The Symmetry Group of Lamé's System and the Associated Guichard Nets for Conformally Flat Hypersurfaces*

João Paulo dos SANTOS and Keti TENENBLAT

Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília-DF, Brazil
E-mail: *j.p.santos@mat.unb.br, k.tenenblat@mat.unb.br*

> **Abstract.** We consider conformally flat hypersurfaces in four dimensional space forms with their associated Guichard nets and Lamé's system of equations. We show that the symmetry group of the Lamé's system, satisfying Guichard condition, is given by translations and dilations in the independent variables and dilations in the dependents variables. We obtain the solutions which are invariant under the action of the 2-dimensional subgroups of the symmetry group. For the solutions which are invariant under translations, we obtain the corresponding conformally flat hypersurfaces and we describe the corresponding Guichard nets. We show that the coordinate surfaces of the Guichard nets have constant Gaussian curvature, and the sum of the three curvatures is equal to zero. Moreover, the Guichard nets are foliated by flat surfaces with constant mean curvature. We prove that there are solutions of the Lamé's system, given in terms of Jacobi elliptic functions, which are invariant under translations, that correspond to a new class of conformally flat hypersurfaces.
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1 Introduction

The investigation of conformally flat hypersurfaces has been of interest for quite some time. Any surface in $\mathbb{R}^3$ is conformally flat, since it can be parametrized by isothermal coordinates. For higher dimensional hypersurfaces, E. Cartan [2] gave a complete classification for the conformally flat hypersurfaces of an $(n+1)$-dimensional space form when $n+1 \geq 5$. He proved that such hypersurfaces are quasi-umbilic, i.e., one of the principal curvatures has multiplicity at least $n-1$. In the same paper, Cartan investigated the case $n+1 = 4$. He showed that the quasi-umbilic surfaces are conformally flat, but the converse does not hold (for a proof see [13]). Moreover, he gave a characterization of the conformally flat 3-dimensional hypersurfaces, with three distinct principal curvatures, in terms of certain integrable distributions. Since then, there has been an effort to obtain a classification of hypersurfaces satisfying Cartan's characterization.

Lafontaine [13] considered hypersurfaces of type $M^3 = M^2 \times I \subset \mathbb{R}^4$. He obtained the following classes of conformally flat hypersurfaces: a) M^3 is a cylinder over a surface, $M^2 \subset \mathbb{R}^3$, with constant curvature; b) M^3 is a cone over a surface in the sphere, $M^2 \subset \mathbb{S}^3$, with constant curvature; c) M^3 is obtained by rotating a constant curvature surface of the hyperbolic space, $M^2 \subset \mathbb{H}^3 \subset \mathbb{R}^4$, where $\mathbb{H}^3$ is the half space model.

Motivated by Cartan's paper, Hertrich-Jeromin [8], established a correspondence between conformally flat three-dimensional hypersurfaces, with three distinct principal curvatures, and

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Guichard nets. These are systems of triply orthogonal surfaces originally considered by C. Guichard in [6], where he referred to those systems as the analogues of isothermal coordinates.

In view of Hertrich-Jeromin results, the problem of classifying conformally flat 3-dimensional hypersurfaces was transferred to the problem of classifying Guichard nets in $\mathbb{R}^3$. These are open sets of $\mathbb{R}^3$, with an orthogonal flat metric $g = \sum_{i=1}^{3} l_i^2 dx_i^2$, where the functions l_i satisfy the Guichard condition, namely,

$$l_1^2 - l_2^2 + l_3^2 = 0,$$

and a system of second-order partial differential equations, which is called Lamé's system (see (2.2)).

Hertrich-Jeromin obtained an example of a Guichard net, starting from surfaces parallel to Dini's helix and he proved that the corresponding conformally flat hypersurface was a new example, since it did not belong to the class described by Lafontaine.

In [20, 21, 22], Suyama extended the previous results by showing that the Guichard nets described by Hertrich-Jeromin are characterized in terms of a differentiable function $\varphi(x_1, x_2, x_3)$ that determines, up to conformal equivalence, the first and second fundamental forms of the corresponding conformally flat hypersurfaces. Moreover, Suyama showed that if φ does not depend on one of the variables, then the hypersurface is conformal to one of the classes described by Lafontaine. He also showed that the function associated to the example given by Hertrich-Jeromin satisfied $\varphi_{,x_1 x_2} = \varphi_{,x_2 x_3} = 0$. Starting with this condition on φ, Suyama obtained a partial classification of such conformally flat hypersurfaces. The complete classification of conformally flat hypersurfaces, satisfying the above condition on the partial derivatives of φ, was obtained by Hertrich-Jeromin and Suyama in [10]. They showed that these hypersurfaces correspond to a special type of Guichard nets. The authors called them cyclic Guichard nets, due to the fact that one of the coordinates curves is contained in a circle.

In this paper, we obtain solutions l_i satisfying Lamé's system and the Guichard condition, which are invariant under the action of the 2-dimensional subgroups of the symmetry group of the system. Moreover, we investigate the properties of the Guichard nets and of the conformally flat hypersurfaces associated to the solutions l_i. We first determine the symmetry group of Lamé's system satisfying the Guichard condition. We prove that the group is given by translations and dilations of the independent variables x_i and dilations of the dependent variables l_i.

We obtain the solutions l_i, $i = 1, 2, 3$, which are invariant under the action of the 2-dimensional translation subgroup, i.e., $l_i(\xi)$, where $\xi = \sum_{i=1}^{3} \alpha_i x_i$. These solutions are given explicitly in Theorem 3 by Jacobi elliptic functions, whenever all the functions l_i are not constant and in Theorem 4 when one of the functions l_i is constant. Moreover, we consider the solutions l_i which are invariant under the 2-dimensional subgroup involving translations and dilations, i.e., $l_i(\eta)$, where $\eta = \sum_{j=1}^{3} a_j x_j / \sum_{k=1}^{3} b_k x_k$. In this case, if we require the functions $l_i(\eta)$ to depend on all three variables, then l_i are constant functions. Otherwise, the solutions $l_i(\eta)$ are given explicitly in Theorem 5. The symmetry subgroup of dilations on the dependent variables is irrelevant for the study of conformally flat hypersurfaces.

Considering the functions l_i which are invariant under the action of translations, we study the corresponding Guichard nets. We show that their coordinate surfaces have constant Gaussian curvature and the sum of the three curvatures is equal to zero. Moreover the Guichard nets are foliated by flat surfaces, with constant mean curvature.

Finally, we investigate the conformally flat hypersurfaces associated to the functions l_i which are invariant under the action of translations. We show that, whenever the basic invariant ξ depends on two variables, the hypersurface is conformal to one of the products considered by

Lafontaine. In this case, the three-dimensional conformally flat hypersurfaces are constructed from flat surfaces contained in the hyperbolic 3-space $\mathbb{H}^3$ or in the sphere $\mathbb{S}^3$. Whenever the basic invariant ξ depends on all three independent variables, then the functions $l_i(\xi)$, which are given in terms of Jacobi elliptic functions, produce a new class of conformally flat hypersurfaces.

In Section 2, we review the correspondence between conformally flat 3-dimensional hypersurfaces with Lamé's system, and Guichard nets.

In Section 3, we obtain the symmetry group of Lamé's system satisfying Guichard condition and the solutions which are invariant under 2-dimensional subgroups of the symmetry group. The motivation and the technique used in this section were inspired by the fact that our system of differential equations is quite similar to the intrinsic generalized wave and sine-Gordon equations and the generalized Laplace and sinh-Gordon equations. The symmetry groups of these systems and the solutions invariant under subgroups were obtained by Tenenblat and Winternitz in [24] and Ferreira [4]. The geometric properties of the submanifolds corresponding to the solutions invariant under the subgroups of symmetries can be found in [1] and [19].

In Sections 4 and 5, we describe the geometric properties of the Guichard nets and of the conformally flat hypersurfaces that are associated to the solutions of Lamé's system which are invariant under the action of the translation group.

The solutions l_i of Lamé's system, satisfying Guichard condition, which are invariant under the subgroup of dilations of the independent variables and the corresponding geometric theory, will be considered in another paper. Such solutions are obtained by solving a (reduced) system of partial differential equations, in contrast to what occurs in this paper, where the Lamé's system is reduced to a system of ordinary differential equations.

2   Lamé's system and conformally flat hypersurfaces

Consider the Minkowski space $\mathbb{R}^6_1$ with coordinates $(x_0, \ldots, x_5)$ and the scalar product $\langle\ ,\ \rangle$ given by

$$\langle\ ,\ \rangle:\ \mathbb{R}^6 \times \mathbb{R}^6 \longrightarrow \mathbb{R},$$

$$(v, w) \mapsto -v_0 w_0 + \sum_{i=1}^{5} v_i w_i.$$

Let $L^5 = \left\{ y \in \mathbb{R}^6_1 \mid \langle y, y \rangle = 0 \right\}$, be the light cone in $\mathbb{R}^6_1$ and consider $m_K \in \mathbb{R}^6_1$, with $\langle m_K, m_K \rangle = K$. Then, it is not difficult to see that, the sets

$$M_K^4 = \left\{ y \in L^5 \mid \langle y, m_K \rangle = -1 \right\},$$

with the metric induced from $\mathbb{R}^6_1$, are complete Riemannian manifolds with constant sectional curvature K. If $K < 0$, then M_K^4 consists of two connected components which can be isometrically identified (see [7, Lemma 1.4.1] for details).

With this approach, consider a Riemannian immersion $f : M^3 \to M_K^4 \subset L^5$, with unit normal n. Then $\langle df, n \rangle \equiv 0$, and n also satisfies $\langle n, m_K \rangle = \langle n, f \rangle = 0$. Let $\tilde{f} : M^3 \to L^5$ be an immersion given by $\tilde{f} = e^u f$, where u is a differentiable function on M. Observe that the metric induced on $\tilde{f}$ is conformal to the metric induced on the immersion f, i.e.,

$$\langle d\tilde{f}, d\tilde{f} \rangle = e^{2u} \langle df, df \rangle.$$

Definition 1. Let $f : M^3 \to L^5$ be an immersion such that the induced metric, $\langle df, df \rangle$, is positive definite. Let n be a unit normal with $\langle f, n \rangle = 0$ and consider differentiable functions u and a on M^3. Then the pair (f, n) is called a *strip* and the pair $(\tilde{f}, \tilde{n})$ given by

$$\tilde{f} = e^u f, \qquad \tilde{n} = n + af$$

is called a *conformal deformation* of the strip (f, n).

Therefore, we can deform a conformally flat immersion in a space form $f : M^3 \to M_K^4 \subset L^5$ to a flat immersion in the light cone $\tilde{f} : M^3 \to L^5$, by considering a conformal deformation, and vice-versa. Hence the problem of investigating conformally flat hypersurfaces in space forms reduces to a problem of studying flat immersions in the light cone $f : M^3 \to L^5$. We say that a conformally flat hypersurface in a space form M_k^4 is *generic* if it has three distinct principal curvatures. Hertrich-Jeromin in [8] established a relation between generic conformally flat hypersurfaces in M_k^4 and Guichard nets [6]. Namely, let e_1, e_2, e_3 be an orthonormal frame tangent to $M^3 \subset M_k^4$, such that e_i are principal directions. Let ω_1, ω_2, ω_3 be the co-frame and let k_1, k_2, k_3 be the principal curvatures. Assume that locally $k_3 > k_2 > k_1$, then the *conformal fundamental forms*

$$\alpha_1 = \sqrt{(k_3 - k_1)(k_2 - k_1)}\omega_1, \qquad \alpha_2 = \sqrt{(k_2 - k_1)(k_3 - k_2)}\omega_2,$$
$$\alpha_3 = \sqrt{(k_3 - k_2)(k_3 - k_1)}\omega_3$$

are closed, if and only if, the hypersurface M^3 is conformally flat. Therefore, when α_i are closed forms, locally there exist x_1, x_2, x_3 such that $\alpha_1 = dx_1$, $\alpha_2 = dx_2$ and $\alpha_3 = dx_3$. By integration, we obtain a special principal coordinate system x_1, x_2, x_3 for a conformally flat hypersurface in M_K^4.

Definition 2. A triply orthogonal coordinate system in a Riemannian 3-manifold (M, g)

$$x = (x_1, x_2, x_3) : (M, g) \to \mathbb{R}^3,$$

where the functions $l_i = \sqrt{g\left(\partial_{x_i}, \partial_{x_i}\right)}$ satisfy the Guichard condition

$$l_1^2 - l_2^2 + l_3^2 = 0, \tag{2.1}$$

is called a *Guichard net*.

Since we can deform a conformally flat immersion in a space form into a flat immersion in the light cone, we can consider Guichard nets for flat immersions $f : M^3 \to L^5$. For such a flat immersion, we express the induced metric $g = \langle df, df \rangle$, in terms of the Guichard net, as

$$g = l_1^2 dx_1^2 + l_2^2 dx_2^2 + l_3^2 dx_3^2.$$

Since the metric is flat, the functions l_i must satisfy the *Lamé's system* [14, pp. 73–78]:

$$\frac{\partial^2 l_i}{\partial x_j \partial x_k} - \frac{1}{l_j}\frac{\partial l_i}{\partial x_j}\frac{\partial l_j}{\partial x_k} - \frac{1}{l_k}\frac{\partial l_i}{\partial x_k}\frac{\partial l_k}{\partial x_j} = 0,$$
$$\frac{\partial}{\partial x_j}\left(\frac{1}{l_j}\frac{\partial l_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{l_i}\frac{\partial l_j}{\partial x_i}\right) + \frac{1}{l_k^2}\frac{\partial l_i}{\partial x_k}\frac{\partial l_j}{\partial x_k} = 0. \tag{2.2}$$

for i, j, k distinct. Moreover, if $f : M^3 \to L^5$ is flat, we can consider M^3 as a subset of the Euclidean space $\mathbb{R}^3$ and f as an isometric immersion. Then we have a Guichard net on an open subset of $\mathbb{R}^3$, by considering as in Definition 2, $x : U \subset \mathbb{R}^3 \to \mathbb{R}^3$, where the functions l_i satisfy the Guichard condition (2.1) and the Lamé's system (2.2). At this point, one can ask if such a Guichard net determines a conformally flat hypersurface in a space form, or equivalently, a flat immersion in L^5. The answer to this question was given by the following fundamental result due to Hertrich-Jeromin [8]:

Theorem 1. *For any generic conformally flat hypersurface of a space form M_K^4, there exists a Guichard net $x : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ on an open set U of the Euclidean space $\mathbb{R}^3$ (uniquely determined up to a Möbius transformation of $\mathbb{R}^3$).*

Conversely, given a Guichard net $x = (x_1, x_2, x_3) : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ for the Euclidean space, with $l_i = \sqrt{g(\partial_{x_i}, \partial_{x_j})}$, where g is the canonical flat metric, there exists a generic conformally flat hypersurface in a space form M_K^4 (in this case, Möbius equivalent Guichard nets are related to conformally equivalent immersions), whose induced metric is given by

$$g = e^{2P(x)}\{l_1^2 dx_1^2 + l_2^2 dx_2^2 + l_3^2 dx_3^2\}, \tag{2.3}$$

where $P(x)$ is a function depending on M_K^4.

The converse is based on the fact that the functions l_i determine the connection forms of a flat immersion $f : M^3 \to L^5$. In fact, these connection forms satisfy the Maurer–Cartan equations if, and only if, the functions l_i satisfy the Guichard condition and the Lamé's system.

Therefore, one way of obtaining generic conformally flat hypersurfaces in space forms M_K^4 is finding solutions of Lamé's system, satisfying the Guichard condition. Then the hypersurfaces are constructed by using Theorem 1. Our objective is to obtain a class of such solutions and to investigate the associated Guichard nets as well as the conformally flat hypersurfaces. We will use the theory of Lie point symmetry groups of differential equations, to obtain the symmetry group of Lamé's system and their solutions invariant under the action of subgroups of the symmetry group. This is the content of the following sections.

3   The symmetry group of Lamé's system

In this section, we obtain the Lie point symmetry group of Lamé's system. We start with a brief introduction of symmetry groups of differential equations. The reader who is familiar to the theory may skip this introduction.

The theory of Lie point symmetry group is an important tool for the analysis of differential equations developed by Lie at the end of the nineteen century [15]. Roughly speaking, Lie point symmetries of a system of differential equations consist of a Lie group of transformations acting on the dependent and independent variables, that transform solutions of the system into solutions.

A standard reference for the theory of symmetry groups of differential equations is Olver's book [17], where a clear approach to the subject is given, with theoretical foundations and a large number of examples and techniques. We will describe here some basic concepts that will be used in this section.

A system S of n-th order differential equations in p independent and q dependent variables is given as a system of equations

$$\Delta_r\big(x, u^{(n)}\big) = 0, \qquad v = 1, \ldots, l, \tag{3.1}$$

involving $x = (x_1, \ldots, x_p)$, $u = (u_1, \ldots, u_q)$ and the derivatives $u^{(n)}$ of u with respect to x up to order n.

A *symmetry group* of the system S is a local Lie group of transformations G acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables for the system, with the property that whenever $u = f(x)$ is a solution of S, and whenever gf is defined for $g \in G$, then $u = gf(x)$ is also a solution of the system. A vector field $\mathbf{v}$ in the Lie algebra $\mathfrak{g}$ of the group G is called an *infinitesimal generator*.

Consider $\mathbf{v}$ as a vector field on $M \subset X \times U$, with corresponding (local) one-parameter group $\exp(\varepsilon \mathbf{v})$, i.e.,

$$\exp(\varepsilon \mathbf{v}) \equiv \Psi(\varepsilon, x),$$

where Ψ is the *flow* generated by $\mathbf{v}$. In this case, $\mathbf{v}$ will be the infinitesimal generator of the action.

The symmetry group of a given system of differential equation, is obtained by using the *prolongation formula* and the *infinitesimal criterion* that are described as follows. Given a vector field on $M \subset X \times U$,

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha},$$

the *n-th prolongation* of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}.$$

It is defined on the corresponding *jet space* $M^{(n)} \subset X \times U^{(n)}$, whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order n. The second summation is taken over all (unordered) multi-indices $J = (j_1, \ldots, j_k)$, with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The coefficient functions ϕ_α^J of $\mathrm{pr}^{(n)}\mathbf{v}$ are given by the following formula:

$$\phi_\alpha^J(x, u^{(n)}) = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha \right),$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i}$, $u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x_i}$ and D_J is given by the total derivatives

$$D_J = D_{j_1} D_{j_2} \cdots D_{j_k},$$

with

$$D_i f(x, u^{(n)}) = \frac{\partial f}{\partial x_i} + \sum_{\alpha_1} \sum_{J} u_{J,i}^\alpha \frac{\partial f}{\partial u_J^\alpha}.$$

We say that the system (3.1) is a system of *maximal rank* over $M \subset X \times U$, if the Jacobian matrix

$$J_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_r}{\partial x_i}, \frac{\partial \Delta_r}{\partial u_{,\mathbf{J}}^\alpha} \right)$$

has rank l, whenever $\Delta_r(x, u^{(n)}) = 0$, where $\mathbf{J} = (j_1, \ldots, j_k)$ is a multi-index that denotes the partial derivatives of u^α.

Suppose that (3.1) is a system of maximal rank. Then the set of all vectors fields $\mathbf{v}$ on M such that

$$\mathrm{pr}^{(n)}\mathbf{v}\left[\Delta_r(x, u^{(n)})\right] = 0, \qquad r = 1, \ldots, l, \qquad \text{whenever} \quad \Delta_r(x, u^{(n)}) = 0, \tag{3.2}$$

is a Lie algebra of infinitesimal generators of a symmetry group for the system. It is shown in [17] that the infinitesimal criterion (3.2) is in fact both a necessary and sufficient condition for a group G to be a symmetry group. Hence, all the connected symmetry groups can be determined by considering this criterion.

Since the prolongation formula is given in terms of ξ^i and ϕ_α and the partial derivatives with respect to both x and u, the infinitesimal criterion provides a system of partial differential equations for the coefficients ξ^i and ϕ_α of $\mathbf{v}$, called the *determining equations*. By solving these equations, we obtain the vector field $\mathbf{v}$ that determines a Lie algebra $\mathfrak{g}$. The symmetry group G is obtained by exponentiating the Lie algebra.

3.1 Obtaining the symmetry group of Lamé's system

From now on, we consider the following notation for derivatives of a function $f = f(x_1, \ldots, x_n)$

$$f_{,x_i} := \frac{\partial f}{\partial x_i} \qquad \text{and} \qquad f_{,x_i x_j} := \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

With this notation, Lamé's system (2.2) is given by

$$l_{i,x_j x_k} - \frac{l_{i,x_j} l_{j,x_k}}{l_j} - \frac{l_{i,x_k} l_{k,x_j}}{l_k} = 0, \tag{3.3}$$

$$\left(\frac{l_{i,x_j}}{l_j} \right)_{,x_j} + \left(\frac{l_{j,x_i}}{l_i} \right)_{,x_i} + \frac{l_{i,x_k} l_{j,x_k}}{l_k^2} = 0, \tag{3.4}$$

where i, j and k are distinct indices in the set $\{1, 2, 3\}$. We will also consider the following notation,

$$\varepsilon_s = \begin{cases} 1 & \text{if } s = 1 \text{ or } s = 3, \\ -1 & \text{if } s = 2. \end{cases} \tag{3.5}$$

We can now rewrite Guichard condition as

$$\varepsilon_i l_i^2 + \varepsilon_j l_j^2 + \varepsilon_k l_k^2 = 0.$$

Next, we introduce auxiliary functions in order to reduce the system of second-order differential equations (3.3) and (3.4), into a first order one. Consider the functions h_{ij}, with $i \neq j$, given by

$$l_{i,x_j} - h_{ij} l_j = 0.$$

With these functions, we rewrite (3.3) and (3.4) as

$$h_{ij,x_k} - h_{ik} h_{kj} = 0, \qquad h_{ij,x_j} + h_{ji,x_i} + h_{ik} h_{jk} = 0.$$

for i, j, k distinct. Since the functions l_1, l_2 and l_3 satisfy Guichard condition, there are other relations involving the derivatives of l_i and h_{ij}. Taking the derivative of Guichard condition with respect to x_i, we have

$$\varepsilon_i l_{i,x_i} + \varepsilon_j h_{ji} l_j + \varepsilon_k h_{ki} l_k = 0,$$

for i, j, k distinct. The derivatives of the above equation with respect to x_j leads to

$$\varepsilon_i h_{ij,x_i} + \varepsilon_j h_{ji,x_j} + \varepsilon_k h_{ki} h_{kj} = 0.$$

Therefore, we summarize the last six equations in the following system of first-order partial differential equations, equivalent to Lamé's system, that we call *Lamé's system of first order*

$$\varepsilon_i l_i^2 + \varepsilon_j l_j^2 + \varepsilon_k l_k^2 = 0, \tag{3.6}$$
$$l_{i,x_j} - h_{ij} l_j = 0, \tag{3.7}$$
$$\varepsilon_i l_{i,x_i} + \varepsilon_j h_{ji} l_j + \varepsilon_k h_{ki} l_k = 0, \tag{3.8}$$
$$h_{ij,x_k} - h_{ik} h_{kj} = 0, \tag{3.9}$$
$$h_{ij,x_j} + h_{ji,x_i} + h_{ik} h_{jk} = 0, \tag{3.10}$$
$$\varepsilon_i h_{ij,x_i} + \varepsilon_j h_{ji,x_j} + \varepsilon_k h_{ki} h_{kj} = 0. \tag{3.11}$$

By considering $x = (x_1, x_2, x_3)$, $l = (l_1, l_2, l_3)$ and h the off-diagonal 3×3 matrix given by h_{ij} in our next two results, we obtain the Lie algebra of the infinitesimal generators and the symmetry group of Lamé's system of first order.

Theorem 2. *Let V be the infinitesimal generator of the symmetry group of Lamé's system of first order* (3.6)–(3.11)*, given by*

$$V = \sum_{i=1}^{3} \xi^i(x, l, h) \frac{\partial}{\partial x_i} + \sum_{i=1}^{3} \eta^i(x, l, h) \frac{\partial}{\partial l_i} + \sum_{i,j=1, i\neq j}^{3} \phi^{ij}(x, l, h) \frac{\partial}{\partial h_{ij}}. \tag{3.12}$$

Then the functions ξ^i, η^i and ϕ^{ij} are given by

$$\xi^i = ax_i + a_i, \qquad \eta^i = cl_i, \qquad \phi^{ij} = -ah_{ij},$$

where $a, c, a_i \in \mathbb{R}$.

The proof of Theorem 2 is very long and technical. It consists of obtaining the functions ξ^i, η^i and ϕ^{ij} by solving the determining equations which are obtained as follows. We apply the first prolongation of V to each equation (3.6)–(3.11) and we eliminate the functional dependence of the derivatives of h and l caused by the system. Then we equate to zero the coefficients of the remaining unconstrained partial derivatives. The complete proof with, all the details, is given in Appendix A.

As a consequence of Theorem 2, by exponentiating V, we obtain the symmetry group of Lamé's system. Observe that the functions ϕ^{ij} do not depend on x and l (see [18] for symmetry group of equivalent systems):

Corollary 1. *The symmetry group of Lamé's system* (3.6)–(3.11) *is given by the following transformations:*

1) *translations in the independent variables: $\tilde{x}_i = x_i + v_i$;*

2) *dilations in the independent variables: $\tilde{x}_i = \lambda x_i$;*

3) *dilations in the dependent variables: $\tilde{l}_i = \rho l_i$;*

where $v_i \in \mathbb{R}$ and $\lambda, \rho \in \mathbb{R} \setminus \{0\}$.

3.2   Group invariant solutions

The knowledge of all the infinitesimal generators $\mathbf{v}$ of the symmetry group of a system of differential equations, allows one to reduce the system to another one with a reduced number of variables. Specifically, if the system has p independent variables and an s-dimensional symmetry subgroup is considered, then the reduced system for the solutions invariant under this subgroup will depend on $p - s$ variables (see Olver [17] for details). Finding all the s-dimensional symmetry subgroups is equivalent to finding all the s-dimensional subalgebras of the Lie algebra of infinitesimal symmetries $\mathbf{v}$. For the remainder of this paper, we will consider the 2-dimensional subgroups of the symmetry group of Lamé's system. The first one will be the translation subgroup and the second one will be the subgroup involving translations and the dilations. The 1-dimensional subgroup given just by dilations and the solutions invariant under this subgroup are being investigated. We will report on our investigation in another paper. We observe that the symmetry subgroup of dilations in the dependent variables (Corollary 1(3)) is irrelevant for the geometric study of conformally flat hypersurfaces due to (2.3).

We start with the 2-dimensional subgroup of translations. The basic invariant of this group is given by

$$\xi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \tag{3.13}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is a non zero vector. We will consider solutions l_i such that

$$l_i(x_1, x_2, x_3) = l_i(\xi), \qquad 1 \leq i \leq 3,$$

where ξ is given by (3.13). For such solutions, Lamé's system reduces to a system of ODEs. We start with two lemmas:

Lemma 1. *Let $l_s(\xi)$, $s = 1, 2, 3$, where $\xi = \sum_{s=1}^{3} \alpha_s x_s$, be a solution of Lamé's system (3.6)–(3.11). Let $i, k \in \{1, 2, 3\}$ be two fixed and distinct indices such that $\alpha_i = \alpha_k = 0$. Then l_i or l_k is constant.*

Proof. Since $\alpha_i = \alpha_k = 0$, it follows from (3.7) that equation (3.10) reduces to

$$\alpha_j^2 \left[\frac{l_{i,\xi}}{l_j} \right]_{,\xi} = 0,$$

which implies $l_{i,\xi} = c_i l_j$, where $c_i \in \mathbb{R}$. Similarly, interchanging i with k, we obtain $l_{k,\xi} = c_k l_j$. Finally, interchanging k with j, we get

$$\alpha_j^2 \frac{l_{i,\xi} l_{k,\xi}}{l_j^2} = \alpha_j^2 c_i c_k = 0.$$

Therefore, we conclude that l_i or l_k is constant. ∎

Lemma 2. *Let $l_s(\xi)$, $s = 1, 2, 3$, where $\xi = \sum_{s=1}^{3} \alpha_s x_s$, be a solution of Lamé's system (3.6)–(3.11). If there exists a unique $j \in \{1, 2, 3\}$ such that l_j is a non zero constant, then $\alpha_j = 0$.*

Proof. Interchanging the indices in (3.9), we obtain the following two equations

$$\alpha_j \alpha_k \left(l_{i,\xi\xi} - \frac{l_{i,\xi} l_{k,\xi}}{l_k} \right) = 0, \tag{3.14}$$

$$\alpha_j \alpha_i \left(l_{k,\xi\xi} - \frac{l_{k,\xi} l_{i,\xi}}{l_i} \right) = 0, \tag{3.15}$$

and an identity.

Similarly, it follows from (3.10) that

$$\alpha_j^2 l_{i,\xi\xi} = 0, \tag{3.16}$$

$$\alpha_j^2 l_{k,\xi\xi} = 0, \tag{3.17}$$

$$\alpha_k^2 \left(\frac{l_{i,\xi}}{l_k} \right)_{,\xi} + \alpha_i^2 \left(\frac{l_{k,\xi}}{l_i} \right)_{,\xi} + \alpha_j^2 \frac{l_{i,\xi} l_{k,\xi}}{l_j} = 0. \tag{3.18}$$

Suppose, by contradiction, that $\alpha_j \neq 0$. It follows from (3.16) and (3.17) that $l_{i,\xi} = c_i$ and $l_{k,\xi} = c_k$, where $c_i \neq 0$ and $c_k \neq 0$, since by hypothesis, l_i and l_k are non constants. Then, it follows from (3.14) and (3.15) that $\alpha_i = \alpha_k = 0$. From (3.18), we obtain $\alpha_j^2 c_i c_k = 0$, which is a contradiction. ∎

The following theorem gives the solutions of Lamé's system, satisfying Guichard condition, which are invariant under the action of the translation group, whenever none of the functions l_i is constant.

Theorem 3. *Let $l_s(\xi)$, $s = 1, 2, 3$, where $\xi = \sum_{s=1}^{3} \alpha_s x_s$, be a solution of Lamé's system (3.6)–(3.11), such that l_s is not constant for all s. Then there exist $c_s \in \mathbb{R} \setminus \{0\}$, such that,*

$$l_{i,\xi} = c_i l_k l_j, \quad i, j, k \quad distinct, \tag{3.19}$$

$$c_1 - c_2 + c_3 = 0, \tag{3.20}$$

$$\alpha_1^2 c_2 c_3 + \alpha_2^2 c_1 c_3 + \alpha_3^2 c_1 c_2 = 0. \tag{3.21}$$

Moreover, the functions $l_i(\xi)$ are given by

$$l_{1,\xi}^2 = c_2(c_2 - c_1)\left(l_1^2 - \frac{\lambda}{c_2}\right)\left(l_1^2 - \frac{\lambda}{c_2 - c_1}\right), \tag{3.22}$$

$$l_2^2 = \frac{c_2}{c_1}\left(l_1^2 - \frac{\lambda}{c_2}\right), \tag{3.23}$$

$$l_3^2 = \frac{c_2 - c_1}{c_1}\left(l_1^2 - \frac{\lambda}{c_2 - c_1}\right), \tag{3.24}$$

where $\lambda \in \mathbb{R}$.

Proof. By hypothesis, we are considering non constant solutions. Then, it follows from Lemma 1, that $\alpha_s \neq 0$ for at least two distinct indices. Suppose that α_j and α_k non zero. From (3.7) and (3.9) we obtain

$$\alpha_j \alpha_k \left\{ \left[\frac{l_{i,\xi}}{l_j}\right]_{,\xi} - \frac{l_{i,\xi}}{l_j}\frac{l_{k,\xi}}{l_k} \right\} = 0,$$

which implies

$$\left[\frac{l_{i,\xi}}{l_j}\right]_{,\xi}\left[\frac{l_{i,\xi}}{l_j}\right]^{-1} = \frac{l_{k,\xi}}{l_k}.$$

Integrating this equation, we obtain $l_{i,\xi} = c_i l_k l_j$, where $c_i \neq 0$.

If $\alpha_i \neq 0$, analogously considering the non zero pairs (α_i, α_j) and (α_i, α_k), we conclude that $l_{k,\xi} = c_k l_i l_j$ and $l_{j,\xi} = c_j l_i l_k$. If $\alpha_i = 0$, then from equation (3.10) we have

$$\left[\frac{l_{i,x_j}}{l_j}\right]_{,x_j} + \frac{l_{i,x_k}}{l_k}\frac{l_{j,x_k}}{l_k} = \alpha_j^2 c_i l_{k,\xi} + \alpha_k^2 c_i l_j \frac{l_{j,\xi}}{l_k} = 0.$$

Since $c_i \neq 0$, we integrate the above expression to obtain

$$\alpha_j^2 l_k^2 + \alpha_k^2 l_j^2 = \lambda_{jk},$$

where λ_{jk} is a constant. This equation and Guichard condition (3.6) lead to

$$l_j^2 = \frac{\alpha_j^2}{\alpha_k^2}\left(\frac{\lambda_{jk}}{\alpha_j^2} - l_k^2\right), \qquad l_i^2 = \frac{\varepsilon_i}{\alpha_k^2}\left[l_k^2\left(\varepsilon_j \alpha_j^2 - \varepsilon_k \alpha_k^2\right) - \varepsilon_j \lambda_{jk}\right]. \tag{3.25}$$

Taking the derivative of the last equation with respect to ξ, we have

$$l_i\left(c_i l_k l_j\right) = \frac{\varepsilon_i}{\alpha_k^2}\left[l_k l_{k,\xi}\left(\varepsilon_j \alpha_j^2 - \varepsilon_k \alpha_k^2\right)\right].$$

If $\varepsilon_j \alpha_j^2 - \varepsilon_k \alpha_k^2 \neq 0$, we conclude that

$$l_{k,\xi} = \frac{c_i \alpha_k^2}{\varepsilon_j \alpha_j^2 - \varepsilon_k \alpha_k^2} l_i l_j = c_k l_i l_j.$$

Applying this expression into the derivative of the first equation in (3.25) with respect to ξ we obtain

$$l_j l_{j,\xi} = -\frac{\alpha_j^2}{\alpha_k^2} l_k l_{k,\xi} = -\frac{\alpha_j^2}{\alpha_k^2} l_k \left(c_k l_i l_j \right),$$

consequently, $l_{j,\xi} = c_j l_i l_k$.

Next, we will show that $\varepsilon_j \alpha_j^2 - \varepsilon_k \alpha_k^2 \neq 0$ to conclude the proof of (3.19). Suppose by contradiction that $\varepsilon_j \alpha_j^2 - \varepsilon_k \alpha_k^2 = 0$, then the first equation of (3.25) can be written as $\varepsilon_j l_j^2 + \varepsilon_k l_k^2 = \frac{\varepsilon_k \lambda_{jk}}{\alpha_j^2}$. Then Guichard condition now implies that l_i is constant, which is a contradiction. The relations between the constants (3.20) and (3.21) follow from a straightforward computation using equations (3.8) and (3.10), respectively.

In order to complete the proof of the theorem, we start with

$$l_{1,\xi} = c_1 l_2 l_3, \tag{3.26}$$
$$l_{2,\xi} = c_2 l_1 l_3, \tag{3.27}$$
$$l_{3,\xi} = c_3 l_1 l_2. \tag{3.28}$$

Multiplying (3.27) by l_2 and integrating we have

$$l_2^2 = \frac{c_2}{c_1} \left(l_1^2 - \frac{\lambda}{c_2} \right), \tag{3.29}$$

where λ is a constant. Therefore, it follows from (3.29) and Guichard condition that

$$l_3^2 = \frac{c_2 - c_1}{c_1} \left(l_1^2 - \frac{\lambda}{c_2 - c_1} \right). \tag{3.30}$$

Using (3.26), (3.29) and (3.30), we conclude that

$$\begin{aligned}
l_{1,\xi}{}^2 &= c_1^2 \left[\frac{c_2}{c_1} \left(l_1^2 - \frac{\lambda}{c_2} \right) \right] \left[\frac{c_2 - c_1}{c_1} \left(l_1^2 - \frac{\lambda}{c_2 - c_1} \right) \right] \\
&= c_2 \left(c_2 - c_1 \right) \left(l_1^2 - \frac{\lambda}{c_2} \right) \left(l_1^2 - \frac{\lambda}{c_2 - c_1} \right). \quad \blacksquare
\end{aligned}$$

In our next theorem, we consider the solutions $l_i(\xi)$ when one of the functions l_i is constant.

Theorem 4. *Let $l_s(\xi)$, $s = 1, 2, 3$, where $\xi = \sum_{s=1}^{3} \alpha_s x_s$, be a solution of Lamé's system (3.6)–(3.11). Suppose that only one of the functions l_s is constant. Then one of the following occur:*

a) *$l_1 = \lambda_1$, $l_2 = \lambda_1 \cosh(b\xi + \xi_0)$, $l_3 = \lambda_1 \sinh(b\xi + \xi_0)$, where $\xi = \alpha_2 x_2 + \alpha_3 x_3$, $\alpha_2^2 + \alpha_3^2 \neq 0$ and $b, \xi_0 \in \mathbb{R}$;*

b) *$l_2 = \lambda_2$, $l_1 = \lambda_2 \cos \varphi(\xi)$, $l_3 = \lambda_2 \sin \varphi(\xi)$, where $\xi = \alpha_1 x_1 + \alpha_3 x_3$, $\alpha_1^2 + \alpha_3^2 \neq 0$ and φ is one of the following:*

    b.1) *$\varphi(\xi) = b\xi + \xi_0$, if $\alpha_1^2 \neq \alpha_3^2$, where $\xi_0, b \in \mathbb{R}$;*

    b.2) *φ is any function of ξ, if $\alpha_1^2 = \alpha_3^2$;*

c) *$l_3 = \lambda_3$, $l_2 = \lambda_3 \cosh(b\xi + \xi_0)$, $l_1 = \lambda_3 \sinh(b\xi + \xi_0)$, where $\xi = \alpha_1 x_1 + \alpha_2 x_2$, $\alpha_1^2 + \alpha_2^2 \neq 0$ and $b, \xi_0 \in \mathbb{R}$.*

Proof. We will consider each case separately:

a) If $l_1 = \lambda_1$, then it follows from Lemma 2 that we must have $\xi = \alpha_2 x_2 + \alpha_3 x_3$. Now Guichard condition implies that $l_2 = \lambda_1 \cosh \varphi(\xi)$ and $l_3 = \lambda_1 \sinh \varphi(\xi)$. In order to determine φ, we use (3.10) with the following indices

$$h_{23,x_3} + h_{32,x_2} + h_{21}h_{31} = 0,$$

to obtain

$$\alpha_3^2 \left(\frac{\lambda_1 \varphi_{,\xi} \sinh \varphi}{\lambda_1 \sinh \varphi} \right)_{,\xi} + \alpha_2^2 \left(\frac{\lambda_1 \varphi_{,\xi} \cosh \varphi}{\lambda_1 \cosh \varphi} \right)_{,\xi} = 0.$$

Since l_2 and l_3 are not constant, we have $\alpha_2^2 + \alpha_3^2 \neq 0$, which implies $\varphi_{,\xi\xi} = 0$. Consequently, $\varphi(\xi) = b\xi + \xi_0$.

b) If $l_2 = \lambda_2$, it follows from Lemma 2 that $\xi = \alpha_1 x_1 + \alpha_3 x_3$. Then Guichard condition implies that $l_1 = \lambda_2 \cos \varphi(\xi)$ and $l_3 = \lambda_2 \sin \varphi(\xi)$. As in the case *a)*, from equation (3.10) we get $\left(\alpha_1^2 - \alpha_3^2 \right) \varphi_{,\xi\xi} = 0$. Since l_1 and l_3 are non constant, we have $\alpha_1^2 + \alpha_3^2 \neq 0$. Then we have two cases to consider:

b.1) If $\alpha_1^2 \neq \alpha_3^2$, then $\varphi(\xi) = b\xi + \xi_0$;

b.2) If $\alpha_1^2 = \alpha_3^2$, then φ can be any function of ξ.

c) The proof is the same as in *a)*.   ∎

Next, we consider the solutions invariant under the 2-dimensional subgroup involving translations and dilations. In this case, the basic invariant is given by

$$\eta = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{b_1 x_1 + b_2 x_2 + b_3 x_3}, \tag{3.31}$$

where the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent. If $f = f(\eta)$ is a function depending on η, then

$$f_{,x_i} = f_{,\eta} \eta_{x_i} = \frac{a_i - b_i \eta}{b_1 x_1 + b_2 x_2 + b_3 x_3} f_{,\eta}.$$

In order to simplify the computations, we will use the following notation:

$$N_i := a_i - b_i \eta \qquad \text{and} \qquad \beta = b_1 x_1 + b_2 x_2 + b_3 x_3. \tag{3.32}$$

Then we have $\eta_{,x_i} = \frac{N_i}{\beta}$.

In order to obtain the solutions of Lamé's system $l_i(\eta)$, which depend on η, we will need some lemmas.

Lemma 3. *Let $l_1(\eta)$, $l_2(\eta)$, $l_3(\eta)$, where η is given by (3.31), be a solution of Lamé's system (3.6)–(3.11). Suppose that for a fixed pair $j, k \in \{1, 2, 3\}$, $j \neq k$, $(a_j, b_j) \neq (0, 0)$ and $(a_k, b_k) \neq (0, 0)$. Then there exists $c_i \in \mathbb{R}$ such that*

$$l_{i,\eta} = c_i \frac{l_k l_j}{N_k N_j}, \qquad i \neq j, k, \tag{3.33}$$

where N_k is given by (3.32).

Proof. From (3.7), we have that $h_{ij} = \frac{l_{i,\eta} N_j}{l_j \beta}$. Then, equation (3.9) can be written as

$$\left[\frac{l_{i,\eta} N_k N_j}{l_j} \right]_\eta - \frac{l_{i,\eta} N_k N_j}{l_j} \frac{l_{k,\eta}}{l_k} = 0,$$

which implies

$$\left(\frac{l_{i,\eta} N_k N_j}{l_k l_j} \right)_{,\eta} = 0.$$

Since $(a_j, b_j) \neq (0,0)$ and $(a_k, b_k) \neq (0,0)$, we have that $N_j \neq 0$, $N_k \neq 0$ and the equation (3.33) holds. ∎

Lemma 4. *Let* $l_1(\eta), l_2(\eta), l_3(\eta)$, *where* η *is given by* (3.31), *be a solution of Lamé's system* (3.6)–(3.11). *If* $(a_i, b_i) = (0,0)$, *for some* $i \in \{1,2,3\}$, *then* l_i *is constant.*

Proof. Since the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent, if $(a_i, b_i) = (0,0)$ we must have $(a_j, b_j) \neq (0,0)$ and $(a_k, b_k) \neq (0,0)$ for i, j, k distinct and we can use Lemma 3. By considering equation (3.10), we have

$$\left(\frac{c_i l_k}{\beta N_k} \right)_{,\eta} + \left(\frac{c_i l_k}{\beta N_k} \right) \frac{l_{j,\eta} N_k}{\beta l_k} = 0,$$

which implies

$$c_i \left[\frac{l_{k,\eta} N_j}{N_k} - \frac{l_k}{N_k^2} (N_k \beta)_{,x_j} + \frac{l_j l_{j,\eta} N_k}{l_k N_j} \right] = 0. \tag{3.34}$$

By interchanging j with k, we have analogously

$$c_i \left[\frac{l_{j,\eta} N_k}{N_j} - \frac{l_j}{N_j^2} (N_j \beta)_{,x_k} + \frac{l_k l_{k,\eta} N_j}{l_j N_k} \right] = 0. \tag{3.35}$$

Suppose by contradiction that $c_i \neq 0$. Then, it follows from (3.34) and (3.35) that

$$\frac{l_k^2}{N_k^2} (N_k \beta)_{x_j} = \frac{l_j^2}{N_j^2} (N_j \beta)_{x_k}.$$

If $a_i = b_i = 0$, we must have

$$(a_k b_j - b_k a_j) \left(\frac{l_k^2}{N_k^2} + \frac{l_j^2}{N_j^2} \right) = 0,$$

which is a contradiction since $(a_k b_j - b_k a_j) \neq 0$. Therefore $c_i = 0$ and l_i is constant. ∎

Lemma 5. *Let* $l_1(\eta), l_2(\eta), l_3(\eta)$, *with* η *given by* (3.31), *be a solution of Lamé's system* (3.6)–(3.11). *If there exists a unique function* l_i *which is a non zero constant, then* $(a_i, b_i) = (0,0)$.

Proof. Suppose by contradiction that $(a_i, b_i) \neq (0,0)$. Since l_j and l_k are not constant, for i, j, k distinct, it follows from Lemma 4, that we must have $(a_j, b_j) \neq (0,0)$ and $(a_k, b_k) \neq (0,0)$. Then, Lemma 3 implies that there are constants c_i, c_j and c_k such that

$$l_{i,\eta} = c_i \frac{l_j l_k}{N_j N_k}, \qquad l_{j,\eta} = c_j \frac{l_k l_i}{N_k N_i} \qquad \text{and} \qquad l_{k,\eta} = c_k \frac{l_k l_i}{N_k N_i}.$$

Using equation (3.10) and interchanging the indices we have

$$c_k \frac{l_i l_j}{N_j} - \frac{(a_k b_i - b_k a_i) l_k}{N_k} = 0, \tag{3.36}$$

$$c_j c_k \frac{l_j l_k}{N_j N_k} - \frac{l_i}{N_i^2} \left[c_j (a_i b_k - a_k b_i) + c_k (a_i b_j - b_i a_j) \right] = 0, \tag{3.37}$$

$$c_j \frac{l_i l_k}{N_k} - \frac{(a_j b_i - b_j a_i) l_j}{N_j} = 0. \tag{3.38}$$

Multiplying equation (3.36) by $c_j \frac{N_k}{l_k}$, (3.37) by $\frac{N_i^2}{l_i}$ and (3.38) by $c_k \frac{N_j}{l_j}$, the sum will reduce to

$$c_j c_k \left[(l_i l_j N_k)^2 + (l_i l_k N_j)^2 + (l_j l_k N_i)^2 \right] = 0,$$

which is a contradiction. Then, we must have $(a_i, b_i) = (0, 0)$ and the lemma is proved. ∎

Remark 1. We observe that when all pairs (a_s, b_s) are different from zero, then the proof of Lemma 5 shows that the solution $l_i(\eta)$ of Lamé's system is constant.

We will now obtain the solutions $l_s(\eta)$, when one pair $(a_s, b_s) = (0, 0)$.

Theorem 5. *Let $l_i(\eta)$, with η given by (3.31), be a solution of Lamé's system invariant under the 2-dimensional subgroup involving translation and dilations. Suppose that one of the pairs $(a_s, b_s) = (0, 0)$. Then one of the following occur:*

a) *If $(a_1, b_1) = (0, 0)$ then $l_1 = \lambda_1$, $l_2 = \lambda_1 \cosh \varphi(\eta)$, $l_3 = \lambda_1 \sinh \varphi(\eta)$, where $\eta = \frac{a_2 x_2 + a_3 x_3}{b_2 x_2 + b_3 x_3}$ and φ is given by*

$$\varphi(\eta) = \frac{C_0}{a_2 b_3 - a_3 b_2} \arctan \left[\frac{b_2^2 + b_3^2}{a_3 b_2 - a_2 b_3} \left(\eta - \frac{a_2 b_2 + a_3 b_3}{b_2^2 + b_3^2} \right) \right] + C_1, \tag{3.39}$$

   where $C_0, C_1 \in \mathbb{R}$.

b) *If $(a_2, b_2) = (0, 0)$ then $l_2 = \lambda_2$, $l_1 = \lambda_2 \cos \varphi(\eta)$, $l_3 = \lambda_2 \sin \varphi(\eta)$, where $\eta = \frac{a_1 x_1 + a_3 x_3}{b_1 x_1 + b_3 x_3}$ and φ is given as follows:*

   b.1) *if $b_1 = b_3 = b$, then*

$$\varphi(\eta) = \frac{D_0}{2b(a_3 - a_1)} \log (2b\eta - a_1 - a_3) + D_1, \tag{3.40}$$

   where $D_0, D_1 \in \mathbb{R}$;
   b.2) *if $b_1 \neq b_3$, then*

$$\varphi(\eta) = \frac{D_2}{2(a_1 b_3 - a_3 b_1)} \log \left[\frac{(b_3 + b_1)\eta - (a_3 + a_1)}{(b_3 - b_1)\eta - (a_3 - a_1)} \right] + D_3, \tag{3.41}$$

   where $D_2, D_3 \in \mathbb{R}$.

c) *If $(a_3, b_3) = (0, 0)$, then $l_3 = \lambda_3$, $l_2 = \lambda_3 \cosh \varphi(\eta)$, $l_1 = \lambda_3 \sinh \varphi(\eta)$, with $\eta = \frac{a_1 x_1 + a_2 x_2}{b_1 x_1 + b_2 x_2}$ and φ is given by*

$$\varphi(\eta) = \frac{E_0}{a_2 b_1 - a_1 b_2} \arctan \left[\frac{b_2^2 + b_1^2}{a_2 b_1 - a_1 b_2} \left(\eta - \frac{a_2 b_2 + a_1 b_1}{b_2^2 + b_1^2} \right) \right] + E_1,$$

   where $E_0, E_1 \in \mathbb{R}$.

Proof. a) If $(a_1, b_1) = (0, 0)$ then Lemma 4 implies that $l_1 = \lambda_1$ and Guichard condition implies that $l_2 = \lambda_1 \cosh \varphi(\eta)$ and $l_3 = \lambda_1 \sinh \varphi(\eta)$. In order to find φ, we use equation (3.10) with the following indices

$$h_{32,x_2} + h_{23,x_3} + h_{31} h_{21} = 0.$$

Since $h_{32} = \frac{\varphi_{,\eta} N_2}{\beta}$, $h_{23} = \frac{\varphi_{,\eta} N_3}{\beta}$ and $h_{31} = h_{21} = 0$, we rewrite the equation above as

$$\left(\frac{\varphi_{,\eta} N_2}{\beta} \right)_{,x_2} + \left(\frac{\varphi_{,\eta} N_3}{\beta} \right)_{,x_3} = 0.$$

By substituting the derivatives, we have the following ODE

$$\varphi_{,\eta\eta}\left[(N_2)^2 + (N_3)^2 \right] - 2\varphi_{,\eta}(b_2 N_2 + b_3 N_3) = 0,$$

whose solution is exactly (3.39).

   b) If $(a_2, b_2) = (0,0)$, then $l_2 = \lambda_2$ and Guichard condition implies that $l_1 = \lambda_2 \cos \varphi(\eta)$ and $l_3 = \lambda_2 \sin \varphi(\eta)$. In order to find φ, we use equation (3.10) with the following indices

$$h_{13,x_3} + h_{31,x_1} + h_{12}h_{32} = 0.$$

By using the same arguments as in a), we have the following ODE

$$\varphi_{,\eta\eta}\left[(N_1)^2 - (N_3)^2 \right] - 2\varphi_{,\eta}(b_1 N_1 - b_3 N_3) = 0,$$

whose solution will depend on b_1 and b_3. If $b_1 = b_3$ we have φ given by (3.40) and if $b_1 \neq b_3$, the solution is given by (3.41).

   c) The arguments when $(a_3, b_3) = (0,0)$ are the same as in a).   ∎

Remark 2. Although our calculation of the symmetry group for the Lamé's system has similar techniques to those used by Tenenblat and Winternitz for the intrinsic generalized wave and sine-Gordon equations in [24], we observe that the solutions invariant under the subgroups are quite different. In fact, when we consider the solutions invariant under the translation subgroup in Theorem 3, the solutions of (3.22) are given by Jacobi elliptic functions that cannot be reduced to elementary functions. Moreover, the only solutions of the Lamé's system, which are invariant under the action of the subgroup involving translation and dilations, that depend on all three variables are constant, in contrast to the solutions in [24]. The main reason is due to Guichard condition.

   In the next two sections, we will deal with the geometric properties of the Guichard nets and of the conformally flat hypersurfaces associated to the solutions invariant under the 2-dimensional translation subgroup. As we will see in Section 5, these are the solutions that will provide a new class of conformally flat hypersurfaces.

4   Geometric properties of the Guichard nets

In this section, we will study the geometric properties of the Guichard nets associated to locally conformally flat hypersurfaces corresponding to the solutions of the Lamé's system $l_i(\xi)$, which are invariant under the translation subgroup. Let $l_1(\xi)$, $l_2(\xi)$, $l_3(\xi)$, with $\xi = \sum_{s=1}^{3} \alpha_s x_s$ be a solution of Lamé's system. Theorem 1 implies that there is a Guichard net $x = (x_1, x_2, x_3) : U \subset \mathbb{R}^3 \to \mathbb{R}^3$, with a Riemannian metric

$$g = l_1^2 dx_1^2 + l_2^2 dx_2^2 + l_3^2 dx_3^2, \tag{4.1}$$

where U is an open set, given by $U = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, \xi_1 < \xi < \xi_2 \right\}$, where ξ_1 and ξ_2 are real constants.

4.1 Level surfaces

In this subsection, we will show that the Guichard nets are foliated by surfaces $\xi = \xi_0$ which are geodesically parallel. Moreover, we will prove that each such surface has flat Gaussian curvature and constant mean curvature that depends on ξ_0.

Definition 3. Let M^n be a Riemannian manifold and let $f : M \to \mathbb{R}$ be a differentiable function. The level submanifolds of f are said to be *geodesically parallel* if $|\operatorname{grad} f|$ is a non zero constant, along each level submanifold.

We have the following theorem

Theorem 6. *Let (U, g), $U \subset \mathbb{R}^3$, be a Riemannian manifold with coordinates (x_1, x_2, x_3) and metric $g = \sum\limits_{s=1}^{3} l_s^2(\xi)dx_i^2$, where $\xi = \sum\limits_{s=1}^{3} \alpha_s x_s$. Then the level surfaces*

$$P_{\xi_0} = \left\{ (x_1, x_2, x_3) \in U; \ \sum_{s=1}^{3} \alpha_s x_s = \xi_0 \right\}, \qquad where \quad \xi_1 < \xi_0 < \xi_2,$$

endowed with the induced metric, are geodesically parallel. Moreover, each level surface has flat Gaussian curvature and constant mean curvature (depending on ξ_0).

Proof. Since at least one α_i is non zero, we can suppose that $\alpha_3 \neq 0$ and we parametrize P_{ξ_0} as

$$X(x_1, x_2) = \left(x_1, x_2, \frac{\xi_0 - \alpha_1 x_1 - \alpha_2 x_2}{\alpha_3} \right).$$

Then $X_{,x_1} = (1, 0, -\alpha_1/\alpha_3)$ and $X_{,x_2} = (0, 1, -\alpha_2/\alpha_3)$. Consequently, the coefficients of the induced metric are constant, since $\xi = \xi_0$ in this surface. Therefore the Gaussian curvature is equal to zero.

Consider now the function $h(x) = \sum\limits_{i=1}^{3} \alpha_i x_i$. Then $P_{\xi_0} = h^{-1}(\xi_0)$. Since h is constant along P_{ξ_0}, it follows that $\operatorname{grad} h$ is normal to P_{ξ_0}. Moreover,

$$g(\operatorname{grad} h, \operatorname{grad} h) = \sum_{j=1}^{3} \frac{\alpha_j^2}{l_j^2},$$

which implies that $|\operatorname{grad} h|$ is constant along P_{ξ_0}. It follows from Definition 3 that the level surfaces $h^{-1}(\xi_0)$ are geodesically parallel.

Now we compute the mean curvatures of P_{ξ_0}. Given $p \in P_{\xi_0}$, let $A : T_p P_{\xi_0} \to T_p P_{\xi_0}$ be the Weingarten operator, i.e., $A\mathbf{v} = -\nabla_{\mathbf{v}} \left(\frac{\operatorname{grad} h}{|\operatorname{grad} h|} \right)(p)$, where ∇ is the Riemannian connection on (U, g). Since $|\operatorname{grad} h|$ is constant along P_{ξ_0}, it follows that

$$A\mathbf{v} = -\frac{1}{|\operatorname{grad} h|} \nabla_{\mathbf{v}} \operatorname{grad} h(p).$$

Then the mean curvature of P_{ξ_0} is given by

$$H = -\frac{\Delta h(p)}{|\operatorname{grad} h|} = \frac{1}{|\operatorname{grad} h|} \sum_{i,k} \frac{\Gamma_{ii}^k(\xi_0)\alpha_k}{l_i^2(\xi_0)},$$

where Γ_{ij}^k are the Christoffel for the connection ∇. Therefore, the mean curvature of P_{ξ_0} is a constant depending on ξ_0. ∎

4.2 Coordinate surfaces

In this subsection, we will use the solutions invariant by the group of translations to show that the coordinate surfaces of the corresponding Guichard net (U, g) have constant Gaussian curvature. Moreover, the values of these curvatures satisfy an algebraic relation.

Theorem 7. *Let (U, g), $U \subset \mathbb{R}^3$, be a Riemannian manifold, with coordinates (x_1, x_2, x_3) and metric $g = \sum_{s=1}^{3} l_s^2(\xi) dx_i^2$, with $\xi = \sum_{s=1}^{3} \alpha_s x_s$. Then each coordinate surface of $U \subset \mathbb{R}^3$, $x_i = \text{const}$, endowed with the induced metric, has constant Gaussian curvature K_i. Moreover,*

$$K_1 + K_2 + K_3 = 0.$$

Proof. Since g is given by (4.1), it follows that the metric induced on each coordinate surface, $x_i = \text{const}$, is

$$g_i = l_j^2(dx_j)^2 + l_k^2(dx_k)^2, \qquad i,\, j,\, k \text{ distinct,}$$

and its Gaussian curvature, K_i, is given by

$$K_i = \frac{1}{l_j l_k}\left(\frac{l_{k,x_i} l_{j,x_i}}{l_i^2}\right). \tag{4.2}$$

Assume that none of the functions l_i is constant and $\xi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, with $\alpha_i \neq 0$, for all i. In this case, we have $l_{i,\xi} = c_i l_j l_k$, where i, j and k are distinct indices in $\{1, 2, 3\}$. Therefore, it follows from (4.2) that the Gaussian curvature of each coordinate surface is given by $K_i = c_j c_k \alpha_i^2$. Moreover, it follows from (3.21) that

$$K_1 + K_2 + K_3 = \alpha_1^2 c_2 c_3 + \alpha_2^2 c_3 c_1 + \alpha_1^3 c_1 c_2 = 0.$$

If only one of the functions l_i is constant, it follows from Lemma 2, that, if l_i is constant, then $\alpha_i = 0$. Then it follows from (4.2) that all the curvatures are equal to zero. In fact, $K_i = 0$, since the functions l_s, for all s, do not depend on x_i. Moreover, for $j \neq i$, $K_j = 0$, since l_i is constant. Hence, the sum $\sum_{j=1}^{3} K_i = 0$ trivially. ∎

5 Conformally flat hypersurfaces

In this section, we describe the generic conformally flat hypersurfaces associated to the solutions of the Lamé's system invariant under the translation group.

It is known that, any locally generic conformally flat hypersurface, in a 4-dimensional space form, has a metric induced by the Guichard net of the form (see [10, 21, 22])

$$g = e^{2P(x)}\big\{\sin^2\varphi(x)dx_1^2 + dx_2^2 + \cos^2\varphi(x)dx_3^2\big\}, \tag{5.1}$$

where $x = (x_1, x_2, x_3)$, or

$$g = e^{2\tilde{P}(x)}\big\{\sinh^2\tilde{\varphi}(x)dx_1^2 + \cosh^2\tilde{\varphi}(x)dx_2^2 + dx_3^2\big\}. \tag{5.2}$$

Suyama classified in [22] the hypersurfaces conformal to the products $M^2 \times I \subset \mathbb{R}^4$ given by Lafontaine in [13], as the hypersurfaces where φ depends only on two variables. Hertrich-Jeromin and Suyama classified in [10] the hypersurfaces where φ has two vanishing mixed derivatives. These conformally flat hypersurfaces are associated to the so called *cyclic Guichard nets*, which are characterized by $\varphi_{,x_1 x_2} = \varphi_{,x_2 x_3} = 0$, when g is of the form (5.1) and by $\varphi_{,x_1 x_3} = \varphi_{,x_2 x_3} = 0$,

when g is given by (5.2). Moreover, the authors showed that all the known cases of conformally flat hypersurfaces, up to now, are associated to cyclic Guichard nets.

We observe that Theorem 5 shows that each solution of the Lamé's system, which is invariant under the action of the 2-dimensional subgroup involving translations and dilations, depends only on two variables. Therefore, the conformally flat hypersurfaces associated to these solutions are conformal to the products $M^2 \times I \subset \mathbb{R}^4$.

We now consider the conformally flat hypersurfaces associated to the solutions invariant under the translation subgroup. We analyse each case separately:

i) $\xi = \alpha_1 x_1 + \alpha_2 x_2$. In this case, we have the solutions $l_1 = \lambda_3 \sinh(\xi + \xi_0)$, $l_2 = \lambda_3 \cosh(\xi + \xi_0)$ and $l_3 = \lambda_3 \neq 0$ (see Theorem 4). Then the associated conformally flat hypersurface has a Guichard net, where the induced metric is given by

$$g = e^{2P(x_1, x_2, x_3)} \big\{ \sinh^2(\xi + \xi_0) dx_1^2 + \cosh^2(\xi + \xi_0) dx_2^2 + dx_3^2 \big\}. \tag{5.3}$$

The hypersurface is conformal to one of the products considered by Lafontaine in [13] that we describe as follows (see [21, 22] for details): Let $\mathbb{H}^3$ be the hyperbolic 3-space, considered as the half space model and as a subset of $\mathbb{R}^4$, i.e.,

$$\mathbb{H}^3 = \big\{ (y^1, y^2, y^3, 0) : y^3 > 0 \big\} \subset \mathbb{R}^4 = \big\{ (y^1, y^2, y^3, y^4) : y^i \in \mathbb{R} \big\},$$

with the metric $g_{ij} = \frac{\delta_{ij}}{y_3^2}$. Consider the rotations of the y^3-axis given by

$$(y^1, y^2, y^3, 0) \to (y^1, y^2, y^3 \cos t, y^3 \sin t),$$

then the hypersurface $M^3 = M^2 \times I$, obtained by the above rotation of a surface of constant curvature $M^2 \subset \mathbb{H}^3$ is a conformally flat hypersurface. One can show that for g given by (5.3), the surface $M^2 \subset \mathbb{H}^3$ is a flat surface, parametrized by lines of curvature whose first and second fundamental forms are given by

$$I = \sinh^2(\xi + \xi_0) dx_1^2 + \cosh^2(\xi + \xi_0) dx_2^2,$$
$$II = \sinh(\xi + \xi_0) \cosh(\xi + \xi_0) \big(dx_1^2 + dx_2^2 \big). \tag{5.4}$$

In order to describe the flat surfaces $M^2 \subset \mathbb{H}^3$, we mention a classification result obtained by the authors in collaboration with Martínez [16]. It is well known that, on a neighbourhood of a non-umbilical point, a flat surface in $\mathbb{H}^3$ can be parametrized by lines of curvature, so that the first and second fundamental forms are given by (for details, see [23, Theorem 2.4, Corollary 2.7])

$$I = \sinh^2 \phi(u, v)(du)^2 + \cosh^2 \phi(u, v)(dv)^2, \tag{5.5}$$
$$II = \sinh \phi(u, v) \cosh \phi(u, v) \big((du)^2 + (dv)^2 \big), \tag{5.6}$$

where ϕ is a harmonic function, i.e. $\phi_{uu} + \phi_{vv} = 0$. The classification result is given as follows:

Theorem 8 ([16]). *Let Σ be a flat surface in $\mathbb{H}^3$ with a local parametrization, in a neighborhood of a nonsingular and nonumbilic point, such that the first and second fundamental forms are diagonal and given by* (5.5) *and* (5.6), *where ϕ is a (Euclidean) harmonic function. Then ϕ is linear, i.e., $\phi = au + bv + c$ if, and only if, Σ is locally congruent to either a helicoidal flat surface (when $(a, b, c) \neq (0, \pm 1, 0)$) or to a "peach front" (when $(a, b, c) = (0, \pm 1, 0)$).*

Helicoidal surfaces arise as a natural generalization of rotational surfaces. They are invariant by a helicoidal group of isometries, i.e., given an axis, we consider a translation along this axis composed with a rotation around it. In the half space model of $\mathbb{H}^3$, up to isometries, we can

consider the y_3-axis, which enables us to write the helicoidal group, relative to this axis, as the composition

$$h_t = \begin{pmatrix} e^{\beta t} & 0 & 0 \\ 0 & e^{\beta t} & 0 \\ 0 & 0 & e^{\beta t} \end{pmatrix} \begin{pmatrix} \cos \alpha t & -\sin \alpha t & 0 \\ \sin \alpha t & \cos \alpha t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

of a rotation around the y_3-axis with *angular pitch* α with a hyperbolic translation of *ratio* β. The "peach front" is a special case of flat surfaces that is not helicoidal. Details about this surface can be found in [11].

The study of flat surfaces in hyperbolic 3-space has received a lot of attention in the last few years, mainly because Galvéz, Martínez and Milán have shown in [5] that flat surfaces in the hyperbolic 3-space admit a Weierstrass representation formula in terms of meromorphic data as in the theory of minimal surfaces in $\mathbb{R}^3$. Namely, if $\psi : M^2 \to \mathbb{H}^3$ is a surface in $\mathbb{H}^3$, for any $p \in M^2$, there exist $G(p), G^*(p) \in \mathbb{C}_\infty$ distinct points in the ideal boundary of $\mathbb{H}^3$ such that the oriented normal geodesic at $\psi(p)$ is the geodesic in $\mathbb{H}^3$ starting from $G^*(p)$ towards $G(p)$. The maps $G, G^* : \Sigma \to \mathbb{C}_\infty$ are called the *hyperbolic Gauss maps* of ψ. It is proved in [5] that, for flat surfaces, they are holomorphic when one considers $\mathbb{C}_\infty$ as the Riemann sphere and M^2 has a complex structure induced by the second fundamental form. Conversely, given two holomorphic functions G and G^*, with $G \neq G^*$, one can recover a flat immersion of a surface in $\mathbb{H}^3$ (for more details see also [3, 11, 12]). This representation formula was the main tool to obtain Theorem 8.

With the previous results we conclude that

Theorem 9. *Let $l_i(\xi)$ be solutions of the Lamé's system, where $\xi = \alpha_1 x_1 + \alpha_2 x_2$. Then the associated conformally flat hypersurfaces are conformal to the product, $M^2 \times I$, where M^2 is locally congruent to either a helicoidal flat surface in $\mathbb{H}^3$ or the "peach front".*

Proof. When $\xi = \alpha_1 x_1 + \alpha_2 x_2$, it follows from Theorem 4 that the solution of Lamé's system is $l_1 = \lambda_3 \sinh(\xi + \xi_0)$, $l_2 = \lambda_3 \cosh(\xi + \xi_0)$ and $l_3 = \lambda_3 \neq 0$ and the corresponding conformally flat hypersurface M^3 has a metric g given by (5.3). Then M^3 is conformal to the product $M^2 \times I$, where M^2 is a flat surface in $\mathbb{H}^3$ with fundamental forms given by (5.4). It follows from Theorem 8 that M^2 is locally congruent to either a helicoidal flat surface in $\mathbb{H}^3$ or to the "peach front". ∎

ii) $\xi = \alpha_1 x_1 + \alpha_3 x_3$. In this case, we have the solution of Lamé's system , $l_1 = \lambda_2 \sin(\xi + \xi_0)$, $l_2 = \lambda_2 \neq 0$ and $l_3 = \lambda_2 \cos(\xi + \xi_0)$ (see Theorem 4 b)). Then the associated conformally flat hypersurface M^3 has a Guichard net, whose induced metric is given by

$$g = e^{2P(x_1, x_2, x_3)} \{ \sin^2(\xi + \xi_0) dx_1^2 + dx_2^2 + \cos^2(\xi + \xi_0) dx_3^2 \}.$$

The hypersurface M^3 is conformal to another class of products $M^2 \times I$ (see [21, 22]). Namely, let $\mathbb{S}^3 \subset \mathbb{R}^4$ be the canonical 3-sphere, then $M^2 \times I = \{ tp : 0 < t < \infty, p \in M^2 \subset \mathbb{S}^3 \}$, is a conformally flat hypersurface, where M^2 is a surface with constant curvature in S^3. In our case, M^3 is conformal to the product $M^2 \times I$, where the surface $M^2 \subset \mathbb{S}^3$ is a flat surface, parametrized by lines of curvature, whose first and second fundamental forms are given by

$$\begin{aligned} I &= \sin^2(\xi + \xi_0) dx_1^2 + \cos^2(\xi + \xi_0) dx_3^2, \\ II &= \sin(\xi + \xi_0) \cos(\xi + \xi_0) (dx_1^2 - dx_3^2). \end{aligned} \tag{5.7}$$

The geometry of these surfaces in $\mathbb{S}^3$ is being studied and it will appear in another paper.

iii) $\xi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $\alpha_i \neq 0$, for all i. In this case, we will show that the solutions $l_i(\xi)$ of the Lamé's system give rise to a new class of conformally flat hypersurfaces, according to the following theorem:

Theorem 10. *Let M^3 be a conformally flat hypersurface in a space form M_K^4, associated to a solution of Lamé's system $l_i(x_1, x_2, x_3) = l_i(\xi)$, with $\xi = \sum\limits_{s=1}^{3} \alpha_s x_s$ and $\alpha_s \neq 0$, for all s, given in terms of elliptic functions by (3.22)–(3.24). Then its first fundamental form g is given by*

$$g = e^{2P(x)}\big\{ \cos^2 \varphi(\xi)(dx_1)^2 + (dx_2)^2 + \sin^2 \varphi(\xi)(dx_3)^2 \big\}, \tag{5.8}$$

where φ satisfies,

$$\varphi_{,\xi}^2 = c(a \cos^2 \varphi - b), \tag{5.9}$$

or g is given by

$$g = e^{2\tilde{P}(x)}\big\{ \sinh^2 \tilde{\varphi}(\xi)(dx_1)^2 + \cosh^2 \tilde{\varphi}(\xi)(dx_2)^2 + (dx_3)^2 \big\} \tag{5.10}$$

where $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}_{,\xi}^2 = c\big(b \cosh^2 \tilde{\varphi} - b\big). \tag{5.11}$$

where $a, b, c \in \mathbb{R} \setminus \{0\}$, $P(x)$ and $\tilde{P}(x)$ are differentiable functions that depend on l_s and M_K^4. In both cases, $\xi \in I \subset \mathbb{R}$, where I is an open interval such that g is positive definite.

Proof. Guichard condition (2.1) implies that we may consider

$$l_1 = l_2 \cos \varphi, \tag{5.12}$$
$$l_3 = l_2 \sin \varphi. \tag{5.13}$$

It follows from Theorem 1 that the metric is given by (5.8). In order to obtain the expression for the derivative of φ with respect to ξ, we consider

$$l_{1,\xi} = l_{2,\xi} \cos \varphi - l_2 \varphi_{,\xi} \sin \varphi.$$

Since $\alpha_s \neq 0$ for all s, the functions l_i are given as in Theorem 3, by (3.22)–(3.24). Hence, using (3.19), we have that

$$c_1 l_2 l_3 = c_2 l_1 l_3 \cos \varphi - \varphi_{,\xi} l_3,$$

for $c_1, c_2 \in \mathbb{R} \setminus \{0\}$, which implies

$$\varphi_{,\xi} = l_2\big(c_2 \cos^2 \varphi - c_1\big). \tag{5.14}$$

By taking the derivative again, it follows from (5.12)–(5.14) and (3.19) that

$$\varphi_{,\xi\xi} = l_{2,\xi}\big(c_2 \cos^2 \varphi - 1\big) - 2c_2 l_2 \cos \varphi \sin \varphi \varphi_{,\xi} = \frac{1}{l_2}\big[l_{2,\xi}\varphi_{,\xi} - 2c_2(l_2 \cos \varphi)(l_2 \sin \varphi)\varphi_{,\xi}\big]$$

$$= \frac{1}{l_2}\big[l_{2,\xi}\varphi_{,\xi} - 2c_2 l_1 l_3 \varphi_{,\xi}\big] = \frac{1}{l_2}\big[l_{2,\xi}\varphi_{,\xi} - 2l_{2,\xi}\varphi_{,\xi}\big] = -\frac{l_{2,\xi}\varphi_{,\xi}}{l_2}.$$

Therefore,

$$\varphi_{,\xi} l_2 = c, \tag{5.15}$$

where $c \in \mathbb{R} \setminus \{0\}$, since $\varphi_{,\xi} \neq 0$. Then, multiplying (5.14) by $\varphi_{,\xi}$ and using (5.15) we have that $\varphi_{,\xi}^2 = c\big(c_2 \cos^2 \varphi - c_1\big)$, i.e., (5.9) holds. The proof of the second part of the theorem is analogous, when we consider $l_2 = l_3 \cosh \varphi$ and $l_1 = l_3 \sinh \varphi$.   ■

Corollary 2. *Let $M^3 \subset M^4$ be a conformally flat hypersurface associated to the solutions of Lamé's system $l_i(\xi)$ with $\xi = \sum\limits_{s=1}^{3} \alpha_s x_s$ and $\alpha_s \neq 0$ for all s, given in terms of elliptic functions by* (3.22)–(3.24). *Then the Guichard net of M^3 is not cyclic.*

Proof. It follows from Theorem 10 that the first fundamental form of M^3 is given by (5.8), where $\varphi(\xi)$ satisfies (5.9) or by (5.10) where $\tilde{\varphi}(\xi)$ satisfies (5.11), $\xi \in I \subset \mathbb{R}$. In the first case, (5.9) implies that $\varphi_{,\xi\xi} = \lambda \sin 2\varphi$, $\lambda \neq 0$ and in the second case, (5.11) implies that $\tilde{\varphi}_{,\xi\xi} = \lambda \sin 2\tilde{\varphi}$, $\lambda \neq 0$. In either case, $\varphi_{,x_i x_j} = \alpha_i \alpha_j \varphi_{,\xi\xi} \neq 0$, $i \neq j$ and $\xi \in I$. Therefore, the Guichard net of M^3 is not cyclic. ∎

We observe that, as a consequence of the results of Hertrich-Jeromin and Suyama, the surfaces M^3 of Corollary 2 provide a new class of conformally flat hypersurfaces.

It is important to observe that Hertrich-Jeromin and Suyama in [9] have independently considered Guichard nets with the ansatz on the function φ such that $\varphi(x_1, x_2, x_3) = \varphi(ax_1 + bx_2 + cx_3)$. They investigated the geometric properties of these Guichard nets, that they called Bianchi-type Guichard nets, as well as the new class of associated conformally flat hypersurfaces.

A Appendix

Proof of Theorem 2. The infinitesimal generator associated to the symmetry group is written as in (3.12). The functions ξ^i, η^i, ϕ^{ij} will be obtained by solving the determining equations that arise when we apply the first prolongation formula

$$\mathrm{pr}^{(1)}V = V + \sum_{i,k} D_k\!\left(\eta^i\right)\frac{\partial}{\partial l_{i,x_k}} + \sum_{i,j,k\,i\neq j} D_k\!\left(\phi^{ij}\right)\frac{\partial}{\partial h_{ij,x_k}}$$
$$- \sum_{i,k,r} D_k(\xi^r)l_{i,x_r}\frac{\partial}{\partial l_{i,x_k}} - \sum_{i,j,k,r} D_k(\xi^r)h_{ij,x_r}\frac{\partial}{\partial h_{ij,x_k}},$$

with

$$D_i = \frac{\partial}{\partial x_i} + \sum_j l_{j,x_i}\frac{\partial}{\partial l_i} + \sum_{j,l} h_{jl,x_i}\frac{\partial}{\partial h_{jl}},$$

on each equation of the system, i.e., when we consider the infinitesimal criterion (3.2). In order to avoid any functional dependence, the following substitutions will be considered

$$l_{i,x_j} = h_{ij}l_j, \qquad i \neq j, \tag{A.1}$$
$$l_{i,x_i} = -\varepsilon_i\varepsilon_j h_{ji}l_j - \varepsilon_i\varepsilon_k h_{ki}l_k \tag{A.2}$$
$$h_{ij,x_k} = h_{ik}h_{kj}, \tag{A.3}$$
$$h_{ij,x_j} = -h_{ji,x_i} - h_{ik}h_{jk}, \qquad i < j, \tag{A.4}$$
$$h_{ij,x_i} = -\varepsilon_i\varepsilon_j h_{ji,x_j} - \varepsilon_i\varepsilon_k h_{ki}h_{kj}, \qquad i < j. \tag{A.5}$$

Fixing i, j and k, distinct indices, we start applying $\mathrm{pr}^{(1)}V$ to equation (3.9). Then the infinitesimal criterion (3.2), gives $\phi^{ij}_{(k)} - \phi^{ik}h_{kj} - h_{ik}\phi^{kj} = 0$, using the prolongation formula, we get

$$\phi^{ij}_{,x_k} + \sum_r \phi^{ij}_{,l_r}l_{r,x_k} + \sum_{r,s}\phi^{ij}_{,h_{rs}}h_{rs,x_k} - \sum_t\left(\xi^t_{,x_k} + \sum_r\xi^t_{,l_r}l_{r,x_k} + \sum_{r,s}\xi^t_{,h_{rs}}h_{rs,x_k}\right)h_{ij,x_t}$$
$$- \phi^{ik}h_{kj} - h_{ik}\phi^{kj} = 0. \tag{A.6}$$

For $i < j$, we apply the substitutions (A.1)–(A.5) and we analyse each term of (A.6) as follows

$$\sum_r \phi^{ij}_{,l_r} l_{r,x_k} = \sum_{r \neq k} \phi^{ij}_{,l_r} h_{rk} l_k - \phi^{ij}_{,l_k} \left(\varepsilon_k \varepsilon_j h_{jk} l_j + \varepsilon_k \varepsilon_i h_{ik} l_i \right), \tag{A.7}$$

$$\sum_{r,s} \phi^{ij}_{,h_{rs}} h_{rs,x_k} = \sum_{r \neq k,\, s \neq k} \phi^{ij}_{,h_{rs}} h_{rk} h_{ks} + \sum_{s<k} \phi^{ij}_{,h_{ks}} h_{ks,x_k} - \sum_{s>k} \phi^{ij}_{,h_{ks}} \left(\varepsilon_k \varepsilon_s h_{sk,x_s} + \varepsilon_k \varepsilon_m h_{mk} h_{ms} \right)$$
$$- \sum_{r<k} \phi^{ij}_{,h_{rk}} \left(h_{kr,x_r} + h_{rn} h_{kn} \right) + \sum_{r>k} \phi^{ij}_{,h_{rk}} h_{rk,x_k}, \tag{A.8}$$

$$\sum_t \left(\xi^t_{,x_k} + \sum_r \xi^t_{,l_r} l_{r,x_k} + \sum_{r,s} \xi^t_{,h_{rs}} h_{rs,x_k} \right) h_{ij,x_t}$$
$$= C^k_k h_{ik} h_{kj} - C^j_k \left(h_{ji,x_i} + h_{ik} h_{jk} \right) - C^i_k \left(\varepsilon_i \varepsilon_j h_{ji,x_j} + \varepsilon_i \varepsilon_k h_{ki} h_{kj} \right), \tag{A.9}$$

where the coefficients C^t_k are given by

$$C^t_k = \xi^t_{,x_k} + \sum_{r \neq k} \xi^t_{,l_r} h_{rk} l_k - \xi^t_{,l_k} \left(\varepsilon_k \varepsilon_j h_{jk} l_j + \varepsilon_k \varepsilon_i h_{ik} l_i \right) + \sum_{r \neq k,\, s \neq k} \xi^t_{,h_{rs}} h_{rk} h_{ks}$$
$$+ \sum_{s<k} \xi^t_{,h_{ks}} h_{ks,x_k} - \sum_{s>k} \xi^t_{,h_{ks}} \left(\varepsilon_k \varepsilon_s h_{sk,x_s} + \varepsilon_k \varepsilon_m h_{mk} h_{ms} \right)$$
$$- \sum_{r<k} \xi^t_{,h_{rk}} \left(h_{kr,x_r} + h_{rn} h_{kn} \right) + \sum_{r>k} \xi^t_{,h_{rk}} h_{rk,x_k}, \tag{A.10}$$

and the indices m and n are such that $\{k, s, m\}$, $s > k$ and $\{k, r, n\}$, $r < k$ are two sets of three distinct numbers.

Now we analyse the coefficients of equation (A.6), considering (A.7)–(A.9). By equating to zero the coefficients of the products $h_{ji,x_j} h_{ks,x_k}$, with $k > s$, we obtain $\xi^i_{,h_{ks}} = 0$. Analogously, for the coefficients of $h_{ji,x_j} h_{sk,x_s}$, with $k < s$, we obtain $\xi^i_{,h_{ks}} = 0$. This implies that

$$\xi^i_{,h_{ks}} = 0, \quad \forall s, \ \ s \neq k, \qquad \text{i.e.} \quad \xi^i_{,h_{kj}} = \xi^i_{,h_{ki}} = 0.$$

Similarly, from the coefficients of $h_{ji,x_j} h_{kr,x_k}$, $r < k$ and $h_{ji,x_j} h_{rk,x_k}$, with $r > k$, we obtain

$$\xi^i_{,h_{rk}} = 0, \quad \forall r, \ \ r \neq k, \qquad \text{i.e.} \quad \xi^i_{,h_{jk}} = \xi^i_{,h_{ik}} = 0,$$

where $i, j, k \in \{1, 2, 3\}$ are distinct and $i < j$. By analysing the coefficients of $h_{ji,x_i} h_{ks,x_k}$ with $k > s$ and $h_{ji,x_i} h_{sk,x_s}$ with $k < s$, we obtain

$$\xi^j_{,h_{ks}} = 0, \quad \forall s, \ \ s \neq k, \qquad \text{i.e.} \quad \xi^j_{,h_{ki}} = \xi^j_{,h_{kj}} = 0.$$

Similarly, the coefficients of $h_{ji,x_i} h_{kr,x_r}$, with $k > r$, and $h_{ji,x_i} h_{rk,x_k}$, with $k < r$, lead to

$$\xi^j_{,h_{rk}} = 0, \quad \forall r, \ \ r \neq k, \qquad \text{i.e.} \quad \xi^j_{,h_{ik}} = \xi^j_{,h_{jk}} = 0.$$

Since $i, j, k \in \{1, 2, 3\}$ are distinct and arbitrary indices, with $i < j$, we conclude that $\xi^m_{,h_{st}} = 0$ for any indices m, s and t, $s \neq t$, i.e., ξ^m depends only on x and l. Therefore, the expression of C^t_k given in (A.10) reduces to

$$C^t_k = \xi^t_{,x_k} + \sum_{r \neq k} \xi^t_{,l_r} h_{rk} l_k - \xi^t_{,l_k} \left(\varepsilon_k \varepsilon_j h_{jk} l_j + \varepsilon_k \varepsilon_i h_{ik} l_i \right),$$

that can be rewritten as

$$C^t_k = \xi^t_{,x_k} + \sum_{r \neq k} \left(\xi^t_{,l_r} l_k - \xi^t_{,l_k} \varepsilon_r \varepsilon_k l_r \right) h_{rk}. \tag{A.11}$$

From (A.8), we have that the coefficients of h_{ks,x_k}, with $s < k$, and the coefficients of h_{sk,x_s}, with $s > k$, lead to $\phi^{ij}_{,h_{ks}} = 0$, $\forall s \neq k$, i.e. $\phi^{ij}_{,h_{ki}} = \phi^{ij}_{,h_{kj}} = 0$.

Considering (A.9), the coefficients of h_{ji,x_i} and h_{ji,x_j} imply that $C^j_k = 0$ and $C^i_k = 0$, respectively. Since $i < j$ and $i, j, k \in \{1, 2, 3\}$ are arbitrary and distinct, we conclude that $C^i_k = C^j_k = 0$, for all i, j, k distinct indices.

Since ξ^m does not depend on h_{st}, the analysis of (A.11) gives us the following system

$$\xi^m_{,x_k} = 0, \qquad \forall \, m \neq k,$$
$$\xi^m_{,l_r} l_k - \varepsilon_r \varepsilon_k \xi^m_{,l_k} l_r = 0, \qquad \forall \, r \neq k.$$

The first equation of this system says that ξ^m depends only on x_m and l. By solving the characteristic system for the second equation, we have that ξ^m depends on x_m and a variable $\zeta = \varepsilon_i l_i^2 + \varepsilon_j l_j^2 + \varepsilon_k l_k^2$. However, Guichard condition implies that $\zeta \equiv 0$, hence ξ^m does not depend on l_s, for all s.

Summarising the conclusions of this first part of the proof, we have that

$$\phi^{st} = \phi^{st}(h_{st}, h_{ts}, x, l), \qquad \text{and} \qquad \xi^m = \xi^m(x_m).$$

We now consider equation (3.7). By applying the prolongation $\mathrm{pr}^{(1)}V$ to (3.7), we have that $\eta^i_{(j)} - \phi^{ij} l_j - h_{ij} \eta^j = 0$, which implies,

$$\eta^i_{,x_j} + \sum_r \eta^i_{,l_r} l_{r,x_j} + \sum_{r,s} \eta^i_{,h_{rs}} h_{rs,x_j} - \xi^j_{,x_j} l_{i,x_j} - \phi^{ij} l_j - h_{ij} \eta^j = 0. \tag{A.12}$$

Observe that by applying the substitution (A.1), we have

$$\sum_r \eta^i_{,l_r} l_{r,x_j} = \sum_{r \neq j} \eta^i_{,l_r} h_{rj} l_j - \eta^i_{,l_j} (\varepsilon_j \varepsilon_i h_{ij} l_i + \varepsilon_j \varepsilon_k h_{kj} l_k).$$

Moreover, by applying the substitutions (A.3), (A.4) and (A.5) we have

$$\sum_{r,s} \eta^i_{,h_{rs}} h_{rs,x_j} = \sum_{r \neq j, \, s \neq j} \eta^i_{,h_{rs}} h_{rj} h_{js} + \sum_{s < j} \eta^i_{,h_{js}} h_{js,x_j} - \sum_{s > j} (\varepsilon_j \varepsilon_s h_{sj,x_s} + \varepsilon_j \varepsilon_m h_{mj} h_{ms})$$
$$- \sum_{r < j} \eta^i_{,h_{rj}} (h_{jr,x_r} + h_{rn} h_{jn}) + \sum_{r > j} \eta^i_{,h_{rj}} h_{rj,x_j}.$$

Therefore, by considering in (A.12), the coefficients of h_{js,x_j}, with $s < j$, and h_{sj,x_s}, with $s > j$, we conclude that $\eta^i_{,h_{js}} = 0$. Similarly, the analysis of the coefficients of h_{jr,x_r}, with $r < j$, and h_{rj,x_j}, with $r > j$, imply that $\eta^i_{,h_{rj}} = 0$. Hence, we conclude that

$$\eta^i_{,h_{jt}} = \eta^i_{,h_{tj}} = 0, \qquad \forall \, t \neq j.$$

Since i and $t \neq j$ are arbitrary, we conclude that η^m does not depend on h_{st}, for any indices, m, s and t with $s \neq t$. Consequently, (A.12) reduces to

$$\eta^i_{,x_j} + \left(\eta^i_{,l_i} l_j - \varepsilon_i \varepsilon_j \eta^i_{,l_j} l_i - \xi^j_{,x_j} l_j - \eta^j \right) h_{ij} + \left(\eta^i_{,l_k} l_j - \varepsilon_j \varepsilon_k \eta^i_{,l_j} l_k \right) h_{kj} - \phi^{ij} l_j = 0. \tag{A.13}$$

Since ϕ^{ij} depends only on x, l, h_{ij} and h_{ji}, we obtain from (A.13) the following system

$$\eta^i_{,l_k} l_j - \varepsilon_j \varepsilon_k \eta^i_{,l_j} l_k = 0, \tag{A.14}$$
$$\eta^i_{,x_j} + \left(\eta^i_{,l_i} l_j - \varepsilon_i \varepsilon_j \eta^i_{,l_j} l_i - \xi^j_{,x_j} l_j - \eta^j \right) h_{ij} - \phi^{ij} l_j = 0. \tag{A.15}$$

By solving the characteristic system for (A.14), we have that $n^i = n^i(x, l_i)$. By taking derivatives of (A.15) with respect to h_{ji} we get

$$\phi^{ij}_{,h_{ji}} = 0. \tag{A.16}$$

On the other hand, by taking the derivatives of (A.15) twice with respect to h_{ij}, we obtain

$$\phi^{ij}_{,h_{ij}h_{ij}} = 0. \tag{A.17}$$

Consequently, it follows from (A.16) and (A.17) that ϕ^{ij} is given by

$$\phi^{ij} = A^{ij}(x, l)h_{ij} + B^{ij}(x, l). \tag{A.18}$$

Therefore, (A.6) reduces to

$$\phi^{ij}_{,x_k} + \sum_r \phi^{ij}_{,l_r} l_{r,x_k} + A^{ij} h_{ij,x_k} - \xi^k_{,x_k} h_{ij,x_k} - \left(A^{ik} h_{ik} + B^{ik}\right) h_{kj} - \left(A^{kj} h_{kj} + B^{kj}\right) h_{ik} = 0.$$

By considering the substitutions (A.1)–(A.5), this equation reduces to

$$\phi^{ij}_{,x_k} + \phi^{ij}_{,l_i} h_{ik} l_k + \phi^{ij}_{,l_j} h_{jk} l_k - \phi^{ij}_{,l_k} (\varepsilon_k \varepsilon_j h_{jk} l_j + \varepsilon_k \varepsilon_i h_{ik} l_i)$$
$$+ A^{ij} h_{ik} h_{kj} - \xi^k_{,x_k} h_{ik} h_{kj} - A^{ik} h_{ik} h_{kj} - B^{ik} h_{kj} - A^{kj} h_{ik} h_{kj} - B^{kj} h_{ik} = 0,$$

which can be rewritten as

$$B^{ij}_{,x_k} + A^{ij}_{,x_k} h_{ij} + \left(B^{ij}_{,l_i} l_k - \varepsilon_k \varepsilon_i B^{ij}_{,l_k} l_i - B^{kj}\right) h_{ik} + \left(B^{ij}_{,l_j} l_k - \varepsilon_k \varepsilon_j B^{ij}_{,l_k} l_j\right) h_{jk}$$
$$- B^{ik} h_{kj} + \left(A^{ij}_{,l_i} l_k - \varepsilon_k \varepsilon_i A^{ij}_{,l_k} l_i\right) h_{ik} h_{ij} + \left(A^{ij}_{,l_j} l_k - \varepsilon_k \varepsilon_j A^{ij}_{,l_k} l_j\right) h_{ij} h_{jk}$$
$$+ \left(A^{ij} - \xi^k_{,x_k} - A^{ik} - A^{kj}\right) h_{ik} h_{kj} = 0.$$

It follows from the coefficients of h_{kj} that $B^{ik} = 0$. The permutation of the indices i, j and k leads to

$$B^{st} = 0, \qquad \forall\, s, t, \quad s \neq t. \tag{A.19}$$

By equating to zero the coefficients of $h_{ik}h_{kj}$ and $h_{ij}h_{jk}$, the following system is obtained

$$A^{ij}_{,l_i} l_k - \varepsilon_k \varepsilon_i A^{ij}_{,l_k} l_i = 0, \qquad A^{ij}_{,l_j} l_k - \varepsilon_k \varepsilon_j A^{ij}_{,l_k} l_j = 0,$$

where we solve the characteristic system to conclude that A^{ij} depends only on x. On the other hand, the coefficient of h_{ij} implies that A^{ij} does not depend on x_k, therefore $A^{ij} = A^{ij}(x_i, x_j)$. Considering the coefficient of $h_{ik}h_{kj}$, we obtain the following equation

$$A^{ij} - \xi^k_{,x_k} - A^{ik} - A^{kj} = 0. \tag{A.20}$$

Therefore, equation (A.13) reduces to

$$\eta^i_{,x_j} + \left(\eta^i_{,l_i} l_j - \xi^j_{,x_j} l_j - \eta^j - A^{ij} l_j\right) h_{ij} = 0.$$

Since η^i does not depend on h_{ij}, we must have

$$\eta^i_{,x_j} = 0,$$
$$\eta^i_{,l_i} l_j - \xi^j_{,x_j} l_j - \eta^j - A^{ij} l_j = 0, \tag{A.21}$$

By applying $\mathrm{pr}^{(1)}V$ to equation (3.10), we have $\phi^{ij}_{(j)} + \phi^{ji}_{(i)} + \phi^{ik}h_{kj} + h_{ik}\phi^{kj} = 0$, which implies that

$$\phi^{ij}_{,x_j} + \phi^{ij}_{,h_{ij}}h_{ij,x_j} - \xi^j_{,x_j}h_{ij,x_j} + \phi^{ji}_{,x_i} + \phi^{ji}_{,h_{ji}}h_{ji,x_i} - \xi^i_{,x_i}h_{ji,x_i} + +\phi^{ik}h_{kj} + h_{ik}\phi^{kj} = 0.$$

Considering the substitution (A.4), for $i < j$, we obtain

$$\phi^{ij}_{,x_j} + \phi^{ji}_{,x_i} + \left(\xi^j_{,x_j} - A^{ij} + A^{ji} - \xi^i_{,x_i}\right)h_{ji,x_i} + \left(\xi^j_{,x_j} - A^{ij} + A^{ik} + A^{kj}\right)h_{ik}h_{jk} = 0.$$

Then, the coefficient of h_{ji,x_i} leads to

$$\xi^j_{,x_j} - A^{ij} + A^{ji} - \xi^i_{,x_i} = 0. \tag{A.22}$$

By applying the prolongation $\mathrm{pr}^{(1)}V$ to (3.11) we have $\varepsilon_i\phi^{ij}_{(i)} + \varepsilon_j\phi^{ji}_{(j)} + \varepsilon_k\phi^{ki}h_{kj} + \varepsilon_k\phi^{kj}h_{ki} = 0$, which implies

$$\varepsilon_i\left(\phi^{ij}_{,x_i} + \phi^{ij}_{,h_{ij}}h_{ij,x_i} - \xi^i_{,x_i}h_{ij,x_i}\right) + \varepsilon_j\left(\phi^{ji}_{,x_j} + \phi^{ji}_{,h_{ji}}h_{ji,x_j} - \xi^j_{,x_j}h_{ji,x_j}\right)$$
$$+ \varepsilon_k\left(\phi^{ki}h_{kj} + \phi^{kj}h_{ki}\right) = 0.$$

Considering the substitution (A.5) with $i < j$, we obtain

$$\varepsilon_i\phi^{ij}_{,x_j} + \varepsilon_j\phi^{ji}_{,x_j} + \varepsilon_j\left(\xi^i_{,x_i} - A^{ij} + A^{ji} - \xi^j_{,x_j}\right)h_{ji,x_j} + \varepsilon_k\left(\xi^i_{,x_i} - A^{ij} + A^{ki} + A^{kj}\right)h_{ki}h_{kj} = 0.$$

From the coefficient of h_{ji,x_j}, we get

$$A^{ij} - \xi^i_{,x_i} - A^{ji} + \xi^j_{,x_j} = 0. \tag{A.23}$$

Therefore, it follows from (A.22) and (A.23) that

$$A^{ji} = A^{ij}. \tag{A.24}$$

Consequently, both equations imply that $\xi^i_{,x_i} = \xi^j_{,x_j}$, which enables us to conclude that

$$\xi^m = ax_m + a_m, \qquad \forall\, 1 \le m \le 3, \tag{A.25}$$

where a and a_m are real constants. Moreover, from (A.25) and (A.20), we have that

$$A^{ij} - a - A^{ik} - A^{kj} = 0 \qquad \text{and} \qquad A^{ik} - a - A^{ij} - A^{jk} = 0.$$

By taking the sum of these equations and using (A.24), we obtain $A^{kj} = -a$. Therefore, it follows from (A.18) and (A.19), that

$$\phi^{st} = -ah_{st}, \qquad \forall\, s \ne t. \tag{A.26}$$

Moreover, from (A.25) and (A.21), we get

$$\eta^i_{,l_i}l_j = \eta^j. \tag{A.27}$$

Since the function η^m depends only on x_m and l_m, we conclude that, $\eta^i_{,l_il_i} = 0$, i.e.,

$$\eta^i = N^i(x_i)l_i + M^i(x_i). \tag{A.28}$$

Hence, it follows from (A.27) and (A.28) that $\eta^i_{,l_i} = \eta^j_{,l_j} = N(x_i)$. Therefore, $N'(x_i) = \eta^j_{,l_jx_i} = 0$, which implies that,

$$\eta^i = cl_i + M^i(x_i). \tag{A.29}$$

Finally, we apply the prolongation $\mathrm{pr}^{(1)}V$ to equation (3.8) to obtain

$$\varepsilon_i \eta^i_{(i)} + \varepsilon_j \phi^{ji} l_j + \varepsilon_j h_{ji} \eta^j + \varepsilon_k \phi^{ki} l_k + \varepsilon_k h_{ki} \eta^k = 0,$$

which implies that

$$\varepsilon_i \eta^i_{,x_i} + \varepsilon_i \eta^i_{,l_i} l_{i,x_i} - \varepsilon_i \xi^i_{,x_i} l_{i,x_i} + \varepsilon_j \phi^{ji} l_j + \varepsilon_j h_{ji} \eta^j + \varepsilon_k \phi^{ki} l_k + \varepsilon_k h_{ki} \eta^k = 0.$$

When we substitute (A.2) for l_{i,x_i} and we consider equations (A.25), (A.26) and (A.29), we obtain

$$\varepsilon_i M^i_{,x_i} + \varepsilon_j h_{ji} M^i + \varepsilon_k h_{ki} M^k = 0.$$

The analysis of the coefficients of h_{ji} and h_{ki} enables us to conclude that $M^i = M^k = 0$, consequently, $\eta^m = cl_m$, $\forall\, 1 \le m \le 3$. This concludes the proof of Theorem 2. ∎

Acknowledgements

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