Universal Low Temperature Asymptotics of the Correlation Functions of the Heisenberg Chain*

Nicolas CRAMPÉ †, *Frank GÖHMANN* ‡ *and Andreas KLÜMPER* ‡

† *LPTA, UMR 5207 CNRS-UM2, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France*
E-mail: *ncrampe@um2.fr*

‡ *Fachbereich C – Physik, Bergische Universität Wuppertal, 42097 Wuppertal, Germany*
E-mail: *goehmann@physik.uni-wuppertal.de*, *kluemper@uni-wuppertal.de*

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Abstract. We calculate the low temperature asymptotics of a function γ that generates the temperature dependence of all static correlation functions of the isotropic Heisenberg chain.

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1 Introduction

Over the past few years the mathematical structure of the static correlation functions of the XXZ chain was largely resolved. After an appropriate regularization by a disorder parameter they all factorize into polynomials in only two functions ρ and ω [8]. These are the one-point function and a special neighbor two-point function which, in turn, can be represented as integrals over solutions of certain linear and non-linear integral equations [2]. This resembles much the situation with free fermions, and what is behind is indeed a remarkable fermionic structure on the space of quasi-local operators acting on the spin chain [5]. It allows us, for instance, to calculate short-range correlators with high numerical precision directly in the thermodynamic limit [1, 12].

The low temperature asymptotics of ρ and ω universally determines the low temperature properties of all static correlation functions. In this short note we obtain the low temperature asymptotics in the special case of the isotropic Hamiltonian

$$\mathcal{H} = J\sum_j \left(\sigma^x_{j-1}\sigma^x_j + \sigma^y_{j-1}\sigma^y_j + \sigma^z_{j-1}\sigma^z_j\right) \tag{1.1}$$

with no magnetic field applied and vanishing disorder parameter. Then $\rho = 1$ and we are left with only one function (and its derivatives) which, up to a trivial factor, is the function γ defined in [3].

2 Definition of the basic function γ

For our purpose here it is convenient to introduce the function γ within the context of a special realization of a six-vertex model (see e.g. [4]) and its associated quantum transfer matrix [10].

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By definition the latter has $2(\mathcal{N}+\mathcal{M})$ vertical lines alternating in direction and carrying spectral parameters

$$\underbrace{u, -u, u, -u, \dots, -u}_{2\mathcal{N}}, \underbrace{u' + \mu_1, \mu_1 - u', u' + \mu_1, \mu_1 - u', \dots, \mu_1 - u'}_{2\mathcal{M}}.$$

The spectral parameter on the horizontal line will be denoted μ_2. We consider this system in the limit $\mathcal{N}, \mathcal{M} \to +\infty$ with the fine tuning $u\mathcal{N} = i\frac{J}{T}$ and $u'\mathcal{M} = i\frac{\delta}{T}$. With an appropriate overall normalization the largest eigenvalue $\Lambda(\mu_2, \mu_1)$ is given by

$$\begin{aligned} \ln(\Lambda(\mu_2, \mu_1)) = \frac{4\pi J}{T} K(\mu_2) + \frac{4\pi\delta}{T} K(\mu_2 - \mu_1) \\ + \int_{-\infty}^{\infty} dt \frac{\ln\left[(1 + \mathfrak{b}(t, \mu_1))(1 + \overline{\mathfrak{b}}(t, \mu_1))\right]}{2\cosh(\pi(\mu_2 - t))}. \end{aligned} \tag{2.1}$$

Let us note that we recover the familiar system of equations, allowing us to study the thermodynamical properties of the Hamiltonian (1.1), by setting $\delta = 0$. The function $K(x)$ is defined as

$$\begin{aligned} K(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{e^{-ikx}}{1 + e^{|k|}} \\ &= \frac{1}{4\pi} \left(\psi\left(1 - i\frac{x}{2}\right) - \psi\left(\frac{1 + ix}{2}\right) - \psi\left(\frac{1 - ix}{2}\right) + \psi\left(1 + i\frac{x}{2}\right) \right), \end{aligned}$$

where ψ is the digamma function. The auxiliary functions $\mathfrak{b}(x, \mu)$ and $\overline{\mathfrak{b}}(x, \mu)$ are solutions of a pair of non-linear integral equations given by

$$\begin{aligned} \ln(\mathfrak{b}(x, \mu_1)) = -\frac{2\pi J}{T\cosh(\pi x)} - \frac{2\pi\delta}{T\cosh(\pi(x - \mu_1))} + \int_{-\infty}^{\infty} dt K(x - t) \ln(1 + \mathfrak{b}(t, \mu_1)) \\ - \int_{-\infty}^{\infty} dt K(x - t + i) \ln(1 + \overline{\mathfrak{b}}(t, \mu_1)) \end{aligned} \tag{2.2}$$

and a similar equation obtained by exchanging $\mathfrak{b} \leftrightarrow \overline{\mathfrak{b}}$ and $i \leftrightarrow -i$ in (2.2). The function γ can now be introduced as

$$\gamma(\mu_1, \mu_2) = -1 + \left(1 + (\mu_1 - \mu_2)^2\right) T \frac{\partial}{\partial\delta} \ln(\Lambda(\mu_2, \mu_1))\Big|_{\delta=0}. \tag{2.3}$$

It has been conjectured [3] that the correlation functions of the isotropic Heisenberg chain at any finite temperature (for vanishing magnetic field) are polynomials in γ and its derivatives evaluated at $(0, 0)$. A similar statement (involving a function ω and its derivative with respect to the disorder parameter) was proved for the anisotropic XXZ chain [5, 8, 2]. Amazingly the isotropic limit seems non-trivial and is still a subject of ongoing work. Here we would only like to mention that the nearest- and next-to-nearest-neighbor two-point functions were expressed in terms of γ in [3] starting from the multiple integral representation for the density matrix of the Heisenberg chain obtained in [7]. The formulae for the longitudinal two-point functions are, for instance,

$$\langle \sigma_1^z \sigma_2^z \rangle_T = -\frac{1}{3}\gamma(0, 0), \tag{2.4}$$

$$\langle \sigma_1^z \sigma_3^z \rangle_T = -\frac{1}{3}\gamma(0, 0) - \frac{1}{6}\gamma_{xx}(0, 0) + \frac{1}{3}\gamma_{xy}(0, 0). \tag{2.5}$$

They will be used below to test our results for the low-temperature expansion. We denoted derivatives with respect to the first (resp. second) argument by the subscript x (resp. y). Similar results for four sites can be obtained from [1] in the isotropic limit. In previous work [13] the high-temperature expansion (up to order 25) of the two-point functions was obtained analytically based on (2.4) and (2.5).

3 Low-temperature expansion

To compute the low-temperature expansion of γ, we follow the line of reasoning of the article [9], where a similar task was performed for the free energy. There are, however, two differences between the usual equations and the ones used in this note: the additional driving term in (2.2) proportional to δ and the shift μ_2 in the kernel of the integration in (2.1).

The computation is based on the introduction of a shift $\mathcal{L} = \frac{1}{\pi}\ln\left(\pi\frac{J}{T}\right)$ in the auxiliary functions:

$$\mathfrak{b}_{\mathcal{L}}(x) = \mathfrak{b}(x+\mathcal{L}) \qquad \text{and} \qquad \widetilde{\mathfrak{b}}_{\mathcal{L}}(x) = \mathfrak{b}(-x-\mathcal{L}).$$

In the low-temperature limit these functions satisfy

$$\begin{aligned}\ln(\mathfrak{b}_{\mathcal{L}}(x,\mu_1)) \sim & -4e^{-\pi x} - 4\frac{\delta}{J}\, e^{-\pi(x-\mu_1)} + \mathcal{D}_{\mathcal{L}}(x) \\ & + \int_{-\mathcal{L}}^{\infty} dt\big[K(x-t)\ln(1+\mathfrak{b}_{\mathcal{L}}(t,\mu_1)) - K(x-t+i)\ln(1+\overline{\mathfrak{b}}_{\mathcal{L}}(t,\mu_1))\big], \end{aligned} \tag{3.1}$$

where $\mathcal{D}_{\mathcal{L}}(x)$ is the rest of the integral which does not contribute to the low-temperature limit, when the magnetic field vanishes (see [9]). A similar relation holds with $\mathfrak{b} \leftrightarrow \overline{\mathfrak{b}}$ and $i \leftrightarrow -i$ exchanged.

In terms of the shifted functions the largest eigenvalue becomes

$$\begin{aligned}\ln(\Lambda(\mu_2,\mu_1)) \sim & \frac{4\pi J}{T}K(\mu_2) + \frac{4\pi\delta}{T}K(\mu_2-\mu_1) \\ & + \frac{T}{J\pi}\int_{-\mathcal{L}}^{\infty} dt e^{\pi(\mu_2-t)}\ln\left[(1+\mathfrak{b}_{\mathcal{L}}(t,\mu_1))(1+\overline{\mathfrak{b}}_{\mathcal{L}}(t,\mu_1))\right] \\ & + \frac{T}{J\pi}\int_{-\mathcal{L}}^{\infty} dt e^{-\pi(\mu_2+t)}\ln\left[(1+\widetilde{\mathfrak{b}}_{\mathcal{L}}(t,\mu_1))(1+\widetilde{\overline{\mathfrak{b}}}_{\mathcal{L}}(t,\mu_1))\right]. \end{aligned}$$

To evaluate these integrals we compute

$$\mathcal{I} = \int_{-\mathcal{L}}^{\infty} dt\big[\ln(1+\mathfrak{b}_{\mathcal{L}}(t,\mu_1))\ln(\mathfrak{b}_{\mathcal{L}}(t,\mu_1))' + \ln(1+\overline{\mathfrak{b}}_{\mathcal{L}}(t,\mu_1))\ln(\overline{\mathfrak{b}}_{\mathcal{L}}(t,\mu_1))'\big]$$

using two different methods. Here the prime stands for the derivative with respect to t. First, we compute it explicitly using the change of variables $z = \ln(\mathfrak{b}_{\mathcal{L}})$ or $z = \ln(\overline{\mathfrak{b}}_{\mathcal{L}})$, respectively, which results in

$$\mathcal{I} = 2\int_{-\infty}^{0} \ln\left(1+e^z\right)dz = \frac{\pi^2}{6}.$$

Second, we replace $\ln(\mathfrak{b}_{\mathcal{L}}(t,\mu_1))$ and $\ln(\overline{\mathfrak{b}}_{\mathcal{L}}(t,\mu_1))$ by their scaling limits (3.1) and simplify the resulting expression by taking into account that the derivative of $K(x)$ is odd and contributions by double integrals cancel pairwise. This way we obtain

$$\mathcal{I} = 4\pi\left(1+\frac{\delta}{J}e^{\pi\mu_1}\right)\int_{-\mathcal{L}}^{\infty} dt e^{-\pi t}\ln\left[(1+\mathfrak{b}(t,\mu_1))(1+\overline{\mathfrak{b}}(t,\mu_1))\right].$$

The same type of manipulation can be performed for the functions $\widetilde{b}$, and a similar result is obtained with μ_1 replaced by $-\mu_1$.

Gathering these findings we obtain the asymptotic form of the largest eigenvalue,

$$\ln(\Lambda(\mu_2,\mu_1)) \sim \frac{4\pi J}{T}K(\mu_2) + \frac{4\pi\delta}{T}K(\mu_2-\mu_1) + \frac{T}{24J}\left(\frac{e^{\pi\mu_2}}{1+\frac{\delta}{J}e^{\pi\mu_1}} + \frac{e^{-\pi\mu_2}}{1+\frac{\delta}{J}e^{-\pi\mu_1}}\right).$$

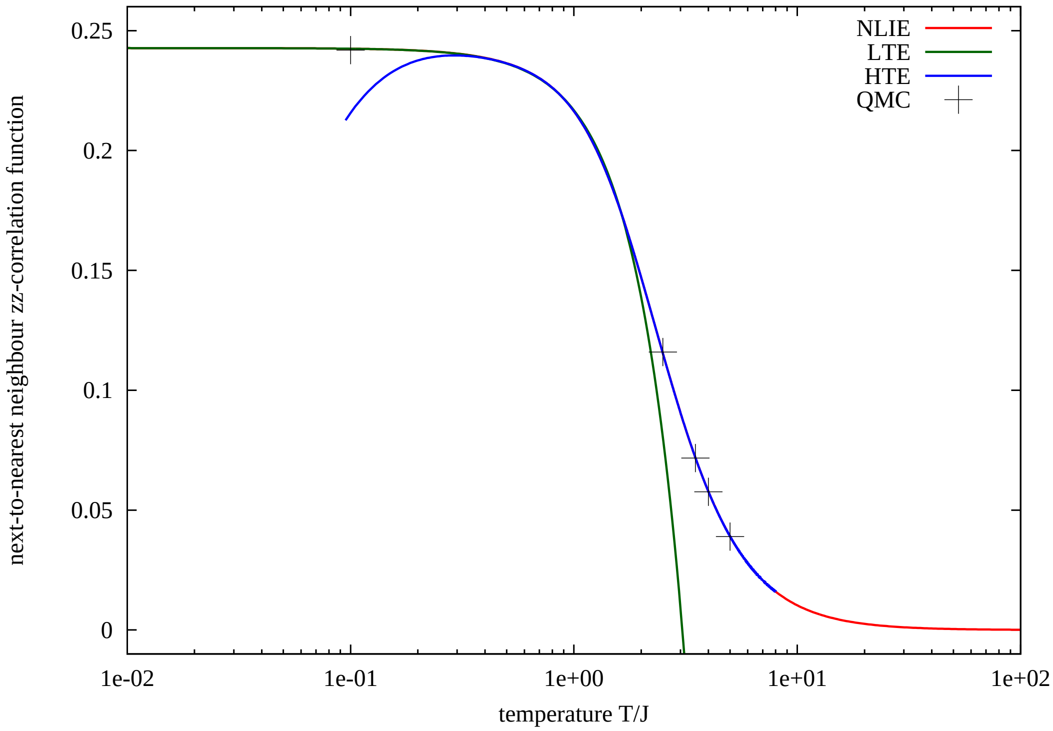

Figure 1. Comparison of the high- and low-temperature expansions (HTE, LTE) of $\langle\sigma_1^z\sigma_3^z\rangle$ with the full numerical solution obtained from the integral equations (NLIE) and with Monte-Carlo data (QMC).

Thus, using (2.3), the function γ behaves asymptotically for small temperatures as

$$\gamma(\mu_1,\mu_2) \sim -1 + \big(1+(\mu_1-\mu_2)^2\big)\left(4\pi K(\mu_2-\mu_1) - \frac{T^2}{12J^2}\cosh(\pi(\mu_1+\mu_2))\right).$$

This is our main result.

Using (2.4) and (2.5), we obtain the low-temperature expansion of the longitudinal correlation functions

$$\langle\sigma_1^z\sigma_2^z\rangle_T \sim \frac{1}{3} - \frac{4}{3}\ln(2) + \frac{T^2}{J^2}\frac{1}{36},$$
$$\langle\sigma_1^z\sigma_3^z\rangle_T \sim \frac{1}{3} - \frac{16}{3}\ln(2) + 3\zeta(3) - \frac{T^2}{J^2}\frac{1}{36}\left(\frac{\pi^2}{2}-4\right).$$

The constant terms (independent of the temperature) in these expansions are in agreement with those originally found in [11, 6]. In the figure we compare the combined low- and high-temperature results for the next-to-nearest neighbor zz-correlation functions with the full numerical curve obtained by implementing the linear and non-linear integral equations that determine γ and its derivatives [3] on a computer. The high-temperature data and some additional Monte-Carlo data are taken from [14]. We find that the numerical curves (NLIE, QMC) are amazingly well approximated by its low- and high-temperature approximations.

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