Symmetry, Integrability and Geometry: Methods and Applications SIGMA **12** (2016), 024, 4 pages

Nijenhuis Integrability for Killing Tensors*

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Received October 30, 2015, in final form February 26, 2016; Published online March 07, 2016
http://dx.doi.org/10.3842/SIGMA.2016.024

Abstract. The fundamental tool in the classification of orthogonal coordinate systems in which the Hamilton–Jacobi and other prominent equations can be solved by a separation of variables are second order Killing tensors which satisfy the Nijenhuis integrability conditions. The latter are a system of three non-linear partial differential equations. We give a simple and completely algebraic proof that for a Killing tensor the third and most complicated of these equations is redundant. This considerably simplifies the classification of orthogonal separation coordinates on arbitrary (pseudo-)Riemannian manifolds.

Key words: integrable systems; separation of variables; Killing tensors; Nijenhuis tensor; Haantjes tensor

2010 Mathematics Subject Classification: 70H06; 53A60; 53B20

It is a natural problem to classify all coordinate systems in which a given partial differential equation can be solved by a separation of variables – the so called *separation coordinates*. For fundamental equations such as the Hamilton–Jacobi and the Schrödinger equation the theory of separation of variables is built on a characterisation of orthogonal separation coordinates by second order Killing tensors, i.e., solutions of the Killing equation

$$\nabla_\alpha K_{\beta\gamma} + \nabla_\beta K_{\gamma\alpha} + \nabla_\gamma K_{\alpha\beta} = 0, \tag{1}$$

which are integrable in the sense that they have simple eigenvalues and the orthogonal complements of each eigenvector field form an integrable distribution. The relation between orthogonal separation coordinates and integrable Killing tensors was observed in 1891 by Paul Stäckel in his Habilitation thesis [10] and then used by Luther P. Eisenhart in 1934 to classify orthogonal separation coordinates on 3-dimensional Euclidean space and on the 3-dimensional sphere [1]. His results were generalised to arbitrary spaces of constant curvature of any dimension by Kalnins and Miller in 1986 [3, 4].

The property of an endomorphism field to be integrable in the above sense has been cast into the form of a system of three non-linear partial differential equations by Nijenhuis in 1950 [5]. Explicitly, an endomorphism K with simple eigenvalues is integrable if and only if it satisfies the *Nijenhuis integrability conditions*

$$0 = N^\delta{}_{[\beta\gamma} g_{\alpha]\delta}, \tag{2a}$$

$$0 = N^\delta{}_{[\beta\gamma} K_{\alpha]\delta}, \tag{2b}$$

$$0 = N^\delta{}_{[\beta\gamma} K_{\alpha]\varepsilon} K^\varepsilon{}_\delta, \tag{2c}$$

where the square brackets stand for antisymmetrisation and N denotes the *Nijenhuis torsion* of K, defined by

$$N(X,Y) := K^2[X,Y] - K[KX,Y] - K[X,KY] + [KX,KY]$$

*This paper is a contribution to the Special Issue on Analytical Mechanics and Differential Geometry in honour of Sergio Benenti. The full collection is available at http://www.emis.de/journals/SIGMA/Benenti.html and given in local coordinates by

$$N^{\alpha}{}_{\beta\gamma} = K^{\alpha}{}_{\delta} K^{\delta}{}_{[\beta;\gamma]} + K^{\delta}{}_{[\beta} K^{\alpha}{}_{\gamma];\delta}\,, \tag{3}$$

where a semicolon denotes a covariant derivative.

Of course, these equations have not been known to Stäckel or Eisenhart. Neither did they play any role in the complete classification of Kalnins and Miller. However, they reveal that the classification of orthogonal separation coordinates is actually an *algebraic geometric problem*. Indeed, the Nijenhuis integrability conditions (2) are algebraic in K and its derivatives and hence constitute a set of homogeneous algebraic equations on the space of second order Killing tensors. Note that, as the space of solutions to the overdetermined linear equation (1), this space is a finite-dimensional vector space. Consequently, the set of Killing tensors satisfying the Nijenhuis integrability conditions is a projective variety. Moreover, this variety comes along with a natural group action, namely the isometry group of the manifold.

To be more precise, orthogonal separation coordinates are in one-to-one correspondence with so-called *Stäckel systems*, i.e., n-dimensional spaces of integrable Killing tensors which mutually commute in the algebraic sense. This leads to the following remarkable observation [8]:

> For any (pseudo-)Riemannian manifold the set of orthogonal separation coordinates carries a very rich structure: It is a projective variety isomorphic to a subvariety in the Grassmannian of n-dimensional subspaces in the space of Killing tensors, equipped with a natural action of the isometry group.

To the best of our knowledge, this point has never been made in the literature and the structure of these varieties had never been elucidated. The reason is probably that a general solution of the equations (2) was deemed intractable [2].

Recently it has been possible to rewrite the Nijenhuis integrability conditions explicitly as algebraic equations for constant curvature manifolds [6] and to solve them in the least non-trivial case, namely for the sphere of dimension three [7]. The outcome, a detailed algebraic geometric description of the variety of integrable Killing tensors as well as the variety of Stäckel systems, has lead to a surprising connection between separation coordinates on spheres on one hand and algebraic curves on the other. More precisely, the set of orthogonal separation coordinates modulo isometries on the n-dimensional sphere is naturally parametrised by the real Deligne–Mumford moduli space $\bar{\mathscr{M}}_{0,n+2}(\mathbb{R})$ of stable algebraic curves of genus 0 with $n+2$ marked points [9]. As a consequence, the well known classification of Kalnins and Miller can be reinterpreted in terms of the geometry and combinatorics of Stasheff polytopes. This also revealed a hitherto unknown operad structure on equivalence classes of separation coordinates on spheres. To date, comparable results are unknown for manifolds other than spheres.

An important step in the explicit solution of the Nijenhuis equations has been the proof that the third of the Nijenhuis conditions is redundant when applied to a Killing tensor. According to a footnote in [2], this had previously been proven by Steve Czapor for Euclidean space in dimension three using Gröbner bases. The author extended this to arbitrary constant curvature manifolds [6]. The purpose of the present paper is to give a simple proof that this result holds in full generality. This will considerably simplify the classification of orthogonal separation coordinates on arbitrary manifolds.

Theorem 1. *For a Killing tensor on an arbitrary Riemannian manifold the third of the Nijenhuis equations* (2) *is redundant. The same holds true on a pseudo-Riemannian manifold.*

Remark 1. For a Killing tensor the first two Nijenhuis equations are in general independent [7].

Remark 2. Note that a Stäckel system contains Killing tensors whose eigenvalues are not simple (the metric for instance). This is why we did not impose simple eigenvalues in the above theorem.

Remark 3. Instead of the Nijenhuis conditions (2), the vanishing of the Haantjes tensor

$$H(X,Y) := K^2N(X,Y) - KN(KX,Y) - KN(X,KY) + N(KX,KY)$$

is often used as a criterion for integrability. Being of order four in K, this condition is of the same complexity as the third of the Nijenhuis equations, while the first two are only of order two and three. Our result therefore shows that for Killing tensors the Nijenhuis conditions are better suited than the Haantjes tensor.

The proof

We will prove the statement pointwise. That is, we fix an arbitrary point p in the manifold and consider the restrictions $K_{\alpha\beta}(p)$ and $(\nabla_\alpha K_{\beta\gamma})(p)$ of the Killing tensor field and its covariant derivative to the tangent space at p. The statement then becomes a purely algebraic statement on these two tensors. For ease of notation we omit to indicate the dependence on the chosen point p.

A Killing tensor is symmetric by definition. Hence at the fixed point we can choose an orthonormal basis of the tangent space in which the Killing tensor is diagonal, i.e., $K^\alpha{}_\beta = \lambda_\alpha\delta^\alpha{}_\beta$ (no sum). In this basis, the Nijenhuis torsion (3) reads

$$N_{\alpha\beta\gamma} = \begin{array}{|c|}\hline \beta \\ \hline \gamma \\ \hline\end{array}(\lambda_\alpha - \lambda_\gamma)K_{\alpha\beta;\gamma},$$

where the Young projector $\begin{array}{|c|}\hline \beta \\ \hline \gamma \\ \hline\end{array}$ stands for antisymmetrisation in β and γ. Substituted into the Nijenhuis integrability conditions (2) we get

$$\begin{aligned}
0 &= N^\delta{}_{[\beta\gamma}\,g_{\alpha]\delta} = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array} N_{\alpha\beta\gamma} = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array}(\lambda_\alpha - \lambda_\gamma)K_{\alpha\beta;\gamma}, \\
0 &= N^\delta{}_{[\beta\gamma}\,K_{\alpha]\delta} = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array}\lambda_\alpha N_{\alpha\beta\gamma} = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array}\lambda_\alpha(\lambda_\alpha - \lambda_\gamma)K_{\alpha\beta;\gamma}, \\
0 &= N^\delta{}_{[\beta\gamma}\,K_{\alpha]\varepsilon}K^\varepsilon{}_\delta = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array}\lambda_\alpha^2 N_{\alpha\beta\gamma} = \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array}\lambda_\alpha^2(\lambda_\alpha - \lambda_\gamma)K_{\alpha\beta;\gamma},
\end{aligned}$$

where the Young projector

$$\begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array}$$

denotes a complete antisymmetrisation in the indices α, β and γ. Using that $K_{\alpha\beta;\gamma}$ is symmetric in α, β and that a complete antisymmetrisation over α, β, γ can be split into an antisymmetrisation in α, β and a subsequent sum over the cyclic permutations of α, β, γ, we can rewrite the preceding equations as

$$0 = (\lambda_\alpha - \lambda_\beta)K_{\alpha\beta;\gamma} + \text{cyclic}, \tag{4a}$$

$$0 = (\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta - \lambda_\gamma)K_{\alpha\beta;\gamma} + \text{cyclic}, \tag{4b}$$

$$0 = (\lambda_\alpha - \lambda_\beta)\big((\lambda_\alpha + \lambda_\beta)^2 - \lambda_\alpha\lambda_\beta - \lambda_\beta\lambda_\gamma - \lambda_\gamma\lambda_\alpha\big)K_{\alpha\beta;\gamma} + \text{cyclic}, \tag{4c}$$

where "+ cyclic" stands for a summation over the cyclic permutations of α, β, γ. These equations are one by one equivalent to the Nijenhuis integrability conditions (2). In the same way we can write the Killing equation as

$$0 = K_{\alpha\beta;\gamma} + \text{cyclic}. \tag{5}$$

Adding appropriate multiples of (4a) to (4b) and (4c), we can simplify (4) to

$$0 = (\lambda_\alpha - \lambda_\beta)K_{\alpha\beta;\gamma} + \text{cyclic}, \tag{6a}$$

$$0 = (\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)K_{\alpha\beta;\gamma} + \text{cyclic}, \tag{6b}$$

$$0 = (\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)^2 K_{\alpha\beta;\gamma} + \text{cyclic}. \tag{6c}$$

We want to prove that (5) together with (4a) and (4b) imply (4c), which is equivalent to prove that (5) together with (6a) and (6b) imply (6c). To this end we write the first three equations in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_\alpha - \lambda_\beta & \lambda_\beta - \lambda_\gamma & \lambda_\gamma - \lambda_\alpha \\ \lambda_\alpha^2 - \lambda_\beta^2 & \lambda_\beta^2 - \lambda_\gamma^2 & \lambda_\gamma^2 - \lambda_\alpha^2 \end{bmatrix} \begin{bmatrix} K_{\alpha\beta;\gamma} \\ K_{\beta\gamma;\alpha} \\ K_{\gamma\alpha;\beta} \end{bmatrix} = 0.$$

The determinant of the coefficient matrix is an antisymmetric cubic polynomial in λ_α, λ_β, λ_γ and hence a multiple of the Vandermode determinant. If the eigenvalues λ_α, λ_β, λ_γ are pairwise different, this implies that $K_{\alpha\beta;\gamma} = K_{\beta\gamma;\alpha} = K_{\gamma\alpha;\beta} = 0$. If exactly two of the eigenvalues are equal, say $\lambda_\alpha \neq \lambda_\beta = \lambda_\gamma$, then we have $K_{\alpha\beta;\gamma} = -\frac{1}{2}K_{\beta\gamma;\alpha} = K_{\gamma\alpha;\beta}$. For three equal eigenvalues the only restriction on $K_{\alpha\beta;\gamma}$ is the Killing equation (5). In all three cases we see that the equation (6c) is also satisfied.

For a pseudo-Riemannian manifold the statement follows from the above and the fact that in the space of symmetric tensors the (complex) diagonalisable ones are dense. Indeed, the above reasoning for $C_{\alpha\beta} = K_{\alpha\beta}(p)$ remains true even if $C_{\alpha\beta}$ is (complex) diagonalisable, but not necessarily the restriction of a Killing tensor to the tangent space at p. By continuity, it is therefore also true if $C_{\alpha\beta}$ is not diagonalisable and therefore also for the restriction of an arbitrary Killing tensor.

Acknowledgements

The author would like to acknowledge the anonymous referees for their contribution to improve the paper.

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