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Configurations of Points and the Symplectic Berry–Robbins Problem

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Abstract. We present a new problem on configurations of points, which is a new version of a similar problem by Atiyah and Sutcliffe, except it is related to the Lie group $\mathrm{Sp}(n)$, instead of the Lie group $\mathrm{U}(n)$. Denote by $\mathfrak{h}$ a Cartan algebra of $\mathrm{Sp}(n)$, and Δ the union of the zero sets of the roots of $\mathrm{Sp}(n)$ tensored with $\mathbb{R}^3$, each being a map from $\mathfrak{h}\otimes\mathbb{R}^3 \to \mathbb{R}^3$. We wish to construct a map $(\mathfrak{h}\otimes\mathbb{R}^3)\backslash\Delta \to \mathrm{Sp}(n)/T^n$ which is equivariant under the action of the Weyl group W_n of $\mathrm{Sp}(n)$ (the symplectic Berry–Robbins problem). Here, the target space is the flag manifold of $\mathrm{Sp}(n)$, and T^n is the diagonal n-torus. The existence of such a map was proved by Atiyah and Bielawski in a more general context. We present an explicit smooth candidate for such an equivariant map, which would be a genuine map provided a certain linear independence conjecture holds. We prove the linear independence conjecture for $n = 2$.

Key words: configurations of points; symplectic; Berry–Robbins problem; equivariant map; Atiyah–Sutcliffe problem

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1 Introduction

We introduce the relevant manifolds.

$$C_n = \{(x_1,\dots,x_n) \in (\mathbb{R}^3\backslash\{0\})^n;\ x_r \neq x_s \text{ and } x_r \neq -x_s \text{ for all } 1 \le r < s \le n\}.$$

We remark that $C_n = (\mathfrak{h}\otimes\mathbb{R}^3)\backslash\Delta$, where $\mathfrak{h}$ is a Cartan algebra of $\mathrm{Sp}(n)$, and Δ the union of the zero sets of the roots of $\mathrm{Sp}(n)$ tensored with $\mathbb{R}^3$, each being a map from $\mathfrak{h}\otimes\mathbb{R}^3 \to \mathbb{R}^3$. We denote by F_n the following flag manifold

$$F_n = \mathrm{Sp}(n)/T^n,$$

where T^n is the diagonal n-dimensional torus in $\mathrm{Sp}(n)$. Let W_n be the Weyl group of the Lie group $\mathrm{Sp}(n)$. It is well known that W_n is a semidirect product

$$W_n = (\mathbb{Z}/(2))^n \rtimes \Sigma_n,$$

where Σ_n is the symmetric group on n elements, and where we think of $\mathbb{Z}/(2)$ as $\{-1,1\}$ with multiplication. The group W_n acts on C_n as follows. The element $(1,\dots,-1,\dots,1) \in (\mathbb{Z}/(2))^n$, with a -1 in the rth position only, acts on $(x_1,\dots,x_n)$ by replacing x_r with $-x_r$, and leaving all other x_s invariant. An element $\sigma \in \Sigma_n$ acts by permuting the n coordinates.

On the other hand, the action of W_n on F_n can be described as follows: an element in $(\mathbb{Z}/(2))^n$ having -1 only in the rth position, multiplies the rth column of each point in F_n by the quaternionic structure j (leaving the other columns invariant), while a permutation σ simply permutes the columns of each point $gT^n \in F_n$.

Having described the main players in our story, we consider the following question, which is a special case of a question asked and solved by Atiyah and Bielawski for an arbitrary Lie group in [3]. Another special case, for the unitary groups, was considered earlier by Berry and Robbins in [5].

Question. *Is there for each $n \geq 2$, a continuous map $f_n : C_n \to F_n$ which is equivariant for the action of W_n?*

Actually, as we wrote earlier, Atiyah and Bielawski have already posed and solved in [3] a more general problem for any compact Lie group G (in our case, $G = \mathrm{Sp}(n)$). But their solution is non-elementary, as it relies on an analysis of the Nahm equations. Here we propose, similar to Atiyah [1, 2]), and Atiyah and Sutcliffe [4], a more elementary construction in the same spirit as those papers, but for the case $G = \mathrm{Sp}(n)$, instead of $G = \mathrm{U}(n)$.

2 The main construction

The stereographic projection is a map $s : S^2 \to \mathbb{C} \cup \{\infty\}$. Denoting by N the point with coordinates $(0, 0, 1) \in S^2$, we define the stereographic projection $s(x, y, z)$ of a point $(x, y, z) \in S^2 \backslash \{N\}$ by

$$s(x, y, z) = \frac{x + iy}{1 - z}$$

and we define $s(N) = \infty$. We first associate to each configuration $\mathbf{x} \in C_n$, n polynomials $p_1, \ldots, p_n$ of $t \in \mathbb{C}$ of degree less than or equal to $2n - 1$, each defined up to a scalar factor only. Namely, we let p_r be a polynomial having as roots the stereographic projections of

$$-\frac{x_r}{\|x_r\|}, \qquad \frac{-x_r + x_s}{\|-x_r + x_s\|} \qquad \text{and} \qquad \frac{-x_r - x_s}{\|-x_r - x_s\|}$$

for all $s \neq r$ $(1 \leq s \leq n)$, where $\|-\|$ denotes the Euclidean norm on $\mathbb{R}^3$. Similarly, we introduce n other polynomials $q_1, \ldots, q_n$, with q_r having as roots the antipodals of the roots of p_r, namely the stereographic projections of $x_r/\|x_r\|$, $(x_r + x_s)/\|x_r + x_s\|$ and $(x_r - x_s)/\|x_r - x_s\|$, for $s \neq r$.

The space of complex polynomials of degree less than or equal to $2n-1$ has a natural structure of a left vector space over the quaternions $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, where j acts as follows. Writing such a polynomial as

$$p(t) = \sum_{r=0}^{2n-1} a_r t^r$$

the quaternionic structure j maps p to jp, defined by

$$(jp)(t) = \sum_{r=0}^{2n-1} a'_r t^r,$$

where

$$a'_r = (-1)^{r+1} \bar{a}_{2n-1-r}.$$

Moreover, since the roots of q_r are the antipodals of those of p_r, it follows that q_r is a constant times jp_r.

If we think of the space of polynomials above with its quaternionic structure as $\mathbb{H}^n$, then the p_r define n column vectors in $\mathbb{H}^n$. In this way we assign to any point in C_n an n-tuple of elements of $\mathbb{H}^n$ defined up to the diagonal action of $(\mathbb{C}^*)^n$.

Suppose now that for every $x \in C_n$ the polynomials $p_1, \dots, p_n$ are $\mathbb{H}$-linearly independent. Then we get a map from C_n into $\mathrm{GL}(n, \mathbb{H})/(\mathbb{C}^*)^n$. Note that W_n acts on $\mathrm{GL}(n, \mathbb{H})/(\mathbb{C}^*)^n$ as follows. An element $(1, \dots, -1, \dots, 1)$ in the $(\mathbb{Z}/(2))^n$ subgroup of W_n, with -1 only in the rth position, has the effect of mapping the equivalence class of a matrix $A \in \mathrm{GL}(n, \mathbb{H})$ to the equivalence class of the matrix obtained from A by replacing its rth column by its image under j. On the other hand, an element σ in the Σ_n subgroup of W_n maps the equivalence class of A to the equivalence class of the matrix obtained from A by permuting its columns via σ. With this action, the map becomes W_n-equivariant, since replacing x_r by $-x_r$ has the effect of replacing p_r by jp_r, and a permutation of the n coordinates of x corresponds to the same permutation of the n polynomials $p_1, \dots, p_n$. If we follow this map by a W_n-equivariant map from $\mathrm{GL}(n, \mathbb{H})/(\mathbb{C}^*)^n$ into $\mathrm{Sp}(n)/T^n$, then this solves the symplectic Berry–Robbins problem (assuming that $p_1, \dots, p_n$ are $\mathbb{H}$-linearly independent). Such a map can be obtained for example using the quaternionic polar decomposition. More specifically, the map from $\mathrm{GL}(n, \mathbb{H})$ into $\mathrm{Sp}(n)$ mapping A to $A(\sqrt{A^*A})^{-1}$, where $*$ denotes the quaternionic conjugate transpose, descends to a W_n-equivariant map from $\mathrm{GL}(n, \mathbb{H})/(\mathbb{C}^*)^n$ into $\mathrm{Sp}(n)/T^n$.

Motivated by the above discussion, we make the following conjecture.

Conjecture 1. *Given any $\mathbf{x} \in C_n$, the $2n$ polynomials p_r and q_r $(1 \le r \le n)$ are linearly independent over $\mathbb{C}$.*

Note that the conjecture is equivalent to the $\mathbb{H}$-linear independence of $p_1, \dots, p_n$. We remark further that our construction is also naturally equivariant under the action of $\mathrm{SO}(3)$, which acts on $\mathbb{R}^3$ in the natural way (which is equivalent to the adjoint action of $\mathrm{SO}(3)$ on its Lie algebra $\mathfrak{so}(3)$), and acts on the polynomials p_r and q_r, which are only defined up to a (non-zero) scalar factor, via the induced action of its double cover $\mathrm{SU}(2)$ on $S^{2n-1}H$, where H is the natural irreducible complex two-dimensional representation space of $\mathrm{SU}(2)$. In the following section, we define a natural determinant function, and then, we prove the conjecture above for $n = 2$.

3 A determinant function

Let $t_r^{\pm}$ and $t_{rs}^{\pm\pm} \in \mathbb{C}P^1$ be the stereographic projections of the normalizations of the following vectors, respectively

$$\pm x_r \quad (1 \le r \le n) \qquad \text{and} \qquad \pm x_r \pm x_s \quad (1 \le r, s \le n \quad \text{and} \quad r \neq s).$$

The Hopf map $h : \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}P^1$ is the natural projection map mapping $v \in \mathbb{C}^2 \backslash \{0\}$ to its $\mathbb{C}^*$-orbit, where $\mathbb{C}^*$ acts on $\mathbb{C}^2 \backslash \{0\}$ by scalar multiplication.

We then choose lifts $\mathbf{u}_r^{\pm} = (u_r^{\pm}, v_r^{\pm})$ and $\mathbf{u}_{rs}^{\pm\pm} = (u_{rs}^{\pm\pm}, v_{rs}^{\pm\pm}) \in \mathbb{C}^2 \backslash \{0\}$ of these roots under the Hopf map h. Since $\mathbf{u}_{sr}^{-\sigma,-\tau}$ and $\mathbf{u}_{rs}^{\sigma,\tau}$ are both lifts of the same point, we require that they be equal. We then form the polynomials p_r and q_r as the following products

$$p_r(t) = (u_r^- t - v_r^-) \prod_{s \neq r} (u_{rs}^{-+} t - v_{rs}^{-+}) \prod_{s \neq r} (u_{rs}^{--} t - v_{rs}^{--}),$$

$$q_r(t) = (u_r^+ t - v_r^+) \prod_{s \neq r} (u_{rs}^{++} t - v_{rs}^{++}) \prod_{s \neq r} (u_{rs}^{+-} t - v_{rs}^{+-}).$$

Thus in particular, once the lifts are chosen, the polynomials p_r and q_r are determined uniquely, in other words, the scalar factor of each of them gets fixed. We now form the complex $2n$ by $2n$ matrix

$$M = (p_1, q_1, \dots, p_n, q_n)$$

having the coefficients of p_r and q_r as column vectors. One then defines the quantity

$$P = \prod_r \det(\mathbf{u}_r^-, \mathbf{u}_r^+) \prod_{r<s} \left(\det(\mathbf{u}_{rs}^{-+}, \mathbf{u}_{rs}^{+-}) \det(\mathbf{u}_{rs}^{--}, \mathbf{u}_{rs}^{++})\right)^2 .$$

Then the determinant function

$$D(x_1, \dots, x_n) = \det(M)/P$$

is independent of the choices of lifts, and is thus well defined. Similar to the Atiyah–Sutcliffe determinant, D is actually invariant under the action of the Weyl group W_n on C_n, and is also invariant under scaling, and rotations in $\mathbb{R}^3$. However, unlike the Atiyah–Sutcliffe determinant, it is not invariant under translations, because the origin in $\mathbb{R}^3$ plays here a special role, and it is always real-valued, because it is the determinant of a $2n$ by $2n$ complex matrix, which represents an n by n quaternionic matrix, and thus is always real (indeed, the complex conjugate of such a $2n$ by $2n$ complex matrix can be shown to be in the same conjugacy class as the complex matrix itself, so they must have equal determinants).

4 The case $n = 2$

We consider here the case $n = 2$. We have two points $x_1, x_2 \in \mathbb{R}^3$ such that $x_1 \neq x_2$ and $x_1 \neq -x_2$. Using a rotation in $\mathbb{R}^3$, we can assume that they both lie on the xy-plane. We think of the xy-plane as the complex plane. Using a rotation in the xy-plane and scaling, we can further assume that $x_1 = 1$ and we then let z be the complex number representing x_2 in the xy-plane. Thus $z \neq 1$ and $z \neq -1$. We let

$$A = \frac{z-1}{|z-1|}, \qquad B = -\frac{z+1}{|z+1|}, \qquad g = -\frac{z}{|z|}.$$

We then have

$$-64D = \begin{vmatrix} AB & 1 & ABg & 1 \\ AB - A - B & \bar{A} + \bar{B} - 1 & -Ag + Bg - AB & -\bar{A} + \bar{B} + \bar{g} \\ 1 - A - B & \bar{A}\bar{B} - \bar{A} - \bar{B} & A - B - g & -\bar{A}\bar{g} + \bar{B}\bar{g} - \bar{A}\bar{B} \\ 1 & -\bar{A}\bar{B} & 1 & -\bar{A}\bar{B}\bar{g} \end{vmatrix}.$$

We then multiply the second column by $-AB$ and add it to the first column, and we multiply the second column by $-ABg$ and add it to the third column, and finally subtract the second column from the fourth one, and get, after expanding the determinant along the first row:

$$64D = \begin{vmatrix} 2(AB - A - B) & -2Ag + ABg - AB & -2\bar{A} + 1 + \bar{g} \\ 0 & -2g + Bg - B + Ag + A & -2\bar{A}\bar{B} + \bar{A}(1-\bar{g}) + \bar{B}(1+\bar{g}) \\ 2 & 1 + g & \bar{A}\bar{B}(1-\bar{g}) \end{vmatrix}.$$

Taking a 2 out from the first column, and using elementary column operations using the first column in order to make the entries in the $(3,2)$ and $(3,3)$ positions vanish, we get

$$\begin{aligned} 32D &= \begin{vmatrix} AB - A - B & -2AB + A(1-g) + B(1+g) & 2\bar{g} - \bar{A}(1+\bar{g}) + \bar{B}(1-\bar{g}) \\ 0 & A(1+g) - B(1-g) - 2g & -2\bar{A}\bar{B} + \bar{A}(1-\bar{g}) + \bar{B}(1+\bar{g}) \\ 1 & 0 & 0 \end{vmatrix} \\ &= |-2AB + A(1-g) + B(1+g)|^2 + |A(1+g) - B(1-g) - 2g|^2 \\ &= 8 + 2|1-g|^2 + 2|1+g|^2 - 2B(1-\bar{g}) - 2\bar{B}(1-g) - 2A(1+\bar{g}) - 2\bar{A}(1+g) + \cdots \\ &\quad + 2(A\bar{B} - \bar{A}B)(\bar{g} - g) - 2A\bar{g}(1+g) - 2\bar{A}g(1+\bar{g}) + 2B\bar{g}(1-g) + 2\bar{B}g(1-\bar{g}). \end{aligned}$$

Using

$$|1+g|^2 + |1-g|^2 = 4$$

we get

$$16D = 8 - 2A(1+\bar{g}) - 2B(1-\bar{g}) + A(1+\bar{g})\bar{B}(1-g) - 2\bar{A}(1+g) - 2\bar{B}(1-g) \\ + \bar{A}(1+g)B(1-\bar{g}).$$

If we let

$$(w_1, w_2) = \frac{1}{2}(w_1\bar{w}_2 + w_2\bar{w}_1), \qquad \det(w_1, w_2) = \frac{i}{2}(w_1\bar{w}_2 - w_2\bar{w}_1),$$

we can then write

$$4D = 2 - (A, 1+g) - (B, 1-g) - \Im(g)\det(A, B),$$

where $\Im(g)$ denotes the imaginary part of g. Therefore, using the definitions of A, B and g in terms of z, we get

$$4D = 2 + \left(\frac{z-1}{|z-1|}, \frac{z}{|z|} - 1\right) + \left(\frac{z+1}{|z+1|}, \frac{z}{|z|} + 1\right) + 2\frac{(\Im z)^2}{|z||z-1||z+1|}.$$

Writing $z = re^{i\theta}$, and after simplification, we get

$$4D = 2 + \frac{(1+r)(1-\cos(\theta))}{|z-1|} + \frac{(1+r)(1+\cos(\theta))}{|z+1|} + \frac{2r(1-\cos(\theta))(1+\cos(\theta))}{|z-1||z+1|}.$$

Using $1+r \geq |z+1|$ and $1+r \geq |z-1|$, and that $1+\cos(\theta)$ and $1-\cos(\theta)$ are both nonnegative,

$$4D \geq 4 + \frac{2r(1-\cos(\theta))(1+\cos(\theta))}{|z-1||z+1|}.$$

Thus

$$D \geq 1 + \frac{r\sin^2(\theta)}{2|z-1||z+1|}.$$

This proves the inequality $D \geq 1$, which in turns implies the linear independence conjecture, for $n = 2$. Moreover, it is not too difficult to see that equality $D = 1$ occurs if and only if $\sin(\theta) = 0$, or, in other words, if the two points x_1 and x_2 lie on the same line through the origin.

5 A stronger conjecture

Similar to Conjecture 2 in [4], we make the following conjecture

Conjecture 2. *For any $n \geq 2$ and for any $\mathbf{x} \in C_n$, we have $D(\mathbf{x}) \geq 1$.*

The author did some numerical testing for this conjecture for $n \leq 10$, by computing $D(x)$ for a number of pseudo-randomly generated configurations of points, and found that in all these cases, the conjecture was verified, thus gathering some numerical evidence for the above conjecture.

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