Nonlocal Symmetries, Telescopic Vector Fields and λ-Symmetries of Ordinary Differential Equations*

Concepción MURIEL and Juan Luis ROMERO

Department of Mathematics, University of Cádiz, 11510 Puerto Real, Spain
E-mail: *concepcion.muriel@uca.es*, *juanluis.romero@uca.es*

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Abstract. This paper studies relationships between the order reductions of ordinary differential equations derived by the existence of λ-symmetries, telescopic vector fields and some nonlocal symmetries obtained by embedding the equation in an auxiliary system. The results let us connect such nonlocal symmetries with approaches that had been previously introduced: the exponential vector fields and the λ-coverings method. The λ-symmetry approach let us characterize the nonlocal symmetries that are useful to reduce the order and provides an alternative method of computation that involves less unknowns. The notion of equivalent λ-symmetries is used to decide whether or not reductions associated to two nonlocal symmetries are strictly different.

Key words: nonlocal symmetries; λ-symmetries; telescopic vector fields; order reductions; differential invariants

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1 Introduction

Local (or Lie point) symmetries have been extensively used in the study of differential equations [41, 42, 45]. For ordinary differential equations (ODEs), a local symmetry can be used to reduce the order by one. The equation can be integrated by quadratures if a sufficiently large solvable algebra of local symmetries is known. There are equations lacking local symmetries that can also be integrated [21, 22]. Several generalizations to the classical Lie method have been introduced with the aim of including these processes of integration. A number of them are based on the existence of nonlocal symmetries, i.e. symmetries with one or more of the coefficient functions containing an integral. Many of them appear in order reduction procedures as hidden symmetries [1, 2, 4, 5, 6, 29, 40]. During the last two decades a considerable number of papers have been devoted to the study of nonlocal symmetries and their role in the integration of differential equations [7, 24], including equations lacking Lie point symmetries [3, 23].

An alternative approach that avoids nonlocal terms is based on the concept of λ-symmetry [34], that uses a vector field $\mathbf{v}$ and certain function λ; the *λ-prolongation* of $\mathbf{v}$ is done by using this function λ. A complete system of invariants for this λ-prolongation can be constructed by derivation of lower order invariants [36]. As a consequence, the order of an ODE invariant under a λ-symmetry can be reduced as for Lie point symmetries. Many of the procedures to reduce the order of ODEs, including equations that lack Lie point symmetries, can be explained by the existence of λ-symmetries [30]. From a geometrical point of view, several studies and interpretations of λ-symmetries have been made by several authors [12, 13, 14, 27] including further extensions of λ-symmetries to systems [15, 28], to partial differential equations [18],

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to variational problems [16, 38] and to difference equations [25]. Several applications of the λ-symmetry approach to relevant equations of the mathematical physics appear in [9, 10, 46].

A nonlocal interpretation of the λ-symmetries was proposed by D. Catalano-Ferraioli in [12] (see also [13] from a theoretical point of view). By embedding the equation into a suitable system (λ-*covering*) determined by the function λ, the λ-symmetries of the ODE can be connected to some standard but generalized symmetries of the system (that in the variables of the ODE involve nonlocal terms).

These techniques have been recently used in [11, 19, 20] to calculate some nonlocal symmetries of ODEs. In this work we show that cited method is essentially included in the framework of the λ-coverings and that the obtained reductions are consequence of the existence of λ-symmetries.

A review of the main results on λ-symmetries that are used in the paper is contained in Section 2, including the study of some new relationships with the telescopic vector fields introduced in [44]. A telescopic vector field can be considered as a λ-prolongation where the two first infinitesimals can depend on the first derivative of the dependent variable. We also prove the existence of a (generalized) λ-symmetry associated to any telescopic vector field that leaves invariant the given equation (Corollary 1).

Motivated by the fact that the reduction procedure associated to the nonlocal symmetries obtained by the λ-covering method uses the method of the differential invariants, we prove in Section 3 the existence of a λ-symmetry associated to a nonlocal symmetry of this type. In Theorem 5 such correspondence between the nonlocal symmetries and the λ-symmetries is explicitly established. In Section 4 we prove that, for some special cases, such nonlocal symmetries are the called exponential vector fields introduced by P. Olver some years ago [41], which are related to λ-symmetries [34].

In Section 5 we show how to construct nonlocal symmetries of exponential type associated to a known λ-symmetry, which recovers the nonlocal interpretation of λ-symmetries given in [12]. This result shows that nonlocal symmetries of exponential type are a kind of prototype of the nonlocal symmetries useful to reduce the order. In fact, this is the usual form of the nonlocal symmetries reported in the references.

In Section 6 we investigate when two reductions associated to two different nonlocal symmetries are strictly different. This problem is, as far as we know, new in the literature and it is difficult to establish in terms of the nonlocal symmetries, because the reduction procedures correspond to different symmetries of different systems (the coverings associated to different functions). To overcome this difficulty we use the corresponding λ-symmetries and the notion of equivalent λ-symmetries introduced in [33] to provide an easy-to-check criterion to know whether or not two order reductions are equivalent.

Finally we collect some examples in Section 7 and prove that the reductions obtained by using nonlocal symmetries are equivalent to reduction procedures derived by λ-symmetries that have been previously reported in the literature.

2 λ-symmetries and order reductions

2.1 The invariants-by-derivation property and λ-prolongations

Let us consider a nth order ordinary differential equation written in the form

$$x_n = F(t, x, x_1, \dots, x_{n-1}), \tag{2.1}$$

where t denotes the independent variable, x is the dependent variable and $x_i = d^i x/dt^i$, for $i = 1, \dots, n$. For $i = 1$, x_1 is sometimes denoted by $x'(t)$. For first-order partial derivatives of a function of several variables we use subscripts of the corresponding independent variable.

We require the functions to be smooth, meaning $\mathcal{C}^\infty$, although most results hold under weaker differentiability requirements.

Let us assume that (t, x) are in some open set $M \subset \mathbb{R}^2$ and denote by $M^{(k)}$ the corresponding jet space of order k, for $k \in \mathbb{N}$. Let us consider the total derivative operator

$$D_t = \partial_t + x_1\partial_x + \cdots + x_i\partial_{x_{i-1}} + \cdots$$

and its restriction to the submanifold defined by the equation,

$$A = \partial_t + x_1\partial_x + \cdots + x_i\partial_{x_{i-1}} + \cdots + F\partial_{x_{n-1}},$$

that will be called the vector field associated to equation (2.1). For an arbitrary (smooth) vector field defined on M

$$\mathbf{v} = \xi(t, x)\partial_t + \eta^0(t, x)\partial_x \tag{2.2}$$

and for $k \in \mathbb{N}$, the usual kth order prolongation [41] of $\mathbf{v}$ is given by

$$\mathbf{v}^{(k)} = \xi\partial_t + \eta^0\partial_x + \sum_{i=1}^{k} \eta^i\partial_{x_i},$$

where, for $1 \le i \le k$,

$$\eta^i = D_t\big(\eta^{i-1}\big) - D_t(\xi)x_i. \tag{2.3}$$

The *infinitesimal Lie point symmetries* of equation (2.1) are the vector fields (2.2) such that $\mathbf{v}^{(n)}$ is tangent to the submanifold defined by equation (2.1). The invariance of (2.1) under $\mathbf{v}^{(n)}$ provides an overdetermined linear system of determining equations for the infinitesimals ξ and η^0. Assuming that a particular nontrivial solution of the system has been derived, an order reduction procedure of the equation can be carried out. Briefly, the first step of the method consists in calculating two invariants of $\mathbf{v}^{(1)}$,

$$z = z(t, x), \qquad \zeta = \zeta(t, x, x_1), \qquad \zeta_{x_1} \neq 0. \tag{2.4}$$

Let us recall that if a zero-order differential invariant $z = z(t, x)$ is known then a first-order invariant $\zeta = \zeta(t, x, x_1)$ can be found by quadrature ([17, Proposition 26.5, p. 97] and [43]). By successive derivations of ζ with respect to z, a complete system of invariants of $\mathbf{v}^{(n)}$

$$\{z, \zeta, \zeta_1, \ldots, \zeta_{n-1}\} \tag{2.5}$$

is constructed, where ζ_{i+1} denotes $D_t\zeta_i/D_tz$, for $i = 1, \ldots, n-2$. Since equation (2.1) is invariant under $\mathbf{v}^{(n)}$, the equation can be written in terms of (2.5) as a $(n-1)$th order equation. This algorithm is usually known as the *method of the differential invariants* to reduce the order.

The prolongation defined in (2.3) is not the only prolongation that lets obtain by derivation a complete system of invariants by using two invariants (2.4) of its first prolongation. This property of vector fields has been called the *invariants-by-derivation* (ID) property [36, Definition 1]. The prolongations with the ID property have been completely characterized in [36] as the so-called λ-*prolongations* [34]. For a function $\lambda = \lambda(t, x, x_1) \in \mathcal{C}^\infty(M^{(1)})$ and a vector field $\mathbf{X} = \rho(t, x)\partial_t + \phi^0(t, x)\partial_x$, the kth order λ-prolongation of $\mathbf{X}$ is the vector field

$$\mathbf{X}^{[\lambda,(k)]} = \rho\partial_t + \phi^0\partial_x + \sum_{i=1}^{k} \phi^{[\lambda,(i)]}\partial_{x_i},$$

where

$$\phi^{[\lambda,(i)]} = D_t\big(\phi^{[\lambda,(i-1)]}\big) - D_t(\rho)x_i + \lambda\big(\phi^{[\lambda,(i-1)]} - \rho x_i\big), \qquad 1 \le i \le k, \tag{2.6}$$

and $\phi^{[\lambda,(0)]} = \phi^0$. For $k \in \mathbb{N}$, the kth order λ-prolongation of $\mathbf{X}$ is characterized [34] as the unique vector field $\mathbf{X}^{[\lambda,(k)]}$ such that

$$\big[\mathbf{X}^{[\lambda,(k)]}, D_t\big] = \lambda \mathbf{X}^{[\lambda,(k)]} + \mu D_t, \qquad \text{where} \quad \mu = -(D_t + \lambda)(\rho). \tag{2.7}$$

Standard prolongations can be considered as a particular case of λ-prolongations for $\lambda = 0$.

We say that the pair $(\mathbf{X}, \lambda)$ defines a $\mathcal{C}^\infty(M^{(1)})$-*symmetry* (or that $\mathbf{X}$ is a λ-*symmetry*) of equation (2.1) if and only if $\mathbf{X}^{[\lambda,(n)]}$ is tangent to the submanifold defined by (2.1). This is equivalent [34] to the property

$$\big[\mathbf{X}^{[\lambda,(n-1)]}, A\big] = \lambda \mathbf{X}^{[\lambda,(n-1)]} + \mu A, \tag{2.8}$$

where $\mu = -(D_t + \lambda)(\rho)$. Obviously, if a vector field $\mathbf{X}$ is a λ-symmetry of equation (2.1) for the function $\lambda = 0$, then $\mathbf{X}$ becomes a Lie point symmetry of the equation.

2.2 λ-symmetries and order reductions

Since λ-prolongations have the ID property, the method of the differential invariants can be used to reduce the order, as well as for Lie point symmetries [34]:

Theorem 1. *If the pair* $(\mathbf{X}, \lambda)$ *defines a* $\mathcal{C}^\infty(M^{(1)})$*-symmetry of equation* (2.1) *and* (2.4) *are invariants of* $\mathbf{X}^{[\lambda,(1)]}$ *then the equation* (2.1) *can be written in terms of* (2.5) *as an ODE of order* $n-1$.

Such method has been successfully applied to reduce the order of a number of ODEs, many of them lacking Lie point symmetries [30]. In fact, many of the known reduction processes can be obtained via the above method as a consequence of the existence of λ-symmetries.

In this context, it is important to recall that the converse of Theorem 1 also holds. Although this result has been proven in [30], we present here an alternative proof that constructs explicitly the λ-symmetry that will be used later in the proof of Theorem 5.

Let us assume that there exist two functions $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ such that equation (2.1) can be written in terms of $\{z, \zeta, \zeta_1, \ldots, \zeta_{n-1}\}$ as an ODE of order $n-1$, denoted by $\Delta(z, \zeta, \ldots, \zeta_{n-1}) = 0$. Let us determine a vector field $\mathbf{X} = \rho(t,x)\partial_t + \phi^0(t,x)\partial_x$ and a function $\lambda = \lambda(t,x,x_1)$ with the conditions $\mathbf{X}(z) = 0$ and $\mathbf{X}^{[\lambda,(1)]}(\zeta) = 0$. The condition $\mathbf{X}(z) = 0$ is satisfied, for instance, if we choose $\rho = -z_x$ and $\phi^0 = z_t$, i.e., we can choose

$$\mathbf{X} = -z_x(t,x)\partial_t + z_t(t,x)\partial_x. \tag{2.9}$$

The function λ may be obtained from the condition $\mathbf{X}^{[\lambda,(1)]}(\zeta) = 0$,

$$\lambda = \frac{z_x\zeta_t - z_t\zeta_x}{D_t(z)\zeta_{x_1}} - \frac{D_t(z_t) + D_t(z_x)x_1}{D_t(z)}. \tag{2.10}$$

By construction, it is clear that (2.4) are invariants of $\mathbf{X}^{[\lambda,(1)]}$ and, by the ID property, the corresponding set (2.5) is a complete system of invariants of $\mathbf{X}^{[\lambda,(n)]}$.

Let us prove that $(\mathbf{X}, \lambda)$ defines a λ-symmetry of (2.1). In order to construct a local system of coordinates on $M^{(n)}$, we complete (2.5) with a function $\alpha = \alpha(t,x)$ functionally independent with $z(t,x)$. Since (2.5) are invariants of $\mathbf{X}^{[\lambda,(n)]}$, in the new coordinates $\mathbf{X}^{[\lambda,(n)]}$ is of the form $\varphi(z,\alpha)\partial_\alpha$, where $\varphi(z,\alpha) = \mathbf{X}(\alpha(t,x))$.

Since $\varphi(z,\alpha)\partial_\alpha(\Delta(z,\zeta,\ldots,\zeta_{n-1})) = 0$, we conclude that the pair $(\mathbf{X}, \lambda)$ given by (2.9) and (2.10) defines a λ-symmetry of (2.1). Therefore the following result, converse of Theorem 1, holds:

Theorem 2. *If there exist two functions $z = z(t, x)$ and $\zeta = \zeta(t, x, x_1)$ such that equation (2.1) can be written in terms of $\{z, \zeta, \zeta_1, \dots, \zeta_{n-1}\}$ as an ODE of order $n-1$ then the pair $(\mathbf{X}, \lambda)$ given by (2.9) and (2.10) defines a λ-symmetry of equation (2.1). The functions z and ζ are invariants of $\mathbf{X}^{[\lambda,(1)]}$.*

Remark 1. We recall that if the pair $(\mathbf{X}, \lambda)$ defines a λ-symmetry of (2.1) and $f = f(t, x)$ is any smooth function, then $(f\mathbf{X}, \widetilde{\lambda})$ is also a λ-symmetry of (2.1) for $\widetilde{\lambda} = \lambda - D_t(f)/f$ (see [34, Lemma 5.1]). Since $(f\mathbf{X})^{[\widetilde{\lambda},(1)]} = f\mathbf{X}^{[\lambda,(1)]}$, it is clear that $(f\mathbf{X})^{[\widetilde{\lambda},(1)]}$ and $\mathbf{X}^{[\lambda,(1)]}$ have the same invariants, if f is a non-null function. We conclude that, in the conditions of Theorem 2, there exist infinitely many λ-symmetries of the equation that also have the same invariants z and ζ.

2.3 Generalized λ-prolongations and telescopic vector fields

The prolongations of vector fields $\mathbf{X}$ defined on $M \subset \mathbb{R}^2$ to the kth jet space $M^{(k)}$ that have the ID property are characterized by (2.7). It is easy to check that the vector fields Y on $M^{(k)}$ that satisfy

$$[Y, D_t] = \lambda Y + \mu D_t, \tag{2.11}$$

for some functions λ and $\mu \in \mathcal{C}^\infty(M^{(k)})$, also have a ID property, in the sense that if $g = g(t, x, \dots, x_i)$ and $h = h(t, x, \dots, x_j)$ are invariants of Y then $h_g = D_t h / D_t g$ is also an invariant of Y. Relation (2.11) implies that Y can be written in the form

$$Y = \rho(t, x, \dots, x_{i_1})\partial_t + \phi^0(t, x, \dots, x_{i_2})\partial_x + \sum_{j=1}^{k} \phi^{[\lambda,(j)]}(t, x, \dots, x_{i_j})\partial_{x_j}, \tag{2.12}$$

where the functions $\phi^{[\lambda,(j)]}$ are defined by recurrence as in (2.6). Even if ρ, ϕ^0 and λ depend on derivatives up to some finite order, formulae (2.6) are well-defined and, formally, we can write $Y = (\rho\partial_t + \phi^0\partial_x)^{[\lambda,(k)]}$.

Let us observe that the class of these vector fields Y contains well-known subclasses of vector fields that have appeared in the literature:

Generalized λ-prolongations. When the infinitesimals ρ and ϕ^0 in (2.12) only depend on (t, x), Y projects onto $\mathbf{X} = \rho\partial_t + \phi^0\partial_x$, that is a vector field defined on $M \subset \mathbb{R}^2$. If $\lambda = \lambda(t, x, \dots, x_s) \in \mathcal{C}^\infty(M^{(s)})$, for some $s > 1$, the vector field Y is the *generalized* λ-prolongation of $\mathbf{X}$, i.e., $Y = \mathbf{X}^{[\lambda,(k)]}$ (see Definition 2.1 in [30]). If a given differential equation is invariant under the λ-prolongation of $\mathbf{X}$, for some function $\lambda \in \mathcal{C}^\infty(M^{(s)})$, we say that $\mathbf{X}$ is a *generalized* λ-symmetry or that the pair $(\mathbf{X}, \lambda)$ defines a $\mathcal{C}^\infty(M^{(s)})$-symmetry of the equation (see [30] for details).

Telescopic vector fields. A class of vector fields that also satisfy (2.11) is formed by the called *telescopic* vector fields [44]. They are defined as the vector fields in $M^{(k)}$ that satisfy the ID property, but now z in (2.4) can depend on x_1, i.e, z and ζ are both independent invariants of first-order

$$\widetilde{z} = \widetilde{z}(t, x, x_1), \qquad \widetilde{\zeta} = \widetilde{\zeta}(t, x, x_1). \tag{2.13}$$

Telescopic vector fields have been characterized in [44], up to a multiplicative factor, as the vector fields in $M^{(k)}$ of the form

$$\tau^{(k)} = \alpha(t, x, x_1)\partial_t + \beta(t, x, x_1)\partial_x + \sum_{i=1}^{k} \gamma^{(i)}(t, x, \dots, x_i)\partial_{x_i}, \tag{2.14}$$

where $\alpha = \alpha(t,x,x_1)$, $\beta = \beta(t,x,x_1)$ and $\gamma^{(1)} = \gamma^{(1)}(t,x,x_1)$ are arbitrary functions such that $\beta - \alpha x_1 \neq 0$ and, for $i = 2,\dots,k$, $\gamma^{(i)} = \gamma^{(i)}(t,x,\dots,x_i)$ is given by

$$\gamma^{(i)} = D_t\big(\gamma^{(i-1)}\big) - D_t(\alpha)x_i + \frac{\gamma^{(1)} + x_1 D_t\alpha - D_t\beta}{\beta - x_1\alpha}\big(\gamma^{(i-1)} - \alpha x_i\big). \tag{2.15}$$

It should be pointed out that the condition $\beta - \alpha x_1 \neq 0$ is necessary to have the ID property or to be a telescopic vector field in the sense given in [44]. In the case $\beta - \alpha x_1 = 0$, if (2.13) are independent invariants of a vector field of the form (2.14), then $\widetilde{\zeta}_1 = \frac{D_t\zeta}{D_t z} = \frac{\widetilde{\zeta}_{x_1}}{\widetilde{z}_{x_1}}$ does not depend on x_2. Since

$$\begin{pmatrix} \widetilde{z}_t & \widetilde{z}_x & \widetilde{z}_{x_1} \\ \widetilde{\zeta}_t & \widetilde{\zeta}_x & \widetilde{\zeta}_{x_1} \\ \widetilde{\zeta}_{1t} & \widetilde{\zeta}_{1x} & \widetilde{\zeta}_{1x_1} \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha x_1 \\ \gamma^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

we conclude that $\{\widetilde{z}, \widetilde{\zeta}, \widetilde{\zeta}_1\}$ cannot be independent invariants of a vector field of the form (2.14).

Now we give some hints on the relationships between telescopic vector fields and λ-prolongations. By using (2.15), the following characterization of telescopic vector fields can easily be checked:

Theorem 3. *A telescopic vector field* (2.14) *satisfies*

$$\big[\tau^{(k)}, D_t\big] = \lambda\tau^{(k)} + \mu D_t, \tag{2.16}$$

where

$$\lambda = \frac{\gamma^{(1)} + x_1 D_t\alpha - D_t\beta}{\beta - x_1\alpha}, \tag{2.17}$$

and $\mu = -(D_t + \lambda)(\alpha)$. *Accordingly, the telescopic vector field* (2.14) *can be written as* $\tau^{(k)} = (\alpha\partial_t + \beta\partial_x)^{[\lambda,(k)]}$ *for the function* $\lambda \in \mathcal{C}^\infty(M^{(2)})$ *given by* (2.17).

Previous theorem shows that a telescopic vector field is a λ-prolongation where the two first infinitesimals can depend on the first derivative of the dependent variable. We point out that a telescopic vector field (2.14) admits a zero-order invariant if and only if $\alpha = 0$ or the ratio β/α does not depend on x_1. In this case the two first infinitesimals of $1/\alpha \cdot \tau^{(k)}$ (resp. of $1/\beta \cdot \tau^{(k)}$ if $\alpha = 0$) do not depend on x_1. If $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$, we can write $\tau^{(k)} = (\alpha\partial_t + \beta\partial_x)^{[\lambda,(k)]}$, where the function λ is given by (2.17) and only depends on (t,x,x_1). In other words, *the telescopic vector fields that admit an invariant of order zero are standard λ-prolongations of vector fields in M, with $\lambda \in \mathcal{C}^\infty(M^{(1)})$.*

Example 1. The telescopic vector field $\tau^{(2)} = x_1\partial_x + x\partial_{x_1} + \gamma^{(2)}\partial_{x_2}$, where $\gamma^{(2)}$ is defined by (2.15), given in [44, equation (48)], is not the λ-prolongation of a vector field on M. However, since $z = t$ is a zero-order invariant, this telescopic vector field is, up to the multiplicative factor x_1, the λ-prolongation of $\mathbf{X} = \partial_x$ for $\lambda = t/x_1$. This pair $(\mathbf{X}, \lambda)$ defines a $\mathcal{C}^\infty(M^{(1)})$-symmetry of equation (46) in [44] associated to this telescopic vector field.

In the general case, as a direct consequence of (2.7), (2.16) and the properties of the Lie bracket, the following relation between telescopic vector fields and λ-prolongations of vector fields in M holds:

Theorem 4. *If* (2.14) *is a telescopic vector field, as defined in* [44], *then* $\beta - x_1\alpha \neq 0$ *and*

$$\tau^{(k)} = \alpha D_t + (\beta - \alpha x_1)\mathbf{X}^{[\lambda,(k)]}, \tag{2.18}$$

where $\mathbf{X} = \partial_x$ *and* $\lambda \in \mathcal{C}^\infty(M^{(2)})$ *is given by*

$$\lambda = \frac{\gamma^{(1)} - \alpha x_2}{\beta - \alpha x_1}. \tag{2.19}$$

If $\alpha = 0$*, then* λ *does not depend on* x_2*, i.e.,* $\lambda \in \mathcal{C}^\infty(M^{(1)})$*.*

As a consequence, from (2.18) we deduce the existence of a $\mathcal{C}^\infty(M^{(2)})$-symmetry associated to a telescopic vector field that leaves invariant the given equation:

Corollary 1. *If an nth order ordinary differential equation* (2.1) *is invariant under a telescopic vector field* (2.14) *then the equation admits the vector field* $\mathbf{X} = \partial_x$ *as* $\mathcal{C}^\infty(M^{(2)})$*-symmetry for the function* λ *given by* (2.19)*.*

For the particular case $n = 2$, the λ-symmetry associated to a telescopic vector field is a $\mathcal{C}^\infty(M^{(1)})$-symmetry. The proof consists of the evaluation of (2.18) and (2.19) on the submanifold defined by $x_2 = F(t, x, x_1)$.

Corollary 2. *If a second-order ordinary differential equation* $x_2 = F(t, x, x_1)$ *is invariant under a telescopic vector field* (2.14) *then the equation admits the vector field* $\mathbf{X} = \partial_x$ *as* $\mathcal{C}^\infty(M^{(1)})$*-symmetry for the function* $\lambda = \lambda(t, x, x_1)$ *given by*

$$\lambda = \frac{\gamma^{(1)} - \alpha F}{\beta - \alpha x_1}. \tag{2.20}$$

In Section 6 we prove that the order reduction procedures of second-order equations associated to the telescopic vector field and to the λ-symmetry are equivalent (see Remark 5). In this sense, the inclusion of first-order derivatives in the two first infinitesimals seems to be irrelevant in order to get different order reductions of second-order equations.

3 Reductions derived from nonlocal symmetries

The following method has been used in [19, 20] (see also [11]) to obtain some nonlocal symmetries of a given second-order ODE

$$x_2 = F(t, x, x_1) \tag{3.1}$$

that lets reduce the order of the equation. That procedure introduces an auxiliary system of the form

$$x_2 = F(t, x, x_1), \qquad w_1 = H(t, x, x_1), \tag{3.2}$$

or its equivalent first-order system (obtained by setting $v = x_1$)

$$x_1 = v, \qquad v_1 = F(t, x, v), \qquad w_1 = H(t, x, v), \tag{3.3}$$

where H is an unknown function to be determined in the procedure.

Let us denote by Δ (resp. Δ_1) the submanifold of the corresponding jet space defined by system (3.2) (resp. (3.3)). Let us observe that, although there is an equivalence between systems (3.2) and (3.3), there is no complete equivalence between the Lie point symmetries of both systems [39]. If

$$\mathbf{v} = \xi(t, x, x_1, w)\partial_t + \eta^0(t, x, x_1, w)\partial_x + \psi^0(t, x, x_1, w)\partial_w \tag{3.4}$$

is a generalized symmetry of (3.2) then

$$\mathbf{v}_1 = \xi(t,x,v,w)\partial_t + \eta^0(t,x,v,w)\partial_x + \varphi^0(t,x,v,w)\partial_v + \psi^0(t,x,v,w)\partial_w, \tag{3.5}$$

where $\varphi^0 = \eta^1\big|_{\Delta_1}$, is a Lie point symmetry of (3.3). Conversely, if the vector field $\mathbf{v}_1$ given by (3.5) is a Lie point symmetry of (3.3) then necessarily $\varphi^0 = \eta^1\big|_{\Delta_1}$ and the vector field $\mathbf{v}$ given by (3.4) is a generalized symmetry of (3.2).

In the sequel, we will only consider generalized symmetries of the form (3.4) of system (3.2) with the condition

$$(\xi_w)^2 + \left(\eta^0_w\right)^2 \neq 0. \tag{3.6}$$

The mentioned procedure consists in determining some function $H = H(t,x,x_1)$ and a vector field $\mathbf{v}$ of the form (3.4) satisfying (3.6) with the following three properties:

a) The vector field (3.5), with $\varphi^0 = \eta^1\big|_{\Delta_1}$, is a Lie point symmetry of (3.3).

b) There exist two functionally independent functions $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ such that

$$\mathbf{v}(z) = 0, \qquad \mathbf{v}^{(1)}(\zeta)\Big|_{\Delta} = 0. \tag{3.7}$$

c) Equation (3.1) can be written in terms of $\{z,\zeta,\zeta_z\}$ as a first-order ODE.

Since strictly speaking ζ is not an invariant of $\mathbf{v}^{(1)}$ and this reduction is not exactly the classical one we prefer to call *semi-classical* to this reduction. In what follows, the term *nonlocal symmetry* will refer to a vector field $\mathbf{v}$ of the form (3.4) with the above-described properties.

Several important aspects about the context of the procedure should be pointed out.

1. In order the procedure works, the pair $(\mathbf{v}, H)$ has to be such that there exist two functions $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ with the characteristics described above. This fact has not been explicitly remarked in the examples presented in [11, 19, 20]. In these examples, for the provided pairs $(\mathbf{v}, H)$, there exist two invariants of that form for $\mathbf{v}^{(1)}$. However, in principle, for any given generalized symmetry of system (3.2) the existence of two invariants of the form $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ is not warranted and therefore the procedure can not be applied to reduce the original equation (3.1).
2. The main aim of the procedure is to obtain two functions $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ such that in terms of $\{z,\zeta,\zeta_z\}$ equation (3.1) can be written as a first-order ODE. In [30] it is proved that this reduction procedure is always the reduction procedure derived from the existence of a λ-symmetry of the equation. An explicit construction of such λ-symmetry is given in Theorem 2.
3. In [12], D. Catalano-Ferraioli considered systems of the form (3.2) in order to obtain a nonlocal interpretation of λ-symmetries as standard (but generalized) symmetries of a suitable system (λ-*covering*). For a system (3.2), symmetries of the form (3.4) are called *semi-classical nonlocal symmetries* in [12].

This shows that the procedure should be clarified from a theoretical point of view and it is interesting to investigate more closely the relationship between the reduction procedure described above and the reduction procedure derived from the existence of a λ-symmetry. It is also interesting to compare the computational aspects of both procedures.

Let us suppose that (3.4) is a (generalized) symmetry of the system (3.2), for some function $H = H(t,x,x_1)$, such that there exist two functionally independent functions $z = z(t,x)$ and

$\zeta = \zeta(t, x, x_1)$ verifying (3.7) and such that equation (3.1) can be written in terms of $\{z, \zeta, \zeta_z\}$ as a first-order ODE.

In order to deal with system (3.2), we denote by $\widetilde{D_t}$ the total derivative vector field corresponding to variables t, x, x_1, w

$$\widetilde{D_t} = \partial_t + x_1\partial_x + x_2\partial_{x_1} + w_1\partial_w + \cdots .$$

Condition (3.7) lets us determine some relationships among functions ξ, η^0, η^1, z and ζ. We distinguish two cases: $\xi \neq 0$ and $\xi = 0$.

Case 1: $\xi \neq 0$. In this case the condition $\mathbf{v}(z) = 0$ implies that

$$\eta^0 = f^0\xi, \qquad \text{where} \quad f^0 = -\frac{z_t}{z_x} = \frac{\eta^0}{\xi}. \tag{3.8}$$

Although, in principle, ξ and η^0 may depend on x_1, w, the function f^0 can not depend on these variables; i.e. $f^0 = f^0(t, x)$. By using (2.3), it can be checked that $\eta^1 = \widetilde{D_t}(\eta^0) - \widetilde{D_t}(\xi x_1) = f^1\xi$, where

$$f^1 = f^1(t, x, x_1, x_2, w, w_1) = \widetilde{D_t} f^0 + \big(f^0 - x_1\big)\frac{\widetilde{D_t}\xi}{\xi}.$$

The condition $\mathbf{v}^{(1)}(\zeta)\big|_\Delta = 0$ can be written as $(\xi\zeta_t + \xi f^0\zeta_x + \xi f^1\zeta_{x_1})\big|_\Delta = 0$. Since $\xi \neq 0$, we have $(\zeta_t + f^0\zeta_x + f^1\zeta_{x_1})\big|_\Delta = 0$ and, therefore, $f^1\big|_\Delta$ can be written in terms of t, x, x_1 as

$$f^1\big|_\Delta = -\frac{\zeta_t + f^0\zeta_x}{\zeta_{x_1}}. \tag{3.9}$$

On the other hand, by Theorem 2, we know that $\mathbf{X} = -z_x\partial_t + z_t\partial_x$ is a λ-symmetry of (3.1) for the function λ given by (2.10). We try to express $\mathbf{X}$ and λ in terms of ξ, η^0. By using (3.8), (3.9) and that $z_t = D_t z - x_1 z_x$, it can be checked that

$$\lambda = \lambda(t, x, x_1) = \left.\frac{\widetilde{D_t}\xi}{\xi}\right|_\Delta - \frac{\widetilde{D_t} z_x}{z_x}. \tag{3.10}$$

Since $\mathbf{X} = -z_x\partial_t + z_t\partial_x$ is a λ-symmetry of equation (3.1) for λ given by (3.10), by Remark 1

$$\widetilde{\mathbf{X}} = -\frac{1}{z_x}\mathbf{X} = \partial_t + f^0\partial_x$$

is a $\widetilde{\lambda}$-symmetry of (3.1) for $\widetilde{\lambda}$ given by $\widetilde{\lambda} = \lambda - D_t(g)/g$, where $g = -1/z_x$. It can be checked that

$$\widetilde{\lambda} = \widetilde{\lambda}(t, x, x_1) = \left.\frac{\widetilde{D_t}\xi}{\xi}\right|_\Delta = \frac{\xi_t + x_1\xi_x + F\xi_{x_1} + H\xi_w}{\xi}. \tag{3.11}$$

This shows that the functions $\widetilde{\lambda}$, f^0 can readily be obtained from ξ, η^0 and H.

Case 2: $\xi = 0$. In this case η^0 has to be non null, $\eta^1 = \widetilde{D_t}\eta^0$ and the condition $\mathbf{v}(z) = 0$ implies that $z_x = 0$. The condition $\mathbf{v}^{(1)}(\zeta)\big|_\Delta = 0$ can be written as $\eta^0\zeta_x + \eta^1\big|_\Delta\,\zeta_{x_1} = 0$. Hence

$$-\frac{\zeta_x}{\zeta_{x_1}} = \frac{\eta^1\big|_\Delta}{\eta^0}, \tag{3.12}$$

where both members depend only on t, x, x_1. By Theorem 2, $\mathbf{X} = z'(t)\partial_x$ is a λ-symmetry for $\lambda = -\frac{\zeta_x}{\zeta_{x_1}} - \frac{z''}{z'}$. By denoting $h = 1/z'(t)$, Remark 1 implies that $\widetilde{\mathbf{X}} = h\mathbf{X} = \partial_x$ is a $\widetilde{\lambda}$-symmetry of equation (3.1) for $\widetilde{\lambda} = \lambda - D_t(h)/h$. By using (3.12), it can be checked that

$$\widetilde{\lambda} = \widetilde{\lambda}(t,x,x_1) = \frac{\widetilde{D_t}\eta^0}{\eta^0}\bigg|_\Delta = \frac{\eta^0_t + x_1\eta^0_x + F\eta^0_{x_1} + H\eta^0_w}{\eta^0}. \tag{3.13}$$

This proves that $\widetilde{\mathbf{X}} = \partial_x$ is a $\widetilde{\lambda}$-symmetry of equation (3.1) for $\widetilde{\lambda}$ given by (3.13).

Thus we have proven the following result:

Theorem 5. *Let us assume that for a given second-order equation* (3.1) *there exists some function $H = H(t,x,x_1)$ such that the corresponding system* (3.2) *admits a nonlocal symmetry* (3.4)*. We also assume that there exist two functionally independent functions $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ such that $\mathbf{v}(z) = 0$, $\mathbf{v}^{(1)}(\zeta)\big|_\Delta = 0$ and that* (3.1) *can be written in terms of $\{z,\zeta,\zeta_z\}$ as a first-order ODE. Then*

(*i*) *If $\xi \neq 0$, the functions η^0/ξ and $\widetilde{\lambda}$, given by* (3.8) *and* (3.11) *respectively, do not depend on w and the pair*

$$\widetilde{\mathbf{X}} = \partial_t + \frac{\eta^0}{\xi}\partial_x, \qquad \widetilde{\lambda} = \frac{\xi_t + \xi_x x_1 + \xi_{x_1}F + \xi_w H}{\xi}$$

defines a λ-symmetry of the equation (3.1).

(*ii*) *If $\xi = 0$, the function $\widetilde{\lambda}$ given by* (3.13) *does not depend on w and the pair*

$$\widetilde{\mathbf{X}} = \partial_x, \qquad \widetilde{\lambda} = \frac{\eta^0_t + \eta^0_x x_1 + \eta^0_{x_1}F + \eta^0_w H}{\eta^0}$$

defines a λ-symmetry of the equation (3.1).

In both cases, $\{z,\zeta,\zeta_z\}$ is a complete system of invariants of $\widetilde{\mathbf{X}}^{[\widetilde{\lambda},(1)]}$.

Example 2. Two examples of reduction of nonlinear oscillators [8, 26] by using the procedure described at the beginning of this section have been reported in [11]. Although in this paper the authors use systems of the form (3.3), the comments we have provided at the beginning of this section let us consider systems of the form (3.2) in its stead. The corresponding systems are of the form

$$x_2 = F^i(t,x,x_1), \qquad w_1 = H^i(t,x,x_1), \qquad i = 1,2,$$

where

$$F^1(t,x,x_1) = \frac{kxx_1^2}{1+kx^2} - \frac{\alpha^2 x}{1+kx^2}, \qquad F^2(t,x,x_1) = \frac{-kxx_1^2}{1+kx^2} - \frac{\alpha^2 x}{(1+kx^2)^3}.$$

For both systems the calculated infinitesimal generators are of the form

$$\mathbf{v}_i = \xi_i\partial_t + \eta^0_i\partial_x + \psi^0_i\partial_w = e^w\partial_x + \frac{H^i}{x_1}e^w\partial_w, \qquad i = 1,2, \tag{3.14}$$

where

$$H^1(t,x,x_1) = -\frac{x(\alpha^2 - kx_1^2)}{(kx^2+1)x_1}, \qquad H^2(t,x,x_1) = \frac{-kx(1+kx^2)^2x_1^2 - \alpha^2 x}{(kx^2+1)^3 x_1}.$$

Since $\xi_i = 0$ and $\eta^0_i = e^w$, for $i = 1,2$, the case (ii) of Theorem 5 let us conclude that the pairs $(\mathbf{X}_i, \lambda_i) = (\partial_x, H^i(t,x,x_1))$, $i = 1,2$, define, respectively, λ-symmetries of the corresponding equations $x_2 = F^i(t,x,x_1)$, for $i = 1,2$.

4 Exponential vector fields and $\boldsymbol{\lambda}$-symmetries

In this section we study the same problem we have considered in Section 3, but for the special case where the function H that appears in (3.2) can be chosen in the form $H = H(t,x)$ and the infinitesimals of $\mathbf{v}$ do not depend on x_1, i.e. $\mathbf{v}$ is a Lie point symmetry of system (3.2); this is the case in most of the examples considered in [11, 19, 20]. Therefore, in this section the system is

$$x_2 = F(t,x,x_1), \qquad w_1 = H(t,x), \tag{4.1}$$

and a Lie point symmetry of (4.1) is

$$\mathbf{v} = \xi(t,x,w)\partial_t + \eta^0(t,x,w)\partial_x + \psi^0(t,x,w)\partial_w. \tag{4.2}$$

We will consider the same two cases as in Section 3: $\xi \neq 0$ and $\xi = 0$.

If $\xi \neq 0$ then, by Theorem 5, $\widetilde{\mathbf{X}} = \partial_t + (\eta^0/\xi)\partial_x$ is a $\widetilde{\lambda}$-symmetry of (3.1) for

$$\widetilde{\lambda} = \widetilde{\lambda}(t,x,x_1) = \frac{\xi_t + \xi_x x_1 + \xi_w H}{\xi}.$$

Since the function $\widetilde{\lambda}$ does not depend on w

$$\left(\frac{\xi_t}{\xi}\right)_w + \left(\frac{\xi_x}{\xi}\right)_w x_1 + \left(\frac{\xi_w}{\xi}\right)_w H = 0. \tag{4.3}$$

Hence, the coefficient of x_1 in (4.3) has to be null and thus

$$\left(\frac{\xi_x}{\xi}\right)_w = \left(\frac{\xi_w}{\xi}\right)_x = 0. \tag{4.4}$$

By derivation of (4.3) with respect to x we deduce

$$\left(\frac{\xi_w}{\xi}\right)_w H_x = 0. \tag{4.5}$$

We need to consider two subcases: $H_x = 0$ and $H_x \neq 0$.

If $H_x = 0$ the function H depends only on t. If $h = h(t)$ is a primitive of $H(t)$ and we denote $\widetilde{\xi}(t,x) = \xi(t,x,h(t))$, $\widetilde{\eta}^0(t,x) = \eta^0(t,x,h(t))$, it is easy to prove that $\widetilde{\mathbf{v}} = \widetilde{\xi}\partial_t + \widetilde{\eta}^0\partial_x$ becomes a Lie point symmetry of equation (3.1). This case will not be considered here in the sequel: if $H_x = 0$ then $\mathbf{v}$ projects on a Lie point symmetry of (3.1).

If $H_x \neq 0$, (4.5) implies that

$$\left(\frac{\xi_w}{\xi}\right)_w = 0 \tag{4.6}$$

and, by (4.3),

$$\left(\frac{\xi_t}{\xi}\right)_w = \left(\frac{\xi_w}{\xi}\right)_t = 0. \tag{4.7}$$

By (4.4), (4.6) and (4.7), $\xi_w/\xi = C$, for some $C \in \mathbb{R}$, and therefore $\xi = e^{Cw}\rho(t,x)$ for some function ρ. By (3.8), $\eta^0 = e^{Cw}\phi^0(t,x)$, where $\phi^0 = f^0\rho$. The condition $\mathbf{v}^{(2)}(w_1 - H(t,x)) = 0$ when $w_1 = H$, implies that

$$\psi^0_t + \psi^0_x x_1 + \psi^0_w H = e^{Cw}\big(\rho H_t + \phi^0 H_x\big). \tag{4.8}$$

By derivation with respect to x_1, we obtain $\psi^0_x = 0$. By derivation of (4.8) with respect to x we deduce that ψ^0 has to be of the form $\psi^0 = e^{Cw}\psi(t) + R(t)$, for some functions $\psi = \psi(t)$ and $R = R(t)$. If we multiply both members of (4.8) by $-e^{Cw}$ then we obtain

$$\psi'(t) + e^{-Cw}R'(t) + C\psi(t)H = \left(\rho H_t + \phi^0 H_x\right)$$

and we deduce that $R(t) = C_1$ for some constant $C_1 \in \mathbb{R}$.

Previous discussion proves that (4.2) has to be of the form

$$\mathbf{v} = e^{Cw}\left(\rho(t,x)\partial_t + \phi^0(t,x)\partial_x + \psi(t)\partial_w\right) + C_1\partial_w.$$

It should be noted that the vector field ∂_w is always a Lie point symmetry of system (4.1). The symmetries of system (4.1) that are proportional to ∂_w are irrelevant for the reduction of the original equation (3.1) because the projection to the space of the variables of the equation is null.

Since $\rho \neq 0$, Theorem 5 proves that the pair

$$\mathbf{X} = \partial_t + \frac{\phi^0}{\rho}\partial_x, \qquad \lambda = \frac{D_t(\rho)}{\rho} + CH$$

defines a λ-symmetry of equation (3.1) and that $\{z, \zeta, \zeta_z\}$ are invariants of $\mathbf{X}^{[\lambda,(1)]}$. By Remark 1, the pair $\widetilde{\mathbf{X}} = \rho\partial_t + \phi^0\partial_x$, $\widetilde{\lambda} = CH$ also defines a λ-symmetry of equation (3.1) and $\widetilde{\mathbf{X}}^{[\widetilde{\lambda},(1)]}$ has the same invariants as $\mathbf{v}$.

A similar argument for the case $\xi = 0$, proves that the pair

$$\mathbf{X} = \partial_x, \qquad \lambda = \frac{D_t(\phi^0)}{\phi^0} + CH$$

is a λ-symmetry of equation (3.1). By Remark 1, the pair $\widetilde{\mathbf{X}} = \phi^0\partial_x$, $\widetilde{\lambda} = CH$ also defines a λ-symmetry of equation (3.1) and $\widetilde{\mathbf{X}}^{[\widetilde{\lambda},(1)]}$ has the same invariants as $\mathbf{X}^{[\lambda,(1)]}$.

Thus we have proven the following result:

Theorem 6. *Let us suppose that for a given second-order equation* (3.1) *there exists some function $H = H(t,x)$ such that the system* (4.1) *admits a Lie point symmetry* (4.2) *satisfying* (3.6). *We assume that $z = z(t,x)$, $\zeta = \zeta(t,x,x_1)$ are two functionally independent functions that verify* (3.7) *and are such that equation* (3.1) *can be written in terms of $\{z, \zeta, \zeta_z\}$ as a first-order ODE. Then*

1. *The vector field* $\mathbf{v}$ *has to be of the form*

$$\mathbf{v} = e^{Cw}\left(\rho(t,x)\partial_t + \phi^0(t,x)\partial_x + \psi(t)\partial_w\right) + C_1\partial_w \tag{4.9}$$

for some $C, C_1 \in \mathbb{R}$.

2. *The pair*

$$\widetilde{\mathbf{X}} = \rho(t,x)\partial_t + \phi^0(t,x)\partial_x, \qquad \widetilde{\lambda} = CH. \tag{4.10}$$

defines a λ-symmetry of the equation (3.1) *and the set $\{z, \zeta, \zeta_z\}$ is a complete system of invariants of $\widetilde{\mathbf{X}}^{[\widetilde{\lambda},(1)]}$.*

Remark 2. It should be observed that the vector field (4.9) can be written in the variables of the equation (3.1) in the form $\mathbf{v}^* = e^{C\int H(t,x)dt}\left(\rho(t,x)\partial_t + \phi^0(t,x)\partial_x\right)$, where the integral $\int H(t,x)dt$ is, formally, the integral of the function $H(t,x)$, once a function $x = f(t)$ has been chosen. These are the exponential vector fields that are considered in the book of P. Olver [41, p. 181] in order to show that not every integration method comes from the classical method of Lie. The relationship between these vector fields and λ-symmetries has been studied in [34]: the λ-symmetry given in (4.10) can be obtained by using Theorem 5.1 in [34].

5 The nonlocal symmetries associated to a λ-symmetry

A natural question is to investigate the converse of the the results provided in Theorems 5 and 6: given a λ-symmetry $\mathbf{X} = \rho(t,x)\partial_t + \phi^0(t,x)\partial_x$, $\lambda = \lambda(t,x,x_1)$ of equation (3.1), is it possible to construct some system (3.2) admitting nonlocal symmetries that let reduce the order of the equation? Let us remember that if the answer is affirmative then, by (3.8), the function $f^0 = \eta^0/\xi$ does not depend on x_1, w. Therefore, motivated by the result presented in Theorem 6, we can try to give an explicit construction of $\mathbf{v}$. We choose $C = 1$, $H = \lambda(t,x,x_1)$ and the vector field

$$\mathbf{v} = e^w \left(\rho(t,x)\partial_t + \phi^0(t,x)\partial_x + \psi(t,x,x_1)\partial_w\right),$$

where ρ and ϕ^0 are the infinitesimal coefficients of $\mathbf{X}$ and $\psi = \psi(t,x,x_1)$ satisfies the condition $\mathbf{v}^{(2)}(w_1 - \lambda)\big|_\Delta = 0$. This equation provides a linear first-order partial differential equation to determine such a function ψ

$$\psi_t + \psi_x x_1 + \psi_{x_1} F + \psi\lambda = D_t(\rho)\lambda + \rho\lambda^2 + \mathbf{X}^{[\lambda,(1)]}(\lambda). \tag{5.1}$$

Now, let us suppose that $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ are two invariants of $\mathbf{X}^{[\lambda,(1)]}$. It can be checked that $\mathbf{v}^{(1)}(z) = e^w\mathbf{X}(z) = 0$ and that $\mathbf{v}^{(1)}(\zeta) = e^w(\rho\zeta_t + \phi^0\zeta_x + \phi^{(1)}\zeta_{x_1})$, where $\phi^{(1)} = \widetilde{D_t}(e^w\phi^0) - \widetilde{D_t}(e^w\rho)x_1 = e^w((D_t + w_1)(\phi^0) - (D_t + w_1)(\rho)x_1)$. Therefore $\mathbf{v}^{(1)}(\zeta)\big|_\Delta = e^w(\mathbf{X}^{[\lambda,(1)]}(\zeta)) = 0$.

Hence, the following result holds:

Theorem 7. *Let $\mathbf{X} = \rho(t,x)\partial_t + \phi^0(t,x)\partial_x$ be a λ-symmetry of equation (3.1) for some $\lambda = \lambda(t,x,x_1)$ and let $\psi = \psi(t,x,x_1)$ be a particular solution of equation (5.1). Then*

a) The vector field

$$\mathbf{v} = e^w \left(\rho(t,x)\partial_t + \phi^0(t,x)\partial_x + \psi(t,x,x_1)\partial_w\right) \tag{5.2}$$

is a nonlocal symmetry of equation (3.1) associated to system (3.2) for $H = \lambda(t,x,x_1)$.

b) If $z = z(t,x)$ and $\zeta = \zeta(t,x,x_1)$ are two invariants of $\mathbf{X}^{[\lambda,(1)]}$ then these functions satisfy (3.7) and equation (3.1) can be written in terms of $\{z,\zeta,\zeta_z\}$ as a first-order ODE.

As a direct consequence of Theorem 7 and Corollary 2, a telescopic vector field that leaves invariant the equation (3.1) has an associated nonlocal symmetry that can explicitly be constructed:

Corollary 3. *Let*

$$\tau^{(2)} = \alpha(t,x,x_1)\partial_t + \beta(t,x,x_1)\partial_x + \gamma^{(1)}(t,x,x_1)\partial_{x_1} + \gamma^{(2)}(t,x,x_1,x_2)\partial_{x_2}$$

be a telescopic vector field that leaves invariant the equation (3.1). Let $\psi = \psi(t,x,x_1)$ be a particular solution of the corresponding equation (5.1) where λ is given by (2.20). Then the vector field $\mathbf{v} = e^w\left(\partial_x + \psi(t,x,x_1)\partial_w\right)$ is a nonlocal symmetry of equation (3.1) associated to system (3.2) for $H = \frac{\gamma^{(1)} - \alpha F}{\beta - \alpha x_1}$.

Remark 3. With the hypothesis of Theorem 7, Remark 1 let us ensure that, for any smooth function $f = f(t,x)$, $\widetilde{\mathbf{X}} = f\mathbf{X} = f\rho\partial_t + f\phi^0\partial_x$ is a $\widetilde{\lambda}$-symmetry of the equation (3.1) for $\widetilde{\lambda} = \lambda - D_t f/f$. Therefore $\widetilde{\mathbf{v}} = e^w(f\rho\partial_t + f\phi^0\partial_x + \widetilde{\psi}\partial_w)$ is a nonlocal symmetry of the system (3.2) obtained by using $\widetilde{H} = \widetilde{\lambda}$ instead of H; in this case, $\widetilde{\psi}$ has to be a particular solution of the linear equation

$$\widetilde{\psi}_t + \widetilde{\psi}_x x_1 + \widetilde{\psi}_{x_1} F + \widetilde{\psi}\widetilde{\lambda} = D_t(f\rho)\widetilde{\lambda} + f\rho\widetilde{\lambda}^2 + \widetilde{\mathbf{X}}^{[\widetilde{\lambda},(1)]}(\widetilde{\lambda}). \tag{5.3}$$

Remark 4. The concept of *semi-classical nonlocal symmetries* was introduced in [12] to give a nonlocal interpretation of λ-symmetries as standard (but generalized) symmetries of a suitable system (λ-covering). The result presented in Theorem 7 corresponds to the particular case $n = 2$ of Proposition 1 in [12], but here the correspondence between λ-symmetries and semi-classical nonlocal symmetries is explicitly established.

As a consequence of Theorems 5 and 7, the nonlocal symmetries of the form (5.2) could be thought as a prototype of the nonlocal symmetries of the equation that are useful to reduce the order of the equation:

Corollary 4. *Let us suppose that for a given second-order equation* (3.1) *there exists some function $H = H(t, x, x_1)$ such that the corresponding system* (3.2) *admits a (generalized) symmetry* $\mathbf{v}$ *of the form* (3.4) *satisfying* (3.6)*. We also assume that there exist two functionally independent functions z, ζ of the form* (2.4) *satisfying* (3.7) *and such that equation* (3.1) *can be written in terms of $\{z, \zeta, \zeta_z\}$ as a first-order ODE. Then there exists a function $\widetilde{H} = \widetilde{H}(t, x, x_1)$ such that the corresponding system* (3.2) *admits a Lie point symmetry $\widetilde{\mathbf{v}}$ of the form* (5.2) *satisfying* (3.6) *and z, ζ are invariants of $\widetilde{\mathbf{v}}^{(1)}$.*

This corollary may be very helpful from a computational point of view, because the form (5.2) provides an *ansatz* to search nonlocal symmetries useful to reduce the order. In fact, this is the form of all nonlocal symmetries reported in the literature (of the class we are considering in this paper); the *ansatz* that is used in [11] to solve the determining equations and obtain the infinitesimals generators (3.14) has the form (5.2).

Although the function ψ is necessary to define the nonlocal symmetry (5.2), its determination requires to obtain a particular solution of the corresponding equation (5.1). However, this function is not necessary either to define the associated λ-symmetry or to reduce the order of the original equation.

6 Equivalent order reductions

A natural question is to know when two reductions associated to two different nonlocal symmetries are equivalent. This problem is apparently new in the literature and it is difficult to establish in terms of the nonlocal symmetries, because we are comparing reduction procedures associated to different symmetries, $\mathbf{v}_i = \xi_i\partial_t + \eta_i^0\partial_x + \psi_i^0\partial_w$, of different systems

$$x_2 = F(t, x, x_1), \qquad w_1 = H_i(t, x, x_1), \qquad i = 1, 2. \tag{6.1}$$

This open problem can be solved if we consider the associated λ-symmetries, because we have a criterion to know when the first integrals associated to different λ-symmetries of the same ODE are functionally dependent [33, 32]. This is used here to know when the reductions procedures associated to different λ-symmetries are equivalent. For the sake of simplicity we consider the case $n = 2$, what is sufficient to deal with the examples presented in this paper.

Let us assume that $(\mathbf{X}_i, \lambda_i) = (\rho_i\partial_t + \phi_i^0\partial_x, \lambda_i)$ define the λ-symmetry associated to $\mathbf{v}_i$ according to Theorem 5, for $i = 1, 2$.

It can be checked that the vector fields $\{A, \mathbf{X}_1^{[\lambda_1,(1)]}, \mathbf{X}_2^{[\lambda_2,(1)]}\}$ are linearly dependent if and only if

$$\lambda_1 + \frac{A(Q_1)}{Q_1} = \lambda_2 + \frac{A(Q_2)}{Q_2}, \tag{6.2}$$

where $Q_i = \phi_i^0 - \rho_i x_1$ is the characteristic of $\mathbf{X}_i$ for $i = 1, 2$. In this case,

$$Q_2\mathbf{X}_1^{[\lambda_1,(1)]} = \begin{vmatrix} \rho_1 & \phi_1^0 \\ \rho_2 & \phi_2^0 \end{vmatrix} A + Q_1\mathbf{X}_2^{[\lambda_2,(1)]}. \tag{6.3}$$

This is a motivation to define an equivalence relationship between pairs of the form $(\mathbf{X}, \lambda)$.

Definition 1. We say that two pairs $(\mathbf{X}_1, \lambda_1)$ and $(\mathbf{X}_2, \lambda_2)$ are A-equivalent and we write $(\mathbf{X}_1, \lambda_1) \stackrel{A}{\sim} (\mathbf{X}_2, \lambda_2)$ if and only if (6.2) is satisfied [32, 33].

By using this definition, we can compare the reduced equations associated to two A-equivalent λ-symmetries $(\mathbf{X}_1, \lambda_1)$ and $(\mathbf{X}_2, \lambda_2)$. We calculate two invariants $z^1 = z^1(t,x)$ and $\zeta^1 = \zeta^1(t, x, x_1)$ of $\mathbf{X}_1^{[\lambda_1,(1)]}$ and write the equation in terms of $\{z^1, \zeta^1, \zeta^1_{z^1}\}$. Let $I_1 = I_1(z^1, \zeta^1)$ denote a first integral of the reduced equation. Therefore such reduced equation can be expressed as $D_{z^1}(I_1(z^1, \zeta^1)) = 0$.

We repeat the procedure with $(\mathbf{X}_2, \lambda_2)$ and express the reduced equation associated to $(\mathbf{X}_2, \lambda_2)$ as $D_{z^2}(I_2(z^2, \zeta^2)) = 0$. By (2.8), it is clear that I_1 (resp. I_2) is a basis of the first integrals common to $\mathbf{X}_1^{[\lambda_1,(1)]}$ and A (resp. to $\mathbf{X}_2^{[\lambda_2,(1)]}$ and A). Since $(\mathbf{X}_1, \lambda_1) \stackrel{A}{\sim} (\mathbf{X}_2, \lambda_2)$ then, by (6.3), I_1 is a first integral of $\mathbf{X}_2^{[\lambda_2,(1)]}$. In consequence, $I_1 = G(I_2)$ for some non null function G and $D_{z^1}(I_1) = D_{z^1}(G(I_2)) = \frac{G'(I_2)}{D_{z^2}(z^1)} D_{z^2}(I_2)$.

Previous discussion proves that the order reductions associated to$A$-equivalent pairs are essentially the same: the reduced equations associated to two A-equivalent λ-symmetries $(\mathbf{X}_1, \lambda_1)$ and $(\mathbf{X}_2, \lambda_2)$ are functionally dependent.

Remark 5. Let us assume that the equation (3.1) is invariant under a telescopic vector field (2.14) and $(\mathbf{X}, \lambda)$ is the corresponding λ-symmetry constructed in Corollary 2. By using (2.16) and (2.18), a similar discussion also proves that the reduced equations associated to the telescopic vector field and to that λ-symmetry are functionally dependent.

We can now give a criterion to know when the reductions procedures associated to different nonlocal symmetries are equivalent:

Theorem 8. *Let $\mathbf{v}_1$, $\mathbf{v}_2$ be two nonlocal symmetries associated to two systems of the form (6.1) that satisfy the same condition as $\mathbf{v}$ in Theorem 5. Let $(\mathbf{X}_i, \lambda_i)$ be the λ-symmetry associated to $\mathbf{v}_i$ according to Theorem 5, for $i = 1, 2$. The reduced equations associated to $\mathbf{v}_i$ are functionally dependent if and only if $(\mathbf{X}_1, \lambda_1) \stackrel{A}{\sim} (\mathbf{X}_2, \lambda_2)$. In this case, we shall say that the pairs $(\mathbf{v}_1, H_1)$ and $(\mathbf{v}_2, H_2)$ are A-equivalent.*

By using (6.3), it can be checked that for any pair $(\mathbf{X}, \lambda)$ we have

$$(\mathbf{X}, \lambda) \stackrel{A}{\sim} \left(\partial_x, \lambda + \frac{A(Q)}{Q} \right). \tag{6.4}$$

The right member in (6.4) will be called the *canonical* pair of the equivalence class $[(\mathbf{X}, \lambda)]$. For $n = 2$, the functions λ of the *canonical* representatives arise as particular solutions of the first-order quasi-linear PDE [31, 33]

$$\lambda_t + x_1 \lambda_x + F \lambda_{x_1} + \lambda^2 = F_x + \lambda F_{x_1}. \tag{6.5}$$

Since two pairs of the form (∂_x, λ_1) and (∂_x, λ_2) are A-equivalent if and only if $\lambda_1 = \lambda_2$, two different particular solutions of (6.5) generate two different A-equivalence classes.

7 Some examples

Let us recall that, by Theorem 5, the construction of a λ-symmetry associated to a known nonlocal symmetry is straightforward.

Conversely, if $\mathbf{X} = \rho\partial_t + \phi^0\partial_x$ is a known λ-symmetry of (3.1) then, by Theorem 7, the vector field (5.2) is a nonlocal symmetry of equation (3.1) associated to system (3.2) for $H = \lambda(t, x, x_1)$. The determination of ψ requires the calculation of a particular solution of the corresponding PDE (5.1). Nevertheless, this function does not take part in the search of invariants of the form (2.4). Most of the examples of nonlocal symmetries reported in [19, 20] correspond to equations with λ-symmetries that had been previously calculated. We show, in an explicit way, the correspondence between these nonlocal symmetries and λ-symmetries and apply the results in Section 3 to deduce the equivalence of the reduction procedures.

Example 3. The equation

$$x_2 = \frac{x_1^2}{x} + nc(t)x^n x_1 + c'(t)x^{n+1} \tag{7.1}$$

had been proposed as an example of an equation integrable by quadratures that lacks Lie point symmetries except for particular choices of function $c(t)$ [22]. In [34] a λ-symmetry of (7.1) was calculated and the integrability of the equation was derived by the reduction process associated to the λ-symmetry.

A slight modification of equation (7.1) has been considered in [19]

$$x_2 = \frac{x_1^2}{x} + (c(t)x^n + b(t))x_1 + (c'(t) - c(t)b(t))\frac{x^{n+1}}{n} + d(t)x. \tag{7.2}$$

Both equations (7.1) and (7.2) are in the class $\mathcal{A}$ (see [35, 37]) of the second-order equation that admit first integrals of the form $A(t, x)x_1 + B(t, x)$. Several characterizations of these equations have been derived. In particular, it has been proven that such equations admit λ-symmetries whose canonical representative is of the form $(\partial_x, \alpha(t, x)x_1 + \beta(t, x))$ and α and β can be calculated directly from the coefficients of the equation. For equation (7.2) such λ-symmetry is given by the pair

$$\mathbf{X}_1 = \partial_x, \qquad \lambda_1 = \frac{x_1}{x} + c(t)x^n. \tag{7.3}$$

By using Theorem 7 we have that the corresponding function H is $H = \lambda_1 = \frac{x_1}{x} + c(t)x^n$ and the nonlocal symmetry is given by

$$\mathbf{v}_1 = e^w(\partial_x + \psi\partial_w), \tag{7.4}$$

where ψ is a particular solution of the corresponding equation (5.1). It may be checked that $\psi(t, x) = (n + 1)/x$ is a particular solution of this PDE.

On the other hand, the nonlocal symmetry calculated in [19] is given by the vector field

$$\mathbf{v} = e^w a(t)x\partial_x + e^w kna(t)\partial_w \tag{7.5}$$

and corresponds to the function $H = c(t)x^n - a'(t)/a(t)$. However, it seems that there has been a mistake in the calculations, because (7.5) is not a Lie point symmetry of the system (16) in [19], unless $k = 1$. A correct expression for (7.5) could be obtained directly from (7.3) by using Remark 3. If we consider $f(t, x) = a(t)x$, the vector field $\widetilde{\mathbf{v}} = e^w(a(t)x\partial_x + \widetilde{\psi}(t, x, x_1)\partial_w)$ is a nonlocal symmetry associated to $\widetilde{H} = \lambda_1 - D_t(f)/f = c(t)x^n - a'(t)/a(t)$. A particular solution of PDE (5.3) is given by $\widetilde{\psi}(t, x, x_1) = a(t)n$.

It is clear, by Theorem 8, that the reduced equations associated to the nonlocal symmetries $(\mathbf{v}, H)$ and $(\widetilde{\mathbf{v}}, \widetilde{H})$ are functionally dependent, because $(\mathbf{X}_1, \lambda_1) \overset{A}{\sim} (f\mathbf{X}_1, \lambda_1 - D_t(f)/f)$, where A is the vector field associated to equation (7.2).

Example 4. The equation

$$x_2 + x + \frac{1}{2x} + \frac{t^2}{4x^3} = 0 \tag{7.6}$$

was proposed in [34] as an example of an equation with trivial Lie point symmetries that can be integrated via the λ-symmetry

$$\mathbf{X} = x\partial_x, \qquad \lambda = \frac{t}{x^2}.$$

Equation (7.6) is a particular case of the family of equations we later considered in Example 2.1 of [33]

$$x_2 - d(t)x + \frac{b'(t)}{2x} + \frac{b(t)^2}{4x^3} = 0. \tag{7.7}$$

These equations admit the λ-symmetry

$$\mathbf{X}_1 = \partial_x, \qquad \lambda_1 = \frac{x_1}{x} + \frac{b(t)}{x^2}. \tag{7.8}$$

Such λ-symmetry was used to construct first integrals of any of the equations in family (7.7). When $b'(t) = 0$, equation (7.7) is the Ermakov–Pinney equation, for which two nonequivalent λ-symmetries and their associated independent first integrals were reported in [31].

The same family (7.7) was considered in [19]. A nonlocal symmetry is given by the vector field

$$\mathbf{v} = e^{Cw}a(t)x\partial_x - e^{Cw}\frac{2a(t)}{k}\partial_w, \tag{7.9}$$

that is associated to the function $H = 1/C(b(t)/x^2 - a'(t)/a(t))$, for $C \in \mathbb{R}\setminus\{0\}$. By Theorem 5, the pair

$$\mathbf{X}_2 = a(t)x\partial_x, \qquad \lambda_2 = \frac{b(t)}{x^2} - \frac{a'(t)}{a(t)} \tag{7.10}$$

defines a λ-symmetry of equation (7.7). The pairs (7.8) and (7.10) are equivalent because (6.2) is satisfied and the associated order reductions are equivalent. Therefore, by Theorem 8, the reduced equation associated to the nonlocal symmetry (7.9) is equivalent to the reduction previously obtained by using the λ-symmetry (7.8).

Example 5. The well-known Painlevé XIV equation

$$x_2 - \frac{x_1^2}{x} + x_1\left(-xq(t) - \frac{s(t)}{x}\right) + s'(t) - q'(t)x^2 = 0 \tag{7.11}$$

has been studied in [31], where it is shown that a λ-symmetry of (7.11) is defined by

$$\mathbf{X} = \partial_x, \qquad \lambda = \frac{x_1}{x} + xq(t) + \frac{s(t)}{x}. \tag{7.12}$$

Equation (7.11) has also been considered in [20] where it has been checked that for $H(t,x) = q(t)x + s(t)/x$ the corresponding system (4.1) admits the generalized symmetry

$$\mathbf{v} = xe^w\partial_x + \beta(t,x,x_1)e^w\partial_w, \tag{7.13}$$

where β is an undetermined functions that satisfies a PDE.

By using Theorem 5, the pair

$$\mathbf{X}_2 = x\partial_x, \qquad \lambda_2 = xq(t) + \frac{s(t)}{x}$$

defines a new λ-symmetry of equation (7.11). However, such λ-symmetry is equivalent to (7.12). Therefore the reduction process associated to the nonlocal symmetry (7.13) can be deduced from the reduction previously obtained by using the λ-symmetry (7.12).

It should be observed that function β is necessary to define the nonlocal symmetry (7.13) and requires a particular solution of the corresponding equation (5.1) for $\lambda = \lambda_2$. Nevertheless, this function is not necessary either to define the associated λ-symmetries or to reduce the order of the original equation.

Example 6. Let us consider the family of equations

$$x_2 + \big(xf'(x) + 2f(x) + c_1\big)x_1 + \big(f^2(x) + c_1 f(x) + c_2\big)x = 0, \tag{7.14}$$

where $f(x)$ is an arbitrary differentiable function and c_1 and c_2 are arbitrary constants. Several well-known equations representing physically important oscillator systems are particular cases of (7.14):

- For $f(x) = kx$ and $c_1 = 0$ equation (7.14) is the modified Emden type equation with additional linear forcing

$$x_2 + 3kxx_1 + k^2x^3 + c_2x = 0.$$

- For $f(x) = kx$ we obtain the generalized modified Emden type equation

$$x_2 + (3kx + c_1)x_1 + k^2x^3 + c_1kx^2 + c_2x = 0.$$

- For $f(x) = kx^2$ equation (7.14) becomes the generalized force-free Duffing–van der Pol oscillator

$$x_2 + \big(4kx^2 + c_1\big)x_1 + k^2x^5 + kc_1x^3 + c_2x = 0.$$

Different choices of $f(x), c_1$ and c_2 generate a wide class of nonlinear ODEs.

Since $\lambda = \frac{x_1}{x} - xf'(x)$ is a particular solution of the corresponding equation (6.5), the pair

$$(\mathbf{X}, \lambda) = \Big(\partial_x, \frac{x_1}{x} - xf'(x)\Big) \tag{7.15}$$

defines a λ-symmetry of the equations in (7.14). Two invariants of $\mathbf{X}^{[\lambda,(1)]}$ are $z = t$ and $\zeta = \frac{x_1}{x} - xf'(x)$. The equations in (7.14) can be written in terms of $\{z, \zeta, \zeta_z\}$ as the first-order ODEs

$$\zeta_z + \zeta^2 + c_1\zeta + c_2 = 0. \tag{7.16}$$

By Theorem 7 the order reduction (7.16) can also be obtained by using the nonlocal symmetry approach. For example, we can construct the λ-covering

$$\begin{aligned} &x_2 + \big(xf'(x) + 2f(x) + c_1\big)x_1 + \big(f^2(x) + c_1 f(x) + c_2\big)x = 0,\\ &w_1 = x_1/x - xf'(x) \end{aligned}$$

and the associated nonlocal symmetry $\mathbf{v} = e^w(\partial_x + \psi(t, x, x_1)\partial_w)$, where ψ is a particular solution of the corresponding PDE (5.1). The infinitesimal ψ is not necessary to obtain the order reduction (7.16).

It should be noted that we can construct infinitely many λ-coverings and nonlocal symmetries associated to the λ-symmetry (7.15). For any functions $\rho = \rho(t, x)$ and $\phi^0 = \phi^0(t, x)$ let $g = g(t, x, x_1)$ denote the function $A(\phi^0 - \rho x_1)/(\phi^0 - \rho x_1)$, where A is the vector field associated to (7.14). The λ-covering

$$x_2 + \big(xf'(x) + 2f(x) + c_1\big)x_1 + \big(f^2(x) + c_1 f(x) + c_2\big)x = 0,$$
$$w_1 = x_1/x - xf'(x) - g(t, x, x_1)$$

admits a nonlocal symmetry of the form $\mathbf{v} = e^w(\rho(t, x)\partial_t + \phi^0(t, x)\partial_x + \overline{\psi}(t, x, x_1)\partial_w)$ (see Theorem 7). The results presented in Section 6 show that the order reductions derived from these nonlocal symmetries are all equivalent and lead to equation (7.16).

8 Conclusions

In this paper, for second-order ODEs, we study the relationships between the reduction method based on generalized symmetries of a covering system and the reduction methods for equations that are invariant under a λ-prolongation or a telescopic vector field. We also discuss the relationships between these two last classes of vector fields.

We first analyze the strong relationships between λ-prolongations and telescopic vector fields. A telescopic vector field can be considered as a λ-prolongation where the two first infinitesimals can depend on the first derivative of the dependent variable. The corresponding reductions methods are also similar: the only difference between the methods is on the dependencies of the two first-order invariants.

It is also proven that the generalized symmetries of a possible covering system that can be used to reduce the order of the given second-order ODE determine nonlocal symmetries of the exponential type; these nonlocal symmetries are associated to λ-symmetries and therefore to telescopic vector fields.

From a computational point of view, the construction of generalized symmetries of a covering system that lets reduce the order of the given equation requires the solution of a nonlinear system of PDEs, whose four unknown functions are the three infinitesimals and the corresponding function H. Nevertheless, by searching a λ-symmetry in a canonical form, only a quasilinear first-order PDE must be solved. Therefore the advantages of using λ-symmetries seems obvious.

As an important consequence of the λ-symmetry approach we have provided a criterion to decide whether or not reductions associated to two nonlocal symmetries are strictly different. This problem is difficult to establish in the context of nonlocal symmetries and had not been considered before.

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