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$\mathcal{A} = \mathcal{U}$ for Locally Acyclic Cluster Algebras[⋆]

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Abstract. This note presents a self-contained proof that acyclic and locally acyclic cluster algebras coincide with their upper cluster algebras.

Key words: cluster algebras; upper cluster algebras; acyclic cluster algebras

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1 Introduction

Cluster algebras are commutative domains with distinguished generators (cluster variables) and certain combinatorial identities between them (mutation relations). They were introduced in [4] to study dual canonical bases for Lie groups, and have since been discovered in a remarkable range of applications (see [5, 8, 9] for surveys).

1.1 Upper cluster algebras

Each cluster algebra $\mathcal{A}$ also determines an *upper cluster algebra* $\mathcal{U}$, with $\mathcal{A} \subseteq \mathcal{U}$. Defined as an intersection of Laurent rings, the upper cluster algebra is more natural than the cluster algebra from a geometric perspective.

Upper cluster algebras were introduced in the seminal paper [1], where the authors sought to show that certain rings are cluster algebras[1]. They instead proved that these rings are upper cluster algebras. Naturally, they also asked when $\mathcal{A} \subseteq \mathcal{U}$ is equality.

To this end, they introduced *acyclic cluster algebras*: a class of elementary cluster algebras which have since proven to be particularly easy to work with. They also defined a *totally coprime* condition; a more technical condition that can depend on the coefficients of the cluster algebra. Using both properties, they were able to trap $\mathcal{A}$ and $\mathcal{U}$ between two simpler algebras (the *lower bound* and the *upper bound*) and 'close the gap' between the bounds to show that $\mathcal{A} = \mathcal{U}$.

Theorem 1 ([1])**.** *If $\mathcal{A}$ is a totally coprime, acyclic cluster algebra, then $\mathcal{A} = \mathcal{U}$.*

1.2 Locally acyclic cluster algebras

The current paper's author became interested in studying when $\mathcal{A} = \mathcal{U}$ for cluster algebras coming from marked surfaces[2]. Fortuitously, the geometric techniques developed for that problem (cluster localization and covers) generalized beyond marked surfaces. In [12], *locally acyclic cluster algebras* were defined to capture those cluster algebras for which these techniques could show $\mathcal{A} = \mathcal{U}$.

[⋆]This paper is a contribution to the Special Issue on New Directions in Lie Theory. The full collection is available at http://www.emis.de/journals/SIGMA/LieTheory2014.html

[1]Specifically, coordinate rings of double Bruhat cells.

[2]$\mathcal{A} = \mathcal{U}$ for marked surfaces was needed to prove that *skein algebras* are cluster algebras (see [11]).

Unfortunately, the proof that $\mathcal{A} = \mathcal{U}$ for locally acyclic cluster algebras which appears in [12, Theorem 4.1] depends an incorrectly stated version of Theorem 1; specifically, the totally coprime hypothesis was omitted. This has led to some confusion about whether $\mathcal{A} = \mathcal{U}$ for locally acyclic cluster algebras[3]. Thankfully, the techniques of [12] can be used to show that $\mathcal{A} = \mathcal{U}$ for locally acyclic cluster algebras without assuming totally coprime or using Theorem 1[4]; this was alluded to in [12, Remark 6.7] but not shown. Since acyclic cluster algebras are locally acyclic, this shows that the totally coprime assumption may be removed from Theorem 1.

Theorem 2. *If $\mathcal{A}$ is an acyclic or locally acyclic cluster algebra, then $\mathcal{A} = \mathcal{U}$.*

The purpose of this note is to present an elementary proof of this fact, which assumes nothing except the Laurent phenomenon. This not only resolves the dependency error in [12], but serves as a short and self-contained introduction to the techniques and effectiveness of cluster localization and locally acyclic cluster algebras. The reader might find this a more straightforward and accessible motivation for the study of locally acyclic cluster algebras than [12], which relies heavily on geometric intuition and techniques.

We also work in the setting of cluster algebras with *normalized coefficients*, which are a bit more general than the cluster algebras with *geometric coefficients* studied in [12].

2 Cluster algebra recollections

We recall the definition of cluster algebras with normalized coefficients. This generalizes cluster algebras with geometric coefficients; see [6] for the appropriate correspondence.

2.1 Seeds and mutation

Let $\mathbb{P}$ be a *semifield*: a torsion-free abelian group (written multiplicatively) equipped with an *auxiliary addition* $\oplus$ which is commutative, associative and distributive over multiplication. Its integral group ring $\mathbb{Z}\mathbb{P}$ will be the *coefficient ring* of the cluster algebra.

Let $\mathcal{F}$ be a field which contains $\mathbb{Z}\mathbb{P}$. A *seed* of rank n in $\mathcal{F}$ is a triple $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ consisting of three parts:

- the *cluster*: $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is an n-tuple in $\mathcal{F}$ which freely generates $\mathcal{F}$ as a field over the fraction field of $\mathbb{Z}\mathbb{P}$,
- the *coefficients*: $\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$ is an n-tuple in $\mathbb{P}$, and
- the *exchange matrix*: B is an integral, skew-symmetrizable[5] $n \times n$ matrix.

A seed $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ may be *mutated* at an index $1 \leq k \leq n$, to produce a new seed $(\mu_k(\mathbf{x}), \mu_k(\mathbf{y}), \mu_k(\mathsf{B}))$ defined as follows:

- $\mu_k(\mathbf{x}) := \{x_1, x_2, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_n\}$, where

$$x_k' := \frac{y_k \prod x_j^{[\mathsf{B}_{jk}]_+} + \prod x_j^{[-\mathsf{B}_{jk}]_+}}{(y_k \oplus 1)x_k}$$

  (here, $[\mathsf{B}_{jk}]_+$ denotes $\max(\mathsf{B}_{jk}, 0)$),

[3] The author apologizes for any confusion this may have caused.

[4] Though, some lemmas here are influenced by lemmas in [1]; cf. Proposition 3.

[5] *Skew-symmtrizable* means there is a diagonal matrix D such that $\mathsf{D}B$ is skew-symmetric.

- $\mu_k(\mathbf{y}) := \{y_1', y_2', \ldots, y_n'\}$, where

$$y_j' := \begin{cases} y_j^{-1} & \text{if} \quad k=j, \\ y_j y_k^{[\mathsf{B}_{kj}]_+}(y_k \oplus 1)^{-\mathsf{B}_{kj}} & \text{if} \quad k \neq j, \end{cases}$$

- $\mu_k(\mathsf{B})$ is defined entry-wise by

$$\mu_k(\mathsf{B})_{ij} = \begin{cases} -\mathsf{B}_{ij} & \text{if} \quad k=i \quad \text{or} \quad k=j, \\ \mathsf{B}_{ij} + \frac{1}{2}(|\mathsf{B}_{ik}|\mathsf{B}_{kj} + \mathsf{B}_{ik}|\mathsf{B}_{kj}|) & \text{otherwise,} \end{cases}$$

Mutating at the same index twice in a row returns to the original seed. Permuting the indices $\{1, 2, \ldots, n\}$ induces a new seed in the obvious way. Two seeds are *mutation-equivalent* if they are related by a sequence of mutations and permutations.

2.2 Cluster algebras

Given a seed $(\mathbf{x}, \mathbf{y}, \mathsf{B})$, the union of all the clusters which appear in mutation-equivalent seeds defines a set of *cluster variables* in the embedding field $\mathcal{F}$. The *cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathsf{B})$ determined by $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ is the unital subring of $\mathcal{F}$ generated by $\mathbb{ZP}$ and the cluster variables. The cluster algebra only depends on the mutation-equivalence class of the initial seed, and so the initial seed $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ will often be omitted from the notation.

We say $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a *cluster* in $\mathcal{A}$ if $\mathbf{x}$ is in some seed $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ of $\mathcal{A}$. A fundamental property of cluster algebras is the Laurent phenonemon, which states that the localization of $\mathcal{A}$ at a cluster $\mathbf{x}$ is the ring of Laurent polynomials in $\mathbf{x}$ over $\mathbb{ZP}$

$$\mathcal{A} \hookrightarrow \mathcal{A}\big[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\big] = \mathbb{ZP}\big[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\big].$$

Every cluster in $\mathcal{A}$ defines such an inclusion. Define the *upper cluster algebra* $\mathcal{U}$ of $\mathcal{A}$ to be the intersection of each of these Laurent rings, taken inside the common fraction field $\mathcal{F}$

$$\mathcal{U} := \bigcap_{\text{clusters } \mathbf{x} \subset \mathcal{A}} \mathbb{ZP}\big[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\big] \subset \mathcal{F}.$$

By the Laurent phenomenon, there is an embedding $\mathcal{A} \subseteq \mathcal{U}$. This inclusion is not always equality (see [1, Proposition 1.26]), but it is an equality in many examples of cluster algebras, and it is hoped to be an equality in many more important examples.

Remark 1. An explanation of the geometric significance of the upper cluster algebra can be found in [10, Section 3.2].

3 Cluster localization and covers

This section reviews the techniques of cluster localization and covers, defined in [12].

3.1 Freezing

'Freezing' a cluster variable in a seed promotes one or more cluster variables to the coefficient ring. The name is motivated by the case of geometric seeds, in which the generators of $\mathbb{P}$ are regarded as 'frozen' cluster variables.

Let $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ be a seed of rank n over a coefficient ring $\mathbb{ZP}$. Define the *freezing* of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ at x_n to be the seed $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$ of rank $n-1$ defined as follows.

- $\mathbb{P}^\dagger := \mathbb{P} \times \mathbb{Z}$, the direct product of $\mathbb{P}$ with a free abelian group $\mathbb{Z}$, whose generator will be denoted x_n. The auxilliary addition is defined by

$$\left(p_1 x_n^a\right) \oplus \left(p_2 x_n^b\right) := (p_1 \oplus p_2) x_n^{\min(a,b)}.$$

It follows that $\mathbb{Z}\mathbb{P}^\dagger \simeq \mathbb{Z}\mathbb{P}[x_n^{\pm 1}]$ as rings.

- There is an obvious isomorphism from the field $\mathcal{F} \simeq \mathbb{Q}(\mathbb{P}, x_1, x_2, \dots, x_n)$ to the field $\mathcal{F}^\dagger \simeq \mathbb{Q}(\mathbb{P}^\dagger, x_1, x_2, \dots, x_{n-1})$. Identify x_i with its image under this isomorphism, and let $\mathbf{x}^\dagger := \{x_1, x_2, \dots, x_{n-1}\}$.
- $y_i^\dagger := y_i x_n^{B_{ni}}$ and $\mathbf{y}^\dagger := \{y_1^\dagger, y_2^\dagger, \dots, y_{n-1}^\dagger\}$.
- Let $\mathsf{B}^\dagger$ be the submatrix of B obtained by deleting the nth row and column.

Freezing an arbitrary variable $x_i \subset \mathbf{x}$ is defined by conjugating the above construction by any permutation that sends x_i to x_n. The following proposition is straightforward.

Proposition 1. *Let $(\mu(\mathbf{x}), \mu(\mathbf{y}), \mu(\mathsf{B}))$ be the mutation of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ at the variable x_i, and let $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$ be the freezing of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ at $x_j \neq x_i$. Then the freezing of $(\mu(\mathbf{x}), \mu(\mathbf{y}), \mu(\mathsf{B}))$ at x_j is the mutation at x_i of $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}')$.*

It follow that the exchange graph[6] of $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$ is the subgraph of the exchange graph of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ which contains the initial seed and avoids mutating at x_i.

Given a subset $S \subset \mathbf{x}$, the *freezing* of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ at S is the iterated freezing of each variable in S in any order. As above, freezing multiple variables S commutes with mutation away from S.

3.2 Cluster localization

The effect of freezing a seed on the corresponding cluster algebras and upper cluster algebras is given by the following sequence of containments.

Lemma 1. *Let $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subset \mathbf{x}$ be a set of variables in a seed $(\mathbf{x}, \mathbf{y}, \mathsf{B})$, with freezing $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$. Let $\mathcal{A}$ and $\mathcal{U}$ be the cluster algebra and upper cluster algebra of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ and let $\mathcal{A}^\dagger$ and $\mathcal{U}^\dagger$ be the cluster algebra and upper cluster algebra of $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$. Then there are inclusions in $\mathcal{F}$*

$$\mathcal{A}^\dagger \subseteq \mathcal{A}\left[(x_{i_1} x_{i_2} \dots x_{i_k})^{-1}\right] \subseteq \mathcal{U}\left[(x_{i_1} x_{i_2} \dots x_{i_k})^{-1}\right] \subseteq \mathcal{U}^\dagger.$$

Proof. The coefficient ring $\mathbb{Z}\mathbb{P}^\dagger = \mathbb{Z}\mathbb{P}\left[x_{i_1}^{\pm 1}, \dots, x_{i_k}^{\pm 1}\right]$, so $\mathbb{Z}\mathbb{P}^\dagger \subset \mathcal{A}\left[(x_{i_1} x_{i_2} \dots x_{i_k})^{-1}\right]$. Since the cluster variables of $\mathcal{A}$ contain the cluster variables of $\mathcal{A}^\dagger$, this implies the first inclusion.

Since the clusters of $\mathcal{A}$ contain the clusters of $\mathcal{A}^\dagger$, there is a containment $\mathcal{U} \subset \mathcal{U}^\dagger$. Furthermore, $(x_{i_1} x_{i_2} \dots x_{i_k})^{-1} \in \mathbb{P}^\dagger \subset \mathcal{U}^\dagger$, which implies the last inclusion.

The middle inclusion follows from the Laurent phenomenon. ■

Each of these inclusions is fairly interesting; but for the purposes at hand, we are most interested in when the first inclusion is equality.

Definition 1. If $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$ is a freezing of $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ at $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ such that

$$\mathcal{A}(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger) = \mathcal{A}(\mathbf{x}, \mathbf{y}, \mathsf{B})\left[(x_{i_1} x_{i_2} \dots x_{i_k})^{-1}\right],$$

we say $\mathcal{A}(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$ is a *cluster localization* of $\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathsf{B})$.

[6]The *exchange graph* of $\mathcal{A}$ is the possibly infinite graph with a vertex for each seed (up to permutation) and an edge for each mutation between seeds.

This is the most natural way a cluster structure can descend to a cluster structure on a localization; hence the name. Whenever $\mathcal{A}(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger) = \mathcal{U}(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$, each inclusion in Lemma 1 is an equality, and so $\mathcal{A}(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger)$ is a cluster localization of $\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathsf{B})$.

Cluster localizations are transitive; if $\mathcal{A}^\dagger$ is a cluster localization of $\mathcal{A}$, and $\mathcal{A}^\ddagger$ is a cluster localization of $\mathcal{A}^\dagger$, then $\mathcal{A}^\ddagger$ is a cluster localization of $\mathcal{A}$.

Example 1. The extreme case $S = \mathbf{x}$ amounts to freezing every variable in a seed. The freezing $(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger) = (\varnothing, \varnothing, \varnothing)$ is rank zero and so

$$\mathcal{A}\big(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger\big) = \mathbb{Z}\mathbb{P}^\dagger \simeq \mathbb{Z}\mathbb{P}\big[x_1^{\pm1}, x_2^{\pm1}, \ldots, x_n^{\pm1}\big] = \mathcal{U}\big(\mathbf{x}^\dagger, \mathbf{y}^\dagger, \mathsf{B}^\dagger\big).$$

This is the inclusion coming from the Laurent phenomenon.

3.3 Covers

For a cluster algebra $\mathcal{A}$, a set $\{\mathcal{A}_i\}_{i\in I}$ of cluster localizations of $\mathcal{A}$ (not necessarily localizations from the same seed) is called a *cover* if, for every prime ideal P in $\mathcal{A}$, there is some $\mathcal{A}_i$ such that $\mathcal{A}_i P \subsetneq \mathcal{A}_i$.

Covers are transitive; that is, if $\{\mathcal{A}_i\}_{i\in I}$ is a cover of $\mathcal{A}$, and $\{\mathcal{A}_{ij}\}_{j\in J_i}$ is a cover of $\mathcal{A}_i$, then $\bigcup_{i\in I}\{\mathcal{A}_{ij}\}_{j\in J_i}$ is a cover of $\mathcal{A}$. However, because there is no 'geometric intersection' for cluster localizations, there is no notion of a common refinement of two covers.

The following is a useful property of covers.

Proposition 2. *If $\{\mathcal{A}_i\}_{i\in I}$ is a cover of $\mathcal{A}$, then $\mathcal{A} = \bigcap_{i\in I}\mathcal{A}_i$.*

Proof. Let $a \in \bigcap_{i\in I}\mathcal{A}_i$. Write each $\mathcal{A}_i$ as $\mathcal{A}[d_i^{-1}]$ for some $d_i \in \mathcal{A}$, so that $a \in \bigcap_{i\in I}\mathcal{A}[d_i^{-1}]$. For each i, there is some n_i such that $d_i^{n_i} a \in \mathcal{A}$. Define the $\mathcal{A}$-ideal

$$J := \{b \in \mathcal{A} \,|\, ba \in \mathcal{A}\}.$$

It follows that $d_i^{n_i} \in J$ and $\mathcal{A}_i J = \mathcal{A}_i$, for all i. If M is a prime $\mathcal{A}$-ideal which contains J, then $\mathcal{A}_i M = \mathcal{A}_i$ for all i. Hence, no prime $\mathcal{A}$-ideals contain J, and so $J = \mathcal{A}$. In particular, $1 \in J$, and so $a \in \mathcal{A}$. ■

The equality $\mathcal{A} = \mathcal{U}$ may be checked locally in a cover.

Lemma 2. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a cover of $\mathcal{A}$. If $\mathcal{A}_i = \mathcal{U}_i$ for each $i \in I$, then $\mathcal{A} = \mathcal{U}$.*

Proof. Lemma 1 implies that $\mathcal{U} \subset \mathcal{U}_i$ for all i, and so $\mathcal{U} \subseteq \bigcap_{i\in I}\mathcal{U}_i$. Then

$$\mathcal{U} \subseteq \bigcap_{i\in I}\mathcal{U}_i = \bigcap_{i\in I}\mathcal{A}_i = \mathcal{A} \subseteq \mathcal{U}$$

and so $\mathcal{A} = \mathcal{U}$. ■

Example 2. Consider the example with initial seed and cluster algebra

$$\left(\{x_1, x_2\}, \varnothing, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right), \qquad \mathcal{A} = \mathbb{Z}\left[x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}\right].$$

The two freezings of the initial seed both define cluster localizations

$$\mathcal{A}[x_1^{-1}] = \mathbb{Z}[x_1^{\pm1}]\left[x_2, \frac{x_1+1}{x_2}\right], \qquad \mathcal{A}[x_2^{-1}] = \mathbb{Z}[x_2^{\pm1}]\left[x_1, \frac{x_2+1}{x_1}\right],$$

which collectively define a cover of $\mathcal{A}$, because no prime ideal in $\mathcal{A}$ can contain both x_1 and x_2. Proposition 2 implies that

$$\mathbb{Z}[x_1^{\pm1}]\left[x_2, \frac{x_1+1}{x_2}\right] \cap \mathbb{Z}[x_2^{\pm1}]\left[x_1, \frac{x_2+1}{x_1}\right] = \mathbb{Z}\left[x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}\right].$$

Since $\mathcal{U}_i = \mathcal{A}_i$ for both cluster localizations, Lemma 2 implies that $\mathcal{U} = \mathcal{A}$.

4 Acyclic and locally acyclic cluster algebras

This section shows that acyclic cluster algebras admit covers by *isolated cluster algebras*.

4.1 Isolated cluster algebras

A cluster algebra is *isolated* if the exchange matrix of any seed (equivalently, every seed) is the zero matrix[7].

Proposition 3. *Let $\mathcal{A}$ be an isolated cluster algebra. Then $\mathcal{A}=\mathcal{U}$.*[8]

Proof. Let $(\mathbf{x},\mathbf{y},\mathsf{B})$ be a seed for $\mathcal{A}$. Let $\mathbf{x}=\{x_1,x_2,\ldots,x_n\}$ and let $x_i' := P_i x_i^{-1}$ denote the mutation of x_i in the initial seed. Since $\mathsf{B}=0$, each of the $P_i\in\mathbb{ZP}$, and mutation does not change $\mathbf{y}$ or B. Hence, mutating at distinct indices is order-independent. It follows that $\{x_1,x_2,\ldots,x_n,x_1',x_2',\ldots,x_n'\}$ is the complete set of cluster variables in $\mathcal{A}$, and the clusters are of the form

$$\{x_i|i\notin I\}\bigcup\{x_i'\,|\,i\in I\}$$

for any subset $I\subset\{1,2,\ldots,n\}$.

Choose some $a\in\mathcal{U}$. Since $a\in\mathbb{ZP}[x_1^{\pm1},x_2^{\pm1},\ldots,x_n^{\pm1}]$, it can be written as

$$a=\sum_{\alpha\in\mathbb{Z}^n}\lambda_\alpha x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},\qquad \lambda_\alpha\in\mathbb{ZP}.$$

For any $I\subset\{1,2,\ldots,n\}$, the element a can be written as

$$a=\sum_{\alpha\in\mathbb{Z}^n}\gamma_{\alpha,I}\left(\prod_{i\notin I}x_i^{\alpha_i}\right)\left(\prod_{i\in I}x_i'^{-\alpha_i}\right)=\sum_{\alpha\in\mathbb{Z}^n}\gamma_{\alpha,I}\left(\prod_{i\in I}P_i^{-\alpha_i}\right)x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}.$$

Since the monomials in $x_1,x_2,\ldots,x_n$ are a basis for $\mathbb{ZP}[x_1^{\pm1},x_2^{\pm1},\ldots,x_n^{\pm1}]$, it follows that $\lambda_\alpha=\gamma_{\alpha,I}\big(\prod_{i\in I}P_i^{-\alpha_i}\big)$ for all α. Hence, a can be written as

$$a=\sum_{\alpha\in\mathbb{Z}}\gamma_\alpha\left(\prod_{i\,|\,\alpha_i\geq0}x_i^{\alpha_i}\right)\left(\prod_{i\,|\,\alpha_i<0}x_i'^{-\alpha_i}\right),$$

where γ_α is $\gamma_{\alpha,I}$ for I the subset where $\alpha_i<0$. This expression for a is clearly in $\mathcal{A}$. ∎

Corollary 1. *If $\mathcal{A}$ admits a cover by isolated cluster algebras, then $\mathcal{A}=\mathcal{U}$.*

Proof. This is an immediate corollary of Proposition 3 and Lemma 2. ∎

4.2 Acyclic cluster algebras

Isolated cluster algebras are too simple to be interesting by themselves, but the key idea of [12] is that many interesting cluster algebras can be covered by isolated cluster algebras.

A cluster algebra is *acyclic* if it has an *acyclic seed*: a seed with an exchange matrix B, such that there is no sequence of indices $i_1,i_2,\ldots,i_\ell\in\{1,2,\ldots,n\}$ with $\mathsf{B}_{i_{j+1}i_j}>0$ for all j and $i_\ell=i_1$.[9] Acyclic cluster algebras form a class of well-behaved cluster algebras which contain many notable examples, including all finite-type cluster algebras.

[7] For a seed of geometric type defined by a quiver, being isolated is equivalent to having no arrows between mutable vertices; this is the origin of the term 'isolated'.

[8] This is a straightforward generalization of the statement and proof of [1, Lemma 6.2].

[9] For a seed of geometric type defined by a quiver, being acyclic is equivalent to having no directed cycles; this is the origin of the term 'acyclic'. Not every seed of an acyclic cluster algebra is an acyclic seed.

Proposition 4. *If $\mathcal{A}$ is acyclic, then it admits a cover by isolated cluster algebras.*

Proof. The proof will be by induction on the rank n of $\mathcal{A}$. If acyclic $\mathcal{A}$ has rank $n \leq 1$, then $\mathcal{A}$ is isolated, and trivially has a cover by isolated cluster algebras.

Assume that every acyclic cluster algebra of rank $< n$ admits a cover by isolated cluster algebras. Let $\mathcal{A}$ be an acyclic cluster algebra of rank n, and let $(\mathbf{x}, \mathbf{y}, \mathsf{B})$ be an acyclic seed. There must be some index $i \in \{1, 2, \dots, n\}$ which is a *sink*; that is, $\mathsf{B}_{ji} \leq 0$ for all j. Otherwise, it would be possible to create arbitrarily long sequences of indices $i_1, i_2, \dots, i_\ell$ with $\mathsf{B}_{i_{j+1}i_j} > 0$; by finiteness, at least one such sequence will have $i_\ell = i_1$.

If $\mathcal{A}$ is isolated, then it trivially has a cover by isolated cluster algebras. If $\mathcal{A}$ is not isolated, then there must be indices i and j such that i is a sink and $\mathsf{B}_{ji} < 0$. The mutation relation at i is then

$$x_i x_i' = \frac{y_i}{y_i \oplus 1} + \frac{1}{y_i \oplus 1} \prod_{k \,|\, \mathsf{B}_{ki} < 0} x_k^{-\mathsf{B}_{ki}}.$$

Since $\frac{y_i}{y_i \oplus 1}$ is invertible in $\mathbb{ZP}$, this can be rewritten as

$$1 = \frac{y_i \oplus 1}{y_i} x_i' x_i - y_i^{-1} \prod_{k \,|\, \mathsf{B}_{ki} < 0} x_k^{-\mathsf{B}_{ki}}.$$

Since x_j appears in the right-hand product, x_i and x_j generate the trivial $\mathcal{A}$-ideal.

Let $\mathcal{A}_i$ and $\mathcal{A}_j$ denote the freezings of $\mathcal{A}$ at the indices i and j, respectively. The freezing of an acyclic seed is an acyclic seed, and so $\mathcal{A}_i$ and $\mathcal{A}_j$ are acyclic cluster algebras of rank $n-1$. By the inductive hypothesis, they admit covers by isolated cluster algebras, and by Corollary 1, $\mathcal{A}_i = \mathcal{U}_i$ and $\mathcal{A}_j = \mathcal{U}_j$. By Lemma 1, these are cluster localizations with $\mathcal{A}_i = \mathcal{A}\big[x_i^{-1}\big]$ and $\mathcal{A}_j = \mathcal{A}_j\big[x_j^{-1}\big]$.

Let P be a prime $\mathcal{A}$-ideal. Since x_i and x_j generate the trivial $\mathcal{A}$-ideal, P cannot contain both elements. If $x_i \notin P$, then $\mathcal{A}_i P \subsetneq \mathcal{A}_i$. If $x_j \notin P$, then $\mathcal{A}_j P \subsetneq \mathcal{A}_j$. Hence, $\mathcal{A}_i$ and $\mathcal{A}_j$ cover $\mathcal{A}$. The union of the covers of $\mathcal{A}_i$ and $\mathcal{A}_j$ by isolated cluster algebras defines a cover $\mathcal{A}$ by isolated cluster algebras. ■

Corollary 2. *If $\mathcal{A}$ is acyclic, then $\mathcal{A} = \mathcal{U}$.*

4.3 Locally acyclic cluster algebras

Many non-acyclic cluster algebras also admit covers by isolated cluster algebras. As an example, every triangulable marked surface determines a cluster algebra (see [2, 3, 7]), and these cluster algebras are locally acyclic as long as there are at least two marked points on the boundary [12, Theorem 10.6]. Since this class is interesting in its own right, we give it a name.

Definition 2 ([12])**.** A cluster algebra is *locally acyclic* if it admits a cover by isolated cluster algebras.

Since isolated cluster algebras are acyclic, and every acyclic cluster algebra admits a cover by isolated cluster algebras, this condition is equivalent to admitting a cover by acyclic cluster algebras; hence the name. By Corollary 1, every locally acyclic cluster algebra has $\mathcal{A} = \mathcal{U}$. More generally, locally acyclic cluster algebras possess any property of isolated cluster algebras which is 'geometrically local'. For example, they are integrally closed, finitely generated and locally a complete intersection; see [12] for details.

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