Rational Solutions of the Sasano System of Type $A_5^{(2)}$

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Abstract. In this paper, we completely classify the rational solutions of the Sasano system of type $A_5^{(2)}$, which is given by the coupled Painlevé III system. This system of differential equations has the affine Weyl group symmetry of type $A_5^{(2)}$.

Key words: affine Weyl group; rational solutions; Sasano system

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1 Introduction

Paul Painlevé and his colleagues [21, 4] intended to find new transcendental functions defined by second order nonlinear differential equations. In general, nonlinear differential equations have moving branch points. If a solution has moving branch points, it is too complicated and is not worth considering. Therefore, they determined the second order nonlinear differential equations with rational coefficients which have no moving branch points. As a result, the standard forms of such equations turned out to be given by the following six equations:

$$
\begin{aligned}
P_{\rm I}: \quad & y'' = 6y^2 + t,\\
P_{\rm II}: \quad & y'' = 2y^3 + ty + \alpha,\\
P_{\rm III}: \quad & y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},\\
P_{\rm IV}: \quad & y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},\\
P_{\rm V}: \quad & y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1},\\
P_{\rm VI}: \quad & y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y'\\
& \qquad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right),
\end{aligned}
$$

where $' = d/dt$ and α, β, γ, δ are all complex parameters. In this article, our concern is with the Bäcklund transformations and special solutions which are given by rational, algebraic functions or classical special functions.

Each of P_J ($J = \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}$) has Bäcklund transformations, which transform solutions into other solutions of the same equation with different parameters. It was shown by Okamoto [17, 18, 19, 20] that the Bäcklund transformation groups of the Painlevé equations except for $P_{\rm I}$ are isomorphic to the extended affine Weyl groups. For $P_{\rm II}$, $P_{\rm III}$, $P_{\rm IV}$, $P_{\rm V}$, and $P_{\rm VI}$, the Bäcklund transformation groups correspond to $A_1^{(1)}$, $A_1^{(1)} \bigoplus A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(3)}$, and $D_4^{(1)}$, respectively.

While generic solutions of the Painlevé equations are "new transcendental functions", there are special solutions which are expressible in terms of rational, algebraic, or classical special functions.

For example, Airault [1] constructed explicit rational solutions of $P_{\rm II}$ and $P_{\rm IV}$ with their Bäcklund transformations. Milne, Clarkson and Bassom [13] treated $P_{\rm III}$, and described their Bäcklund transformations and exact solution hierarchies, which are given by rational, algebraic, or certain Bessel functions. Bassom, Clarkson and Hicks [2] dealt with $P_{\rm IV}$, and described their Bäcklund transformations and exact solution hierarchies, which are expressed by rational functions, the parabolic cylinder functions or the complementary error functions. Clarkson [3] studied some rational and algebraic solutions of $P_{\rm III}$ and showed that these solutions are expressible in terms of special polynomials defined by second order, bilinear differential-difference equations which are equivalent to Toda equations.

Furthermore, the rational solutions of P_J $(J = {\rm II}, {\rm III}, {\rm IV}, {\rm V}, {\rm VI})$ were classified by Yablonski and Vorobev [25, 24], Gromak [6, 5], Murata [14, 15], Kitaev, Law and McLeod [8], Mazzoco [11] and Yuang and Li [26].

Noumi and Yamada [16] discovered the equation of type $A_l^{(1)}$ $(l \geq 2)$, whose Bäcklund transformation group is isomorphic to the extended affine Weyl group $\tilde{W}(A_l^{(1)})$. The Noumi and Yamada systems of types $A_2^{(1)}$ and $A_3^{(1)}$ correspond to the fourth and fifth Painlevé equations, respectively. Moreover, we [9, 10] classified the rational solutions of the Noumi and Yamada systems of types $A_4^{(1)}$ and $A_5^{(1)}$.

Sasano [22] found the coupled Painlevé V and VI systems which have the affine Weyl group symmetries of types $D_5^{(1)}$ and $D_6^{(1)}$. In addition, he [23] obtained the equation of the affine Weyl group symmetry of type $A_5^{(2)}$, which is defined by

$$A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}\ \begin{cases} tq_1' = 2q_1^2p_1 - q_1^2 + (\alpha_0+\alpha_1+\alpha_3)q_1 - t + 4tp_2 + 2q_1q_2p_2, \\ tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0+\alpha_1+\alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2, \\ tq_2' = 2q_2^2p_2 - q_2^2 + (\alpha_0+\alpha_1+\alpha_3)q_2 - t + 4tp_1 + 2q_1p_1q_2, \\ tp_2' = -2q_2p_2^2 + 2q_2p_2 - (\alpha_0+\alpha_1+\alpha_3)p_2 + \alpha_1 - 2q_1p_1p_2, \\ \alpha_0+\alpha_1+2\alpha_2+\alpha_3 = 1/2, \end{cases}$$

where $' = d/dt$. This system of differential equations is also expressed by the Hamiltonian system:

$$t\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}, \qquad t\frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}, \qquad t\frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \qquad t\frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2},$$

where the Hamiltonian H is given by

$$\begin{aligned} H &= q_1^2p_1^2 - q_1^2p_1 + (\alpha_0+\alpha_1+\alpha_3)q_1p_1 - \alpha_0q_1 - tp_1 \\ &\quad + q_2^2p_2^2 - q_2^2p_2 + (\alpha_0+\alpha_1+\alpha_3)q_2p_2 - \alpha_1q_2 - tp_2 + 4tp_1p_2 + 2q_1p_1q_2p_2. \end{aligned}$$

Let us note that Mazzocco and Mo [12] studied the Hamiltonian structure of the $P_{\rm II}$ hierarchy, and Hone [7] studied the coupled Painlevé systems from the similarity reduction of the Hirota–Satsuma system and another gauge-related system, and presented their Bäcklund transformations and special solutions.

$A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$ has the Bäcklund transformations s_0, s_1, s_2, s_3, π, which are given by

$$s_0: \quad (*) \to \left(q_1 + \frac{\alpha_0}{p_1}, p_1, q_2, p_2, t; -\alpha_0, \alpha_1, \alpha_2+\alpha_0, \alpha_3\right),$$

$$
\begin{aligned}
s_1: \quad & (*) \to \left(q_1, p_1, q_2 + \frac{\alpha_1}{p_2}, p_2, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3\right), \\
s_2: \quad & (*) \to \left(q_1, p_1 - \frac{\alpha_2 q_2}{q_1 q_2 + t}, q_2, p_2 + \frac{\alpha_2 q_1}{q_1 q_2 + t}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + 2\alpha_2\right), \\
s_3: \quad & (*) \to \left(q_1 + \frac{\alpha_3}{p_1 + p_2 - 1}, p_1, q_2 + \frac{\alpha_3}{p_1 + p_2 - 1}, p_2, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3\right), \\
\pi: \quad & (*) \to (q_2, p_2, q_1, p_1, t; \alpha_1, \alpha_0, \alpha_2, \alpha_3),
\end{aligned}
$$

with the notation $(*) = (q_1, p_1, q_2, p_2, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3)$. The Bäcklund transformation group $\langle s_0, s_1, s_2, s_3, \pi \rangle$ is isomorphic to the affine Weyl group of type $A_5^{(2)}$.

Our main theorem is as follows:

Theorem 1.1. *For a rational solution of $A_5^{(2)}(\alpha_j)_{0 \le j \le 3}$, by some Bäcklund transformations, the solution and parameters can be transformed so that*

$$
\begin{aligned}
&(q_1, p_1, q_2, p_2) = (0, 1/4, 0, 1/4) \qquad \text{and} \\
&(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha_3/2, \alpha_3/2, \alpha_2, \alpha_3) = (\alpha_3/2, \alpha_3/2, 1/4 - \alpha_3, \alpha_3),
\end{aligned}
$$

respectively. Furthermore, for $A_5^{(2)}(\alpha_j)_{0 \le j \le 3}$, there exists a rational solution if and only if one of the following occurs:

$$
\begin{aligned}
&(1) \quad -2\alpha_0 + \alpha_3 \in \mathbb{Z}, \quad -2\alpha_1 + \alpha_3 \in \mathbb{Z}, \\
&(2) \quad -2\alpha_0 + \alpha_3 \in \mathbb{Z}, \quad 2\alpha_1 + \alpha_3 \in \mathbb{Z}, \\
&(3) \quad 2\alpha_0 + \alpha_3 \in \mathbb{Z}, \quad -2\alpha_1 + \alpha_3 \in \mathbb{Z}, \\
&(4) \quad 2\alpha_0 + \alpha_3 \in \mathbb{Z}, \quad 2\alpha_1 + \alpha_3 \in \mathbb{Z}, \\
&(5) \quad -2\alpha_0 + \alpha_3 \in \mathbb{Z}, \quad \alpha_3 - 1/2 \in \mathbb{Z}, \\
&(6) \quad -2\alpha_1 + \alpha_3 \in \mathbb{Z}, \quad \alpha_3 - 1/2 \in \mathbb{Z}.
\end{aligned}
$$

This paper is organized as follows. In Section 2, for $A_5^{(2)}(\alpha_j)_{0 \le j \le 3}$, we determine meromorphic solutions at $t = \infty$. Then, we find that the constant terms $a_{\infty,0}$, $c_{\infty,0}$ of the Laurent series of q_1, q_2 at $t = \infty$ are given by

$$
a_{\infty,0} := -2\alpha_0 + \alpha_3, \qquad c_{\infty,0} := -2\alpha_1 + \alpha_3,
$$

respectively.

In Section 3, for $A_5^{(2)}(\alpha_j)_{0 \le j \le 3}$, we determine meromorphic solutions at $t = 0$. Then, we see that the constant terms $a_{0,0}$, $c_{0,0}$ of the Laurent series of q_1, q_2 at $t = 0$ are given by the parameters α_0, α_1, α_2, α_3.

In Section 4, for $A_5^{(2)}(\alpha_j)_{0 \le j \le 3}$, we treat meromorphic solutions at $t = c \in \mathbb{C}^*$, where in this paper, $\mathbb{C}^*$ means the set of nonzero complex numbers. Then, we observe that q_1, q_2 have both a pole of order of at most one at $t = c$ and the residues of q_1, q_2 at $t = c$ are expressed by nc $(n \in \mathbb{Z})$. Thus, it follows that

$$
a_{\infty,0} - a_{0,0} \in \mathbb{Z}, \qquad c_{\infty,0} - c_{0,0} \in \mathbb{Z}, \tag{1.1}
$$

which gives a necessary condition for $A_5^{(2)}(\alpha_j)_{0 \le j \le 3}$ to have a rational solution.

In Section 5, using the meromorphic solution at $t = \infty, 0$, we first compute the constant terms of the Laurent series of the Hamiltonian at $t = \infty, 0$. Furthermore, by the meromorphic solution at $= c \in \mathbb{C}^*$, we calculate the residue of H at $t = c$.

In Section 6, by equation (1.1), we obtain the necessary conditions for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$ to have rational solutions, which are given in our main theorem. Furthermore, we show that if there exists a rational solution for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, the parameters can be transformed so that $-2\alpha_0+\alpha_3\in\mathbb{Z}$, $-2\alpha_1+\alpha_3\in\mathbb{Z}$.

In Section 7, we define shift operators, and for a rational solution of $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, we transform the parameters to

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(\alpha_3/2,\alpha_3/2,\alpha_2,\alpha_3).$$

In Section 8, we determine rational solutions of $A_5^{(2)}(\alpha_3/2,\alpha_3/2,\alpha_2,\alpha_3)$ and prove our main theorem.

In Appendix A, using the shift operators, we give examples of rational solutions.

2 Meromorphic solutions at $t=\infty$

In this section, for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, we treat meromorphic solutions at $t=\infty$. For the purpose, in this paper, we define the coefficients of the Laurent series of q_1, p_1, q_2, p_2 at $t=\infty$ by $a_{\infty,k}$, $b_{\infty,k}$, $c_{\infty,k}$, $d_{\infty,k}$, $k\in\mathbb{Z}$.

2.1 The case where q_1, p_1, q_2, p_2 are all holomorphic at $t=\infty$

Proposition 2.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t=\infty$. Then,*

$$\begin{cases} q_1=(-2\alpha_0+\alpha_3)+\cdots, \\ p_1=1/4+(-2\alpha_1+\alpha_3)(-2\alpha_1-\alpha_3)t^{-1}/4+\cdots, \\ q_2=(-2\alpha_1+\alpha_3)+\cdots, \\ p_2=1/4+(-2\alpha_0+\alpha_3)(-2\alpha_0-\alpha_3)t^{-1}/4+\cdots. \end{cases}$$

Proposition 2.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t=\infty$. Then, it is unique.*

Proof. We set

$$\begin{aligned} q_1&=a_{\infty,0}+a_{\infty,-1}t^{-1}+\cdots+a_{\infty,-(k-1)}t^{-(k-1)}+a_{\infty,-k}t^{-k}+a_{\infty,-(k+1)}t^{-(k+1)}+\cdots, \\ p_1&=1/4+b_{\infty,-1}t^{-1}+\cdots+b_{\infty,-(k-1)}t^{-(k-1)}+b_{\infty,-k}t^{-k}+b_{\infty,-(k+1)}t^{-(k+1)}+\cdots, \\ q_2&=c_{\infty,0}+c_{\infty,-1}t^{-1}+\cdots+c_{\infty,-(k-1)}t^{-(k-1)}+c_{\infty,-k}t^{-k}+c_{\infty,-(k+1)}t^{-(k+1)}+\cdots, \\ p_2&=1/4+d_{\infty,-1}t^{-1}+\cdots+d_{\infty,-(k-1)}t^{-(k-1)}+d_{\infty,-k}t^{-k}+d_{\infty,-(k+1)}t^{-(k+1)}+\cdots, \end{aligned}$$

where $a_{\infty,0}$, $b_{\infty,-1}$, $c_{\infty,0}$, $d_{\infty,-1}$ all have been determined.

Comparing the coefficients of the terms t^{-k} $(k\ge 1)$ in

$$\begin{aligned} tp_1'&=-2q_1p_1^2+2q_1p_1-(\alpha_0+\alpha_1+\alpha_3)p_1+\alpha_0-2p_1q_2p_2, \\ tp_2'&=-2q_2p_2^2+2q_2p_2-(\alpha_0+\alpha_1+\alpha_3)p_2+\alpha_1-2q_1p_1p_2, \end{aligned}$$

we have

$$\begin{aligned} 3a_{\infty,-k}/8-c_{\infty,-k}/8&=-kb_{\infty,-k}+(\alpha_0+\alpha_1+\alpha_3)b_{\infty,-k} \\ &\quad+2\sum a_{\infty,-l}b_{\infty,-m}b_{\infty,-n}-2\sum a_{\infty,-l}b_{\infty,-m}+2\sum c_{\infty,-l}b_{\infty,-m}d_{\infty,-n}, \end{aligned}$$

$$-a_{\infty,-k}/8+3c_{\infty,-k}/8=-kd_{\infty,-k}+(\alpha_0+\alpha_1+\alpha_3)d_{\infty,-k}$$
$$+2\sum c_{\infty,-l}d_{\infty,-m}d_{\infty,-n}-2\sum c_{\infty,-l}d_{\infty,-m}+2\sum a_{\infty,-l}b_{\infty,-m}d_{\infty,-n},$$

where the first and third sums extend over nonnegative integers l, m, n such that $l+m+n=k$ and $0\le l<k$, and the second sums extend over nonnegative integers l, m such that $l+m=k$ and $m\ge 1$. Therefore, $a_{\infty,-k}$, $c_{\infty,-k}$ are both inductively determined.

Comparing the coefficients of the terms t^{-k} $(k\ge 1)$ in

$$tq_1'=2q_1^2p_1-q_1^2+(\alpha_0+\alpha_1+\alpha_3)q_1-t+4tp_2+2q_1q_2p_2,$$
$$tq_2'=2q_2^2p_2-q_2^2+(\alpha_0+\alpha_1+\alpha_3)q_2-t+4tp_1+2q_1p_1q_2,$$

we obtain

$$4d_{\infty,-(k+1)}=-ka_{\infty,-k}-(\alpha_0+\alpha_1+\alpha_3)a_{\infty,-k}$$
$$-2\sum a_{\infty,-l}a_{\infty,-m}b_{\infty,-n}+\sum a_{\infty,-l}a_{\infty,-m}-2\sum a_{\infty,-l}c_{\infty,-m}d_{\infty,-n},$$
$$4b_{\infty,-(k+1)}=-kc_{\infty,-k}-(\alpha_0+\alpha_1+\alpha_3)c_{\infty,-k}$$
$$-2\sum c_{\infty,-l}c_{\infty,-m}d_{\infty,-n}+\sum c_{\infty,-l}c_{\infty,-m}-2\sum c_{\infty,-l}a_{\infty,-m}b_{\infty,-n},$$

where the first and third sums extend over nonnegative integers l, m, n such that $l+m+n=k$, and the second sums extend over nonnegative integers l, m such that $l+m=k$. Therefore, $b_{\infty,-(k+1)}$, $d_{\infty,-(k+1)}$ are both inductively determined, which proves the proposition. ■

2.2 The case where one of (q_1,p_1,q_2,p_2) has a pole at $t=\infty$

In this subsection, we deal with the case in which one of (q_1,p_1,q_2,p_2) has a pole at $t=\infty$. For the purpose, by π, we have only to consider the following two cases:

(1) q_1 has a pole at $t=\infty$ and p_1, q_2, p_2 are all holomorphic at $t=\infty$,

(2) p_1 has a pole at $t=\infty$ and q_1, q_2, p_2 are all holomorphic at $t=\infty$.

2.2.1 The case where q_1 has a pole at $t=\infty$

Proposition 2.3. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1 has a pole at $t=\infty$ and p_1, q_2, p_2 are all holomorphic at $t=\infty$.*

2.2.2 The case where p_1 has a pole at $t=\infty$

Proposition 2.4. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that p_1 has a pole at $t=\infty$ and q_1, q_2, p_2 are all holomorphic at $t=\infty$.*

2.3 The case where two of (q_1,p_1,q_2,p_2) have a pole at $t=\infty$

In this subsection, we deal with the case in which two of (q_1,p_1,q_2,p_2) has a pole at $t=\infty$. For the purpose, by π, we have only to consider the following four cases:

(1) q_1, p_1 have both a pole at $t=\infty$ and q_2, p_2 are both holomorphic at $t=\infty$,

(2) q_1, q_2 have both a pole at $t=\infty$ and p_1, p_2 are both holomorphic at $t=\infty$,

(3) q_1, p_2 have both a pole at $t=\infty$ and p_1, q_2 are both holomorphic at $t=\infty$,

(4) p_1, p_2 have both a pole at $t=\infty$ and q_1, q_2 are both holomorphic at $t=\infty$.

2.3.1 The case where q_1, p_1 have a pole at $t = \infty$

Proposition 2.5. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1, p_1 have both a pole at $t=\infty$ and q_2, p_2 are both holomorphic at $t=\infty$.*

2.3.2 The case where q_1, q_2 have a pole at $t = \infty$

Proposition 2.6. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1, q_2 have both a pole at $t=\infty$ and p_1, p_2 are both holomorphic at $t=\infty$.*

2.3.3 The case where q_1, p_2 have a pole at $t = \infty$

By direct calculation, we can obtain the following two lemmas:

Lemma 2.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, $q_1 \equiv 0$. Then, one of the following occurs:*

$$
\begin{aligned}
&(1) \quad \alpha_0 = \frac{1}{4}, \qquad \alpha_3 = \frac{1}{2}, \qquad \textit{and}\\
&\qquad (q_1,p_1,q_2,p_2) = \left(0, \frac{1}{4} + \frac{(4\alpha_1-1)(4\alpha_1+1)}{16t}, -2\alpha_1 + \frac{1}{2}, \frac{1}{4}\right),\\
&(2) \quad \alpha_0 = \frac{\alpha_3}{2}, \qquad \alpha_1 = \frac{\alpha_3}{2}, \qquad \textit{and} \qquad (q_1,p_1,q_2,p_2) = \left(0, \frac{1}{4}, 0, \frac{1}{4}\right),\\
&(3) \quad \alpha_0 = \frac{\alpha_3}{2}, \qquad \alpha_1 = -\frac{\alpha_3}{2}, \qquad \textit{and} \qquad (q_1,p_1,q_2,p_2) = \left(0, \frac{1}{4}, 2\alpha_3, \frac{1}{4}\right).
\end{aligned}
$$

Lemma 2.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, $q_2 \equiv 0$. Then, one of the following occurs:*

$$
\begin{aligned}
&(1) \quad \alpha_1 = \frac{1}{4}, \qquad \alpha_3 = \frac{1}{2}, \qquad \textit{and}\\
&\qquad (q_1,p_1,q_2,p_2) = \left(-2\alpha_0 + \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4} + \frac{(4\alpha_0-1)(4\alpha_0+1)}{16t}\right),\\
&(2) \quad \alpha_0 = \frac{\alpha_3}{2}, \qquad \alpha_1 = \frac{\alpha_3}{2}, \qquad \textit{and} \qquad (q_1,p_1,q_2,p_2) = \left(0, \frac{1}{4}, 0, \frac{1}{4}\right),\\
&(3) \quad \alpha_0 = -\frac{\alpha_3}{2}, \qquad \alpha_1 = \frac{\alpha_3}{2}, \qquad \textit{and} \qquad (q_1,p_1,q_2,p_2) = \left(2\alpha_3, \frac{1}{4}, 0, \frac{1}{4}\right).
\end{aligned}
$$

By Lemma 2.2, we find that $q_2 \not\equiv 0$. Now, let us assume that q_1 has a pole of order n_0 ($n_0 \ge 1$) and p_2 has a pole of order n_3 ($n_3 \ge 1$).

Lemma 2.3. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_2 have both a pole at $t=\infty$ and p_1, q_2 are both holomorphic at $t=\infty$. Then, $n_0 \neq n_3$.*

Proof. We suppose that $n_0 = n_3$. Especially, we treat the case where $n_0 = n_3 = 1$ and show contradiction. If $n_0 = n_3 > 1$, we can prove contradiction in the same way.

Comparing the coefficients of the terms t^2, t in

$$
\begin{aligned}
tq_1' &= 2q_1^2p_1 - q_1^2 + (\alpha_0+\alpha_1+\alpha_3)q_1 - t + 4tp_2 + 2q_1q_2p_2,\\
tp_1' &= -2q_1p_1^2 + 2q_1p_1 - (\alpha_0+\alpha_1+\alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,
\end{aligned}
$$

we have

$$
2a_{\infty,1}^2 b_{\infty,0} - a_{\infty,1}^2 + 4d_{\infty,1} + 2a_{\infty,1}c_{\infty,0}d_{\infty,1} = 0,
$$

$$-2a_{\infty,1}b_{\infty,0}^2 + 2a_{\infty,1}b_{\infty,0} - 2b_{\infty,0}c_{\infty,0}d_{\infty,1} = 0, \tag{2.1}$$

respectively.

Comparing the coefficients of the terms t, t^2 in

$$tq_2' = 2q_2^2p_2 - q_2^2 + (\alpha_0 + \alpha_1 + \alpha_3)q_2 - t + 4tp_1 + 2q_1p_1q_2,$$
$$tp_2' = -2q_2p_2^2 + 2q_2p_2 - (\alpha_0 + \alpha_1 + \alpha_3)p_2 + \alpha_1 - 2q_1p_1p_2,$$

we obtain

$$2c_{\infty,0}^2d_{\infty,1} - 1 + 4b_{\infty,0} + 2a_{\infty,1}b_{\infty,0}c_{\infty,0} = 0,$$
$$-2c_{\infty,0}d_{\infty,1}^2 - 2a_{\infty,1}b_{\infty,0}d_{\infty,1} = 0, \tag{2.2}$$

which implies that $b_{\infty,0} = 1/4$. Furthermore, from the second equation in (2.1) and the first equation in (2.2), it follows that $a_{\infty,1} = 0$, which is impossible. ■

Lemma 2.4. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_2 have both a pole at $t = \infty$ and p_1, q_2 are both holomorphic at $t = \infty$. Then, $n_0 < n_3$.*

Proposition 2.7. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1, p_2 have both a pole at $t = \infty$ and p_1, q_2 are both holomorphic at $t = \infty$.*

Proof. We treat the case where $(n_0, n_3) = (1, 2)$ and show contradiction. The other cases can be proved in the same way.

Comparing the coefficients of the terms t^3 in

$$tq_1' = 2q_1^2p_1 - q_1^2 + (\alpha_0 + \alpha_1 + \alpha_3)q_1 - t + 4tp_2 + 2q_1q_2p_2,$$

we have $d_{\infty,2} = 0$, which is impossible. ■

2.3.4 The case where p_1, p_2 have a pole at $t = \infty$

Proposition 2.8. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that p_1, p_2 have both a pole at $t = \infty$ and q_1, q_2 are both holomorphic at $t = \infty$.*

2.4 The case where three of (q_1, p_1, q_2, p_2) have a pole at $t = \infty$

In this subsection, considering π, we treat the following two cases:

(1) q_1, p_1, q_2 all have a pole at $t = \infty$ and p_2 is holomorphic at $t = \infty$,

(2) q_1, p_1, p_2 all have a pole at $t = \infty$ and q_2 is holomorphic at $t = \infty$.

2.4.1 The case where q_1, p_1, q_2 have a pole at $t = \infty$

Proposition 2.9. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1, p_1, q_2 all have a pole at $t = \infty$ and p_2 is holomorphic at $t = \infty$.*

2.4.2 The case where q_1, p_1, p_2 have a pole at $t=\infty$

By Lemma 2.2, let us note that $q_2 \not\equiv 0$.

Lemma 2.5. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, p_2 all have a pole at $t=\infty$ and q_2 is holomorphic at $t=\infty$. Moreover, assume that q_1, p_1, p_2 has a pole of order n_0, n_1, n_3 $(n_0,n_1,n_3\ge 1)$ at $t=\infty$, respectively. Then, $n_3 \ge n_0+n_1$.*

Proof. Considering that

$$tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0+\alpha_1+\alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,$$

we can prove the lemma. ∎

Therefore, we define the nonnegative integer k by $n_3 = n_0+n_1+k$.

Lemma 2.6. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, p_2 all have a pole at $t=\infty$ and q_2 is holomorphic at $t=\infty$. Then, $c_{\infty,0}=c_{\infty,-1}=\cdots=c_{\infty,-(k-1)}=0$, $a_{\infty,n_0}b_{\infty,1}+c_{\infty,-k}d_{\infty,n_3}=0$, and $n_0-k\ge 1$.*

Proof. Considering that

$$tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0+\alpha_1+\alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,$$

we find that $c_{\infty,0}=c_{\infty,-1}=\cdots=c_{\infty,-(k-1)}=0$, $a_{\infty,n_0}b_{\infty,1}+c_{\infty,-k}d_{\infty,n_3}=0$. Furthermore, considering that

$$tq_1' = 2q_1^2p_1 - q_1^2 + (\alpha_0+\alpha_1+\alpha_3)q_1 - t + 4tp_2 + 2q_1q_2p_2,$$

we can show the lemma. ∎

Proposition 2.10. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1, p_1, p_2 all have a pole at $t=\infty$ and q_2 is holomorphic at $t=\infty$.*

Proof. We treat the case where $n_1=1$. The other cases can be proved in the same way. Comparing the coefficients of the terms t^{n_0+1} in

$$tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0+\alpha_1+\alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,$$

we have

$$-2a_{\infty,n_0}b_{\infty,0} - 2a_{\infty,n_0-1}b_{\infty,1} + 2a_{\infty,n_0} - 2c_{\infty,-k}d_{\infty,n_3-1} - 2c_{\infty,-k-1}d_{\infty,n_3} = 0.$$

If $n_0-k\ge 3$, comparing the coefficients of the terms t^{2n_0} in

$$tq_1' = 2q_1^2p_1 - q_1^2 + (\alpha_0+\alpha_1+\alpha_3)q_1 - t + 4tp_2 + 2q_1q_2p_2,$$

we obtain

$$2b_{\infty,1}a_{\infty,n_0-1} + 2a_{\infty,n_0}b_{\infty,0} - a_{\infty,n_0} + 2c_{\infty,-k}d_{\infty,n_3-1} + 2c_{\infty,-k-1}d_{\infty,n_3} = 0.$$

Then, it follows that $a_{\infty,n_0}=0$, which is impossible.

If $n_0-k=2$, comparing the coefficients of the terms t^2, t^{3n_0-1} in

$$tq_2' = 2q_2^2p_2 - q_2^2 + (\alpha_0+\alpha_1+\alpha_3)q_2 - t + 4tp_1 + 2q_1p_1q_2,$$
$$tp_2' = -2q_2p_2^2 + 2q_2p_2 - (\alpha_0+\alpha_1+\alpha_3)p_2 + \alpha_1 - 2q_1p_1p_2,$$

we have

$$2d_{\infty,n_3}c_{\infty,-k}c_{\infty,-k-1} + 2d_{\infty,n_3-1}c_{\infty,-k}^2 + 4b_{\infty,1}$$
$$+ 2c_{\infty,-k}a_{\infty,n_0}b_{\infty,0} + 2c_{\infty,-k}a_{\infty,n_0-1}b_{\infty,1} = 0,$$
$$-2c_{\infty,-k}d_{\infty,n_3-1} - 2c_{\infty,-k-1}d_{\infty,n_3} - 2a_{\infty,n_0}b_{\infty,0} - 2a_{\infty,n_0-1}b_{\infty,1} = 0,$$

respectively. Then, it follows that $b_{\infty,1} = 0$, which is impossible.

If $n_0 - k = 1$, comparing the coefficients of the terms t^2 in

$$tq_2' = 2q_2^2p_2 - q_2^2 + (\alpha_0 + \alpha_1 + \alpha_3)q_2 - t + 4tp_1 + 2q_1p_1q_2,$$

we obtain

$$2c_{\infty,-k}^2d_{\infty,n_3} + 4b_{\infty,1} + 2a_{\infty,n_0}b_{\infty,1}c_{\infty,-k} = 4b_{\infty,1} = 0,$$

which is impossible. ■

2.5 The case where all of (q_1, p_1, q_2, p_2) have a pole at $t = \infty$

Lemma 2.7. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, q_2, p_2 all have a pole at $t = \infty$. Moreover, assume that q_1, p_1, q_2, p_2 have a pole of order n_0, n_1, n_2, n_3 $(n_0, n_1, n_2, n_3 \ge 1)$ at $t = \infty$, respectively. Then, $n_0 + n_1 = n_2 + n_3$.*

Proof. Considering

$$tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0 + \alpha_1 + \alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,$$

we can show the lemma. ■

Therefore, we see that $n_0 + n_1 = n_2 + n_3 \ge 2$.

Proposition 2.11. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists no solution such that q_1, p_1, q_2, p_2 all have a pole at $t = \infty$.*

Proof. We treat the case where $n_0 + n_1 = n_2 + n_3 = 2$. The other cases can be proved in the same way.

Comparing the coefficients of the term t^3 in

$$tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0 + \alpha_1 + \alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,$$

we have $a_{\infty,1}b_{\infty,1} + c_{\infty,1}d_{\infty,1} = 0$.

Comparing the coefficients of the term t^2 in

$$tq_1' = 2q_1^2p_1 - q_1^2 + (\alpha_0 + \alpha_1 + \alpha_3)q_1 - t + 4tp_2 + 2q_1q_2p_2,$$
$$tp_1' = -2q_1p_1^2 + 2q_1p_1 - (\alpha_0 + \alpha_1 + \alpha_3)p_1 + \alpha_0 - 2p_1q_2p_2,$$
$$tq_2' = 2q_2^2p_2 - q_2^2 + (\alpha_0 + \alpha_1 + \alpha_3)q_2 - t + 4tp_1 + 2q_1p_1q_2,$$
$$tp_2' = -2q_2p_2^2 + 2q_2p_2 - (\alpha_0 + \alpha_1 + \alpha_3)p_2 + \alpha_1 - 2q_1p_1p_2,$$

we obtain

$$2a_{\infty,1}a_{\infty,0}b_{\infty,1} + 2b_{\infty,0}a_{\infty,1}^2 - a_{\infty,1}^2 + 4d_{\infty,1} + 2a_{\infty,1}c_{\infty,1}d_{\infty,0} + 2a_{\infty,1}c_{\infty,0}d_{\infty,1} = 0,$$
$$-2a_{\infty,1}b_{\infty,0} - 2a_{\infty,0}b_{\infty,1} + 2a_{\infty,1} - 2c_{\infty,1}d_{\infty,0} - 2c_{\infty,0}d_{\infty,1} = 0,$$
$$2c_{\infty,1}c_{\infty,0}d_{\infty,1} + 2d_{\infty,0}c_{\infty,1}^2 - c_{\infty,1}^2 + 4b_{\infty,1} + 2c_{\infty,1}a_{\infty,1}b_{\infty,0} + 2c_{\infty,1}a_{\infty,0}b_{\infty,1} = 0,$$
$$-2c_{\infty,1}d_{\infty,0} - 2c_{\infty,0}d_{\infty,1} + 2c_{\infty,1} - 2a_{\infty,1}b_{\infty,0} - 2a_{\infty,0}b_{\infty,1} = 0, \tag{2.3}$$

respectively. Based on the second and fourth equations of (2.3), we have $a_{\infty,1} = c_{\infty,1}$. From the first and second equations of (2.3), we obtain $a_{\infty,1}^2 + 4d_{\infty,1} = 0$. From the third and fourth equations of (2.3), we have $c_{\infty,1}^2 + 4b_{\infty,1} = 0$.

Therefore, since $a_{\infty,1}b_{\infty,1} + c_{\infty,1}d_{\infty,1} = 0$, it follows that $a_{\infty,1}b_{\infty,1} = 0$, which is impossible. ■

2.6 Summary

Proposition 2.12. *For $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a meromorphic solution at $t = \infty$. Then, q_1, p_1, q_2, p_2 are uniquely expanded as follows:*

$$\begin{cases} q_1 = (-2\alpha_0 + \alpha_3) + \cdots, \\ p_1 = 1/4 + (-2\alpha_1 + \alpha_3)(-2\alpha_1 - \alpha_3)t^{-1}/4 + \cdots, \\ q_2 = (-2\alpha_1 + \alpha_3) + \cdots, \\ p_2 = 1/4 + (-2\alpha_0 + \alpha_3)(-2\alpha_0 - \alpha_3)t^{-1}/4 + \cdots. \end{cases}$$

3 Meromorphic solution at $t = 0$

In this section, we treat meromorphic solutions at $t = 0$. Then, in the same way as Proposition 2.12, we can show the following proposition:

Proposition 3.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a meromorphic solution at $t = 0$. Then, one of the following occurs:*

(1) *q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$,*

(2) *p_1 has a pole of order one at $t = 0$ and q_1, q_2, p_2 are all holomorphic at $t = 0$,*

(3) *p_2 has a pole of order one at $t = 0$ and q_1, p_1, q_2 are all holomorphic at $t = 0$.*

In this paper, we define the coefficients of the Lauren series of q_1, p_1, q_2, p_2 at $t = 0$ by $a_{0,k}$, $b_{0,k}$, $c_{0,k}$, $d_{0,k}$, $k \in \mathbb{Z}$. In this section, we prove that the constant terms of q_1, q_2 at $t = 0$, $a_{0,0}$, $c_{0,0}$ are zero, or expressed by the parameters, α_j $(0 \le j \le 3)$.

3.1 The case where q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$

Proposition 3.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$. Then, one of the following occurs:*

(1) $a_{0,0} = 0$, $-(\alpha_0 + \alpha_1 + \alpha_3)b_{0,0} + \alpha_0 = 0$, $c_{0,0} = 0$, $-(\alpha_0 + \alpha_1 + \alpha_3)d_{0,0} + \alpha_1 = 0$,

(2) $a_{0,0} = 0$, $(-\alpha_0 + \alpha_1 - \alpha_3)b_{0,0} + \alpha_0 = 0$, $c_{0,0} = \alpha_0 - \alpha_1 + \alpha_3$, $(-\alpha_0 + \alpha_1 - \alpha_3)d_{0,0} - \alpha_1 = 0$,

(3) $a_{0,0} = -\alpha_0 + \alpha_1 + \alpha_3$, $(\alpha_0 - \alpha_1 - \alpha_3)b_{0,0} - \alpha_0 = 0$, $c_{0,0} = 0$, $(\alpha_0 - \alpha_1 - \alpha_3)d_{0,0} + \alpha_1 = 0$,

(4) $a_{0,0} = -\alpha_0 - \alpha_1 + \alpha_3$, $(\alpha_0 + \alpha_1 - \alpha_3)b_{0,0} - \alpha_0 = 0$, $c_{0,0} = -\alpha_0 - \alpha_1 + \alpha_3$, $(\alpha_0 + \alpha_1 - \alpha_3)d_{0,0} - \alpha_1 = 0$.

3.2 The case where p_1 has a pole at $t = 0$

Proposition 3.3. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that p_1 has a pole at $t = 0$ and q_1, q_2, p_2 are all holomorphic at $t = 0$. Then,*

$$\begin{cases} q_1 = (-8\alpha_0 - 8\alpha_3 + 6)t/\{(4\alpha_1 - 1)(4\alpha_1 + 1)\} + \cdots, \\ p_1 = (4\alpha_1 - 1)(4\alpha_1 + 1)t^{-1}/16 + \cdots, \\ q_2 = (-2\alpha_1 + 1/2) + \cdots, \\ p_2 = 1/4 + \cdots. \end{cases}$$

3.3 The case where p_2 has a pole at $t = 0$

Proposition 3.4. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that p_2 has a pole at $t=0$ and q_1, p_1, q_2 are all holomorphic at $t=0$. Then,*

$$\begin{cases} q_1 = (-2\alpha_0 + 1/2) + \cdots, \\ p_1 = 1/4 + \cdots, \\ q_2 = (-8\alpha_1 - 8\alpha_3 + 6)t/\{(4\alpha_0 - 1)(4\alpha_0 + 1)\} + \cdots, \\ p_2 = (4\alpha_0 - 1)(4\alpha_0 + 1)t^{-1}/16 + \cdots. \end{cases}$$

4 Meromorphic solution at $t = c \in \mathbb{C}^*$

In this section, we deal with meromorphic solutions at $t = c \in \mathbb{C}^*$, where $\mathbb{C}^*$ means the set of nonzero complex numbers.

Proposition 4.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a meromorphic solution at $t = c \in \mathbb{C}^*$ such that some of (q_1, p_1, q_2, p_2) have a pole at $t = c$. Then, one of the following occurs:*

(1) *q_1 has a pole at $t = c$ and p_1, q_2, p_2 are all holomorphic at $t = c$,*

(2) *q_2 has a pole at $t = c$ and q_1, p_1, p_2 are all holomorphic at $t = c$,*

(3) *q_1, q_2 have both a pole at $t = c$ and p_1, p_2 are both holomorphic at $t = c$,*

(4) *q_1, p_2 have both a pole at $t = c$ and p_1, q_2 are both holomorphic at $t = c$,*

(5) *p_1, q_2 have both a pole at $t = c$ and q_1, p_2 are both holomorphic at $t = c$,*

(6) *p_1, p_2 have both a pole at $t = c$ and q_1, q_2 are both holomorphic at $t = c$,*

(7) *q_1, p_1, q_2, p_2 all have a pole at $t = c$.*

4.1 The case where q_1 has a pole at $t = c \in \mathbb{C}^*$

Proposition 4.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1 has a pole at $t = c \in \mathbb{C}^*$ and p_1, q_2, p_2 are all holomorphic at $t = c$. Then, either of the following occurs:*

$$(1)\ \begin{cases} q_1 = c(t-c)^{-1} + \cdots, \\ p_1 = -\dfrac{\alpha_0}{c}(t-c) + \cdots, \end{cases} \qquad (2)\ \begin{cases} q_1 = -c(t-c)^{-1} + \cdots, \\ p_1 = 1 + \dfrac{\alpha_1 + \alpha_3}{c}(t-c) + \cdots, \\ q_2 = O(t-c), \\ p_2 = O(t-c). \end{cases}$$

4.2 The case where q_2 has a pole at $t = c \in \mathbb{C}^*$

Proposition 4.3. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_2 has a pole at $t = c \in \mathbb{C}^*$ and q_1, p_1, p_2 are all holomorphic at $t = c$. Then, either of the following occurs:*

$$(1)\ \begin{cases} q_2 = c(t-c)^{-1} + \cdots, \\ p_2 = -\dfrac{\alpha_1}{c}(t-c) + \cdots, \end{cases} \qquad (2)\ \begin{cases} q_1 = O(t-c), \\ p_1 = O(t-c), \\ q_2 = -c(t-c)^{-1} + \cdots, \\ p_2 = 1 + \dfrac{\alpha_0 + \alpha_3}{c}(t-c) + \cdots. \end{cases}$$

4.3 The case where q_1, q_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.4. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, q_2 have both a pole at $t = c \in \mathbb{C}^*$ and p_1, p_2 are both holomorphic at $t = c$. Then, either of the following occurs:*

$$(1)\ \begin{cases} q_1 = -c(t-c)^{-1} + \cdots , \\ p_1 = b_{c,0} + b_{c,1}(t-c) + \cdots , \\ q_2 = -c(t-c)^{-1} + \cdots , \\ p_2 = d_{c,0} + d_{c,1}(t-c) + \cdots , \end{cases} \qquad (2)\ \begin{cases} q_1 = c(t-c)^{-1} + \cdots , \\ p_1 = -\dfrac{\alpha_0}{c}(t-c) + \cdots , \\ q_2 = c(t-c)^{-1} + \cdots , \\ p_2 = -\dfrac{\alpha_1}{c}(t-c) + \cdots , \end{cases}$$

where $b_{c,0} + d_{c,0} = 1$ and $b_{c,1} + d_{c,1} = \frac{\alpha_3}{c}$.

4.4 The case where q_1, p_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.5. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_2 have both a pole at $t = c \in \mathbb{C}^*$ and p_1, q_2 are both holomorphic at $t = c$. Then, one of the following occurs:*

$$(1)\ \begin{cases} q_1 = 4c(t-c)^{-1} + 8/3 + \cdots , \\ p_1 = 0 - \alpha_0/\{5c\} \cdot (t-c) + \cdots , \\ q_2 = (t-c) + (3\alpha_0 - \alpha_1 - \alpha_3 + 2)/\{2c\} \cdot (t-c)^2 + \cdots , \\ p_2 = c(t-c)^{-2} - 4\alpha_0/5 \cdot (t-c)^{-1} + \cdots , \end{cases}$$

$$(2)\ \begin{cases} q_1 = -c(t-c)^{-1} + (-1/4 - \alpha_0) + \cdots , \\ p_1 = 0 + \alpha_0/\{5c\} \cdot (t-c) \\ q_2 = (t-c) + (-3\alpha_0 - \alpha_1 - \alpha_3 + 2)/\{2c\} \cdot (t-c)^2 + \cdots , \\ p_2 = c(t-c)^{-2} + 4\alpha_0/5 \cdot (t-c)^{-1} + \cdots , \end{cases}$$

$$(3)\ \begin{cases} q_1 = c(t-c)^{-1} + (3/4 - \alpha_0) + \cdots , \\ p_1 = 1/2 - 1/\{12c\} \cdot (t-c) + \cdots , \\ q_2 = -(t-c) + [(\alpha_1 + \alpha_3)/c - 3/\{4c\}](t-c)^2 + \cdots , \\ p_2 = -c/2 \cdot (t-c)^{-2} - 1/6(t-c)^{-1} + \cdots . \end{cases}$$

4.5 The case where p_1, q_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.6. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that p_1, q_2 have both a pole at $t = c \in \mathbb{C}^*$ and q_1, p_2 are both holomorphic at $t = c$. Then, one of the following occurs:*

$$(1)\ \begin{cases} q_1 = (t-c) + (3\alpha_1 - \alpha_0 - \alpha_3 + 2)/\{2c\} \cdot (t-c)^2 + \cdots , \\ p_1 = c(t-c)^{-2} - 4\alpha_1/5 \cdot (t-c)^{-1} + \cdots , \\ q_2 = 4c(t-c)^{-1} + 8/3 + \cdots , \\ p_2 = 0 - \alpha_1/\{5c\} \cdot (t-c) + \cdots , \end{cases}$$

$$(2)\ \begin{cases} q_1 = (t-c) + (-3\alpha_1 - \alpha_0 - \alpha_3 + 2)/\{2c\} \cdot (t-c)^2 + \cdots , \\ p_1 = c(t-c)^{-2} + 4\alpha_1/5 \cdot (t-c)^{-1} + \cdots , \\ q_2 = -c(t-c)^{-1} + (-1/4 - \alpha_1) + \cdots , \\ p_2 = 0 + \alpha_1/\{5c\} \cdot (t-c) + \cdots , \end{cases}$$

$$(3)\quad \begin{cases} q_1 = -(t-c) + [(\alpha_0+\alpha_3)/c - 3/\{4c\}](t-c)^2 + \cdots, \\ p_1 = -c/2 \cdot (t-c)^{-2} - 1/6(t-c)^{-1} + \cdots, \\ q_2 = c(t-c)^{-1} + (3/4-\alpha_1) + \cdots, \\ p_2 = 1/2 - 1/\{12c\} \cdot (t-c) + \cdots. \end{cases}$$

4.6 The case where p_1, p_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.7. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that p_1, p_2 have both a pole at $t = c \in \mathbb{C}^*$ and q_1, q_2 are both holomorphic at $t = c$. Then,*

$$\begin{cases} q_1 = (-4d_{c,-1}) + a_{c,1}(t-c) + \cdots, \\ p_1 = b_{c,-1}(t-c)^{-1} + (3/8 + 2b_{c,-1}^2/c) + \cdots, \\ q_2 = (-4b_{c,-1}) + c_{c,1}(t-c) + \cdots, \\ p_2 = d_{c,-1}(t-c)^{-1} + (3/8 + 2d_{c,-1}^2/c) + \cdots, \end{cases}$$

where the coefficients satisfy

$$16b_{c,-1}d_{c,-1} + c = 0, \qquad a_{c,1}b_{c,-1}c + c_{c,1}d_{c,-1}c + \frac{c}{2}(\alpha_0+\alpha_1+\alpha_3) = \frac{c}{2}.$$

4.7 The case where q_1, p_1, q_2, p_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 4.8. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, q_2, p_2 all have a pole at $t = c \in \mathbb{C}^*$. Then,*

$$\begin{cases} q_1 = -2c(t-c)^{-1} + (\sqrt{c} - 4/3) + a_{c,1}(t-c) + \cdots, \\ p_1 = \sqrt{c}/4 \cdot (t-c)^{-1} + 1/2 + b_{c,1}(t-c) + \cdots, \\ q_2 = -2c(t-c)^{-1} + (-\sqrt{c} - 4/3) + c_{c,1}(t-c) + \cdots, \\ p_2 = -\sqrt{c}/4 \cdot (t-c)^{-1} + 1/2 + d_{c,1}(t-c) + \cdots, \end{cases}$$

where the coefficients satisfy

$$b_{c,1} + d_{c,1} = \alpha_3/\{2c\}, \qquad a_{c,1}\sqrt{c} - c_{c,1}\sqrt{c} = 2 + 2\alpha_3 - 2\alpha_0 - 2\alpha_1.$$

4.8 Summary

Proposition 4.9.

(1) *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a meromorphic solution at $t = c \in \mathbb{C}^*$. Then, q_1, q_2 have both a pole of order at most one at $t = c$ and the residues of q_1, q_2 at $t = c$ are expressed by nc $(n \in \mathbb{Z})$.*

(2) *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution. Then, $a_{\infty,0} - a_{0,0} \in \mathbb{Z}$, $c_{\infty,0} - c_{0,0} \in \mathbb{Z}$.*

Proof. Case (1) is obvious. Let us prove case (2). From the discussions in Sections 2, 3 and 4, it follows that

$$q_1 = a_{\infty,0} + \sum_{j=1}^{m_1} \frac{n_j c_j}{t - c_j}, \qquad q_2 = c_{\infty,0} + \sum_{k=1}^{m_2} \frac{n'_k c'_k}{t - c'_k}, \qquad n_j, n'_k \in \mathbb{Z},$$

where m_1, m_2 are both positive integers and $c_k \in \mathbb{C}^*$ $(1 \le k \le m_1)$ and $c'_j \in \mathbb{C}^*$ $(1 \le j \le m_2)$ are poles of q_1 and q_2, respectively. If q_1 or q_2 is holomorphic in $\mathbb{C}^*$, then its second sum is considered to be zero.

Considering the constant terms of the Taylor series of q_1, q_2 at $t = 0$, we can prove the proposition. ■

5 The Laurent series of the Hamiltonian H

In this section, for a meromorphic solution at $t=\infty, 0$, we first compute the constant terms $h_{\infty,0}$, $h_{0,0}$ of the Laurent series of the Hamiltonian H at $t=\infty, 0$. Moreover, for a meromorphic solution at $t=c\in\mathbb{C}^*$, we calculate the residue of H at $t=c$.

5.1 The Laurent series of H at $t=\infty$

Proposition 5.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$, there exists a meromorphic solution at $t=\infty$. Then,*

$$h_{\infty,0}=\frac{3}{4}(\alpha_0+\alpha_1+\alpha_3)^2-\frac{1}{2}(-2\alpha_0+\alpha_3)(-2\alpha_1+\alpha_3)-3(\alpha_0+\alpha_1)\alpha_3.$$

5.2 The Laurent series of H at $t=0$

5.2.1 The case where q_1, p_1, q_2, p_2 are all holomorphic at $t=0$

Proposition 5.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$, there exists a solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t=0$. Then,*

$$h_{0,0}=\begin{cases}0 & \text{if case (1) occurs in Proposition 3.2,}\\ -\alpha_1(\alpha_0+\alpha_3) & \text{if case (2) occurs in Proposition 3.2,}\\ -\alpha_0(\alpha_1+\alpha_3) & \text{if case (3) occurs in Proposition 3.2,}\\ -\alpha_3(\alpha_0+\alpha_1) & \text{if case (4) occurs in Proposition 3.2.}\end{cases}$$

5.2.2 The case where p_1 has a pole at $t=0$

Proposition 5.3. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$, there exists a solution such that p_1 has a pole at $t=0$ and q_1, q_2, p_2 are all holomorphic at $t=0$. Then,*

$$h_{0,0}=-\frac{1}{4}(\alpha_0+\alpha_1+\alpha_3)^2+\alpha_1^2+\frac{3}{16}.$$

5.2.3 The case where p_2 has a pole at $t=0$

Proposition 5.4. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$, there exists a solution such that p_2 has a pole at $t=0$ and q_1, p_1, q_2 are all holomorphic at $t=0$. Then,*

$$h_{0,0}=-\frac{1}{4}(\alpha_0+\alpha_1+\alpha_3)^2+\alpha_0^2+\frac{3}{16}.$$

5.3 The Laurent series of H at $t=c\in\mathbb{C}^*$

5.3.1 The case where q_1 has a pole at $t=c\in\mathbb{C}^*$

Proposition 5.5. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$, there exists a solution such that q_1 has a pole at $t=c\in\mathbb{C}^*$ and p_1, q_2, p_2 are all holomorphic at $t=c$. Then, H is holomorphic at $t=c$.*

5.3.2 The case where q_2 has a pole at $t=c\in\mathbb{C}^*$

Proposition 5.6. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\leq j\leq 3}$, there exists a solution such that q_2 has a pole at $t=c\in\mathbb{C}^*$ and q_1, p_1, p_2 are all holomorphic at $t=c$. Then, H is holomorphic at $t=c$.*

5.3.3 The case where q_1, q_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 5.7. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, q_2 have both a pole at $t = c \in \mathbb{C}^*$ and p_1, p_2 are both holomorphic at $t = c$. Then, H is holomorphic at $t = c$.*

5.3.4 The case where q_1, p_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 5.8. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, q_2 have both a pole at $t = c \in \mathbb{C}^*$ and p_1, p_2 are both holomorphic at $t = c$. Then, H has a pole of order one at $t = c$ and*

$$\operatorname*{Res}_{t=c} H = \begin{cases} c & \text{if case (1) occurs in Proposition 4.5,} \\ c & \text{if case (2) occurs in Proposition 4.5,} \\ c/2 & \text{if case (3) occurs in Proposition 4.5.} \end{cases}$$

5.3.5 The case where p_1, q_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 5.9. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that p_1, q_2 have both a pole at $t = c \in \mathbb{C}^*$ and q_1, p_2 are both holomorphic at $t = c$. Then, H has a pole of order one at $t = c$ and*

$$\operatorname*{Res}_{t=c} H = \begin{cases} c & \text{if case (1) occurs in Proposition 4.6,} \\ c & \text{if case (2) occurs in Proposition 4.6,} \\ c/2 & \text{if case (3) occurs in Proposition 4.6.} \end{cases}$$

5.3.6 The case where p_1, p_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 5.10. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that p_1, p_2 have both a pole at $t = c \in \mathbb{C}^*$ and q_1, q_2 are both holomorphic at $t = c$. Then, H has a pole of order one at $t = c$ and $\operatorname*{Res}_{t=c} H = c/4$.*

5.3.7 The case where q_1, p_1, q_2, p_2 have a pole at $t = c \in \mathbb{C}^*$

Proposition 5.11. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a solution such that q_1, p_1, q_2, p_2 all have a pole at $t = c \in \mathbb{C}^*$. Then, H has a pole of order one at $t = c$ and $\operatorname*{Res}_{t=c} H = c/4$.*

5.4 Summary

Proposition 5.12.

(1) *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a meromorphic solution at $t = c \in \mathbb{C}^*$. Then, the residue of H at $t = c$ is expressed by $nc/4$ ($n \in \mathbb{Z}$).*

(2) *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution. Then, $4(h_{\infty,0} - h_{0,0}) \in \mathbb{Z}$.*

Proof. Case (1) is obvious. Case (2) can be proved in the same way as Proposition 4.9. ■

6 Necessary condition ... (1)

6.1 The case where q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$

6.1.1 The case where $a_{0,0} = 0$, $c_{0,0} = 0$

Proposition 6.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$. Moreover, assuming that $a_{0,0} = 0$, $c_{0,0} = 0$, then, $-2\alpha_0 + \alpha_3 \in \mathbb{Z}$, $-2\alpha_1 + \alpha_3 \in \mathbb{Z}$.*

Proof. The proposition follows from Propositions 2.12, 4.9. ■

6.1.2 The case where $a_{0,0} = 0$, $c_{0,0} \neq 0$

Proposition 6.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$. Moreover, assuming that $a_{0,0} = 0$, $c_{0,0} \neq 0$, then, $-2\alpha_0 + \alpha_3 \in \mathbb{Z}$, $2\alpha_1 + \alpha_3 \in \mathbb{Z}$.*

Proof. The proposition follows from Propositions 2.12, 3.2 and 4.9. ■

6.1.3 The case where $a_{0,0} \neq 0$, $c_{0,0} = 0$

Proposition 6.3. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$. Moreover, assuming that $a_{0,0} \neq 0$, $c_{0,0} = 0$, then, $2\alpha_0 + \alpha_3 \in \mathbb{Z}$, $-2\alpha_1 + \alpha_3 \in \mathbb{Z}$.*

Proof. The proposition follows from Propositions 2.12, 3.2 and 4.9. ■

6.1.4 The case where $a_{0,0} \neq 0$, $c_{0,0} \neq 0$

Proposition 6.4. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$. Moreover, assuming that $a_{0,0} \neq 0$, $c_{0,0} \neq 0$, then, $2\alpha_0 + \alpha_3 \in \mathbb{Z}$, $2\alpha_1 + \alpha_3 \in \mathbb{Z}$.*

Proof. From Propositions 2.12, 3.2 and 4.9, it follows that $\alpha_0 - \alpha_1 \in \mathbb{Z}$.

If $\alpha_0 \neq 0$, by Proposition 3.2, we find that $s_0(q_1, p_1, q_2, p_2)$ is a rational solution of $A_5^{(2)}(-\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3)$ such that all of $s_0(q_1, p_1, q_2, p_2)$ are holomorphic at $t = 0$ and $a_{0,0} = 0$, $c_{0,0} \neq 0$. Then, from Proposition 6.2, we obtain the necessary condition. If $\alpha_1 \neq 0$, by s_1 and Proposition 6.3, we obtain the necessary condition in the same way.

If $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 \neq 0$, by Proposition 3.2, we see that $s_2(q_1, p_1, q_2, p_2)$ is a rational solution of $A_5^{(2)}(\alpha_2, \alpha_2, -\alpha_2, \alpha_3 + 2\alpha_2)$ such that all of $s_0(q_1, p_1, q_2, p_2)$ are holomorphic at $t = 0$ and $a_{0,0} \neq 0$, $c_{0,0} \neq 0$. Based on the above discussion, considering that $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 = 1/2$, we can obtain the necessary condition.

The remaining case is that $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1/2$. We prove that for $A_5^{(2)}(0, 0, 0, 1/2)$, there exists no rational solution such that q_1, p_1, q_2, p_2 are all holomorphic at $t = 0$ and $a_{0,0} \neq 0$, $c_{0,0} \neq 0$. If there exists such a rational solution, by Proposition 3.2, we find that $b_{0,0} = d_{0,0} = 0$. Then, $s_3(q_1, p_1, q_2, p_2)$ is a rational solution of $A_5^{(2)}(0, 0, 1/2, -1/2)$ such that all of $s_3(q_1, p_1, q_2, p_2)$ are holomorphic at $t = 0$ and $a_{0,0} = c_{0,0} = 0$. Therefore, it follows from Proposition 6.2 that $-2 \cdot 0 + (-1/2) \in \mathbb{Z}$, which is impossible. ■

6.2 The case where p_1 has a pole at $t = 0$

Proposition 6.5. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that p_1 has a pole at $t=0$ and q_1, q_2, p_2 are all holomorphic at $t=0$. Then, $-2\alpha_0+\alpha_3\in\mathbb{Z}$, $\alpha_3-1/2\in\mathbb{Z}$.*

Proof. The proposition follows from Propositions 2.12, 3.3 and 4.9. ∎

By $s_1 s_2$, we can prove the following corollary.

Corollary 6.1. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that p_1 has a pole at $t=0$ and q_1, q_2, p_2 are all holomorphic at $t=0$. Then, by some Bäcklund transformations, the parameters can be transformed so that $-2\alpha_0+\alpha_3\in\mathbb{Z}$, $-2\alpha_1+\alpha_3\in\mathbb{Z}$.*

6.3 The case where p_2 has a pole at $t = 0$

Proposition 6.6. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that p_2 has a pole at $t=0$ and q_1, p_1, q_2 are all holomorphic at $t=0$. Then, $-2\alpha_1+\alpha_3\in\mathbb{Z}$, $\alpha_3-1/2\in\mathbb{Z}$.*

Proof. The proposition follows from Propositions 2.12, 3.4 and 4.9. ∎

By $s_0 s_2$, we can prove the following corollary.

Corollary 6.2. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution such that p_2 has a pole at $t=0$ and q_1, p_1, q_2 are all holomorphic at $t=0$. Then, by some Bäcklund transformations, the parameters can be transformed so that $-2\alpha_0+\alpha_3\in\mathbb{Z}$, $-2\alpha_1+\alpha_3\in\mathbb{Z}$.*

6.4 Summary

Proposition 6.7. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution. Then, one of the following occurs:*

$$
\begin{array}{lll}
(1) & -2\alpha_0+\alpha_3\in\mathbb{Z}, & -2\alpha_1+\alpha_3\in\mathbb{Z}, \\
(2) & -2\alpha_0+\alpha_3\in\mathbb{Z}, & 2\alpha_1+\alpha_3\in\mathbb{Z}, \\
(3) & 2\alpha_0+\alpha_3\in\mathbb{Z}, & -2\alpha_1+\alpha_3\in\mathbb{Z}, \\
(4) & 2\alpha_0+\alpha_3\in\mathbb{Z}, & 2\alpha_1+\alpha_3\in\mathbb{Z}, \\
(5) & -2\alpha_0+\alpha_3\in\mathbb{Z}, & \alpha_3-1/2\in\mathbb{Z}, \\
(6) & -2\alpha_1+\alpha_3\in\mathbb{Z}, & \alpha_3-1/2\in\mathbb{Z}.
\end{array}
$$

Corollary 6.3. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution. Then, by some Bäcklund transformations, the parameters can be transformed so that $-2\alpha_0+\alpha_3\in\mathbb{Z}$, $-2\alpha_1+\alpha_3\in\mathbb{Z}$.*

7 Necessary condition . . . (2)

7.1 Shift operators

In order to transform the parameters to the standard form, let us construct shift operators.

Proposition 7.1. *Let the shift operators T_0, T_1, T_2 be defined by*

$$T_0 = \pi s_2 s_3 s_2 s_1 s_0, \qquad T_1 = s_0 T_0 s_0, \qquad T_2 = s_2 T_0 s_2,$$

respectively. Then,

$$T_0(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha_0 + 1/2, \alpha_1 + 1/2, \alpha_2 - 1/2, \alpha_3),$$
$$T_1(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha_0 - 1/2, \alpha_1 + 1/2, \alpha_2, \alpha_3),$$
$$T_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha_0, \alpha_1, \alpha_2 + 1/2, \alpha_3 - 1),$$

respectively.

7.2 The properties of Bäcklund transformations

Proposition 7.2.

(1) *If $p_1 \equiv 0$ for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, then $\alpha_0 = 0$.*

(2) *If $p_2 \equiv 0$ for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, then $\alpha_1 = 0$.*

(3) *If $q_1 q_2 + t \equiv 0$ for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, then $\alpha_2 = 0$.*

(4) *If $p_1 + p_2 - 1 \equiv 0$ for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, then $\alpha_3 = 0$.*

By this proposition, we can consider s_0 as the identical transformation, if $p_0 \equiv 0$. In the same way, we consider each of s_1, s_2, s_3 as the identical transformation, if $p_2 \equiv 0$, or if $q_1 q_2 + t \equiv 0$, or if $p_1 + p_2 - 1 \equiv 0$, respectively.

7.3 Reduction of the parameters to the standard form

By Corollary 6.3, using T_0, we can transform the parameters to $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha_3/2, \alpha_3/2, \alpha_2, \alpha_3)$.

Proposition 7.3. *Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution. Then, by some Bäcklund transformations, the parameters can be transformed so that $-2\alpha_0 + \alpha_3 = 0$, $-2\alpha_1 + \alpha_3 = 0$.*

8 Classification of rational solutions

8.1 Rational solution of $A_5^{(2)}(\alpha_3/2, \alpha_3/2, \alpha_2, \alpha_3)$

Proposition 8.1. *For $A_5^{(2)}(\alpha_3/2, \alpha_3/2, \alpha_2, \alpha_3)$, there exists a rational solution and $(q_1, p_1, q_2, p_2) = (0, 1/4, 0, 1/4)$. Moreover, it is unique.*

Proof. The proposition follows from the direct calculation and Proposition 2.2. ■

8.2 Proof of main theorem

Let us prove our main theorem.

Proof. Suppose that for $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$, there exists a rational solution. Then, from Proposition 6.7, we find that the parameters satisfy one of the conditions in the theorem. Moreover, from Proposition 7.3, we see that the parameters can be transformed so that $-2\alpha_0 + \alpha_3 = -2\alpha_1 + \alpha_3 = 0$.

From Proposition 8.1, it follows that for $A_5^{(2)}(\alpha_3/2, \alpha_3/2, \alpha_2, \alpha_3)$, there exists a unique rational solution such that $(q_1, p_1, q_2, p_2) = (0, 1/4, 0, 1/4)$, which proves the main theorem. ■

A Examples of rational solutions

In this appendix, we give examples of rational solutions of $A_5^{(2)}(\alpha_j)_{0\le j\le 3}$. For the purpose, we use the shift operators, T_0, T_1, T_2, and the seed rational solution,

$$(q_1,p_1,q_2,p_2)=(0,1/4,0,1/4) \qquad \text{for} \quad A_5^{(2)}(\alpha_3/2,\alpha_3/2,\alpha_2,\alpha_3).$$

Then, we obtain the following examples of rational solutions:
for $A_5^{(2)}(\alpha_3/2+1/2,\alpha_3/2+1/2,\alpha_2-1/2,\alpha_3)$,

$$(q_1,p_1,q_2,p_2)=\left(-1,\frac{1}{4}+\frac{\alpha_3}{2(t+4\alpha_3^2)}-\frac{4\alpha_3+1}{4(t+1)},-1,\frac{1}{4}-\frac{\alpha_3}{2(t+4\alpha_3^2)}+\frac{4\alpha_3+1}{4(t+1)},\right);$$

for $A_5^{(2)}(\alpha_3/2-1/2,\alpha_3/2+1/2,\alpha_2,\alpha_3)$,

$$q_1=2\alpha_3-\cfrac{1}{1+\cfrac{\alpha_3(1-2\alpha_3)}{t}}+\cfrac{-2\alpha_3+2}{1+\cfrac{2\alpha_3+1}{-t+\alpha_3(2\alpha_3+1)-\cfrac{1}{1+\cfrac{\alpha_3(-2\alpha_3+1)}{t}}}},$$

$$p_1=\frac{1}{4}+\cfrac{2\alpha_3+1}{-4t+4\alpha_3(2\alpha_3+1)-\cfrac{1}{1-\cfrac{\alpha_3(2\alpha_3-1)}{t}}},$$

$$q_2=-\cfrac{1}{1-\cfrac{\alpha_3(2\alpha_3-1)}{t}},$$

$$p_2=\frac{1}{4}+\frac{\alpha_3(2\alpha_3-1)}{t}-\cfrac{2\alpha_3+1}{\cfrac{-4}{1+\cfrac{\alpha_3(1-2\alpha_3)}{t}}+\cfrac{4t}{2\alpha_3-\cfrac{1}{1+\cfrac{\alpha_3(1-2\alpha_3)}{t}}}};$$

for $A_5^{(2)}(\alpha_3/2,\alpha_3/2,\alpha_2+1/2,\alpha_3-1)$,

$$(q_1,p_1,q_2,p_2)=\Bigg(-1,\frac{1}{4}-\frac{2\alpha_3-1}{4\{t+(2\alpha_3-1)^2\}}+\frac{4\alpha_3-3}{2(t+1)},-1,$$
$$\frac{1}{4}+\frac{2\alpha_3-1}{4\{t+(2\alpha_3-1)^2\}}-\frac{4\alpha_3-3}{2(t+1)}\Bigg).$$

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