partial}{\partial t},$$
$$A_3 = \frac{1-t-t_1}{t_1-1}\frac{\partial}{\partial t}, \qquad A_4 = \frac{\partial}{\partial t}.$$

Summarizing all computations above one gets the result

$$D[Y_+,Y_-] = [Y_+,Y_-] - \frac{1}{t_1-1}Y_- - \frac{1}{t_1}Y_+ + \left(1 + \frac{t-1}{t_1} + \frac{t}{t_1-1}\right)\frac{\partial}{\partial t},$$

which implies $D([Y_+,Y_-]-Y_++Y_-) = [Y_+,Y_-]-Y_++Y_-$. Now apply Lemma 2 to the function $X = [Y_+,Y_-] - Y_+ + Y_-$ to get $X \equiv 0$. ■

7 How to find the invariants?

In this section two examples of the equations with the invariants are shown. Start with a simple one.

Example 1. Consider a linear equation of the form

$$t_{uv} = t_u + t_v - t + 1$$

so that $f(t,t_u,t_v) = t_u + t_v - t + 1$ and $g(t_{-v},t,t_u) = t_u - t + t_{-v} - 1$. It is easy to see that the algebra is of one dimension

$$X_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial t_u} + \frac{\partial}{\partial t_2} + \cdots, \qquad X_2 \equiv 0.$$

The first integral of the least order for the equation $X_1F = 0$ can be taken as $G_0 = t_1 - t$. Evidently it solves the equation $\bar{D}G_0 = G_0+1$. Now, the invariant is to be taken as $I = t_1-t-v$, because $\bar{D}I = t_{uv} - t_v - v - 1 = t_u - t - v = I$. Any other v-invariant of the equation can be represented as $F = F(I, DI, D^2I, \ldots, D^kI)$.

Example 2. Return to the discrete Liouville equation discussed above. Find the intersection of the kernels of the operators X_1 and X_{-1}, hence they constitute the basis of the characteristic algebra of the Liouville equation (13). To this end first solve the equation $X_1F = 0$ which is reduced to the following infinite system of the ordinary differential equations:

$$\frac{dt}{1} = \frac{dt_1}{t_1/(t-1)} = \frac{dt_2}{t_1t_2/(t-1)(t_1-1)} = \cdots.$$

So that the invariants of the vector field X_1 are $I_0 = \frac{t_1}{t-1}$, $I_1 = D(I_0)$, $I_2 = D(I_1)$, $\dots$. Change the variables in the vector fields by setting $\tilde{t} = t$, $\tilde{t}_1 = I_0$, $\tilde{t}_2 = I_1$, $\dots$. Then one gets

$$X_1 = \frac{\partial}{\partial t} \qquad \text{and} \qquad X_{-1} = \frac{\partial}{\partial t} - \frac{\tilde{t}_1 + 1}{t(t-1)}\frac{\partial}{\partial \tilde{t}_1} - \frac{\tilde{t}_2 + 1}{\tilde{t}_1 t(t-1)}\frac{\partial}{\partial \tilde{t}_2} - \cdots.$$

Now solve the equation $\frac{d\tilde{t}_2}{d\tilde{t}_1} = \frac{\tilde{t}_2+1}{\tilde{t}_1(\tilde{t}_1+1)}$ to find the common solution $F_0 = \left(\frac{t_2}{t_1-1} + 1\right)\left(\frac{t-1}{t_1} + 1\right)$ of both equations $X_1F = 0$ and $X_{-1}F = 0$. The invariant is

$$I(t, t_1, t_2) = F_0(t, t_1, t_2) = \left(\frac{t_2}{t_1 - 1} + 1\right)\left(\frac{t-1}{t_1} + 1\right)$$

(see also [5]). Evidently, the basis of common invariants can be chosen as follows: $F_0, D(F_0)$, $D^2(F_0), \dots$. It can be easily verified by the direct computation that F_0 solves the equation $\bar{D}F_0 = F_0$.

Corollary 1. *In the case of the Liouville equation each solution of the system of the equations $X_1F = 0$, $X_{-1}F = 0$ gives the v-invariant.*

Proof. Each solution of this system can be represented as $F = F(F_0, D(F_0), D^2(F), \dots)$, hence

$$\begin{aligned}\bar{D}F\big(F_0, D(F_0), D^2(F), \dots\big) &= F\big(\bar{D}(F_0), \bar{D}(D(F_0)), \bar{D}(D^2(F)), \dots\big) \\ &= F\big(F_0, D(F_0), D^2(F), \dots\big).\end{aligned}$$ ∎

8 Discrete potentiated Korteweg–de Vries equation

Discrete equation of the form

$$(t_{uv} - t + p + q)(t_u - t_v + q - p) = q^2 - p^2 \tag{15}$$

is called the discrete potentiated KdV equation (DPKdV). The constant parameters p, q are removed by a combination of the shift and scaling transformations of the form $t(u,v) = h\tilde{t}(u,v) - pv - qu$, where $h^2 = q^2 - p^2$ if $q^2 - p^2 > 0$ and $h^2 = -q^2 + p^2$ if $q^2 - p^2 < 0$. After this transformation the equation (15) takes one of the forms

$$(t_{uv} - t)(t_u - t_v) = \pm 1.$$

The algebraic properties of the equation do not depend on the choice of the sign before the unity, we will take the upper sign. Study the problem how to describe the characteristic Lie algebra for the equation

$$(t_{uv} - t)(t_u - t_v) = 1. \tag{16}$$

Represent the equation (16) as $t_{uv} = f(t, t_u, t_v)$, where $f(t, t_u, t_v) = t + \frac{1}{t_u - t_v}$. Set $x = \frac{\partial f}{\partial t_v} = \frac{1}{(t_v - t_u)^2}$, then $x = (f - t)^2$. Below we use the upper index to denote the shift with respect to v and lower index to denote the shift with respect to u so that $\bar{D}^{-k}D^j x = x_j^k$. Then due to the general rule the operator X_1 is written as

$$X_1 = \frac{\partial}{\partial t} + x^1\frac{\partial}{\partial t_1} + x^1x_1^1\frac{\partial}{\partial t_2} + \cdots + x^1x_1^1\cdots x_j^1\frac{\partial}{\partial t_{j+1}} + \cdots.$$

Theorem 4. *Characteristic Lie algebra of the DPKdV equation (16) is of infinite dimension.*

Proof. In the case of the DPKdV equation one gets $\frac{\partial f}{\partial t} = 1$, $\frac{\partial f}{\partial t_1} = -x$, $\frac{\partial f}{\partial t_v} = x$. The formula (10) is specified

$$\bar{D}n_{k+1,j+1} = n_{k,j} - x_j n_{k,j+1} + x_j \bar{D}n_{k+1,j}.$$

It is more convenient to write it as

$$n_{k+1,j+1} = \bar{D}^{-1}n_{k,j} - x_j^1 \bar{D}^{-1}n_{k,j+1} + x_j^1 n_{k+1,j},$$

remind that $D^{-1}x_j = x_j^1$. It is clear now that the coefficients are polynomials of the finite number of the dynamical variables in the set $\{x_j^i\}_{i,j=0}^{\infty}$. Order the variables in this set according to the following rule: $\mathrm{ord}(x_j^n) > \mathrm{ord}(x_p^n)$ if $j > p$ and $\mathrm{ord}(x_j^m) > \mathrm{ord}(x_p^n)$ if $m > n$.

Lemma 4. *For any positive k, j the coefficient $n_{k,j}$ for the operator X_k in the DPKdV case is represented as*

$$n_{k,j} = m_{k,j} x_{j-1}^k + r_{k,j}$$

where $m_{k,j}$ and $r_{k,j}$ are polynomials of the variables with the orders less than the order of x_{j-1}^k, moreover, $m_{k,j}$ is a monomial.

Lemma can be easily proved by using the induction method. It allows to complete the proof of the theorem. The vector fields $\{X_k\}_{k=1}^{\infty}$ are all linearly independent. ■

Due to the theorem the DPKdV equation do not admit any v-invariant. It is not surprising, because the equation can be integrated by the inverse scattering method or, in other words, it is S-integrable, it is well known in the case of partial differential equations that only C-integrable equations (Liouville type) admit such kind objects called x- and y-integrals.

9 Conclusion

The notion of the characteristic algebra for discrete equations is introduced. It is proved that the equation is Darboux integrable if and only if its characteristic algebras in both directions are of finite dimension. The notion can evidently be generalized to the systems of discrete hyperbolic equations. It would be useful to compute the algebras for the periodically closed discrete Toda equation, or for the finite field discrete Toda equations found in [4] and [8], corresponding in the continuum limit to the simple Lie algebras of the classical series A and C.

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