Supersymmetry Transformations for Delta Potentials⋆

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Abstract. We make a detailed study of the first and second-order SUSY partners of a one-dimensional free Hamiltonian with a singular perturbation proportional to a Dirac delta function. It is shown that the second-order transformations increase the spectral manipulation possibilities offered by the standard first-order supersymmetric quantum mechanics.

Key words: first and second-order SUSY; singular potentials

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1 Introduction

The study of one-dimensional Hamiltonians with a point interaction has received renewed attention during the past two decades [1, 2, 3, 4, 5, 6, 7]. In general, a point interaction is described by a potential concentrated either in a single or a discrete number of points as it happens, e.g., for the Dirac delta or its derivative. Mathematically, in order to define these potentials, we use the theory of extensions of symmetric operators with equal deficiency indices. These extensions have domains which are characterized by some matching conditions for the wave functions at the points supporting the interaction [2, 8, 9, 10, 11]. In particular, the Dirac delta barrier or well have been extensively studied in this way with or without other interactions [12], with or without mass discontinuities at the singular points etc. [13, 14].

On the other hand, supersymmetric quantum mechanics (SUSY QM) has emerged as the standard technique for generating new potentials with known spectra departing from an initial one [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. The method has been applied successfully to regular one-dimensional potentials defined on the full real line [42, 43], on the positive semi-axis [44, 45] or in a finite interval [46]. Although there are some works dealing with SUSY QM applied to point potentials [47, 48, 49, 50, 51, 52, 53], however the corresponding study has been done just for particular first-order SUSY transformations, without analyzing the full possibilities of spectral manipulation offered by the method. It is interesting to note as well that a point potential may appear as hidden supersymmetries [54, 55].

Now, it is the appropriate time for studying the behavior of point potentials with bound states under SUSY QM. Due to the calculation complexity, we shall focus our attention to first and second-order transformations, which anyway are interesting by themselves [56, 57, 58, 59].

We shall restrict the discussion to the following one-dimensional Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x), \qquad V_0(x) := -a\delta(x), \qquad a > 0, \tag{1}$$

which is mathematically well defined and self-adjoint provided that we use as its domain $\mathcal{D}$ the subspace of the Sobolev space $W_2^2(\mathbb{R}/\{0\})$ such that for any $\psi(x) \in \mathcal{D}$, one has:

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}, \tag{2}$$

where $\psi(0+)$, $\psi'(0+)$ and $\psi(0-)$, $\psi'(0-)$ are the right and left limits of $\psi(x)$, $\psi'(x)$ at the origin respectively [2].

In order to achieve our goal, we have organized this paper as follows: in Section 2, we will study the solutions of the stationary Schrödinger equation for the Hamiltonian H_0 given by (1). In Section 3 we will apply the first-order SUSY techniques, in Section 4 we will analyse the second-order transformations and in Section 5 we will present our conclusions.

2   Solution of the Schrödinger equation

Let us evaluate in the first place the general solution of the stationary Schrödinger equation for an arbitrary $\epsilon = -k^2/2 < 0$:

$$H_0 u(x) = \epsilon u(x), \tag{3}$$

with H_0 given in (1). There is one solution vanishing for $x \to -\infty$, denoted $u_+(x)$, of the form

$$u_+(x) = e^{kx}H(-x) + (\alpha\, e^{kx} + \beta e^{-kx})H(x), \qquad k > 0, \tag{4}$$

where $H(x)$ is the Heaviside step function, and α, β are constants to be determined from the discontinuity equations (2). We need as well the derivative of $u_+(x)$,

$$u'_+(x) = ku_+(x) - 2k\beta e^{-kx}H(x) + (\alpha + \beta - 1)\delta(x). \tag{5}$$

From equations (4) and (5) it turns out that

$$u_+(0+) = \alpha + \beta, \qquad u_+(0-) = 1, \qquad u'_+(0+) = k(\alpha - \beta), \qquad u'_+(0-) = k. \tag{6}$$

On the other hand, using equations (2) and (6), we obtain:

$$\alpha + \beta = 1, \qquad \alpha - \beta = 1 - 2\tilde{a},$$

where $\tilde{a} = a/k$. Hence:

$$\alpha = 1 - \tilde{a}, \qquad \beta = \tilde{a}.$$

Inserting these expressions in equations (4) and (5), we finally get

$$u_+(x) = e^{kx} - \tilde{a}(e^{kx} - e^{-kx})H(x), \tag{7}$$
$$u'_+(x) = ku_+(x) - 2ae^{-kx}H(x).$$

Note that the Hamiltonian H_0 in equation (1) is invariant under the change $x \to -x$. Thus, we can find a second linearly independent solution $u_-(x)$ for the same $\epsilon = -k^2/2$, vanishing now for $x \to \infty$, by applying this transformation to $u_+(x)$:

$$u_-(x) = \tilde{a}(e^{kx} - e^{-kx})H(-x) + e^{-kx}. \tag{8}$$

Moreover:

$$u'_-(x) = -ku_-(x) + 2ae^{kx}H(-x).$$

Finally, the general solution of equation (3) for $\epsilon = -k^2/2 < 0$ is a linear combination of both (7) and (8) which, up to an unessential constant factor, becomes:

$$u(x) = u_+(x) + Du_-(x) = e^{kx} + De^{-kx} - \tilde{a}(e^{kx} - e^{-kx})[H(x) - DH(-x)], \tag{9}$$

where D is a constant. The corresponding derivative is given by:

$$u'(x) = -ku(x) + 2ke^{kx}[1 - \tilde{a}H(x) + D\tilde{a}H(-x)]. \tag{10}$$

Note that, up to normalization, both solutions $u_\pm(x)$ lead to the same bound state for $k_0 = a$:

$$\psi_0(x) = \sqrt{a}[e^{k_0x}H(-x) + e^{-k_0x}H(x)].$$

The corresponding eigenvalue becomes

$$E_0 = -\frac{a^2}{2},$$

which coincides with the result derived in [11].

On the other hand, the scattering states for $\epsilon = \kappa^2/2 > 0$ can be simply obtained from the solutions given in equations (7), (8) by the substitution $k \to -i\kappa$, $\kappa > 0$. In particular, for a probability flux approaching the singularity from $-\infty$ the corresponding scattering state arises in this way from the $u_-(x)$ of equation (8), which (up to unessential constant factor) leads to

$$\psi(x) = \left[e^{i\kappa x} + \frac{ia}{\kappa - ia}e^{-i\kappa x}\right]H(-x) + \frac{\kappa}{\kappa - ia}e^{i\kappa x}H(x). \tag{11}$$

It is clear now that the reflection R and transmition T coefficients become the standard ones (see, e.g., [60]):

$$R = \left|\frac{ia}{\kappa - ia}\right|^2 = \frac{a^2}{\kappa^2 + a^2}, \qquad T = \left|\frac{\kappa}{\kappa - ia}\right|^2 = \frac{\kappa^2}{\kappa^2 + a^2}. \tag{12}$$

3 First-order SUSY transformations

Let us start with the initial Schrödinger Hamiltonian H_0 given in (1). As it is well known (see, e.g., [37, 41] and the references cited there), its first-order SUSY partner,

$$H_1 = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x),$$

is intertwined with H_0 in the way

$$H_1A_1^+ = A_1^+H_0, \tag{13}$$

where

$$A_1^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \frac{u'}{u}\right).$$

Here, the transformation function $u(x)$ is the seed solution given in (9), associated to the factorization energy $\epsilon = -k^2/2$ and satisfying equation (3). The SUSY partner potential $V_1(x)$ of $V_0(x)$ is given by:

$$V_1(x) = V_0(x) - [\ln u(x)]''. \tag{14}$$

We assume the standard restriction $\epsilon \leq E_0 \Rightarrow k \geq k_0$, in order to avoid the creation of new singularities in $V_1(x)$ with respect to those of $V_0(x)$. Note that, from equation (14) and (3) we have:

$$V_1(x) = V_0(x) - \frac{u''(x)}{u(x)} + \left[\frac{u'(x)}{u(x)}\right]^2 = -V_0(x) + 2\epsilon + \left[\frac{u'(x)}{u(x)}\right]^2. \tag{15}$$

Hence, a straightforward calculation using equations (9), (10) leads to:

$$\left[\frac{u'(x)}{u(x)}\right]^2 = k^2 - \frac{4k^2(1-\tilde{a})}{u^2(x)}\left[D + \tilde{a}D^2 H(-x) + \tilde{a}H(x)\right]. \tag{16}$$

As $\epsilon = -k^2/2$, equations (15), (16) give:

$$V_1(x) = a\delta(x) - \frac{4k^2(1-\tilde{a})[D + \tilde{a}D^2 H(-x) + \tilde{a}H(x)]}{\{e^{kx} + De^{-kx} + 2\tilde{a}\sinh(kx)[DH(-x) - H(x)]\}^2}. \tag{17}$$

Note that the denominator of equation (17) never vanishes for $x \in (-\infty, \infty)$ and $D \geq 0$. Moreover, it can be seen that the delta term in $V_1(x)$ is now repulsive (since $a > 0$).

A straightforward consequence of the intertwining relationship (13) is that for any eigenfunction ψ of H_0 associated to the eigenvalue E ($H_0\psi = E\psi$) such that $A_1^+\psi \neq 0$, it turns out that $\psi^{(1)} \propto A_1^+\psi \propto W(u,\psi)/u$ is a corresponding eigenfunction of H_1 associated to E. Moreover, if ψ satisfies as well equation (2) it turns out that $\psi^{(1)}$ now obeys:

$$\begin{pmatrix} \psi^{(1)}(0+) \\ \psi^{(1)'}(0+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix} \begin{pmatrix} \psi^{(1)}(0-) \\ \psi^{(1)'}(0-) \end{pmatrix},$$

which is consistent with the fact that the intensity of the delta term in $V_1(x)$ has an opposite sign compared with $V_0(x)$ and the second term of $V_1(x)$ has just a finite discontinuity at $x = 0$ (see equation (17)).

Concerning the spectrum of H_1, let us note in the first place that A_1^+ transforms the scattering eigenfunctions of H_0 into the corresponding ones of H_1. In particular, the wavefunction $\psi(x)$ given in equation (11), when transformed by acting on it with A_1^+, produces an expression $\psi^{(1)}(x)$ which is a bit large to be presented here. However, for large values of $|x|$ that expression reduces to the following scattering one (up to a constant factor):

$$\psi^{(1)}(x) \underset{|x|\to\infty}{\to} \left[e^{i\kappa x} + \left(\frac{a}{a+i\kappa}\right)\left(\frac{-k+i\kappa}{k+i\kappa}\right)e^{-i\kappa x}\right]H(-x)$$
$$+ \left(\frac{i\kappa}{a+i\kappa}\right)\left(\frac{-k+i\kappa}{k+i\kappa}\right)e^{i\kappa x}H(x).$$

This means that the initial reflection and transmission coefficients are unchanged under the first-order SUSY transformation (compare equation (12)). We thus conclude that the continuous spectrum of H_0 belongs as well to the spectrum of H_1.

Let us note that the differences in the spectra of H_1 and H_0 rely in general in the modifications produced by a non-singular SUSY transformation on the discrete part of the initial spectrum. For first-order transformations, these changes can be classified according to the essentially different combinations of the parameters $D \geq 0$ and $k \geq k_0$ which characterize the seed eigenfunction $u(x)$. We can find three different situations.

(i) *Creation of a new ground state at $\epsilon < E_0$.* This case appears for $D > 0$, $k > k_0$. Here, the eigenfunction $\psi_\epsilon^{(1)} \propto 1/u(x)$ of H_1 associated to ϵ is square-integrable. Moreover, since the mapped initial ground state $\psi_0^{(1)} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{E_0 - \epsilon}} \frac{W(u, \psi_0)}{u}$ is as well a normalized eigenfunction of H_1 with eigenvalue E_0, then $\mathrm{Sp}(H_1) = \{\epsilon, E_0\} \cup [0, \infty) = \{\epsilon\} \cup \mathrm{Sp}(H_0)$.

(ii) *Isospectral transformations.* These are achieved from the previous case either by taking $D \to 0$ or $D \to \infty$. Since in both situations $u(x)$ goes to zero at one of the ends of the x-domain, it turns out that $\psi_\epsilon^{(1)} \propto 1/u(x)$ is no longer square-integrable, although $\psi_0^{(1)}$ is. Thus, $\mathrm{Sp}(H_1) = \{E_0\} \cup [0, \infty) = \mathrm{Sp}(H_0)$.

(iii) *Deleting E_0.* This situation arises from the previous one by taking $k = k_0 = a$ ($\tilde{a} = a/k = 1$). Since $u(x) \propto \psi_0(x)$ is square-integrable, then $\psi_\epsilon^{(1)} \propto 1/u(x)$ is not normalizable, and then $\mathrm{Sp}(H_1) = [0, \infty)$. From equation (17), it is clear that now

$$V_1(x) = a\delta(x). \tag{18}$$

This means that, by deleting the bound state of the attractive delta well $V_0(x) = -a\delta(x)$, $a > 0$, which is placed at $E_0 = -a^2/2$, we recover the repulsive delta barrier of equation (18), a standard result well known in the literature.

4 Second-order SUSY transformations

In this section it will be illustrated, by means of the delta-well potential, the advantages for manipulating spectra of the second-order SUSY transformations [56, 57, 58, 59] compared with the first-order ones. It is nowadays known that the second-order SUSY partners H_2 of the initial Hamiltonian H_0 can be generated either by employing two eigenfunctions $u_1(x)$, $u_2(x)$ of H_0, not necessarily physical, associated to two different factorization energies $\epsilon_{1,2}$, $\epsilon_1 \neq \epsilon_2$ [37, 41] or by an appropriate eigenfunction $u_1(x)$ in the limit when $\epsilon_2 \to \epsilon_1$ (the so called confluent case [61, 62]). In both situations the two Hamiltonians H_0, H_2 are intertwined by a second-order operator in the way

$$H_2 B_2^+ = B_2^+ H_0,$$

where

$$B_2^+ = \frac{1}{2}\left(-\frac{d}{dx} + \frac{u_2^{(1)\prime}}{u_2^{(1)}}\right)\left(-\frac{d}{dx} + \frac{u_1'}{u_1}\right), \qquad u_2^{(1)} = \frac{w(x)}{u_1(x)},$$

the new Hamiltonian H_2 takes the standard Schrödinger form

$$H_2 = -\frac{1}{2}\frac{d^2}{dx^2} + V_2(x),$$

and the second-order SUSY partner $V_2(x)$ of the initial potential $V_0(x)$ is given by

$$V_2(x) = V_0(x) - [\ln w(x)]''. \tag{19}$$

the real function $w(x)$ being proportional in general to the Wronskian of two generalized eigenfunctions of H_0 [63]. An explicit classification of the several second-order SUSY transformations is next given.

4.1   Confluent case [61, 62]

Let us consider in the first place the limit $\epsilon_2 \to \epsilon_1 \equiv \epsilon = -k^2/2 < 0$, taking as seed the Schrödinger solution $u_+(x)$ vanishing as $x \longmapsto -\infty$, which means to take the $u(x)$ given in equation (9) with $D = 0$, namely:

$$u(x) = e^{kx} - 2\tilde{a}\sinh(kx)H(x). \tag{20}$$

In this case the real function $w(x)$ appearing in equation (19) takes the form [62]

$$w(x) := w_0 + \int_{-\infty}^{x} u^2(y)\,dy.$$

An explicit calculation for $x \le 0$ leads to:

$$\int_{-\infty}^{x} u^2(y)\,dy = \int_{-\infty}^{x} e^{2ky}\,dy = \frac{e^{2kx}}{2k}.$$

On the other hand, for $x > 0$ it turns out that:

$$\begin{aligned}
\int_{-\infty}^{x} u^2(y)\,dy &= \int_{-\infty}^{0} u^2(y)\,dy + \int_{0}^{x} u^2(y)\,dy \\
&= \frac{\tilde{a}}{k} + \frac{(1-2\tilde{a})}{2k}e^{2kx} + \frac{\tilde{a}^2}{k}\sinh(2kx) + 2\tilde{a}(1-\tilde{a})x.
\end{aligned}$$

By combining these two results, we obtain

$$w(x) = w_0 + \frac{e^{2kx}}{2k} + \left[\frac{\tilde{a}}{k} - \frac{\tilde{a}}{k}e^{2kx} + \frac{\tilde{a}^2}{k}\sinh(2kx) + 2\tilde{a}(1-\tilde{a})x\right]H(x). \tag{21}$$

The second-order SUSY partner potential of $V_0(x)$ becomes now

$$V_2(x) = -a\delta(x) + \frac{u^4(x)}{w^2(x)} - \frac{2u(x)u'(x)}{w(x)}. \tag{22}$$

Note that, since $u'(0+) = k - 2a$ and $u'(0-) = k$, then $u'(x)$ and consequently the potential difference $\Delta V(x) = V_2(x) - V_0(x)$ have a finite discontinuity at $x = 0$.

   In order to avoid the arising of extra singularities for $V_2(x)$ with respect to $V_0(x)$ we have to take $w_0 \ge 0$. Concerning the spectrum of H_2, a similar calculation as in the first-order case shows that the scattering eigenfunctions of H_0 are mapped into the corresponding ones of H_2, i.e., the energy interval $[0, \infty)$ belongs to $\mathrm{Sp}(H_2)$. As for the discrete part of the spectrum, several possibilities of spectral manipulation emerge, according to how we choose k and w_0.

(i) *Creating a new bound state at $\epsilon \ne E_0$.* This case appears by taking $w_0 > 0$ and $k \ne k_0 = a$. Since

$$\lim_{|x|\to\infty} \psi_\epsilon^{(2)} \propto \lim_{|x|\to\infty} \frac{u(x)}{w(x)} \propto e^{-k|x|},$$

the eigenfunction $\psi_\epsilon^{(2)}$ of H_2 associated to ϵ is square-integrable, i.e., a new bound state has been created at ϵ, either below the ground state for $k > k_0$ or above it for $k < k_0$. The last option is illustrated in Fig. 1, where we have plotted the potential difference $\Delta V(x)$ as a function of x for $a = 2$, $k = 1$, $w_0 = 1$, i.e., a new level was created at $\epsilon = -1/2 > E_0 = -2$ (see the two gray horizontal lines in the same graph). Note the existence of a finite discontinuity in $\Delta V(x)$ at $x = 0$, induced by a similar discontinuity of $u'(x)$ at the same point.
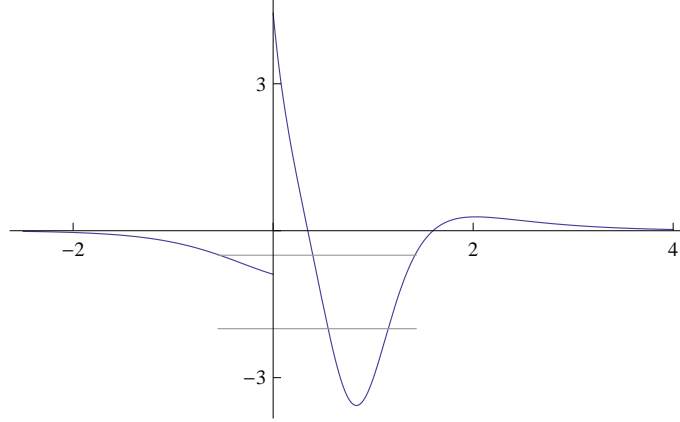
Figure 1. Potential difference $\Delta V(x)$ as function of x (blue lines) induced by the confluent second order SUSY transformation for $a = 2$, $k = 1$, $w_0 = 1$. A new level was created at $\epsilon = -1/2$, above the initial ground state $E_0 = -2$ (gray horizontal lines).

(ii) *Isospectral transformations.* They arise in the first place as a limit of the previous case for $\epsilon \neq E_0$ and $w_0 \to 0$. Note that the long explicit expression for the $V_2(x)$ of (22) which would appear if we would substitute explicitly the $u(x)$ and $w(x)$ of equations (20), (21) becomes strongly simplified in this limit:

$$V_2(x) = -a\delta(x) + H(x)\Big(8\tilde{a}k^2 e^{2kx}[(\tilde{a}-1)e^{2kx} - \tilde{a}]\{(\tilde{a}-1)[2kx(\tilde{a}-1)+1-2\tilde{a}]e^{2kx}$$
$$+ \tilde{a}[2kx(\tilde{a}-1)+2\tilde{a}-3]\}\Big)\Big/\big\{(\tilde{a}-1)^2 e^{4kx} + 2\tilde{a}[1-2kx(\tilde{a}-1)]e^{2kx} - \tilde{a}^2\big\}^2.$$

Since now

$$\lim_{x \to -\infty} \frac{u(x)}{w(x)} = \infty,$$

it turns out that $\epsilon \notin \text{Sp}(H_2) = \{E_0\} \cup [0, \infty) = \text{Sp}(H_0)$.

An alternative way to produce isospectral transformations is to use the single bound state of H_0 for evaluating $w(x)$. The corresponding formula is achieved from equation (21) by taking $k = k_0 = a$, $\tilde{a} = 1$, which leads to:

$$w(x) = w_0 + \frac{e^{2ax}}{2a} - \frac{2}{a}\sinh^2(ax)H(x). \tag{23}$$

Hence

$$V_2(x) = -a\delta(x) - \frac{8w_0 a^3 e^{2ax}}{(2w_0 a + e^{2ax})^2}H(-x) + \frac{8a^2 e^{2ax}(1+w_0 a)}{[2(1+w_0 a)e^{2ax} - 1]^2}H(x). \tag{24}$$

Note that now $w(x)$ does not have any node for

$$w_0 \in \left(-\infty, -\frac{1}{a}\right) \cup (0, \infty).$$

Moreover, in this domain it turns out that

$$\lim_{|x| \to \infty} \frac{u(x)}{w(x)} \propto e^{-a|x|},$$

i.e., $\psi_\epsilon^{(2)} \propto u(x)/w(x)$ is square-integrable $\Rightarrow \text{Sp}(H_2) = \{E_0\} \cup [0, \infty) = \text{Sp}(H_0)$.

(iii) *Deleting the ground state of H_0.* By taking now the limit of equation (23) for $w_0 \to 0$ or $w_0 \to -1/a$, it turns out that $\lim\limits_{x \to -\infty} u(x)/w(x) = \infty$ or $\lim\limits_{x \to \infty} u(x)/w(x) = \infty$ respectively. In both cases $\psi_\epsilon^{(2)}$ is not square-integrable and then

$$E_0 \notin \mathrm{Sp}(H_2) = [0, \infty).$$

This result means that we have deleted the ground state of H_0 in order to obtain H_2. For $w_0 \to 0$ the potential of equation (24) becomes

$$V_2(x) = -a\delta(x) + \frac{8a^2 e^{2ax}}{(2e^{2ax} - 1)^2} H(x). \tag{25}$$

On the other hand, for $w_0 \to -1/a$ the corresponding potential $V_2(x)$ is obtained from the previous one by the change $x \to -x$.

Let us remark that, although the final spectra of the SUSY partner Hamiltonians of H_0 are the same when deleting its ground state in the first-order and in the confluent second-order transformations, however the potentials $V_1(x)$ and $V_2(x)$ are physically different (compare equations (18) and (25)). In particular, note the opposite signs of the coefficients of the Dirac delta function for both potentials.

4.2   Complex case [64, 65, 66]

Let us assume that $k = k_R + ik_I$ is complex with $k_R > 0$, $k_I \in \mathbb{R}$, and suppose that the two involved factorization energies are now given by $\epsilon = -k^2/2$ and $\bar{\epsilon}$, where $\bar{z}$ denotes the complex conjugate of z. Since we need to avoid the arising of extra singularities in the new potential, we will take a Schrödinger seed solution vanishing at one of the ends of the x-domain in the form given in equation (20) with $k \in \mathbb{C}$, namely,

$$u(x) = e^{kx} - \frac{2a}{k}\sinh(kx)H(x), \qquad \bar{u}(x) = e^{\bar{k}x} - \frac{2a}{\bar{k}}\sinh(\bar{k}x)H(x). \tag{26}$$

To compute now the second-order SUSY partner potential $V_2(x)$, we have to obtain in the first place the Wronskian $W(u, \bar{u})$ and then the real function

$$w(x) = \frac{W(u, \bar{u})}{2(\epsilon - \bar{\epsilon})}.$$

This calculation is cumbersome but otherwise straightforward, which leads to:

$$w(x) = \frac{e^{2k_R x}}{2k_R} + \left\{ -\frac{a}{|k|^2}\left[\cosh(2k_R x) - \cos(2k_I x)\right] \right.$$
$$\left. + \frac{a(a - k_R)}{|k|^2 k_R k_I}\left[k_I \sinh(2k_R x) - k_R \sin(2k_I x)\right] \right\} H(x). \tag{27}$$

Then, $V_2(x)$ will be given by

$$V_2(x) = -a\delta(x) + \frac{|u(x)|^4}{w^2(x)} - \frac{[u(x)\bar{u}'(x) + \bar{u}(x)u'(x)]}{w(x)}, \tag{28}$$

with $u(x)$, $\bar{u}(x)$ and $w(x)$ as given in equations (26) and (27). An illustration of the potential difference $\Delta V(x)$ as function of x for $a = 2$, $k = 1/100 + i/10$ is given in Fig. 2.
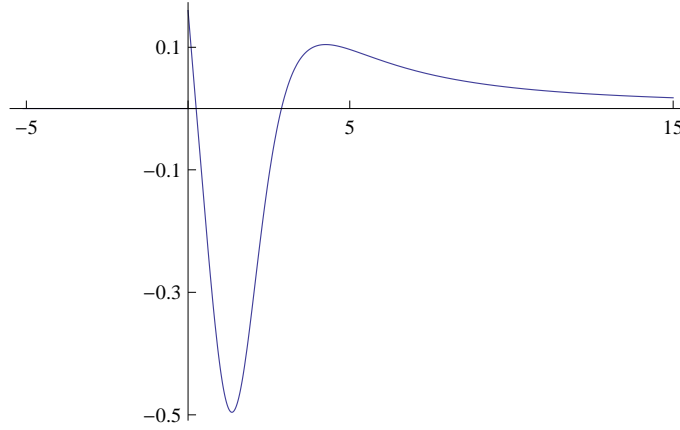
Figure 2. Potential difference $\Delta V(x)$ as function of x induced by the complex second order SUSY transformation for $a = 2$, $k = 1/100 + i/10$. The two potentials $V_2(x)$ and $V_0(x)$ are isospectral.

Note that these equations become highly simplified if $k_R = a$:

$$w(x) = \frac{e^{2ax}}{2a} - \left(\frac{a}{a^2 + k_I^2}\right)[\cosh(2ax) - \cos(2k_I x)]\,H(x), \tag{29}$$

$$|u(x)|^2 = e^{2ax} - \left(\frac{2a}{a^2 + k_I^2}\right)[a\sinh(2ax) + k_I \sin(2k_I x)]\,H(x). \tag{30}$$

Moreover, for the particular choice $k_R = a$ we get a more compact expression for the new potential $V_2(x)$ than for a generic k_R that would appear if we would substitute the $u(x)$ and $w(x)$ of equations (26), (27) in equation (28):

$$V_2(x) = -a\delta(x) + H(x)\Big(4a^2 e^{2ax}[2(a^2 - k_I^2)\cos(2k_I x)(a^2 - k_I^2 e^{4ax}) \tag{31}$$
$$- 4ak_I \sin(2k_I x)(a^2 + k_I^2 e^{4ax}) + 8a^2 k_I^2 e^{2ax}]\Big)\Big/\big[a^2 - k_I^2 e^{4ax} - 2a^2 e^{2ax}\cos(2k_I x)\big]^2.$$

Let us remark that, for the general case characterized by equations (26) and (27) as well as the particular ones described by equations (29)–(31), the scattering states of H_0 are mapped into the corresponding ones of H_2, and the same happens for the bound state. Thus, it turns out that the spectrum of H_2 will be equal to $\mathrm{Sp}(H_0) = \{E_0\} \cup [0, \infty)$, i.e., the complex second-order SUSY transformations which produce a real final potential are strictly isospectral.

4.3   Real case

Let us take now two seed solutions u_1, u_2 in the form given in equation (9), associated to the pair of real factorization energies $\epsilon_2 < \epsilon_1 \Rightarrow k_2 > k_1$. Their explicit forms, and the corresponding derivatives, are given by:

$$u_i(x) = e^{k_i x} + D_i e^{-k_i x} + 2\tilde{a}_i \sinh(k_i x)[D_i H(-x) - H(x)], \tag{32}$$
$$u_i'(x) = -k_i u_i(x) + 2k_i e^{k_i x}\left[1 - \tilde{a}_i H(x) + D_i \tilde{a}_i H(-x)\right], \qquad i = 1, 2,$$

where $\tilde{a}_i = a/k_i$. Similarly as in the complex case, the calculation of the Wronskian $w(x) \equiv W(u_1, u_2)$ of the two involved Schrödinger seed solutions is once again cumbersome, but a convenient compact expression reads:

$$w(x) = (k_1 - k_2)u_1 u_2 + 2k_2 u_1 e^{k_2 x}\left[1 - \tilde{a}_2 H(x) + D_2 \tilde{a}_2 H(-x)\right]$$
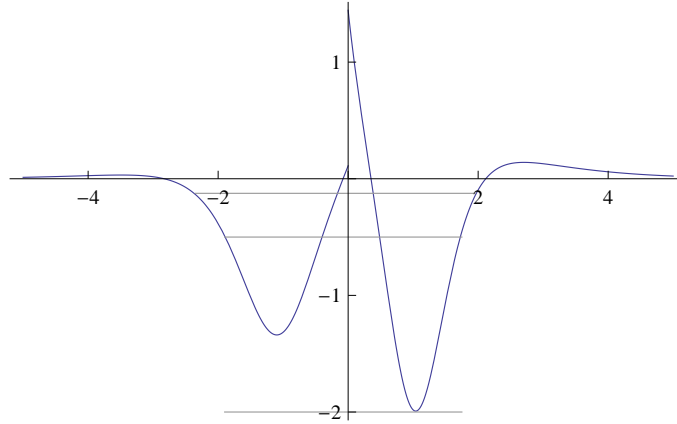
Figure 3. Potential difference $\Delta V(x)$ as function of x (blue lines), induced by a real second order SUSY transformation for $a = 2$, $k_2 = 1$, $k_1 = 1/2$, $D_1 = -1/2$, $D_2 = 1$. Note that two new levels were created above $E_0 = -2$, at the positions $\epsilon_2 = -1/2$ and $\epsilon_1 = -1/8$ (gray horizontal lines).

$$- 2k_1 u_2 e^{k_1 x} \left[1 - \tilde{a}_1 H(x) + D_1 \tilde{a}_1 H(-x) \right].$$

By employing this equation, it is straightforward to calculate the new potential through:

$$V_2(x) = -a\delta(x) + \left(\frac{w'}{w} \right)^2 - \frac{w''}{w} = -a\delta(x) + \left[\frac{(k_1^2 - k_2^2)u_1 u_2}{w} \right]^2 + \frac{(k_1^2 - k_2^2)(u_1 u_2' + u_1' u_2)}{w}.$$

Concerning the spectrum of H_2, once again the scattering states of H_0 are mapped into the corresponding ones of H_2. As for the discrete part of the spectrum, several possibilities are worth of study.

(i) *Creating two new levels.* Let us suppose first that $\epsilon_1 \neq E_0 \neq \epsilon_2$. In order that $w(x)$ do not have nodes, the two factorization energies must be placed either both below (for $k_2 > k_1 > a$) or both above E_0 (for $k_1 < k_2 < a$). Moreover, according to the chosen ordering $\epsilon_2 < \epsilon_1$, the solution $u_2(x)$ must have one extra node with respect to $u_1(x)$ [37]. In the domain $k_2 > k_1 > a$ ($\epsilon_2 < \epsilon_1 < E_0$) this can be achieved by taking $D_2 < 0$ and $D_1 > 0$ while for $k_1 < k_2 < a$ ($E_0 < \epsilon_2 < \epsilon_1$) it must be taken $D_2 > 0$ and $D_1 < 0$. With this choice of parameters, it turns out that the two eigenfunctions of H_2 associated to ϵ_1 and ϵ_2, $\psi_{\epsilon_1}^{(2)} \propto u_2/w$ and $\psi_{\epsilon_2}^{(2)} \propto u_1/w$, are square-integrable. Thus,

$$\mathrm{Sp}(H_2) = \{\epsilon_2, \epsilon_1\} \cup \mathrm{Sp}(H_0),$$

i.e., two new levels have been created for H_2, either both below the ground state of H_0 (for $k_2 > k_1 > a$) or both above E_0 (for $k_1 < k_2 < a$). An illustration of the last situation is shown in Fig. 3, where we have plotted the potential difference $\Delta V(x)$ for $a = 2$, $k_2 = 1$, $k_1 = 1/2$, $D_1 = -1/2$, $D_2 = 1$. As a result of the transformation, two new levels were created above the ground state energy of H_0 at the positions $\epsilon_2 = -1/2$ and $\epsilon_1 = -1/8$ (see the gray horizontal lines at Fig. 3).

(ii) *Creating one new level.* This case arises from the previous one for $D_2 \to 0$. Now it turns out that $\psi_{\epsilon_2}^{(2)}$ is not square-integrable anymore, meaning that

$$\mathrm{Sp}(H_2) = \{\epsilon_1\} \cup \mathrm{Sp}(H_0).$$

Thus, in order to generate H_2 a new level has been created at ϵ_1, above E_0 for $k_1 < a$ and below it for $k_1 > a$.

(iii) *Isospectral transformations.* These can be achieved from case (i) for $D_1 = D_2 \to 0$, where both $\psi_{\epsilon_1}^{(2)}$ and $\psi_{\epsilon_2}^{(2)}$ cease to be square-integrable so that $\epsilon_i \notin \mathrm{Sp}(H_2)$, $i = 1, 2$. Hence,

$$\mathrm{Sp}(H_2) = \mathrm{Sp}(H_0).$$

(iv) *Moving the level E_0.* This procedure is obtained from case (i), e.g., by taking $\epsilon_2 = E_0$, $D_2 \to 0$, $u_2(x) \propto \psi_0(x)$, and $u_1(x)$ as given in equation (32) with $D_1 < 0$, $\epsilon_1 > E_0$. With this choice it can be shown that $\psi_{\epsilon_2}^{(2)} \propto u_1/w$ is not square-integrable but $\psi_{\epsilon_1}^{(2)}$ does, meaning that

$$\mathrm{Sp}(H_2) = \{\epsilon_1\} \cup [0, \infty).$$

In a way, the level E_0 has been moved up to ϵ_1 for generating H_2.

(v) *Deleting the level E_0.* This can be achieved as a limit of the previous case for $D_1 \to 0$. Now it turns out that $\lim_{x \to 0} u_2/w = \infty$, i.e., $\epsilon_1 \notin \mathrm{Sp}(H_2)$, and hence

$$\mathrm{Sp}(H_2) = [0, \infty).$$

5   Conclusions

We have employed the first and second-order supersymmetric quantum mechanics for generating new potentials with modified spectra departing from the delta well potential. The first-order transformation allowed us to change just the ground state energy level, while the second-order transformations enlarged the possibilities of spectral control, including the option of manipulating the excited state levels. On the other hand, it is important to remember that the first-order transformations induced in the new potential a delta term with an opposite sign compared with the initial one (physically the delta term changed from attractive to repulsive). Meanwhile, the second-order transformations generated a delta term with exactly the same sign as the initial one (the attractive nature was preserved under the transformation). These physical differences should be taken into account in the determination of the most appropriate transformation for building a potential model. We can conclude that supersymmetric quantum mechanics is a powerful mathematical tool, which is quite useful for implementing the spectral design in physics.

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