Macdonald Polynomials of Type C_n with One-Column Diagrams and Deformed Catalan Numbers

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Received January 31, 2018, in final form September 11, 2018; Published online September 20, 2018
https://doi.org/10.3842/SIGMA.2018.101

Abstract. We present an explicit formula for the transition matrix $\mathcal{C}$ from the type C_n degeneration of the Koornwinder polynomials $P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)$ with one column diagrams, to the type C_n monomial symmetric polynomials $m_{(1^r)}(x)$. The entries of the matrix $\mathcal{C}$ enjoy a set of three term recursion relations, which can be regarded as a (a,c,t)-deformation of the one for the Catalan triangle or ballot numbers. Some transition matrices are studied associated with the type (C_n,C_n) Macdonald polynomials $P^{(C_n,C_n)}_{(1^r)}(x\,|\,b;q,t) = P_{(1^r)}\big(x\,|\,b^{1/2},-b^{1/2},q^{1/2}b^{1/2},-q^{1/2}b^{1/2}\,|\,q,t\big)$. It is also shown that the q-ballot numbers appear as the Kostka polynomials, namely in the transition matrix from the Schur polynomials $P^{(C_n,C_n)}_{(1^r)}(x\,|\,q;q,q)$ to the Hall–Littlewood polynomials $P^{(C_n,C_n)}_{(1^r)}(x\,|\,t;0,t)$.

Key words: Koornwinder polynomial; Catalan number

2010 Mathematics Subject Classification: 33D52; 33D45

1 Introduction

The aim of this article is to investigate the transition matrix $\mathcal{C}$, which describes the expansion of the type C_n degeneration of the Koornwinder polynomials [10] $P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)$ with one column diagrams, in terms of the type C_n monomial symmetric polynomials $m_{(1^r)}(x)$. As for our convention of notation, see Section 3. On this course, we found that certain deformations appear, associated with the Catalan triangle or ballot numbers, and binomial coefficients. We refer the readers to [21] concerning the Catalan triangle numbers, and [1, 6] for the q-Catalan and q-ballot numbers. For simplicity, write $P^{(C_n)}_{(1^r)} = P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)$.

Theorem 1.1. *Let $n\in\mathbb{Z}_{>0}$. Let $\mathbf{P}^{(n)}$ and $\mathbf{m}^{(n)}$ be the infinite column vectors*

$$\mathbf{P}^{(n)} = {}^t\big(P^{(C_n)}_{(1^n)},\dots,P^{(C_n)}_{(1)},P^{(C_n)}_{\varnothing},0,0,0,\dots\big),$$
$$\mathbf{m}^{(n)} = {}^t\big(m_{(1^n)},\dots,m_{(1)},m_{\varnothing},0,0,0,\dots\big).$$

There exist a unique infinite transition matrix $\mathcal{C}=(\mathcal{C})_{i,j\in\mathbb{Z}_{\geq 0}}$ satisfying the conditions

$$\mathcal{C} \text{ is upper triangular, namely } i>j \text{ implies } \mathcal{C}_{ij}=0, \tag{1.1a}$$
$$\mathcal{C} \text{ is even, namely } i+j \text{ is odd implies } \mathcal{C}_{ij}=0, \tag{1.1b}$$

This paper is a contribution to the Special Issue on Elliptic Hypergeometric Functions and Their Applications. The full collection is available at https://www.emis.de/journals/SIGMA/EHF2017.html

$\mathcal{C}_{ij}$ *are rational functions in* a*,* c *and* t *which do not depend on* n *and we have* $\mathbf{P}^{(n)} = \mathcal{C}\mathbf{m}^{(n)}$ *for all* $n \geq 1$ *(stability).* (1.1c)

This transition matrix $\mathcal{C}$ *is uniquely characterized by the* (a,c,t)*-deformed Catalan triangle type three term recursion relations*

$$\mathcal{C}_{0,0} = 1, \qquad \mathcal{C}_{i-1,i-1} = \mathcal{C}_{i,i}, \qquad i = 1,2,3,\dots, \tag{1.2a}$$

$$f(t)\mathcal{C}_{1,j-1} = \mathcal{C}_{0,j}, \qquad j = 2,4,6,\dots, \tag{1.2b}$$

$$\mathcal{C}_{i-1,j-1} + f\big(t^{i+1}\big)\mathcal{C}_{i+1,j-1} = \mathcal{C}_{i,j}, \qquad i+j \text{ even},\ 0 < i < j, \tag{1.2c}$$

where we have used the notation

$$f(s) = \frac{(1-1/s)\big(1-t^2/sa^2c^2\big)\big(1+t/sa^2\big)\big(1+t/sc^2\big)}{\big(1-t/s^2a^2c^2\big)\big(1-t^3/s^2a^2c^2\big)}. \tag{1.3}$$

A proof of this is presented in Section 2.5. The solution to the three term recursion relations (1.2a), (1.2b) and (1.2c) for $\mathcal{C}_{i,j}$ given in terms of the function $f(s)$ is presented in Proposition 7.3.

Consider the Macdonald polynomials of types (C_n, C_n) and (D_n, D_n) [14, 20, 22]

$$P^{(C_n,C_n)}_{(1^r)}(x\,|\,b;q,t) = P_{(1^r)}\big(x\,|\,b^{1/2}, -b^{1/2}, q^{1/2}b^{1/2}, -q^{1/2}b^{1/2}\,|\,q,t\big),$$

$$P^{(D_n,D_n)}_{(1^r)}(x\,|\,q,t) = P_{(1^r)}\big(x\,|\,1,-1,q^{1/2},-q^{1/2}\,|\,q,t\big).$$

Corollary 1.2. *When* $b = t = q$*, the Macdonald polynomials of type* (C_n, C_n) *become the Schur polynomials* $s_\lambda(x) = s^{(C_n)}_\lambda(x)$ *of type* C_n*. In this case we have* $f\big(t^{i+1}\big) = 1$ *for* $i \geq 0$*, indicating that the recursion relations* (1.2a)*–*(1.2c) *reduces to the ones for the ordinary Catalan triangle (or ballot) numbers. Therefore it holds that*

$$s^{(C_n)}_{(1^r)}(x) = P^{(C_n,C_n)}_{(1^r)}(x\,|\,q;q,q) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{n-r+1}{n-r+k+1}\binom{n-r+2k}{k} m_{(1^{r-2k})}(x), \tag{1.4}$$

where $\binom{m}{j} = \frac{m(m-1)\cdots(m-j+1)}{j!}$ *denotes the ordinary binomial coefficient.*

Corollary 1.3. *When* $b = 1$ *and* $t = q$*, the Macdonald polynomials of type* (C_n, C_n) *become the Schur polynomials* $s_\lambda(x) = s^{(D_n)}_\lambda(x)$ *of type* D_n*. (See Remark* 1.4 *below.) In this case we have* $f(t) = 2$ *and* $f\big(t^{i+1}\big) = 1$ *for* $i > 0$*, and the recursion relations* (1.2a)*–*(1.2c) *reduces to the ones for (the half of) the ordinary Pascal triangle. We have*

$$s^{(D_n)}_{(1^r)}(x) = P^{(D_n,D_n)}_{(1^r)}(x\,|\,q,q) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{n-r+2k}{k} m_{(1^{r-2k})}(x). \tag{1.5}$$

Remark 1.4. To be precise, when $\ell(\lambda) = n$, the polynomial $P^{(C_n,C_n)}_\lambda(x\,|\,1;q,t)$ (or m_λ) has to be further decomposed in terms of the type D_n Macdonald (or monomial) polynomials [20, 22], since the Weyl group is smaller than the one for C_n. Such a decomposition is easy but takes some space for a separate treatment. Therefore throughout in this paper, we do not go into the actual details, leaving this to the interested reader.

The first few terms of (1.4) and (1.5) read

$$\begin{pmatrix} s^{(C_n)}_{(1^n)} \\ s^{(C_n)}_{(1^{n-1})} \\ s^{(C_n)}_{(1^{n-2})} \\ s^{(C_n)}_{(1^{n-3})} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & 1 & & 2 & & 5 & & 14 & & \cdots \\ & 1 & & 2 & & 5 & & 14 & & 42 & \\ & & 1 & & 3 & & 9 & & 28 & & \cdots \\ & & & 1 & & 4 & & 14 & & 48 & \\ & & & & \ddots & & \ddots & & \ddots & & \end{pmatrix} \begin{pmatrix} m_{(1^n)} \\ m_{(1^{n-1})} \\ m_{(1^{n-2})} \\ m_{(1^{n-3})} \\ \vdots \end{pmatrix},$$

$$\begin{pmatrix} s^{(D_n)}_{(1^n)} \\ s^{(D_n)}_{(1^{n-1})} \\ s^{(D_n)}_{(1^{n-2})} \\ s^{(D_n)}_{(1^{n-3})} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 & 20 & 70 & \cdots \\ & 1 & 3 & 10 & 35 & 126 \\ & & 1 & 4 & 15 & 56 & \cdots \\ & & & 1 & 5 & 21 & 84 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} m_{(1^n)} \\ m_{(1^{n-1})} \\ m_{(1^{n-2})} \\ m_{(1^{n-3})} \\ \vdots \end{pmatrix}.$$

As an application of our results obtained in this paper, we calculate the transition matrix from the Schur polynomials to the Hall–Littlewood polynomials, namely the Kostka polynomials, associated with one column diagrams.

Definition 1.5. Let $K^{(C_n)}_{(1^r)(1^{r-2j})}(t)$ and $K^{(D_n)}_{(1^r)(1^{r-2j})}(t)$ be the transition coefficients defined by

$$s^{(C_n)}_{(1^r)}(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} K^{(C_n)}_{(1^r)(1^{r-2j})}(t) P^{(C_n,C_n)}_{(1^{r-2j})}(x \,|\, t; 0, t),$$
$$s^{(D_n)}_{(1^r)}(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} K^{(D_n)}_{(1^r)(1^{r-2j})}(t) P^{(D_n,D_n)}_{(1^{r-2j})}(x \,|\, 0, t).$$

Theorem 1.6. *The $K^{(C_n)}_{(1^r)(1^{r-2j})}(t)$ and $K^{(D_n)}_{(1^r)(1^{r-2j})}(t)$ are polynomials in t with nonnegative integral coefficients. We have*

$$K^{(C_n)}_{(1^r)(1^{r-2j})}(t) = t^{2j} \frac{[n-r+1]_{t^2}}{[n-r+j+1]_{t^2}} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} = \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} - \begin{bmatrix} n-r+2j \\ j-1 \end{bmatrix}_{t^2},$$
$$K^{(D_n)}_{(1^r)(1^{r-2j})}(t) = t^j \frac{1+t^{n-r}}{1+t^{n-r+2j}} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2}$$
$$= t^{n-r+j} \begin{bmatrix} n-r+2j-1 \\ j-1 \end{bmatrix}_{t^2} + t^j \begin{bmatrix} n-r+2j-1 \\ j \end{bmatrix}_{t^2}.$$

Here we have used the notation for the q-integer $[n]_q$, the q-factorial $[n]_q!$ and the q-binomial coefficient $\begin{bmatrix} m \\ j \end{bmatrix}_q$ as

$$[n]_q = \frac{1-q^n}{1-q}, \qquad [n]_q! = [1]_q[2]_q \cdots [n]_q, \qquad \begin{bmatrix} m \\ j \end{bmatrix}_q = \prod_{k=1}^{j} \frac{[m-k+1]_q}{[k]_q} = \frac{[m]_q!}{[j]_q![m-j]_q!}.$$

As for our proof of this, see Section 8.3.

Remark 1.7. Note that the $K^{(C_n)}(t)$'s are essentially given by the t^2-deformed ballot numbers [1] (the case $n = r$ corresponds to the t^2-deformation of the Catalan numbers [6]), and the $K^{(C_n)}(t)$'s by a version of the t-deformed binomial numbers.

First few entries of $K^{(C_n)}(t)$ read

$$\begin{pmatrix} 1 & t^2 & & t^4+t^8 & & t^6+t^{10}+t^{12}+t^{14}+t^{18} \\ & 1 & t^2+t^4 & & t^4+t^6+t^8+t^{10}+t^{12} & & \cdots \\ & & 1 & t^2+t^4+t^6 & & t^4+t^6+2t^8+t^{10}+2t^{12}+t^{14}+t^{16} \\ & & & 1 & t^2+t^4+t^6+t^8 & & \cdots \\ & & & \ddots & & \ddots \end{pmatrix},$$

and for $K^{(D_n)}(t)$ we have

$$\begin{pmatrix} 1 & 2t & & 2t^2+2t^4+2t^6 & & \cdots \\ & 1 & t+t^2+t^3 & & \begin{matrix} t^2+t^3+t^4+t^5+2t^6 \\ +t^7+t^8+t^9+t^{10} \end{matrix} & \\ & & 1 & t+2t^3+t^5 & & \cdots \\ & & & 1 & t+t^3+t^4+t^5+t^7 & \\ & & & \ddots & & \ddots \end{pmatrix}.$$

The present article is organized as follows. In Section 2, several transition formulas obtained in this paper are summarized for the convenience of reading. Then we present a proof of our main result Theorem 1.1. In Sections 3 and 4, we use Mimachi's kernel function identity to have a description of the Koornwinder polynomials and the Macdonald polynomials of type (C_n, C_n) with one column diagrams. Sections 5 and 6 form the core of the technical part of this article. In Section 5, Bressoud's matrix inversion is applied to invert the formula for the Macdonald polynomials of type (C_n, C_n) with one column diagrams. In Section 6, the four term relations for $B(s,j)$ and $\widetilde{B}(s,j)$ are derived. In Section 7 is given the basic properties for the transition matrix $\mathcal{C}$. In Section 8, we study some degenerate cases, including the calculation of the Kostka polynomials. Some conjectures are presented in Section 9, concerning the asymptotically free type eigenfunctions for type C_n when $b = t$. It is quite conceivable that Theorem 1.1 admits an elliptic generalization in terms of the BC_n abelian functions [18, 19], but we have not yet attempted to formulate such a generalization.

Throughout the paper, we use the standard notation (see [7])

$$(z;q)_\infty = \prod_{k=0}^{\infty}\left(1-q^k z\right), \qquad (z;q)_k = \frac{(z;q)_\infty}{(q^k z;q)_\infty}, \qquad k \in \mathbb{Z},$$

$$(a_1, a_2, \dots, a_r; q)_k = (a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k, \qquad k \in \mathbb{Z},$$

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, b_2, \dots, b_r; q)_n} z^n,$$

$${}_{r+1}W_r(a_1; a_4, a_5, \dots, a_{r+1}; q, z) = {}_{r+1}\phi_r\left[\begin{matrix} a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4, \dots, a_{r+1} \\ a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix}; q, z\right].$$

2 Collection of transition formulas and proof of Theorem 1.1

In this section, we collect several transformation formulas which we need to establish Theorem 1.1, giving brief explanations about our ideas and methods for their derivations.

2.1 Koornwinder polynomials $P_{(1^r)}(x \,|\, a, b, c, d \,|\, q, t)$ with one column diagrams

In [5], we studied some explicit formulas for the Koornwinder polynomials [10] with one-row diagrams. The results were interpreted as certain summations over the sets of tableaux of types C_n and D_n. While using the same technique as in [5], but replacing the Cauchy type kernel function by Mimachi's dual-Cauchy type one (as to the kernel functions, see [9, 15]), we can study an explicit formula for the Koornwinder polynomials with one column diagrams. Mimachi's kernel function [15] intertwines the action of the Koornwinder operator of type BC_n to the one for BC_1 (namely for the Askey–Wilson operator) which in turn acts on the Askey–Wilson eigenfunction. To perform explicit calculations based on this idea, as in the one-row diagram case, we need the fourfold summation formula for the Askey–Wilson eigenfunction [8]. The details will be given in Sections 3 and 4.

Specializing the parameters of the Koornwinder polynomials, we obtain the Macdonald polynomials of types C_n and D_n with one column diagram. In these particular limits, the fourfold summation (for the Askey–Wilson eigenfunction) reduces to a twofold one. In this way, we have explicit expressions for the Macdonald polynomials of types C_n and D_n with one column diagrams.

Let $n \in \mathbb{Z}_{>0}$ and $x = (x_1, \dots, x_n)$ be a sequence of variables. Let $P_{(1^r)}(x \,|\, a, b, c, d \,|\, q, t)$ be the Koornwinder polynomial with one column diagram (1^r), $r \in \mathbb{Z}_{\geq 0}$. (See Section 3, as to our notation.)

Definition 2.1. Define the symmetric Laurent polynomial $E_r(x)$'s by expanding the generating function $E(x \,|\, y)$ as

$$E(x \,|\, y) = \prod_{i=1}^{n}(1 - yx_i)(1 - y/x_i) = \sum_{r \geq 0}(-1)^r E_r(x) y^r.$$

Note that we have $E_{2n-r}(x) = E_r(x)$ for $0 \leq r \leq n$ and $E_r(x) = 0$ for $r > 2n$.

Theorem 2.2. *We have the following fourfold summation formula for the BC_n Koornwinder polynomial $P_{(1^r)}(x \,|\, a, b, c, d \,|\, q, t)$ with one column diagram*

$$P_{(1^r)}(x \,|\, a, b, c, d \,|\, q, t) = \sum_{k,l,i,j \geq 0} (-1)^{i+j} E_{r-2k-2l-i-j}(x) \widehat{c}\,'_e\big(k, l; t^{n-r+1+i+j}\big) \widehat{c}_o\big(i, j; t^{n-r+1}\big),$$

where

$$\widehat{c}\,'_e(k, l; s) = \frac{\big(tc^2/a^2; t^2\big)_k \big(sc^2t; t^2\big)_k \big(s^2c^4/t^2; t^2\big)_k}{\big(t^2; t^2\big)_k \big(sc^2/t; t^2\big)_k \big(s^2a^2c^2/t; t^2\big)_k} \frac{\big(1/c^2; t\big)_l (s/t; t)_{2k+l}}{(t; t)_l \big(sc^2; t\big)_{2k+l}} \frac{1 - st^{2k+2l-1}}{1 - st^{-1}} a^{2k} c^{2l},$$

$$\widehat{c}_o(i, j; s) = \frac{(-a/b; t)_i (scd/t; t)_i}{(t; t)_i (-sac/t; t)_i} \frac{(s; t)_{i+j} (-sac/t; t)_{i+j} \big(s^2a^2c^2/t^3; t\big)_{i+j}}{(s^2abcd/t^2; t)_{i+j} (sac/t^{3/2}; t)_{i+j} \big(-sac/t^{3/2}; t\big)_{i+j}}$$

$$\times \frac{(-c/d; t)_j (sab/t; t)_j}{(t; t)_j (-sac/t; t)_j} b^i d^j.$$

Corollary 2.3. *Degenerating Koornwinder's parameters as $(a, b, c, d) \to (a, -a, c, -c)$ we have*

$$\begin{aligned} P_{(1^r)}(x \,|\, a, -a, c, -c \,|\, q, t) = \sum_{\substack{k,l \geq 0 \\ 2k+2l \leq r}} & E_{r-2k-2l}(x) \frac{\big(1/c^2; t\big)_l (s/t; t)_{2k+l}}{(t; t)_l \big(sc^2; t\big)_{2k+l}} \frac{1 - st^{2k+2l-1}}{1 - st^{-1}} c^{2l} \\ & \times \frac{\big(tc^2/a^2; t^2\big)_k \big(sc^2t; t^2\big)_k \big(s^2c^4/t^2; t^2\big)_k}{\big(t^2; t^2\big)_k \big(sc^2/t; t^2\big)_k \big(s^2a^2c^2/t; t^2\big)_k} a^{2k}, \end{aligned} \tag{2.1}$$

where $s = t^{n-r+1}$.

The following formula (2.2) can be derived from (2.1) by applying the Bressoud matrix inversion technique [4, 12]. See Section 5, for details.

Theorem 2.4. *We have*

$$\begin{aligned} E_r(x) = \sum_{\substack{k,l \geq 0 \\ 2k+2l \leq r}} & P_{(1^{r-2l-2k})}(x \,|\, a, -a, c, -c \,|\, q, t) \frac{\big(c^2; t\big)_l}{(t; t)_l} \frac{\big(st^l; t\big)_{l+2k}}{\big(st^{l-1}c^2; t\big)_{l+2k}} \\ & \times \frac{\big(a^2/tc^2; t^2\big)_k}{\big(t^2; t^2\big)_k} \frac{\big(s^2t^{4l-2}c^4; t^2\big)_k}{\big(s^2t^{4l}c^4; t^2\big)_k} \frac{\big(s^2t^{4l+2k-2}c^4; t^2\big)_k}{\big(s^2t^{4l+2k-3}a^2c^2; t^2\big)_k} \big(tc^2\big)^k, \end{aligned} \tag{2.2}$$

where $s = t^{n-r+1}$.

This is proved in (5.4b) of Theorem 5.5.

2.2 The coefficients $B(s,j)$ and $\widetilde{B}(s,j)$

By using the q-analogue of Bailey's transformation [7, p. 99, equation (3.10.14)], one can rewrite the twofold summations in (2.1) and (2.2) as sums having certain ${}_4\phi_3$ series as their coefficients.

Definition 2.5. Let $B(s,j)$ and $\widetilde{B}(s,j)$ be the rational functions in s defined by

$$B(s,j)=(-1)^j s^{-j}\frac{\left(s^2/t^2;t^2\right)_j}{\left(t^2;t^2\right)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}{}_4\phi_3\left[\begin{matrix}-sa^2,-sc^2,s^2t^{2j-2},t^{-2j}\\ -s,-st,s^2a^2c^2/t\end{matrix};t^2,t^2\right],$$

$$\widetilde{B}(s,j)=\left(st^{j-1}\right)^{-j}\frac{\left(t^{2j}s^2;t^2\right)_j}{\left(t^2;t^2\right)_j}{}_4\phi_3\left[\begin{matrix}-t^{-2j+2}/sa^2,-t^{-2j+2}/sc^2,t^{-2j+2}/s^2,t^{-2j}\\ -t^{-2j+1}/s,-t^{-2j+2}/s,t^{-4j+5}/s^2a^2c^2\end{matrix};t^2,t^2\right].$$

Theorem 2.6. *The formulas* (2.1) *and* (2.2) *can be recast as* (*see Theorem* 5.7)

$$P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}B\left(t^{n-r+1},j\right)E_{r-2j}(x), \tag{2.3a}$$

$$E_r(x)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\widetilde{B}\left(t^{n-r+1},j\right)P_{(1^{r-2j})}(x\,|\,a,-a,c,-c\,|\,q,t). \tag{2.3b}$$

Definition 2.7. Let $f(s)$ be the function defined in (1.3). For ease of notation, define

$$F(s,l)=f\left(s/t^l\right)=\frac{\left(1-t^l/s\right)\left(1-t^{l+2}/sa^2c^2\right)\left(1+t^{l+1}/sa^2\right)\left(1+t^{l+1}/sc^2\right)}{\left(1-t^{2l+1}/s^2a^2c^2\right)\left(1-t^{2l+3}/s^2a^2c^2\right)}.$$

We summarize the basic properties for the functions $B(s,i)$'s and $\widetilde{B}(s,i)$'s. See Theorem 6.1 and Proposition 6.2.

Theorem 2.8. *We have the four term relations*

$$B(s,i)+F(s,-1)B\left(st^2,i-1\right)=B(st,i)+B(st,i-1),$$
$$\widetilde{B}(s,i)+F(s,2-2i)\widetilde{B}(s,i-1)=\widetilde{B}\left(st^{-1},i\right)+\widetilde{B}(st,i-1),$$

and the inversion relations

$$\sum_{k=0}^{i}B(s,k)\widetilde{B}\left(st^{2k},i-k\right)=\delta_{i,0},\qquad \sum_{k=0}^{i}\widetilde{B}(s,k)B\left(st^{2k},i-k\right)=\delta_{i,0}.$$

2.3 Transition matrix from $E_r(x)$ to the BC_n interpolation polynomial $E_r(x;a|t)$

Let $\langle z;w\rangle=z+z^{-1}-w-w^{-1}$.

Definition 2.9 ([9, equation (5.1)])**.** Set

$$E_r(x;a\,|\,t)=\sum_{1\le i_1<\cdots<i_r\le n}\left\langle x_{i_1};t^{i_1-1}a\right\rangle\left\langle x_{i_2};t^{i_2-2}a\right\rangle\cdots\left\langle x_{i_r};t^{i_r-r}a\right\rangle.$$

These Laurent polynomials $E_r(x;a\,|\,t)$, $r=0,1,\dots,n$, are essentially the BC_n interpolation polynomials of Okounkov [17] attached to single columns (1^r).

Theorem 2.10 ([9, Theorem 5.1])**.** *Let $s = t^{n-r+1}$. We have*

$$P_{(1^r)}(x \,|\, a, b, c, d \,|\, q, t) = \sum_{l=0}^{r} \frac{(s, sab/t, sac/t, sad/t; t)_l}{t^{l(l-1)/2}(as/t)^l \big(t, s^2abcd/t^2; t\big)_l} E_{r-l}(x; a|t).$$

Theorem 2.11. *We have*

$$E_r(x; a|t) = \sum_{l=0}^{r} (-1)^l \frac{(s, sab/t, sac/t, sad/t; t)_l}{(as/t)^l \big(t, t^{l-3}s^2abcd; t\big)_l} P_{(1^{r-l})}(x|a, b, c, d|q, t).$$

A proof of this is given by using Krattenthaler's matrix inversion as follows. An infinite-dimensional matrix $(f_{ij})_{i,j\in\mathbb{Z}_{\geq 0}}$ is said to be lower-triangular if $f_{ij} = 0$ unless $i \geq j$. Two infinite-dimensional lower-triangular matrices $(f_{ij})_{i,j\in\mathbb{Z}_{\geq 0}}$ and $(g_{ij})_{i,j\in\mathbb{Z}_{\geq 0}}$ are said to be mutually inverse if $\sum_{i\geq j\geq k} f_{ij}g_{jk} = \delta_{i,k}$.

Theorem 2.12 ([11])**.** *Let $\mathcal{N}(x, y; q)$ be the infinite lower-triangle matrix with entries*

$$\mathcal{N}_{i,j}(x, y; q) = y^{i-j} \frac{(x/y; q)_{i-j}}{(q; q)_{i-j}} \frac{1}{\big(xq^{i+j}; q\big)_{i-j} \big(yq^{2j+1}; q\big)_{i-j}},$$

Then $\mathcal{N}(x, y; q)$ and $\mathcal{N}(y, x; q)$ are mutually inverse.

If two infinite matrices (f_{ij}) and (g_{ij}) are mutually inverse, then the conjugated ones $(f_{ij}d_i/d_j)$ and $(g_{ij}d_i/d_j)$ are also mutually inverse for any sequence (d_r) with nonzero entries.

Definition 2.13. Let $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$. Set

$$g(\boldsymbol{u}, v)_r = v^{-r}\big(u_1t^{-r}; t\big)_r (u_2; t)_r (u_3; t)_r (u_4; t)_r.$$

Note that we have

$$g(\boldsymbol{u}, v)_r / g(\boldsymbol{u}, v)_{r-i} = v^{-i}\big(u_1t^{-r}; t\big)_i \big(u_2t^{r-i}; t\big)_i \big(u_3t^{r-i}; t\big)_i \big(u_4t^{r-i}; t\big)_i.$$

Let $\widetilde{\mathcal{N}}(\boldsymbol{u}, v, x, y; t)$ be the conjugation of the matrix $\mathcal{N}(x, yv; t)$ by the sequence $(g(\boldsymbol{u}, v)_r)$ with entries given by

$$\begin{aligned}\widetilde{\mathcal{N}}_{r,r-i}(\boldsymbol{u}, v, x, y; t) &= \mathcal{N}_{r,r-i}(x, yv; t) \times g(\boldsymbol{u}, v)_r / g(\boldsymbol{u}, v)_{r-i} \\ &= y^i \frac{(x/yv; t)_i}{(t; t)_i} \frac{\big(u_1t^{-r}; t\big)_i \big(u_2t^{r-i}; t\big)_i \big(u_3t^{r-i}; t\big)_i \big(u_4t^{r-i}; t\big)_i}{\big(xt^{2r-i}; t\big)_i \big(yvt^{2r-2i+1}; t\big)_i}.\end{aligned}$$

Proof of Theorem 2.11. It follows Krattenthaler's matrix inversion and

$$\begin{aligned}&\widetilde{\mathcal{N}}_{r,r-l}\big(t^{n-1}, t^{-n+1}/ab, t^{-n+1}/ac, t^{-n+1}/ad, t^{-2n+2}/a^2bcd, a/t, 0; t\big) \\ &\quad = \frac{(s, sab/t, sac/t, sad/t; t)_l}{t^{l(l-1)/2}(as/t)^l \big(t, s^2abcd/t^2; t\big)_l}, \\ &\widetilde{\mathcal{N}}_{r,r-l}\big(t^{n-1}, t^{-n+1}/ab, t^{-n+1}/ac, t^{-n+1}/ad, t^{-2n+2}/a^2bcd, 0, a/t; t\big) \\ &\quad = (-1)^l \frac{(s, sab/t, sac/t, sad/t; t)_l}{(as/t)^l \big(t, t^{l-3}s^2abcd; t\big)_l}.\end{aligned}$$

∎

Theorem 2.14. *We have*

$$E_r(x;a|t)=\sum_{m=0}^{r}(-1)^m e\big(t^{n-r+1},m\big)E_{r-m}(x),$$

where

$$e(s,m)=(t/as)^m\frac{(s;t)_m\big({-sa^2t^{-m}};t^2\big)_m}{(t;t)_m}\\ \times {}_4\phi_3\left[\begin{matrix}t^{-m},t^{-m+1},-t^{-m+1}/s,-t^{-m+2}/s\\ -t^{-m+2}/a^2s,-t^{-m}a^2s,t^{-2m+4}/s^2\end{matrix};t^2,t^2\right]. \qquad (2.4)$$

Proof. For simplicity of display, we write

$$b(s,i):=(-1)^i s^{-i}\frac{\big(s^2;t^2\big)_i}{(t^2;t^2)_i}\frac{1-s^2t^{4i-2}}{1-s^2t^{2i-2}}, \qquad (2.5)$$

$${}_4\phi_3(s,i,k):=\frac{\big({-sa^2},-sc^2,s^2t^{2i-2},t^{-2i};t^2\big)_k}{\big(t^2,-s,-st,s^2a^2c^2t^{-1};t^2\big)_k}. \qquad (2.6)$$

Then we have $B(s,i)=b(s,i)\sum_{k=0}^{i}{}_4\phi_3(s,i,k)$. From Theorems 2.6 and 2.9 written for the case $b=-a$ and $d=-c$, we have

$$E_r(x;a\,|\,t)=\sum_{l=0}^{r}(-1)^l\frac{\big(s,-sa^2/t,sac/t,-sac/t;t\big)_l}{(as/t)^l\big(t,t^{l-3}s^2a^2c^2;t\big)_l}\sum_{j=0}^{\lfloor\frac{r-l}{2}\rfloor}B\big(st^l,j\big)E_{r-l-2j}(x)\\ =\sum_{m=0}^{r}(-1)^m e(s,m)E_{r-m}(x),$$

where

$$e(s,m)=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\frac{\big(s,-sa^2/t,sac/t,-sac/t;t\big)_{m-2j}}{(as/t)^{m-2j}\big(t,t^{m-2j-3}s^2a^2c^2;t\big)_{m-2j}}B\big(st^{m-2j},j\big).$$

Changing the order of the summation, we have

$$e(s,m)=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\frac{\big(s,-sa^2/t,sac/t,-sac/t;t\big)_{m-2j}}{(as/t)^{m-2j}\big(t,t^{m-2j-3}s^2a^2c^2;t\big)_{m-2j}}b\big(st^{m-2j},j\big)\sum_{k=0}^{j}{}_4\phi_3\big(st^{m-2j},j,k\big)\\ =\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{k=0}^{\lfloor\frac{m-2i}{2}\rfloor}\frac{\big(s,-sa^2/t,sac/t,-sac/t;t\big)_{m-2i-2k}}{(as/t)^{m-2i-2k}\big(t,t^{m-2i-2k-3}s^2a^2c^2;t\big)_{m-2i-2k}}\\ \times b\big(st^{m-2i-2k},i+k\big){}_4\phi_3\big(st^{m-2i-2k},i+k,k\big). \qquad (2.7)$$

Simplifying the expression, we have

$$\sum_{k=0}^{\lfloor\frac{m-2i}{2}\rfloor}\frac{\big(s,-sa^2/t,sac/t,-sac/t;t\big)_{m-2i-2k}}{(as/t)^{m-2i-2k}\big(t,t^{m-2i-2k-3}s^2a^2c^2;t\big)_{m-2i-2k}}\\ \times b\big(st^{m-2i-2k},i+k\big){}_4\phi_3\big(st^{m-2i-2k},i+k,k\big)$$

$$= \frac{\big(s, -sa^2/t, sac/t, -sac/t; t\big)_{m-2i}}{(as/t)^{m-2i}\big(t, t^{m-2i-3}s^2a^2c^2; t\big)_{m-2i}} b\big(st^{m-2i}, i\big) \tag{2.8}$$
$$\times\, {}_6W_5\big(t^{-2m+4i+3}/s^2a^2c^2; -t^{-m+2i+2}/sc^2, t^{-m+2i}, t^{-m+2i+1}; t^2, -t^{m-2i+2}/a^2s\big).$$

By using [7, p. 42, equation (2.4.2)], we have the factorized expression for this ${}_6W_5$-series as

$$\frac{\big(s^2a^2c^2t^{m-2i-3}; t\big)_{m-2i}}{\big(s^2a^2c^2/t^2; t^2\big)_{m-2i}} \frac{\big(-sa^2t^{-m+2i}; t^2\big)_{m-2i}}{\big(-sa^2/t; t\big)_{m-2i}}. \tag{2.9}$$

Then simplifying the factors again, we have (2.4) from (2.7), (2.8) and (2.9). ■

2.4 The coefficients $C(s, j)$ and Catalan triangle three term relations

We have (in Lemma 3.3 below) the expansion of $E_r(x)$ in terms of the monomial symmetric polynomials as

$$E_r(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \binom{n-r+2j}{j} m_{(1^{r-2j})}(x), \tag{2.10}$$

where $\binom{m}{j}$ denotes the ordinary binomial coefficient. In view of this, we are naturally led to the following definition.

Definition 2.15. Let $s = t^{m+1}$ for $m \in \mathbb{C}$ and $C(s, j)$ be the function of s defined by

$$C(s, j) := \sum_{i=0}^{j} B(s, i) \binom{m+2j}{j-i}.$$

Theorem 2.16. *The Koornwinder polynomial $P_{(1^r)}(x \,|\, a, -a, c, -c \,|\, q, t)$ is expanded in terms of the monomial symmetric polynomials as*

$$P_{(1^r)}(x \,|\, a, -a, c, -c \,|\, q, t) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} C\big(t^{n-r+1}, j\big) m_{(1^{r-2j})}(x).$$

Proof. It follows from (2.3a) and (2.10). ■

Theorem 2.17. *We have the three term relation (see Proposition 7.1)*

$$C(s, j) + F(s, -1) C\big(st^2, j-1\big) = C(st, j), \tag{2.11}$$

and the specialization formula for $s = 1$, i.e., for $m = -1$ (see Proposition 7.2)

$$C(1, j) = \delta_{j,0}. \tag{2.12}$$

Definition 2.18. Define the infinite upper triangular matrix $\mathcal{C} = (\mathcal{C}_{ij})_{i,j \in \mathbb{Z}_{\geq 0}}$ by setting for $r, i \geq 0$

$$\mathcal{C}_{r,r+2i} = C\big(t^{r+1}, i\big), \qquad \mathcal{C}_{r,r+2i+1} = 0.$$

Theorem 2.19. *The $\mathcal{C}_{ij}$'s satisfy the recursion relations in Theorem 1.1*

$$\mathcal{C}_{0,0} = 1, \qquad \mathcal{C}_{i-1,i-1} = \mathcal{C}_{i,i}, \qquad i = 1, 2, 3, \dots, \tag{2.13a}$$
$$F(1, -1)\mathcal{C}_{1,j-1} = \mathcal{C}_{0,j}, \qquad j = 2, 4, 6, \dots, \tag{2.13b}$$
$$\mathcal{C}_{i-1,j-1} + F\big(t^i, -1\big)\mathcal{C}_{i+1,j-1} = \mathcal{C}_{i,j}, \qquad i + j \text{ even}, \quad 0 < i < j. \tag{2.13c}$$

Proof. We have $\mathcal{C}_{0,0}=1$. When $i+j$ is even and $0\le i\le j$, we have $\mathcal{C}_{ij}=C\big(t^{i+1},(j-i)/2\big)$. Therefore from (2.11) we have

$$\begin{aligned}\mathcal{C}_{i-1,j-1}+F\big(t^i,-1\big)\mathcal{C}_{i+1,j-1}&=C\big(t^i,(j-i)/2\big)+F\big(t^i,-1\big)C\big(t^{i+2},(j-i)/2-1\big)\\&=C\big(t^{i+1},(j-i)/2\big)=\mathcal{C}_{ij},\end{aligned}\tag{2.14}$$

giving the three term recursion relation (2.13c) for $0<i<j$ in (2.14). When $0<i=j$, noting that $\mathcal{C}_{i+1,i-1}=0$ from the upper triangularity, we have (2.13a). When $i=0$ and $j\in 2\mathbb{Z}_{>0}$, we have from (2.12) that $\mathcal{C}_{-1,j-1}=C(1,(j-2)/2)=0$, hence (2.13b) holds. ∎

2.5 Proof of the main theorem

Now we are ready to present a proof of our main theorem.

Proof of Theorem 1.1. The transition matrix $\mathcal{C}$ is even and upper triangular. In view of Theorem 2.16 and $C\big(t^{n-r+1},j\big)=\mathcal{C}_{n-r,n-r+2j}$, we have for any $n>0$ and $0\le r\le n$

$$P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\mathcal{C}_{n-r,n-r+2j}m_{(1^{r-2j})}(x),$$

indicating the stabilized transition formula (1.1c). The three term recursion relation (1.2a), (1.2b) and (1.2c) are shown in Theorem 2.17. ∎

3 Koornwinder's q-difference operator, Koornwinder polynomials and Mimachi's kernel function

We briefly recall some basic properties concerning the Koornwinder polynomials [10] and the Mimachi's kernel function identity [15].

3.1 Koornwinder's operator and Mimachi's kernel function

Let a, b, c, d, q, t be complex parameters. We assume that $|q|<1$. Set $\alpha=(abcd/q)^{1/2}$ for simplicity. Let $x=(x_1,\dots,x_n)$ be a sequence of independent indeterminates. The Weyl group of type BC_n is denoted by $W_n(\simeq\mathbb{Z}_2^n\rtimes\mathfrak{S}_n)$. Let $\mathbb{C}\big[x_1^\pm,x_2^\pm,\dots,x_n^\pm\big]^{W_n}$ be the ring of W_n-invariant Laurent polynomials in x. For a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ of length n, i.e., $\lambda_i\in\mathbb{Z}_{\ge0}$ and $\lambda_1\ge\cdots\ge\lambda_n$, we denote by $m_\lambda=m_\lambda(x)$ the monomial symmetric polynomial being defined as the orbit sums of monomials

$$m_\lambda=\frac{1}{|\operatorname{Stab}(\lambda)|}\sum_{\mu\in W_n\cdot\lambda}\prod_i x_i^{\mu_i},$$

where $\operatorname{Stab}(\lambda)=\{s\in W_n\,|\,s\lambda=\lambda\}$.

Koornwinder's q-difference operator $\mathcal{D}_x=\mathcal{D}_x(a,b,c,d\,|\,q,t)$ [10] reads

$$\begin{aligned}\mathcal{D}_x={}&\sum_{i=1}^n\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{\alpha t^{n-1}\big(1-x_i^2\big)\big(1-qx_i^2\big)}\prod_{j\ne i}\frac{(1-tx_ix_j)(1-tx_i/x_j)}{(1-x_ix_j)(1-x_i/x_j)}\big(T_{q,x_i}^{+1}-1\big)\\&+\sum_{i=1}^n\frac{(1-a/x_i)(1-b/x_i)(1-c/x_i)(1-d/x_i)}{\alpha t^{n-1}\big(1-1/x_i^2\big)\big(1-q/x_i^2\big)}\prod_{j\ne i}\frac{(1-tx_j/x_i)(1-t/x_ix_j)}{(1-x_j/x_i)(1-1/x_ix_j)}\big(T_{q,x_i}^{-1}-1\big),\end{aligned}$$

where we have used the notation $T_{q,x}^{\pm1}f(x_1,\dots,x_i,\dots,x_n)=f\big(x_1,\dots,q^{\pm1}x_i,\dots,x_n\big)$.

The Koornwinder polynomial $P_\lambda(x) = P_\lambda(x\,|\,a,b,c,d\,|\,q,t) \in \mathbb{C}\big[x_1^{\pm 1},\dots,x_n^{\pm 1}\big]^{W_n}$ is uniquely characterized by the conditions

(*a*) $P_\lambda(x) = m_\lambda(x) + \text{lower order terms w.r.t. the dominance ordering}$,

(*b*) $\mathcal{D}_x P_\lambda = d_\lambda P_\lambda$.

The eigenvalue d_λ is explicitly written as

$$d_\lambda = \sum_{j=1}^{n} \big\langle abcdq^{-1}t^{2n-2j}q^{\lambda_j}\big\rangle\big\langle q^{\lambda_j}\big\rangle = \sum_{j=1}^{n} \big\langle \alpha t^{n-j}q^{\lambda_j};\alpha t^{n-j}\big\rangle,$$

where we used the notations $\langle x\rangle = x^{1/2} - x^{-1/2}$ and $\langle x;y\rangle = \langle xy\rangle\langle x/y\rangle = x + x^{-1} - y - y^{-1}$ for simplicity of display.

Definition 3.1. Define the involution $\widetilde{*}$ of the parameters by

$$\widetilde{a} = a, \qquad \widetilde{b} = b, \qquad \widetilde{c} = c, \qquad \widetilde{d} = d, \qquad \widetilde{q} = t, \qquad \widetilde{t} = q.$$

We write $\widetilde{\mathcal{D}}_x = \mathcal{D}_x(a,b,c,d\,|\,t,q)$ and $\widetilde{P}_\lambda(x) = P_\lambda(x\,|\,a,b,c,d\,|\,t,q)$ for short.

Theorem 3.2 ([15, Lemma 3.2])**.** *Let n and m be positive integers, and let $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_m)$ be two sets of independent indeterminates. Mimachi's kernel function*

$$\Psi(x;y) = (y_1y_2\cdots y_m)^{-n}\prod_{i=1}^{n}\prod_{j=1}^{m}(1-y_jx_i)(1-y_j/x_i),$$

enjoys the kernel function identity

$$\langle t\rangle\mathcal{D}_x\Psi(x;y) + \langle q\rangle\widetilde{\mathcal{D}}_y\Psi(x;y) = \big\langle t^n\big\rangle\big\langle q^m\big\rangle\big\langle abcdt^{n-1}q^{m-1}\big\rangle\Psi(x;y).$$

When we apply Mimachi's kernel function, the following lemmas will be used. Recall that the generating function $E(x|y)$ is introduced in Definition 2.1

$$E(x\,|\,y) = \prod_{i=1}^{n}(1-yx_i)(1-y/x_i) = \sum_{r\geq 0}(-1)^r E_r(x)y^r.$$

Lemma 3.3. *We have*

$$E_r(x) = \sum_{k=0}^{\lfloor \frac{r}{2}\rfloor}\binom{n-r+2k}{k} m_{(1^{r-2k})}(x),$$

where $\binom{m}{j}$ denotes the ordinary binomial coefficient.

Proof. For an integer s satisfying $0 \leq s \leq n$, we can find that the coefficient of the monomial $x_1x_2\cdots x_s$ in $E(x\,|\,y) = \prod_{i=1}^{n}\big(1-(x_i+1/x_i)y+y^2\big)$ is $(-1)^s y^s\big(1+y^2\big)^{n-s}$. Hence we have

$$\begin{aligned} E(x\,|\,y) &= \sum_{s=0}^{n}\sum_{k=0}^{n-s} m_{(1^s)}(x)(-1)^s y^{s+2k}\binom{n-s}{k} \\ &= \sum_{r=0}^{n}(-1)^r y^r \sum_{k=0}^{\lfloor \frac{r}{2}\rfloor}\binom{n-r+2k}{k} m_{(1^{r-2k})}(x). \end{aligned}$$ ■

Lemma 3.4. *Let $\lambda = (\lambda_1,\dots,\lambda_m)$ be a partition satisfying the condition $\lambda_1 \leq n$. We have*

$$\prod_{i=1}^{m} E_{\lambda_i}(x) = m_{\lambda'}(x) + \text{ lower order terms.} \tag{3.1}$$

Proof. Note that for any partitions λ and μ, we have $m_\lambda m_\mu = m_{\lambda+\mu} + \text{lower order terms}$. By using Lemma 3.3, we have $E_r(x) = m_{(1^r)} + \text{lower order terms}$. Hence we have (3.1). ■

3.2 Asymptotically free eigenfunction $f(x;s)$ for $\mathcal{D}_x$ and reproduction formula

Let $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ and $s=(s_1,\dots,s_n)$ be a sequence of complex parameters as $s_i=t^{-n+i}q^{-\lambda_i}$, $i=1,\dots,n$. We use the shorthand notations $x^\lambda=\prod_i x_i^{\lambda_i}$ and $x^{-\lambda}=\prod_i x_i^{-\lambda_i}$. Let $f(x;s)\in x^{-\lambda}\mathbb{C}[[x_1/x_2,\dots,x_{n-1}/x_n,x_n]]$ be the infinite series satisfying the conditions

$$f(x;s)=x^{-\lambda}\sum_{\beta\in Q^+}c_\beta(s)x^\beta,\qquad c_0(s)=1,$$

$$\mathcal{D}_xf(x;s)=\sum_{i=1}^n\langle\alpha s_i^{-1};\alpha t^{n-i}\rangle f(x;s),$$

where Q^+ denotes the positive cone of the root lattice of type BC_n. To be more explicit, corresponding to the simple roots $\alpha_1=\varepsilon_1-\varepsilon_2,\alpha_2=\varepsilon_2-\varepsilon_2,\dots,\alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\alpha_n=\varepsilon_n$, we have $x^{\alpha_1}=x_1/x_2,\dots,x^{\alpha_{n-1}}=x_{n-1}/x_n,x^{\alpha_n}=x_n$. Assuming the genericity of the eigenvalue, one can show that $f(x;s)$ is determined uniquely.

Definition 3.5. The adjoint $\mathcal{D}_x^*$ of $\mathcal{D}_x$ is defined to be

$$\mathcal{D}_x^*=\sum_{i=1}^n\left(T_{q,x_i}^{-1}-1\right)\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{\alpha t^{n-1}\left(1-x_i^2\right)\left(1-qx_i^2\right)}\prod_{j\neq i}\frac{(1-tx_ix_j)(1-tx_i/x_j)}{(1-x_ix_j)(1-x_i/x_j)}$$
$$+\sum_{i=1}^n\left(T_{q,x_i}^{+1}-1\right)\frac{(1-a/x_i)(1-b/x_i)(1-c/x_i)(1-d/x_i)}{\alpha t^{n-1}\left(1-1/x_i^2\right)\left(1-q/x_i^2\right)}\prod_{j\neq i}\frac{(1-tx_j/x_i)(1-t/x_ix_j)}{(1-x_j/x_i)(1-1/x_ix_j)}.$$

Definition 3.6. Denote by $V(x)$ the Weyl denominator of type C_n

$$V(x)=\prod_{k=1}^n x_k^{-n+k-1}\prod_{i=1}^n\left(1-x_i^2\right)\prod_{1\le i<j\le n}(1-x_ix_j)(1-x_i/x_j).$$

Definition 3.7. Define the involution $\overline{*}$ of the parameters by

$$\overline{a}=q/a,\qquad\overline{b}=q/b,\qquad\overline{c}=q/c,\qquad\overline{d}=q/d,\qquad\overline{q}=q,\qquad\overline{t}=q/t.$$

Write for simplicity the composition of the two involutions $\widetilde{*}$ and $\overline{*}$ as $\widehat{*}=\overline{\widetilde{*}}$. Note that $\widehat{*}$ is not an involution but has order 6

$$\widehat{a}=t/a,\qquad\widehat{b}=t/b,\qquad\widehat{c}=t/c,\qquad\widehat{d}=t/d,\qquad\widehat{q}=t,\qquad\widehat{t}=t/q.$$

Proposition 3.8 ([8, Proposition 6.2])**.** *We have*

$$V(x)^{-1}\mathcal{D}_x^*V(x)-\overline{\mathcal{D}}_x=\sum_{j=1}^n\langle\overline{\alpha}\overline{t}^{n-j};\alpha t^{n-j}\rangle.$$

Theorem 3.9. *Let $n\ge m$ be positive integers, and $x=(x_1,\dots,x_n),y=(y_1,\dots,y_m)$ be sequences of independent indeterminates. Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition satisfying $\ell(\lambda)\le m$ and $\lambda_1\le n$. Set*

$$s_i=\widehat{t}^{\,-m+i}\,\widehat{q}^{\,-\lambda_{m+1-i}+m+1-i+n},\qquad 1\le i\le m.\tag{3.2}$$

Let $\widehat{f}(y;s)$ be the formal series in y uniquely characterized by $\widehat{c}_0(s)=1$ and

$$\widehat{f}(y;s)=\prod_{i=1}^m y_i^{-\lambda_{m+1-i}+m+1-i+n}\sum_{\beta\in Q^+}\widehat{c}_\beta(s)y^\beta,$$

$$\widehat{\mathcal{D}}_y \widehat{f}(y;s) = \sum_{i=1}^{m} \langle \widehat{\alpha} s_i^{-1}; \widehat{\alpha}\widehat{t}^{m-i} \rangle \widehat{f}(y;s).$$

Then we have

$$P_{\lambda'}(x \,|\, a,b,c,d \,|\, q,t) = (-1)^{|\lambda|} \big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y}, \tag{3.3}$$

where the notation $[\cdots]_{1,y}$ *denotes the constant term in* y*, and* λ' *is the conjugate diagram of* λ*.*

Proof. Firstly, we show that the product $\Psi(x;y)V(y)\widehat{f}(y;s)$ has a non-vanishing constant term in y. Write

$$\prod_{i=1}^{m} \big(1-y_i^2\big) \cdot \prod_{1\le i<j\le m} (1-y_iy_j)(1-y_i/y_j) \cdot \sum_{\beta\in Q^+} \widehat{c}_\beta(s) y^\beta = \sum_{\beta\in Q^+} \widehat{c}'_\beta(s) y^\beta,$$

for short. Noting that we have $\ell(\lambda') \le n$ from the assumption $\lambda_1 \le n$, we have

$$\begin{aligned}
\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y} &= \left[\prod_{i=1}^{m} y_i^{-\lambda_{m+1-i}} \prod_{i=1}^{m} E(x\,|\,y_i) \cdot \sum_{\beta\in Q^+} \widehat{c}'_\beta(s) y^\beta\right]_{1,y} \\
&= (-1)^{|\lambda|} \sum_{\beta=\sum k_i\alpha_i \in Q^+} (-1)^{\sum k_i} \widehat{c}'_\beta(s) E_{\lambda_m - k_1}(x) \prod_{i=2}^{m} E_{\lambda_{m+1-i}+k_{i-1}-k_i}(x) \\
&= (-1)^{|\lambda|} m_{\lambda'}(x) + \text{lower order terms} \neq 0.
\end{aligned}$$

In the last step, we have used Lemma 3.4.

Next, we can show that the constant term satisfies the eigenvalue equation as

$$\begin{aligned}
&\left(\mathcal{D}_x - \frac{\langle t^n\rangle\langle q^m\rangle\langle abcdt^{n-1}q^{m-1}\rangle}{\langle t\rangle}\right) \big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y} \\
&\quad= \left[\left(-\frac{\langle q\rangle}{\langle t\rangle}\widetilde{\mathcal{D}}_y\Psi(x;y)\right) V(y)\widehat{f}(y;s)\right]_{1,y} = -\frac{\langle q\rangle}{\langle t\rangle}\big[\Psi(x;y)\big(\widetilde{\mathcal{D}}^*_y V(y)\widehat{f}(y;s)\big)\big]_{1,y} \\
&\quad= -\frac{\langle q\rangle}{\langle t\rangle}\left[\Psi(x;y)V(y)\left(\left(\widehat{\mathcal{D}}_y + \sum_{i=1}^{m}\langle \widehat{\alpha}\widehat{t}^{m-i}; \widetilde{\alpha}\widetilde{t}^{m-i}\rangle\right)\widehat{f}(y;s)\right)\right]_{1,y} \\
&\quad= -\frac{\langle q\rangle}{\langle t\rangle}\left(\sum_{i=1}^{m}\langle \widehat{\alpha}s_i^{-1}; \widehat{\alpha}\widehat{t}^{m-i}\rangle + \langle \widehat{\alpha}\widehat{t}^{m-i}; \widetilde{\alpha}\widetilde{t}^{m-i}\rangle\right)\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y} \\
&\quad= -\frac{\langle q\rangle}{\langle t\rangle}\left(\sum_{i=1}^{m}\langle \widehat{\alpha}s_i^{-1}; \widetilde{\alpha}\widetilde{t}^{m-i}\rangle\right)\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y}.
\end{aligned}$$

Here we have used Theorem 3.2, Proposition 3.8, and the property $\langle x;y\rangle + \langle y;z\rangle = \langle x;z\rangle$.

To check that the eigenvalue is the desired one, we prepare some lemmas.

Lemma 3.10. *The eigenvalue of the Koornwinder polynomial* $P_{\lambda'}(x)$ *can be recast as*

$$\begin{aligned}
\sum_{l=1}^{m}\sum_{i=\lambda_{l+1}+1}^{\lambda_l} \big\langle \alpha t^{n-i}q^l; \alpha t^{n-i}\big\rangle &= \sum_{l=1}^{m}\frac{1}{\langle t\rangle}\big\langle q^l\big\rangle\big\langle t^{\lambda_l-\lambda_{l+1}}\big\rangle\big\langle \alpha^2 q^l t^{2n-1-\lambda_l-\lambda_{l+1}}\big\rangle \\
&= \frac{\langle q^m\rangle}{\langle t\rangle}\big(\alpha q^{m/2}t^{-1/2+n} + \alpha^{-1}q^{-m/2}t^{1/2-n}\big) \\
&\quad - \frac{\langle q\rangle}{\langle t\rangle}\sum_{l=1}^{m}\big(\alpha q^{l-1/2}t^{-1/2+n-\lambda_l} + \alpha^{-1}q^{-l+1/2}t^{1/2-n+\lambda_l}\big).
\end{aligned}$$

Lemma 3.11. *We have*

$$\frac{1}{\langle t\rangle}\langle t^n\rangle\langle q^m\rangle\langle abcdt^{n-1}q^{m-1}\rangle+\sum_{l=1}^{m}\frac{\langle q\rangle}{\langle t\rangle}\big(\alpha q^{1/2+m-l}t^{-1/2}+\alpha^{-1}q^{-1/2-m+l}t^{1/2}\big)$$
$$=\frac{\langle q^m\rangle}{\langle t\rangle}\big(\alpha q^{m/2}t^{-1/2+n}+\alpha^{-1}q^{-m/2}t^{1/2-n}\big).$$

Using Lemmas 3.10 and 3.11, and by noting $\widetilde{\alpha}=\alpha q^{1/2}t^{-1/2}$, $\widehat{\alpha}=\alpha^{-1}q^{-1/2}t^{3/2}$ and (3.2), we can show that

$$\frac{1}{\langle t\rangle}\langle t^n\rangle\langle q^m\rangle\langle abcdt^{n-1}q^{m-1}\rangle-\frac{\langle q\rangle}{\langle t\rangle}\sum_{i=1}^{m}\big\langle\widehat{\alpha}s_i^{-1};\widetilde{\alpha}\widetilde{t}^{m-i}\big\rangle$$
$$=\sum_{l=1}^{m}\sum_{i=\lambda_{l+1}+1}^{\lambda_l}\big\langle\alpha t^{n-i}q^l;\alpha t^{n-i}\big\rangle+\frac{\langle q\rangle}{\langle t\rangle}\sum_{l=1}^{m}\big(\alpha q^{l-1/2}t^{-1/2+n-\lambda_l}+\alpha^{-1}q^{-l+1/2}t^{1/2-n+\lambda_l}\big)$$
$$-\sum_{l=1}^{m}\frac{\langle q\rangle}{\langle t\rangle}\big(\alpha q^{1/2+m-l}t^{-1/2}+\alpha^{-1}q^{-1/2-m+l}t^{1/2}\big)-\frac{\langle q\rangle}{\langle t\rangle}\sum_{i=1}^{m}\big\langle\widehat{\alpha}s_i^{-1};\widetilde{\alpha}\widetilde{t}^{m-i}\big\rangle$$
$$=\sum_{l=1}^{m}\sum_{i=\lambda_{l+1}+1}^{\lambda_l}\big\langle\alpha t^{n-i}q^l;\alpha t^{n-i}\big\rangle.$$

Therefore we have

$$\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y}=(-1)^{|\lambda|}m_{\lambda'}(x)+\text{lower order terms},$$
$$\mathcal{D}_x\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y}=\sum_{i=1}^{m}\big\langle\alpha t^{n-i}q^{\lambda'_i};\alpha t^{n-i}\big\rangle\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y},$$

thereby proving $P_{\lambda'}(x\,|\,a,b,c,d\,|\,q,t)=(-1)^{|\lambda|}\big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y}$. ■

3.3 Macdonald polynomials of types (C_n,C_n), (C_n,B_n) and (D_n,D_n)

We consider some degenerations of the Koornwinder polynomials to the Macdonald polynomials. As for the details, we refer the readers to [10, 13, 22].

Setting the parameters as $(a,b,c,d;q,t)\to\big(b^{1/2},-b^{1/2},q^{1/2}b^{1/2},-q^{1/2}b^{1/2};q,t\big)$ in the Koornwinder polynomial $P_\lambda(x)$, we obtain the Macdonald polynomials of type (C_n,C_n)

$$P_\lambda^{(C_n,C_n)}(x\,|\,b;q,t)=P_\lambda\big(x\,|\,b^{1/2},-b^{1/2},q^{1/2}b^{1/2},-q^{1/2}b^{1/2}\,|\,q,t\big).$$

We obtain the Macdonald polynomials of type (C_n,B_n) as

$$P_\lambda^{(C_n,B_n)}(x\,|\,b;q,t)=P_\lambda\big(x\,|\,b^{1/2},-b^{1/2},q^{1/2},-q^{1/2}\,|\,q,t\big),$$

and the Macdonald polynomials of type (D_n,D_n) as

$$P_\lambda^{(D_n,D_n)}(x\,|\,q,t)=P_\lambda\big(x\,|\,1,-1,q^{1/2},-q^{1/2}\,|\,q,t\big).$$

Note that $P_\lambda^{(D_n,D_n)}(x\,|\,q,t)=P_\lambda^{(C_n,C_n)}(x\,|\,1;q,t)=P_\lambda^{(C_n,B_n)}(x\,|\,1;q,t)$.

Note that setting the parameters as $(a,b,c,d;q,t)\to(a,-a,c,-c;q,t)$ and the application of the twist $\widehat{*}$ on $(a,-a,c,-c;q,t)$ gives

$$(t/a,-t/a,c/t,-c/t;t,t/q). \tag{3.4}$$

4 Koornwinder polynomial with one column diagram

When we apply Theorem 3.9 to the simplest case $m = 1$, namely when we plug the BC_1 asymptotically free eigenfunction $\widehat{f}(y;s)$ into the formula (3.3), we have the Koornwinder polynomials $P_{(1^r)}(x)$ with one column diagrams. Note that in $m = 1$ case, $\widehat{f}(y;s)$ does not have the parameter t. To execute the explicit calculation based on this, we need to recall the fourfold series expansion of the Askey–Wilson polynomials [8].

Let D denote the Askey–Wilson q-difference operator [2]

$$D = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}\big(T_{q,x}^{+1}-1\big) + \frac{(1-a/x)(1-b/x)(1-c/x)(1-d/x)}{(1-1/x^2)(1-q/x^2)}\big(T_{q,x}^{-1}-1\big).$$

Let $s \in \mathbb{C}$ be a parameter. Introduce λ satisfying $s = q^{-\lambda}$. Then we have $T_{q,x}x^{-\lambda} = sx^{-\lambda}$. Let $f(x;s) = f(x;s\,|\,a,b,c,d\,|\,q)$ be a formal series in x

$$f(x;s) = x^{-\lambda}\sum_{n\geq 0} c_n x^n, \qquad c_0 \neq 0,$$

satisfying the q-difference equation

$$Df(x;s) = \left(s + \frac{abcd}{qs} - 1 - \frac{abcd}{q}\right) f(x;s). \tag{4.1}$$

With the normalization $c_0 = 1$, (4.1) determines the coefficients $c_n = c_n(s\,|\,a,b,c,d\,|\,q)$ uniquely as rational functions in a, b, c, d, q and s. We call $f(x;s) = f(x;s\,|\,a,b,c,d\,|\,q)$ the asymptotically free eigenfunction associated with the Askey–Wilson operator D.

Definition 4.1 ([8, Definition 3.1])**.** Set

$$\Phi(x;s\,|\,a,b,c,d\,|\,q) = \sum_{k,l,m,n\geq 0} c_e\big(k,l;q^{m+n}s\,|\,a,c\,|\,q\big)c_o(m,n;s\,|\,a,b,c,d\,|\,q)x^{2k+2l+m+n},$$

where

$$c_e(k,l;s) = \frac{\big(qa^2/c^2;q^2\big)_k\big(q^3s/c^2;q^2\big)_k\big(q^2s^2/c^4;q^2\big)_k}{(q^2;q^2)_k\big(qs/c^2;q^2\big)_k\big(q^3s^2/a^2c^2;q^2\big)_k}\big(q^2/a^2\big)^k \times \frac{\big(c^2/q;q\big)_l(s;q)_{2k+l}}{(q;q)_l\big(q^2s/c^2;q\big)_{2k+l}}\big(q^2/c^2\big)^l,$$

$$c_o(m,n;s) = \frac{(-b/a;q)_m(qs/cd;q)_m}{(q;q)_m(-qs/ac;q)_m} \times \frac{(s;q)_{m+n}(-qs/ac;q)_{m+n}\big(qs^2/a^2c^2;q\big)_{m+n}}{\big(q^2s^2/abcd;q\big)_{m+n}\big(q^{1/2}s/ac;q\big)_{m+n}\big(-q^{1/2}s/ac;q\big)_{m+n}}(q/b)^m \times \frac{(-d/c;q)_n(qs/ab;q)_n}{(q;q)_n(-qs/ac;q)_n}(q/d)^n.$$

Theorem 4.2 ([8, Theorem 1.2, Proposition 4.3])**.** *The asymptotically free eigenfunction $f(x;s)$ associated with the Askey–Wilson operator D is expressed as the following fourfold summation*

$$f(x;s) = x^{-\lambda}\Phi(x;s\,|\,a,b,c,d\,|\,q).$$

Lemma 4.3 ([8, Lemma 5.1])**.** *We have*

$$\big(1-x^2\big)\sum_{k,l\geq 0} c_e(k,l;s)x^{2k+2l} = \sum_{k,l\geq 0} c'_e(k,l;s\,|\,a,c\,|\,q)x^{2k+2l},$$

where

$$\begin{aligned} c'_e(k,l;s\,|\,a,c\,|\,q) ={}& \frac{\big(qa^2/c^2;q^2\big)_k\big(q^3s/c^2;q^2\big)_k\big(q^2s^2/c^4;q^2\big)_k}{\big(q^2;q^2\big)_k\big(qs/c^2;q^2\big)_k\big(q^3s^2/a^2c^2;q^2\big)_k}\big(q^2/a^2\big)^k \\ &\times \frac{\big(c^2/q^2;q\big)_l(s/q;q)_{2k+l}}{(q;q)_l\big(q^2s/c^2;q\big)_{2k+l}}\frac{1-q^{2k+2l-1}s}{1-q^{-1}s}\big(q^2/c^2\big)^l. \end{aligned} \tag{4.2}$$

4.1 Koornwinder polynomial with one column diagram $P_{(1^r)}(x\,|\,a,b,c,d\,|\,q,t)$

We move on to the proof of Theorem 2.2. Recall that n is a positive integer, $x=(x_1,\dots,x_n)$ is a sequence of variables, and $P_{(1^r)}(x\,|\,a,b,c,d\,|\,q,t)$ denotes the Koornwinder polynomial with one column diagram (1^r).

Proof of Theorem 2.2. We consider the following special case of Theorem 3.9 above

$$\begin{aligned} &x=(x_1,\dots,x_n), \qquad n\in\mathbb{Z}_{>0}, \qquad y=(y_1), \qquad m=1, \\ &\Psi(x;y) = y^{-n}\prod_{i=1}^{n}(1-yx_i)(1-y/x_i) = y^{-n}\sum_{r\geq 0}(-1)^rE_r(x)y^r, \\ &\lambda=(r), \qquad r\in\mathbb{Z}_{\geq 0} \quad \text{and} \quad r\leq n, \qquad s=(s_1)=t^{n-r+1}, \qquad V(y)=y^{-1}\big(1-y^2\big), \\ &\widehat{f}(y;s) = y^{-r+1+n}\widehat{\Phi}(y;s) = y^{-r+1+n}\sum_{k,l,i,j\geq 0}\widehat{c}_e\big(k,l;t^{i+j}s\big)\widehat{c}_o(i,j;s)y^{2k+2l+i+j}, \end{aligned}$$

where

$$\widehat{c}_e(k,l;s) = c_e(k,l;s\,|\,t/a,t/c\,|\,t), \qquad \widehat{c}_o(i,j;s) = c_o(i,j;s\,|\,t/a,t/b,t/c,t/d\,|\,t).$$

Then calculating the constant term in y, we have

$$\begin{aligned} \big[\Psi(x;y)V(y)\widehat{f}(y;s)\big]_{1,y} ={}& \Bigg[\Bigg(\sum_{r\geq 0}(-1)^rE_r(x)y^{-n+r}\Bigg)y^{-1}\big(1-y^2\big) \\ &\times\Bigg(y^{-r+1+n}\sum_{k,l,i,j\geq 0}\widehat{c}_e\big(k,l;t^{i+j}s\big)\widehat{c}_o(i,j;s)y^{2k+2l+i+j}\Bigg)\Bigg]_{1,y} \\ ={}& (-1)^r\sum_{k,l,i,j\geq 0}(-1)^{-i-j}E_{r-2k-2l-i-j}(x)\widehat{c}'_e\big(k,l;t^{n-r+1+i+j}\big)\widehat{c}_o\big(i,j;t^{n-r+1}\big) \\ ={}& (-1)^rP_{(1^r)}(x|a,b,c,d|q,t), \end{aligned} \tag{4.3}$$

where $\widehat{c}'_e(k,l;s) = c'_e(k,l;s\,|\,t/a,t/c\,|\,t)$ (see (4.2) above). This proves Theorem 2.2. ■

4.2 Koornwinder polynomial $P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)$ with one column diagram

In view of (3.4), we need $\Phi(x;s\,|\,a,b,c,d\,|\,q)$ written for the parameters

$$(t/a,-t/a,t/c,-t/c;t,t/q).$$

Note that in this case we have the simplification of the coefficient as $\widehat{c}_o(m,n;s) = \delta_{m,0}\delta_{n,0}$. Hence we have a twofold summation formula for the $\Phi(x;s)$. By Lemma 4.3, we have

$$\begin{aligned}&\left(1-y^2\right)\widehat{\Phi}(y;s) = \left(1-y^2\right)\Phi(y;s\,|\,t/a,-t/a,t/c,-t/c\,|\,t) \qquad (4.4)\\ &= \sum_{k,l\geq 0}\frac{\left(tc^2/a^2;t^2\right)_k\left(stc^2;t^2\right)_k\left(s^2c^4/t^2;t^2\right)_k}{\left(t^2;t^2\right)_k\left(sc^2/t;t^2\right)_k\left(s^2a^2c^2/t;t^2\right)_k}\frac{\left(1/c^2;t\right)_l(s/t;t)_{2k+l}}{(t;t)_l\left(sc^2;t\right)_{2k+l}}\frac{1-st^{2k+2l-1}}{1-st^{-1}}a^{2k}c^{2l}y^{2k+2l}.\end{aligned}$$

Write $s=t^{n-r+1}$ for simplicity. Plugging (4.4) in (4.3), we have

$$\begin{aligned}P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t) &= \sum_{0\leq 2k+2l\leq r}E_{r-2k-2l}(x)\frac{\left(1/c^2;t\right)_l(s/t;t)_{2k+l}}{(t;t)_l\left(sc^2;t\right)_{2k+l}}\frac{1-st^{2k+2l-1}}{1-st^{-1}}c^{2l}\\ &\quad\times\frac{\left(tc^2/a^2;t^2\right)_k\left(stc^2;t^2\right)_k\left(s^2c^4/t^2;t^2\right)_k}{\left(t^2;t^2\right)_k\left(sc^2/t;t^2\right)_k\left(s^2a^2c^2/t;t^2\right)_k}a^{2k}. \qquad (4.5)\end{aligned}$$

This proves the formula in Corollary 2.3.

5 Transition matrices $\mathcal{B}(s)$, $\widetilde{\mathcal{B}}(s)$ and Bressoud's matrix inversion

5.1 Bressoud's matrix inversion

Theorem 5.1 ([4, p. 1, Theorem], [12, p. 5, Corollary])**.** *Let $\mathcal{M}(u,v;x,y;q)$ be the infinite even lower-triangle matrix with nonzero entries given by*

$$\mathcal{M}_{r,r-2i}(u,v;x,y;q) = y^iv^i\frac{(x/y;q)_i}{(q;q)_i}\frac{\left(uq^{r-2i};q\right)_{2i}}{\left(uxq^{r-i};q\right)_i\left(uyq^{r-2i+1};q\right)_i}, \qquad (5.1)$$

for $r,i\in\mathbb{Z}_{\geq 0}$, $i\leq[\frac{r}{2}]$. The we have

$$\mathcal{M}(u,v;x,y;q)\mathcal{M}(u,v;y,z;q) = \mathcal{M}(u,v;x,z;q). \qquad (5.2)$$

In particular, $\mathcal{M}(u,v;x,y;q)$ and $\mathcal{M}(u,v;y,x;q)$ are mutually inverse.

Definition 5.2. Set

$$d_r = \frac{\left(t^2v^{1/2};t\right)_r}{\left(u^{1/2};t\right)_r}\left(u^{1/4}/v^{3/4}\right)^r.$$

Let $\widetilde{\mathcal{M}}(u,v;x,y;t)$ denotes the conjugation of the matrix $\mathcal{M}\left(u,v;x,y;t^2\right)$ by the (d_r) with entries

$$\begin{aligned}\widetilde{\mathcal{M}}_{r,r-2i}(u,v;x,y;t) &= \mathcal{M}_{r,r-2i}\left(u,v;x,y;t^2\right)d_r/d_{r-2i}\\ &= \frac{\left(x/y;t^2\right)_i}{\left(t^2;t^2\right)_i}\frac{\left(v^{1/2}t^{r-2i+2};t\right)_{2i}}{\left(u^{1/2}t^{r-2i};t\right)_{2i}}\frac{\left(ut^{2r-4i};t^2\right)_{2i}}{\left(uxt^{2r-2i};t^2\right)_i\left(uyt^{2r-4i+2};t^2\right)_i}\left(yu^{1/2}/v^{1/2}\right)^i.\end{aligned}$$

5.2 Transition matrices $\mathcal{B}(s)$ and $\widetilde{\mathcal{B}}(s)$

Definition 5.3. Let $\mathcal{B}(s)$ and $\widetilde{\mathcal{B}}(s)$ be even lower triangular matrices defined by

$$\begin{aligned}\mathcal{B}(s) &= \widetilde{\mathcal{M}}\left(t^2/s^2c^4,1/s^2t^4;c^2/ta^2,1/t^2;t\right)\mathcal{M}\left(1/s,t;1/c^2,1;t\right),\\ \widetilde{\mathcal{B}}(s) &= \mathcal{M}\left(1/s,t;1,1/c^2;t\right)\widetilde{\mathcal{M}}\left(t^2/s^2c^4,1/s^2t^4;1/t^2,c^2/ta^2;t\right).\end{aligned}$$

Proposition 5.4. *The $\mathcal{B}(s)$ and $\widetilde{\mathcal{B}}(s)$ are mutually inverse.*

Proof. This follows from Bressoud's matrix inversion (5.2). ■

Theorem 5.5. *We have*

$$P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)=\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\mathcal{B}_{r,r-2k}(t^n)E_{r-2k}(x), \tag{5.3a}$$

$$E_r(x)=\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\widetilde{\mathcal{B}}_{r,r-2k}(t^n)P_{(1^{r-2k})}(x\,|\,a,-a,c,-c\,|\,q,t). \tag{5.3b}$$

Writing the coefficients explicitly, these read

$$\begin{aligned}P_{(1^r)}(x\,|\,a,-a,c,-c\,|\,q,t)&=\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\sum_{l=0}^{\lfloor\frac{r-2k}{2}\rfloor}E_{r-2k-2l}(x)\frac{(1/c^2;t)_l(st^{2k-1};t)_l(st^{2k};t)_{2l}}{(t;t)_l(sc^2t^{2k};t)_l(st^{2k-1};t)_{2l}}c^{2l}\\&\quad\times\frac{(tc^2/a^2;t^2)_k(stc^2;t^2)_k(s^2c^4/t^2;t^2)_k(s;t)_{2k}}{(t^2;t^2)_k(sc^2/t;t^2)_k(s^2a^2c^2/t;t^2)_k(sc^2;t)_{2k}}a^{2k},\end{aligned} \tag{5.4a}$$

$$\begin{aligned}E_r(x)&=\sum_{l=0}^{\lfloor\frac{r}{2}\rfloor}\sum_{k=0}^{\lfloor\frac{r-2l}{2}\rfloor}P_{(1^{r-2l-2k})}(x\,|\,a,-a,c,-c\,|\,q,t)\frac{(c^2;t)_l}{(t;t)_l}\frac{(st^l;t)_{l+2k}}{(st^{l-1}c^2;t)_{l+2k}}\\&\quad\times\frac{(a^2/tc^2;t^2)_k}{(t^2;t^2)_k}\frac{(s^2t^{4l-2}c^4;t^2)_k}{(s^2t^{4l}c^4;t^2)_k}\frac{(s^2t^{4l+2k-2}c^4;t^2)_k}{(s^2t^{4l+2k-3}a^2c^2;t^2)_k}(tc^2)^k,\end{aligned} \tag{5.4b}$$

where $s=t^{n-r+1}$. In particular form (5.4b), we have Theorem 2.4.

Proof. Clearly, (4.5) and (5.4a) are the same. We show that (5.4a) and (5.3a) are the same. By (5.1) and $s=t^{n-r+1}$, we have

$$\begin{aligned}\frac{(1/c^2;t)_l(st^{2k-1};t)_l(st^{2k};t)_{2l}}{(t;t)_l(sc^2t^{2k};t)_l(st^{2k-1};t)_{2l}}c^{2l}&=\frac{(1/c^2;t)_l}{(t;t)_l}\frac{(s^{-1}t^{-2k-2l+1};t)_{2l}}{(s^{-1}t^{-2k-l+1}/c^2;t)_l(s^{-1}t^{-2k-2l+2};t)_l}t^l\\&=\mathcal{M}_{r-2k,r-2k-2l}(t^{-n},t,1/c^2,1;t),\end{aligned}$$

and

$$\begin{aligned}&\frac{(tc^2/a^2;t^2)_k(stc^2;t^2)_k(s^2c^4/t^2;t^2)_k(s;t)_{2k}}{(t^2;t^2)_k(sc^2/t;t^2)_k(s^2a^2c^2/t;t^2)_k(sc^2;t)_{2k}}a^{2k}\\&\quad=\frac{(tc^2/a^2;t^2)_k}{(t^2;t^2)_k}\frac{(s^{-1}t^{-2k+1};t)_{2k}}{(s^{-1}t^{-2k+2}/c^2;t)_{2k}}\frac{(s^{-2}t^{-4k+4}/c^4;t^2)_{2k}}{(s^{-2}t^{-2k+3}/a^2c^2;t^2)_k(s^{-2}t^{-4k+4}/c^4;t^2)_k}(t/c^2)^k\\&\quad=\widetilde{\mathcal{M}}_{r,r-2k}(t^{-2n+2}/c^4,t^{-2n-4},c^2/ta^2,1/t^2;t).\end{aligned}$$

As for (5.3b) and (5.4b), we have

$$\begin{aligned}&\widetilde{\mathcal{M}}_{r,r-2k-2l}(t^{-2n+2}/c^4,t^{-2n-4},1/t^2,c^2/ta^2;t)\mathcal{M}_{r-2k,r-2k-2l}(t^{-n},t,1,1/c^2;t)\\&\quad=\frac{(a^2/tc^2;t^2)_k}{(t^2;t^2)_k}\frac{(s^{-1}t^{-2l-2k+1};t)_{2k}}{(s^{-1}t^{-2l-2k+2}/c^2;t)_{2k}}\frac{(s^{-2}t^{-4l-4k+4}/c^4;t^2)_{2k}}{(s^{-2}t^{-4l-2k+2}/c^4;t^2)_k(s^{-2}t^{-4l-4k+5}/c^2a^2;t^2)_k}\\&\quad\quad\times(t^2/a^2)^k\frac{(c^2;t)_l}{(t;t)_l}\frac{(s^{-1}t^{-2l+1};t)_l}{(s^{-1}t^{-2l+2}/c^2;t)_l}(t/c^2)^l\\&\quad=\frac{(c^2;t)_l}{(t;t)_l}\frac{(st^l;t)_{l+2k}}{(st^{l-1}c^2;t)_{l+2k}}\frac{(a^2/tc^2;t^2)_k}{(t^2;t^2)_k}\frac{(s^2t^{4l-2}c^4;t^2)_k}{(s^2t^{4l}c^4;t^2)_k}\frac{(s^2t^{4l+2k-2}c^4;t^2)_k}{(s^2t^{4l+2k-3}a^2c^2;t^2)_k}(tc^2)^k.\end{aligned}$$

■

5.3 Entries of $\mathcal{B}(s)$ and $\widetilde{\mathcal{B}}(s)$ in terms of ${}_4\phi_3$ series

Recall that we have defined $B(s,j)$ and $\widetilde{B}(s,j)$ in Definition 2.5 as

$$B(s,j) = (-1)^j s^{-j} \frac{(s^2/t^2;t^2)_j}{(t^2;t^2)_j} \frac{1-s^2t^{4j-2}}{1-s^2t^{-2}} {}_4\phi_3\left[\begin{matrix}-sa^2, -sc^2, s^2t^{2j-2}, t^{-2j}\\ -s, -st, s^2a^2c^2/t\end{matrix}; t^2, t^2\right],$$
$$\widetilde{B}(s,j) = (st^{j-1})^{-j} \frac{(t^{2j}s^2;t^2)_j}{(t^2;t^2)_j} {}_4\phi_3\left[\begin{matrix}-t^{-2j+2}/sa^2, -t^{-2j+2}/sc^2, t^{-2j+2}/s^2, t^{-2j}\\ -t^{-2j+1}/s, -t^{-2j+2}/s, t^{-4j+5}/s^2a^2c^2\end{matrix}; t^2, t^2\right].$$

Proposition 5.6. *We have*

$$B(s,j) = (-1)^j t^j s^{-j} \frac{(s^2/t^2;t^2)_j}{(t^2;t^2)_j} \frac{1-st^{2j-1}}{1-st^{-1}} {}_4\phi_3\left[\begin{matrix}-sa^2/t, -sc^2/t, s^2t^{2j-2}, t^{-2j}\\ -s, -s/t, s^2a^2c^2/t\end{matrix}; t^2, t^2\right], \tag{5.5a}$$
$$\widetilde{B}(s,j) = t^j \left(st^{j-1}\right)^{-j} \frac{(s^2t^{2j};t^2)_j}{(t^2;t^2)_j} \frac{1+st^{-1}}{1+st^{2j-1}}$$
$$\times {}_4\phi_3\left[\begin{matrix}-t^{-2j+3}/sa^2, -t^{-2j+3}/sc^2, t^{-2j+2}/s^2, t^{-2j}\\ -t^{-2j+2}/s, -t^{-2j+3}/s, t^{-4j+5}/s^2a^2c^2\end{matrix}; t^2, t^2\right]. \tag{5.5b}$$

Proof. This follows from the Sears transformation [7, p. 49, equation (2.10.4)]. ■

Theorem 5.7. *We have*

$$\mathcal{B}_{r,r-2i}(s) = B\left(st^{-r+1}, i\right), \tag{5.6a}$$
$$\widetilde{\mathcal{B}}_{r,r-2i}(s) = \widetilde{B}\left(st^{-r+1}, i\right). \tag{5.6b}$$

Proof. Recall that the bibasic hypergeometric series Φ (see [7, p. 99, equation (3.9.1)]) is defined by

$$\Phi\left[\begin{matrix}a_1,\dots,a_{r+1} : c_1,\dots,c_s\\ b_1,\dots,b_r : d_1,\dots,d_s\end{matrix}; q, p; z\right] = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n} \frac{(c_1,\dots,c_s;p)_n}{(d_1,\dots,d_s;p)_n} z^n.$$

We use the q-analogue of Bailey's transformation [7, p. 99, equation (3.10.14)]:

$$\Phi\left[\begin{matrix}a^2, at^2, -at^2, b^2, c^2 : -at/w, t^{-i}\\ a, -a, a^2t^2/b^2, a^2t^2/c^2 : w, -at^{i+1}\end{matrix}; t^2, t; \frac{awt^{i+1}}{b^2c^2}\right]$$
$$= \frac{(-at, at^2/w, w/at;t)_i}{(-t, at/w, w;t)_i} {}_5\phi_4\left[\begin{matrix}at, at^2, a^2t^2/b^2c^2, a^2t^2/w^2, t^{-2i}\\ a^2t^2/b^2, a^2t^2/c^2, at^{2-i}/w, at^{3-i}/w\end{matrix}; t^2, t^2\right]. \tag{5.7}$$

When $a = c^2$, (5.7) becomes

$$\Phi\left[\begin{matrix}a^2, -at^2, b^2 : -at/w, t^{-i}\\ -a, a^2t^2/b^2 : w, -at^{i+1}\end{matrix}; t^2, t; \frac{wt^{i+1}}{b^2}\right]$$
$$= \frac{(-at, at^2/w, w/at;t)_i}{(-t, at/w, w;t)_i} {}_4\phi_3\left[\begin{matrix}at, at^2/b^2, a^2t^2/w^2, t^{-2i}\\ a^2t^2/b^2, at^{2-i}/w, at^{3-i}/w\end{matrix}; t^2, t^2\right]. \tag{5.8}$$

Replacing the parameters in (5.8) as

$$\left(a, w, b^2\right) \to \left(-sc^2/t, c^2t^{-i+1}, tc^2/a^2\right),$$

we can prove (5.6a) as

$$\begin{aligned}\mathcal{B}_{r,r-2i}(st^{r-1}) &= \sum_{k=0}^{i} \frac{(1/c^2;t)_{i-k}}{(t;t)_{i-k}} \frac{(st^{2k-1};t)_{i-k}}{(sc^2t^{2k};t)_{i-k}} \frac{(st^{2k};t)_{2i-2k}}{(st^{2k-1};t)_{2i-2k}} c^{2i-2k} \\ &\quad\times \frac{(tc^2/a^2;t^2)_k (stc^2;t^2)_k (s^2c^4/t^2;t^2)_k (s;t)_{2k}}{(t^2;t^2)_k (sc^2/t;t^2)_k (s^2a^2c^2/t;t^2)_k (sc^2;t)_{2k}} a^{2k} \\ &= \frac{(1/c^2;t)_i}{(t;t)_i} \frac{(s/t;t)_i}{(sc^2;t)_i} \frac{(s;t)_{2i}}{(s/t;t)_{2i}} c^{2i} \\ &\quad\times \sum_{k=0}^{i} \frac{(t^{-i};t)_k}{(c^2t^{-i+1};t)_k} \frac{(st^{i-1};t)_k}{(sc^2t^i;t)_k} \frac{(tc^2/a^2;t^2)_k (stc^2;t^2)_k (s^2c^4/t^2;t^2)_k}{(t^2;t^2)_k (sc^2/t;t^2)_k (s^2a^2c^2/t;t^2)_k} a^{2k}t^k \\ &= \frac{(1/c^2;t)_i}{(t;t)_i} \frac{(s/t;t)_i}{(sc^2;t)_i} \frac{(s;t)_{2i}}{(s/t;t)_{2i}} c^{2i} \Phi \left[\begin{matrix} tc^2/a^2, sc^2t, s^2c^4/t^2 : st^{i-1}, t^{-i} \\ sc^2/t, s^2a^2c^2/t : c^2t^{-i+1}, sc^2t^i \end{matrix}; t^2, t; a^2t\right] \\ &= \frac{(1/c^2;t)_i}{(t;t)_i} \frac{(s/t;t)_i}{(sc^2;t)_i} \frac{(s;t)_{2i}}{(s/t;t)_{2i}} c^{2i} \\ &\quad\times (-1/sc^2)^i \frac{(sc^2, -s/t;t)_i (-s;t)_{2i}}{(1/c^2, -t;t)_i (-s/t;t)_{2i}} {}_4\phi_3 \left[\begin{matrix} -sa^2, -sc^2, s^2t^{2i-2}, t^{-2i} \\ -s, -st, s^2a^2c^2/t \end{matrix}; t^2, t^2\right] \\ &= B(s,i).\end{aligned}$$

When $at = c^2$, (5.7) becomes

$$\begin{aligned}&\Phi \left[\begin{matrix} a^2, at^2, -at^2, b^2 : -at/w, t^{-i} \\ a, -a, a^2t^2/b^2 : w, -at^{i+1} \end{matrix}; t^2, t; \frac{wt^i}{b^2}\right] \\ &\quad= \frac{(-at, at^2/w, w/at;t)_i}{(-t, at/w, w;t)_i} {}_4\phi_3 \left[\begin{matrix} at^2, at/b^2, a^2t^2/w^2, t^{-2i} \\ a^2t^2/b^2, at^{2-i}/w, at^{3-i}/w \end{matrix}; t^2, t^2\right].\end{aligned} \tag{5.9}$$

Replacing the parameters in (5.9) as

$$(a, w, b^2) \to (-t^{-2i+1}/sc^2, t^{-i+1}/c^2, a^2/tc^2),$$

we can prove (5.6b) as

$$\begin{aligned}\widetilde{\mathcal{B}}_{r,r-2i}(st^{r-1}) &= \sum_{k=0}^{i} \frac{(c^2, t^{-2i+2k+1}/s;t)_{i-k}}{(t, t^{-2i+2k+2}/sc^2;t)_{i-k}} (t/c^2)^{i-k} \frac{(a^2/tc^2;t^2)_k}{(t^2;t^2)_k} \frac{(t^{-2i+1}/s;t)_{2k}}{(t^{-2i+2}/sc^2;t)_{2k}} \\ &\quad\times \frac{(t^{-4i+4}/s^2c^4;t^2)_{2k}}{(t^{-4i+2k+2}/s^2c^4;t^2)_k (t^{-4i+5}/s^2a^2c^2;t^2)_k} (t^2/a^2)^k \\ &= \frac{(c^2, t^{-2i+1}/s;t)_i}{(t, t^{-2i+2}/sc^2;t)_i} (t/c^2)^i \\ &\quad\times \Phi \left[\begin{matrix} a^2/tc^2, t^{-4i+2}/s^2c^4, t^{-2i+3}/sc^2, -t^{-2i+3}/sc^2 : t^{-i+1}/s, t^{-i} \\ t^{-4i+5}/s^2a^2c^2, t^{-2i+1}/sc^2, -t^{-2i+1}/sc^2 : t^{-i+2}/sc^2, t^{-i+1}/c^2 \end{matrix}; t^2, t; t^2/a^2\right] \\ &= \frac{(-s/t, s;t)_{2i}}{(t^2, s^2;t^2)_i} (t/s)^i t^{-i(i-1)} {}_4\phi_3 \left[\begin{matrix} -t^{-2i+3}/sc^2, -t^{-2i+3}/sa^2, t^{-2i+2}/s^2, t^{-2i} \\ t^{-4i+5}/s^2a^2c^2, -t^{-2i+2}/s, -t^{-2i+3}/s \end{matrix}; t^2, t^2\right] \\ &= \widetilde{B}(s,i).\end{aligned}$$

■

6 Four term relations for $B(s,i)$ and $\widetilde{B}(s,i)$

6.1 Four term relations

Recall that we have defined $f(s)$ in (1.3) and have introduced the notation $F(s,l)$ in (2.7) as

$$F(s,l)=f\big(s/t^l\big)=\frac{\big(1-t^l/s\big)\big(1-t^{l+2}/sa^2c^2\big)\big(1+t^{l+1}/sa^2\big)\big(1+t^{l+1}/sc^2\big)}{\big(1-t^{2l+1}/s^2a^2c^2\big)\big(1-t^{2l+3}/s^2a^2c^2\big)}.$$

Theorem 6.1. *We have*

$$B(s,i)+F(s,-1)B\big(st^2,i-1\big)=B(st,i)+B(st,i-1), \tag{6.1a}$$

$$\widetilde{B}(s,i)+F(s,2-2i)\widetilde{B}(s,i-1)=\widetilde{B}\big(st^{-1},i\big)+\widetilde{B}(st,i-1). \tag{6.1b}$$

Proof. Recall the shorthand notation (2.5), (2.6), and $B(s,i)=b(s,i)\sum_{k=0}^{i}{}_4\phi_3(s,i,k)$. For (6.1a), we need to show

$$\sum_{k=0}^{i}{}_4\phi_3(s,i,k)+F(s,-1)\frac{b(st^2,i-1)}{b(s,i)}\sum_{k=0}^{i-1}{}_4\phi_3\big(st^2,i-1,k\big)$$
$$=\frac{b(st,i)}{b(s,i)}\sum_{k=0}^{i}{}_4\phi_3(st,i,k)+\frac{b(st,i-1)}{b(s,i)}\sum_{k=0}^{i-1}{}_4\phi_3(st,i-1,k). \tag{6.2}$$

Firstly, we have

$$\begin{aligned}\text{l.h.s. of (6.2)}&=1+\sum_{k=1}^{i}\left({}_4\phi_3(s,i,k)+F(s,-1)\frac{b(s,i-1)}{b(s,i)}{}_4\phi_3\big(st^2,i-1,k-1\big)\right)\\&=1+\sum_{k=1}^{i}\frac{\big(-sa^2,-sc^2,st^2,s^2t^{2i-2},t^{-2i};t^2\big)_k}{\big(t^2,s,-s,-st,s^2a^2c^2t;t^2\big)_k}t^{2k}\\&={}_5\phi_4\left[\begin{matrix}-sa^2,-sc^2,st^2,s^2t^{2i-2},t^{-2i}\\ s,-s,-st,s^2a^2c^2t\end{matrix};t^2,t^2\right].\end{aligned} \tag{6.3}$$

Nextly, by setting

$$\overline{{}_4\phi_3}(s,i,k):=\frac{\big(t^{-2i},s^2t^{2i-2},-sc^2/t,-sa^2/t;t^2\big)_k}{\big(t^2,s^2a^2c^2/t,-s,-s/t;t^2\big)_k}t^{2k},$$

we can write

$$\sum_{k=0}^{i}{}_4\phi_3(s,i,k)=\frac{1+s^{-1}t}{1+s^{-1}t^{-2i+1}}t^{-i}\sum_{k=0}^{i}\overline{{}_4\phi_3}(s,i,k).$$

Hence we have

$$\begin{aligned}\text{r.h.s. of (6.2)}&=\sum_{k=0}^{i}\left(\frac{b(st,i)}{b(s,i)}\frac{1+s^{-1}}{1+s^{-1}t^{-2i}}t^{-i}\,\overline{{}_4\phi_3}(st,i,k)\right.\\&\qquad\left.+\frac{b(st,i-1)}{b(s,i)}\frac{1+s^{-1}}{1+s^{-1}t^{-2i+2}}t^{-i+1}\,\overline{{}_4\phi_3}(st,i-1,k)\right)\\&=\sum_{k=0}^{i}\frac{\big(-sa^2,-sc^2,st^2,s^2t^{2i-2},t^{-2i};t^2\big)_k}{\big(t^2,s,-s,-st,s^2a^2c^2t;t^2\big)_k}t^{2k}\end{aligned}$$

$$= {}_5\phi_4\left[\begin{matrix}-sa^2,-sc^2,st^2,s^2t^{2i-2},t^{-2i}\\ s,-s,-st,s^2a^2c^2t\end{matrix};t^2,t^2\right]=\text{l.h.s. of (6.3)}.$$

Now we turn to (6.1b). Set

$$\widetilde{b}(s,i):=\frac{\left(-st^{-1},s;t\right)_{2i}}{\left(t^2,s^2;t^2\right)_i}\left(s^{-1}t\right)^i t^{-i(i-1)}, \tag{6.4a}$$

$$ {}_4\widetilde{\phi}_3(s,i,k):=\frac{\left(-s^{-1}t^{-2i+3}/c^2,-s^{-1}t^{-2i+3}/a^2,s^{-2}t^{-2i+2},t^{-2i};t^2\right)_i}{\left(s^{-2}t^{-4i+5}/a^2c^2,-s^{-1}t^{-2i+2},-s^{-1}t^{-2i+3};t^2\right)_i}t^{2k}, \tag{6.4b}$$

$$\overline{{}_4\widetilde{\phi}_3}(s,i,k):=\frac{\left(t^{-2i},s^{-2}t^{-2i+2},-s^{-1}t^{-2i+2}/a^2,-s^{-1}t^{-2i+2}/a^2;t^2\right)_k}{\left(s^{-2}t^{-4i+5}/a^2c^2,-s^{-1}t^{-2i+2},-s^{-1}t^{-2i+1};t^2\right)_k}t^{2k}, \tag{6.4c}$$

for simplicity. Then we can write

$$\widetilde{B}(s,i)=\widetilde{b}(s,i)\sum_{k=0}^{i}{}_4\widetilde{\phi}_3(s,i,k),$$

$$\sum_{k=0}^{i}{}_4\widetilde{\phi}_3(s,i,k)=\frac{1+s^{-1}t^{-2i+1}}{1+s^{-1}t}t^i\sum_{k=0}^{i}\overline{{}_4\widetilde{\phi}_3}(s,i,k).$$

We shall show

$$\sum_{k=0}^{i}{}_4\widetilde{\phi}_3(s,i,k)+F(s,2-2i)\frac{\widetilde{b}(s,i-1)}{\widetilde{b}(s,i)}\sum_{k=0}^{i-1}{}_4\widetilde{\phi}_3(s,i-1,k)$$
$$=\frac{\widetilde{b}(st^{-1},i)}{\widetilde{b}(s,i)}\sum_{k=0}^{i}{}_4\widetilde{\phi}_3\left(st^{-1},i,k\right)+\frac{\widetilde{b}(st,i-1)}{\widetilde{b}(s,i)}\sum_{k=0}^{i-1}{}_4\widetilde{\phi}_3(st,i-1,k). \tag{6.5}$$

We have

$$\begin{aligned}\text{l.h.s. of (6.5)}&=1+\sum_{k=1}^{i}\left({}_4\widetilde{\phi}_3(s,i,k)+F(s,2-2i)\frac{\widetilde{b}(s,i-1)}{\widetilde{b}(s,i)}{}_4\widetilde{\phi}_3(s,i-1,k-1)\right)\\
&=1+\sum_{k=1}^{i}\frac{\left(t^{-2i},s^{-2}t^{-2i+2},-s^{-1}t^{-2i+3}/a^2,-s^{-1}t^{-2i+3}/c^2,s^{-1}t^{-2i+3};t^2\right)_k}{\left(t^2,s^{-2}t^{-4i+7}/a^2c^2,-s^{-1}t^{-2i+3},-s^{-1}t^{-2i+2},s^{-1}t^{-2i+1};t^2\right)_k}t^{2k}\\
&={}_5\phi_4\left[\begin{matrix}t^{-2i},s^{-2}t^{-2i+2},-s^{-1}t^{-2i+3}/a^2,-s^{-1}t^{-2i+3}/c^2,s^{-1}t^{-2i+3}\\ s^{-2}t^{-4i+7}/a^2c^2,-s^{-1}t^{-2i+3},-s^{-1}t^{-2i+2},s^{-1}t^{-2i+1}\end{matrix};t^2,t^2\right].\end{aligned} \tag{6.6}$$

On the other hand, we have

$$\begin{aligned}\text{r.h.s. of (6.5)}&=\sum_{k=0}^{i}\left(\frac{\widetilde{b}(st^{-1},i)}{\widetilde{b}(s,i)}\frac{1+s^{-1}t^{-2i+2}}{1+s^{-1}t^2}t^i\,\overline{{}_4\widetilde{\phi}_3}\left(st^{-1},i,k\right)\right.\\
&\qquad\left.+\frac{\widetilde{b}(st,i-1)}{\widetilde{b}(s,i)}\frac{1+s^{-1}t^{-2i+2}}{1+s^{-1}}t^{i-1}\,\overline{{}_4\widetilde{\phi}_3}(st,i-1,k)\right)\\
&=\sum_{k=1}^{i}\frac{\left(t^{-2i},s^{-2}t^{-2i+2},-s^{-1}t^{-2i+3}/a^2,-s^{-1}t^{-2i+3}/c^2,s^{-1}t^{-2i+3};t^2\right)_k}{\left(t^2,s^{-2}t^{-4i+7}/a^2c^2,-s^{-1}t^{-2i+3},-s^{-1}t^{-2i+2},s^{-1}t^{-2i+1};t^2\right)_k}t^{2k}\\
&={}_5\phi_4\left[\begin{matrix}t^{-2i},s^{-2}t^{-2i+2},-s^{-1}t^{-2i+3}/a^2,-s^{-1}t^{-2i+3}/c^2,s^{-1}t^{-2i+3}\\ s^{-2}t^{-4i+7}/a^2c^2,-s^{-1}t^{-2i+3},-s^{-1}t^{-2i+2},s^{-1}t^{-2i+1}\end{matrix};t^2,t^2\right]\\
&=\text{r.h.s. of (6.6)}.\end{aligned}$$

■

6.2 Another proof of Theorem 5.4

As an application of the four term relations $B(s,i)$ and $\widetilde{B}(s,i)$, we present another proof of Theorem 5.4, providing an amusing complementary argument based on the Bressoud matrix inversion.

Proposition 6.2. *The four terms relations in Theorem 6.1 imply that*

$$\sum_{k=0}^{i} B(s,k)\widetilde{B}\big(st^{2k}, i-k\big) = \delta_{i,0}, \qquad \sum_{k=0}^{i} \widetilde{B}(s,k)B\big(st^{2k}, i-k\big) = \delta_{i,0},$$

hence, that the matrices $\mathcal{B}(s)$ and $\widetilde{\mathcal{B}}(s)$ are mutually inverse.

Proof. Set

$$\text{l.h.s.}(s,i) := \sum_{k=0}^{i} B(s,k)\widetilde{B}\big(st^{2k}, i-k\big),$$

for simplicity.

First we show that for $i \geq 0$ we have the difference equation

$$\text{l.h.s.}(s,i) - \text{l.h.s.}(s/t,i) = 0. \tag{6.7}$$

We prove this by induction. The case $i = 0$ is clearly correct. Suppose that it is valid for $i-1$. Then we have

$$\begin{aligned}\text{l.h.s.}(s,i) - \text{l.h.s.}(s/t,i) = {}& \sum_{k=0}^{i} B(s,k)\big(\widetilde{B}\big(st^{2k-1}, i-k\big) \\ &- F(s,2-2i)\widetilde{B}\big(st^{2k}, i-k-1\big) + \widetilde{B}\big(st^{2k+1}, i-k-1\big)\big) \\ &- \sum_{k=0}^{i} \big(B(s,k) - F(s,0)B(st,k-1) + B(s,k-1)\big)\widetilde{B}\big(st^{2k-1}, i-k\big) \\ = {}& -F(s,2-2i)\,\text{l.h.s.}(s,i-1) + F(s,0)\,\text{l.h.s.}(st,i-1) = 0.\end{aligned}$$

By definition l.h.s.(s,i) is a rational function in s, and it satisfies the difference equation (6.7). Therefore, l.h.s.(s,i) must be a constant. We have l.h.s.$(s,0) = 1$. Then we can check that for $i > 0$ l.h.s.$(1,i) = 0$ (hence l.h.s.$(s,i) = 0$) by using the following lemma as

$$\text{l.h.s.}(s,i) = \sum_{k=0}^{i} B(1,k)\widetilde{B}\big(t^{2k}, i-k\big) = B(1,0)\widetilde{B}(1,i) - B(1,1). \qquad \blacksquare$$

Lemma 6.3. *We have*

$$B(1,j) = \begin{cases} 1, & j = 0, \\ -1, & j = 1, \\ 0, & j > 1, \end{cases} \tag{6.8}$$

$$\widetilde{B}(1,i) - \widetilde{B}\big(t^2, i-1\big) = 0, \qquad i > 1. \tag{6.9}$$

Proof. The (6.8) follows from the definition of $B(s,i)$. By noting

$$\widetilde{b}(s,i)_4\widetilde{\phi}_3(s,i,k) - \widetilde{b}\big(st^2, i-1\big)_4\widetilde{\phi}_3\big(st^2, i-1, k\big) = s^{-i}t^{-i(i-1)}\frac{\big(t^{2i+2}s^2;t^2\big)_i}{\big(t^2;t^2\big)_i}\frac{1-s}{1-st^{2i}}$$

$$\times \frac{\left(-t^{-2i+2}/sa^2;t^2\right)_k}{\left(-t^{-2i}/s;t^2\right)_k} \frac{\left(-t^{-2i+2}/sc^2;t^2\right)_k}{\left(-t^{-2i+1}/s;t^2\right)_k} \frac{\left(t^{-2i}/s^2;t^2\right)_k}{\left(t^{-4i+5}/s^2a^2c^2;t^2\right)_k} \frac{\left(t^{-2i};t^2\right)_k}{\left(t^2;t^2\right)_k} t^{2k},$$

where we used the notation (6.4a) and (6.4b), we have (6.9) from the identity

$$\begin{aligned}\widetilde{B}(s,i) - \widetilde{B}\left(st^2, i-1\right) &= s^{-i}t^{-i(i-1)} \frac{\left(t^{2i+2}s^2;t^2\right)_i}{\left(t^2;t^2\right)_i} \frac{1-s}{1-st^{2i}} \\ &\times {}_4\phi_3 \left[\begin{matrix} -t^{-2i+2}/sa^2, -t^{-2i+2}/sc^2, t^{-2i}/s^2, t^{-2i} \\ -t^{-2i}/s, -t^{-2i+1}/s, t^{-4i+5}/s^2a^2c^2 \end{matrix} ; t^2, t^2 \right]. \qquad \blacksquare\end{aligned}$$

7 Transition matrix $\mathcal{C}$ and (a, c, t)-deformation of Catalan triangle three term recursion relations

7.1 Coefficient $C(s, j)$

Recall that in Definition 2.15, we have defined the function $C(s, j)$ as

$$C(s,j) := \sum_{i=0}^{j} B(s,i) \binom{m+2j}{j-i}. \tag{7.1}$$

Then (5.3a), (5.6a), and (7.1) imply (Theorem 2.16)

$$P_{(1^r)}(x \,|\, a, -a, c, -c \,|\, q, t) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} C\left(t^{n-r+1}, k\right) m_{(1^{r-2k})}(x).$$

7.2 Deformed Catalan triangle recursion relations

Proposition 7.1. *We have the three term relation*

$$C(s,j) + F(s,-1)C\left(st^2, j-1\right) = C(st, j). \tag{7.2}$$

Proof. We have

$$\begin{aligned} &C(s,j) + F(s,-1)C\left(st^2, j-1\right) \\ &\quad = \sum_{i=0}^{j} B(s,i)\binom{m+2j}{j-i} + \sum_{i=0}^{j-1} F(s,-1)B\left(st^2,i\right)\binom{m+2j}{j-1-i} \\ &\quad = \binom{m+2j}{j} + \sum_{i=1}^{j} (B(st,i) + B(st,i-1))\binom{m+2j}{j-i} \\ &\quad = \binom{m+2j}{j} + \binom{m+2j}{j-1} + B(st,j) + \sum_{i=1}^{j-1} B(st,i)\left(\binom{m+2j}{j-i} + \binom{m+2j}{j-i-1}\right) \\ &\quad = \sum_{i=0}^{j} B(st,i)\binom{m+1+2j}{j-i} = C(st,j). \qquad \blacksquare \end{aligned}$$

Proposition 7.2. *We have*

$$C(1,j) = \delta_{j,0}.$$

Hence the three term relation (7.2) for $s = 1$ reads

$$F(1,-1)C\left(t^2, j-1\right) = C(t,j).$$

Proof. We have $C(1,0)=1$. From Lemma 6.3, we have for $j>0$

$$C(1,j)=B(1,0)\binom{-1+2j}{j}+B(1,1)\binom{-1+2j}{j-1}=\binom{-1+2j}{j}-\binom{-1+2j}{j-1}=0. \qquad\blacksquare$$

7.3 Solution to the deformed Catalan triangle recursion relations

Theorem 7.3. *We have $C(t^{r+1},0)=1$ for $r\in\mathbb{Z}_{\geq 0}$, and for $i\in\mathbb{Z}_{>0}, r\in\mathbb{Z}_{\geq 0}$ we have*

$$C\big(t^{r+1},i\big)=\sum_{(d_1,\dots,d_i)\in\mathcal{P}[r,i]}F\big(t^{r+1},d_1\big)F\big(t^{r+1},d_2\big)\cdots F\big(t^{r+1},d_i\big), \tag{7.3}$$

where $\mathcal{P}[r,i]$ denotes the finite set defined by

$$\mathcal{P}[r,i]=\big\{(d_1,d_2,\dots,d_i)\in\mathbb{Z}^i\,|\,0\leq d_1\leq r,\ d_k-1\leq d_{k+1}\leq r \text{ for } 1\leq k<i\big\}.$$

We prepare some lemmas.

Lemma 7.4. *For $r\in\mathbb{Z}_{\geq 0}$, we have*

$$C\big(t^{r+1},i+1\big)=\sum_{k=0}^{r}F\big(t^k,-1\big)C\big(t^{k+2},i\big).$$

Proof. The case $r=0$ holds since $C(t,i+1)=F(1,-1)C\big(t^2,i\big)$. Then we can show the induction step as

$$\begin{aligned}C\big(t^{r+2},i+1\big)&=C\big(t^{r+1},i+1\big)+F\big(t^{r+1},-1\big)C\big(t^{r+3},i\big)\\&=\sum_{k=0}^{r}F\big(t^k,-1\big)C\big(t^{k+2},i\big)+F\big(t^{r+1},-1\big)C\big(t^{r+3},i\big)=\sum_{k=0}^{r+1}F\big(t^k,-1\big)C\big(t^{k+2},i\big).\end{aligned} \qquad\blacksquare$$

Lemma 7.5. *We have*

$$\begin{aligned}\mathcal{P}[r,i+1]=\big\{&(d,d_1,d_2,\dots,d_i)\in\mathbb{Z}^i\,|\\&0\leq d_1\leq r,\ (d_1-d+1,\dots,d_i-d+1)\in\mathcal{P}[r-d+1,i+1]\big\}.\end{aligned}$$

Proof of Proposition 7.3. We prove (7.3) by induction on i. It holds for $i=0$, since we have $C\big[t^{r+1},0\big]=1$, $r\in\mathbb{Z}_{\geq 0}$. The induction step is shown as follows. Lemmas 7.4 and 7.5 and the induction hypothesis give us

$$\begin{aligned}C\big(t^{r+1},i+1\big)&=\sum_{k=0}^{r}F\big(t^k,-1\big)C\big(t^{k+2},i\big)=\sum_{d=0}^{r}F\big(t^{r+1},d\big)C\big(t^{r-d+2},i\big)\\&=\sum_{d=0}^{r}F\big(t^{r+1},d\big)\sum_{(d_1,\dots,d_i)\in\mathcal{P}[r-d+1,i]}F\big(t^{r+1},d_1\big)F\big(t^{r+1},d_2\big)\cdots F\big(t^{r+1},d_i\big)\\&=\sum_{(d,d_1,\dots,d_i)\in\mathcal{P}[r,i+1]}F\big(t^{r+1},d\big)F\big(t^{r+1},d_1\big)F\big(t^{r+1},d_2\big)\cdots F\big(t^{r+1},d_i\big).\end{aligned} \qquad\blacksquare$$

8 Some degenerations of Macdonald polynomials of types C_n and D_n with one column diagrams and Kostka polynomials

This section is devoted to the study of several degenerations of our formulas for the Macdonald polynomial $P_{(1^r)}^{(C_n,C_n)}(x\,|\,b;q,t)$ (see Section 3.3).

8.1 Some degenerations of $B(s,j)$ and $\widetilde{B}(s,j)$

8.1.1 (C_n, C_n) case

Proposition 8.1. *When $a = t^{1/2}$, $c = q^{1/2}t^{1/2}$ in the equations (5.5a) and (5.5b), we have*

$$B(s,j) = \frac{(1/qt;t^2)_j(s^2/t^2,t^2)_j}{(t^2;t^2)_j(s^2qt;t^2)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}(qt)^j,$$
$$\widetilde{B}(s,j) = \frac{(qt;t^2)_j(s^2t^{2j},t^2)_j}{(t^2;t^2)_j(s^2qt^{2j-1};t^2)_j}.$$

Proof. Setting $a = t^{1/2}$, $c = q^{1/2}t^{1/2}$, we have by the Saalschütz summation formula [7, p. 17, equation (1.7.2)] that

$$\begin{aligned} B(s,j) &= (-1)^j s^{-j}\frac{(s^2/t^2;t^2)_j}{(t^2;t^2)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}\,{}_3\phi_2\left[\begin{matrix}-sqt, s^2t^{2j-2}, t^{-2j}\\ -s, s^2qt\end{matrix};t^2,t^2\right]\\ &= (-1)^j s^{-j}\frac{(s^2/t^2;t^2)_j}{(t^2;t^2)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}\frac{(1/qt;t^2)_j(-t^{-2j+2}/s;t^2)_j}{(-s;t^2)_j(t^{-2j+2}/s^2qt;t^2)_j}\\ &= \frac{(1/qt;t^2)_j(s^2/t^2,t^2)_j}{(t^2;t^2)_j(s^2qt;t^2)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}(qt)^j \end{aligned}$$

and

$$\begin{aligned} \widetilde{B}(s,j) &= \left(st^{j-1}\right)^{-j}\frac{(t^{2j}s^2;t^2)_j}{(t^2;t^2)_j}\,{}_3\phi_2\left[\begin{matrix}-t^{-2j+1}/sq, t^{-2j+2}/s^2, t^{-2j}\\ -t^{-2j+2}/s, t^{-4j+3}/s^2q\end{matrix};t^2,t^2\right]\\ &= \left(st^{j-1}\right)^{-j}\frac{(t^{2j}s^2;t^2)_j}{(t^2;t^2)_j}\frac{(qt;t^2)_j(-s;t^2)_j}{(-t^{-2j+2}/s;t^2)_j(s^2qt^{2j-1};t^2)_j} = \frac{(qt;t^2)_j(s^2t^{2j},t^2)_j}{(t^2;t^2)_j(s^2qt^{2j-1};t^2)_j}. \qquad \blacksquare \end{aligned}$$

Corollary 8.2. *When $a = q^{1/2}$, $c = q$, $t = q$, we have*

$$B(s,j) = \begin{cases} 1, & j=0,\\ -1, & j=1,\\ 0, & j>1,\end{cases} \qquad \widetilde{B}(s,j) = 1, \qquad j \geq 0.$$

Corollary 8.3. *Let $m \in \mathbb{C}$. We have*

$$\lim_{q\to 0} B\left(t^{m+1},j\right)\big|_{a=t^{1/2},\,c=q^{1/2}t^{1/2}} = (-1)^j t^{j(j-1)}\frac{[m+2j]_{t^2}}{[m]_{t^2}}\begin{bmatrix} m+j-1\\ j\end{bmatrix}_{t^2},$$
$$\lim_{q\to 0} \widetilde{B}\left(t^{m+1},j\right)\big|_{a=t^{1/2},\,c=q^{1/2}t^{1/2}} = \begin{bmatrix} m+2j\\ j\end{bmatrix}_{t^2}.$$

8.1.2 (D_n, D_n) case

Proposition 8.4. *If $a = 1$, $c = q^{1/2}$, we have*

$$B(s,j) = \frac{(t/q;t^2)_j(s^2/t^2,t^2)_j}{(t^2;t^2)_j(s^2q/t;t^2)_j}\frac{1-st^{2j-1}}{1-s/t}q^j,$$
$$\widetilde{B}(s,j) = \frac{(q/t;t^2)_j(s^2t^{2j},t^2)_j}{(t^2;t^2)_j(s^2qt^{2j-3};t^2)_j}\frac{1+s/t}{1+st^{2j-1}}t^j.$$

Proof. When $a=1$, $c=q^{1/2}$, we have

$$\begin{aligned}B(s,j)&=(-1)^j s^{-j}\frac{\left(s^2/t^2;t^2\right)_j}{\left(t^2;t^2\right)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}{}_3\phi_2\left[\begin{matrix}-sq,s^2t^{2j-2},t^{-2j}\\ -st,s^2q/t\end{matrix};t^2,t^2\right]\\ &=(-1)^j s^{-j}\frac{\left(s^2/t^2;t^2\right)_j}{\left(t^2;t^2\right)_j}\frac{1-s^2t^{4j-2}}{1-s^2t^{-2}}\frac{\left(t/q,t^2\right)_j\left(-t^{-2j+3}/s;t^2\right)_j}{\left(-st;t^2\right)_j\left(t^{-2j+3}/s^2q;t^2\right)_j}\\ &=\frac{\left(t/q;t^2\right)_j\left(s^2/t^2,t^2\right)_j}{\left(t^2;t^2\right)_j\left(s^2q/t;t^2\right)_j}\frac{1-st^{2j-1}}{1-s/t}q^j,\end{aligned}$$

and

$$\begin{aligned}\widetilde{B}(s,j)&=t^j\left(st^{j-1}\right)^{-j}\frac{\left(s^2t^{2j};t^2\right)_j}{\left(t^2;t^2\right)_j}\frac{1+st^{-1}}{1+st^{2j-1}}{}_3\phi_2\left[\begin{matrix}-t^{-2j+3}/sq,t^{-2j+2}/s^2,t^{-2j}\\ -t^{-2j+2}/s,t^{-4j+5}/s^2q\end{matrix};t^2,t^2\right]\\ &=t^j\left(st^{j-1}\right)^{-j}\frac{\left(s^2t^{2j};t^2\right)_j}{\left(t^2;t^2\right)_j}\frac{1+st^{-1}}{1+st^{2j-1}}\frac{\left(q/t;t^2\right)_j\left(-s;t^2\right)_j}{\left(-t^{-2j+2}/s;t^2\right)_j\left(s^2qt^{2j-3};t^2\right)_j}\\ &=\frac{\left(q/t;t^2\right)_j\left(s^2t^{2j},t^2\right)_j}{\left(t^2;t^2\right)_j\left(s^2qt^{2j-3};t^2\right)_j}\frac{1+s/t}{1+st^{2j-1}}t^j.\end{aligned}$$

■

Corollary 8.5. *When $a=1$, $c=q^{1/2}$, $t=q$, we have*

$$B(s,j)=\delta_{j,0},\qquad \widetilde{B}(s,j)=\delta_{j,0}.$$

Corollary 8.6. *Let $m\in\mathbb{C}$. We have*

$$\begin{aligned}\lim_{q\to0}B\left(t^{m+1},j\right)\big|_{a=1,c=q^{1/2}}&=(-1)^jt^{j^2}\frac{[m+2j]_t}{[m]_t}\begin{bmatrix}m+j-1\\ j\end{bmatrix}_{t^2},\\ \lim_{q\to0}\widetilde{B}\left(t^{m+1},j\right)\big|_{a=1,c=q^{1/2}}&=t^j\frac{1+t^m}{1+t^{m+2j}}\begin{bmatrix}m+2j\\ j\end{bmatrix}_{t^2}.\end{aligned}$$

8.2 Explicit formulas for $P^{(C_n,C_n)}_{(1^r)}(x\,|\,t;q,t)$ and $P^{(D_n,D_n)}_{(1^r)}(x\,|\,q,t)$

Using the formulas obtained in the previous subsection, we give some explicit transition formulas for the polynomials $P^{(C_n,C_n)}_{(1^r)}(x\,|\,t;q,t)$ and $P^{(D_n,D_n)}_{(1^r)}(x\,|\,q,t)=P^{(C_n,C_n)}_{(1^r)}(x\,|\,1;q,t)$.

Theorem 8.7. *We have*

$$\begin{aligned}P^{(C_n,C_n)}_{(1^r)}(x\,|\,t;q,t)&=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\left(1/qt;t^2\right)_j\left(t^{2n-2r},t^2\right)_j}{\left(t^2;t^2\right)_j\left(qt^{2n-2r+3};t^2\right)_j}\frac{1-t^{2n-2r+4j}}{1-t^{2n-2r}}(qt)^jE_{r-2j}(x),\\ E_r(x)&=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\left(qt;t^2\right)_j\left(t^{2n-2r+2j+2},t^2\right)_j}{\left(t^2;t^2\right)_j\left(qt^{2n-2r+2j+1};t^2\right)_j}P^{(C_n,C_n)}_{(1^{r-2j})}(x\,|\,t;q,t),\\ P^{(D_n,D_n)}_{(1^r)}(x\,|\,q,t)&=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\left(t/q;t^2\right)_j\left(t^{2n-2r},t^2\right)_j}{\left(t^2;t^2\right)_j\left(qt^{2n-2r+1};t^2\right)_j}\frac{1-t^{n-r+2j}}{1-t^{n-r}}q^jE_{r-2j}(x),\\ E_r(x)&=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\left(q/t;t^2\right)_j\left(t^{2n-2r+2j+2},t^2\right)_j}{\left(t^2;t^2\right)_j\left(qt^{2n-2r+2j-1};t^2\right)_j}\frac{1+t^{n-r}}{1+t^{n-r+2j}}t^jP^{(D_n,D_n)}_{(1^{r-2j})}(x\,|\,q,t).\end{aligned}$$

Corollary 8.8. *Setting $t = q$, we have the formula for the Schur polynomials $s^{(C_n)}_{(1^r)}(x) = P^{(C_n,C_n)}_{(1^r)}(x\,|\,q;q,q)$ and $s^{(D_n)}_{(1^r)}(x) = P^{(D_n,D_n)}_{(1^r)}(x\,|\,q,q)$:*

$$s^{(C_n)}_{(1^r)}(x) = E_r(x) - E_{r-2}(x), \qquad E_r(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} s^{(C_n)}_{(1^{r-2k})}(x), \qquad s^{(D_n)}_{(1^r)}(x) = E_r(x).$$

Hence, from Lemma 3.3, we have

$$\begin{aligned} s^{(C_n)}_{(1^r)}(x) &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \left(\binom{n-r+2j}{j} - \binom{n-r+2j}{j-1} \right) m_{(1^{r-2j})}(x) \\ &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \frac{n-r+1}{n-r+j+1} \binom{n-r+2j}{j} m_{(1^{r-2j})}(x), \\ s^{(D_n)}_{(1^r)}(x) &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \binom{n-r+2j}{j} m_{(1^{r-2j})}(x). \end{aligned}$$

8.3 Hall–Littlewood polynomials $P^{(C_n,C_n)}_{(1^r)}(x\,|\,t;0,t)$, $P^{(D_n,D_n)}_{(1^r)}(x\,|\,0,t)$ and Kostka polynomials

Using the transition formulas we have established, we can study the Kostka polynomials associated with one column diagrams for types C_n and D_n. Setting $b = t$, $q = 0$ for (C_n, C_n) (or $b = 1$, $q = 0$ for (D_n, D_n)) in $P^{(C_n,C_n)}_{(1^r)}(x\,|\,b;q,t)$, we have the type C_n (or type D_n) Hall–Littlewood polynomials with one column diagrams.

Theorem 8.9. *We have*

$$\begin{aligned} P^{(C_n,C_n)}_{(1^r)}(x\,|\,t;0,t) &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^j t^{j(j-1)} \frac{[n-r+2j]_{t^2}}{[n-r]_{t^2}} \begin{bmatrix} n-r+j-1 \\ j \end{bmatrix}_{t^2} E_{r-2j}(x), \\ E_r(x) &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} P^{(C_n,C_n)}_{(1^{r-2j})}(x\,|\,t;0,t), \\ P^{(D_n,D_n)}_{(1^r)}(x\,|\,0,t) &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^j t^{j^2} \frac{[n-r+2j]_t}{[n-r]_t} \begin{bmatrix} n-r+j-1 \\ j \end{bmatrix}_{t^2} E_{r-2j}(x), \\ E_r(x) &= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} t^j \frac{1+t^{n-r}}{1+t^{n-r+2j}} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} P^{(D_n,D_n)}_{(1^{r-2j})}(x\,|\,0,t). \end{aligned}$$

Then, applying the formulas for the Schur polynomials in Corollary 8.8, we can calculate the Kostka polynomials (i.e., the transition coefficients from the Schur polynomials to the Hall–Littlewood polynomials) of types C_n and D_n associated with one column diagrams as follows.

Theorem 8.10. *We have*

$$s^{(C_n)}_{(1^r)}(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} t^{2j} \frac{[n-r+1]_{t^2}}{[n-r+j+1]_{t^2}} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} P^{(C_n,C_n)}_{(1^{r-2j})}(x\,|\,t;0,t)$$

$$= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \left(\begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} - \begin{bmatrix} n-r+2j \\ j-1 \end{bmatrix}_{t^2} \right) P^{(C_n,C_n)}_{(1^{r-2j})}(x \,|\, t; 0, t), \tag{8.1a}$$

$$s^{(D_n)}_{(1^r)}(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} t^j \frac{1+t^{n-r}}{1+t^{n-r+2j}} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} P^{(D_n,D_n)}_{(1^{r-2j})}(x \,|\, 0, t)$$

$$= \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \left(t^{n-r+j} \begin{bmatrix} n-r+2j-1 \\ j-1 \end{bmatrix}_{t^2} + t^j \begin{bmatrix} n-r+2j-1 \\ j \end{bmatrix}_{t^2} \right) P^{(D_n,D_n)}_{(1^{r-2j})}(x \,|\, 0, t). \tag{8.1b}$$

Hence we have Theorem 2.5.

Remark 8.11. The expansion coefficient of (8.1a) (times t^{-2j}) is identified with the q-ballot (when $m = 0$, q-Catalan) number [1, 6]

$$q^{-j} \left(\begin{bmatrix} m+2j \\ j \end{bmatrix}_q - \begin{bmatrix} m+2j \\ j-1 \end{bmatrix}_q \right) = \frac{[m+1]_q}{[m+j+1]_q} \begin{bmatrix} m+2j \\ j \end{bmatrix}_q,$$

by the replacement $m \to n-r$, $q \to t^2$. The case $m = 0$ gives us the q-Catalan number. It is known that the q-Catalan or q-ballot number is a polynomial in q with positive integral coefficients (see [1, 6]).

The expansion coefficient of (8.1b) is identified with the following version of the q-binomial number

$$q^j \frac{1+q^{m-2j}}{1+q^m} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} = q^{m-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_{q^2} + q^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2},$$

by the replacement $m \to n-r+2j$, $q \to t$. Note that this is also a polynomial in q with positive integral coefficients.

9 Some conjectures about Macdonald polynomials of type C_n

9.1 Asymptotically free eigenfunctions for the Macdonald operator of type A_{n-1}

First we recall some facts about the asymptotically free eigenfunctions for the case A_{n-1}. Let $n \in \mathbb{Z}_{>0}$, and $q, t \in \mathbb{C}$ be generic parameters. Let $x = (x_1, \dots, n_n)$ be a sequence of independent indeterminates. Macdonald's difference operator of type A_{n-1} is defined by

$$D^{(A_{n-1})} = \sum_{i=1}^{n} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}.$$

For a partition λ with $\ell(\lambda) \le n$, the Macdonald symmetric polynomial $P_\lambda(x; q, t) \in \mathbb{C}[x_1, \dots, x_n]^{S_n}$ exists uniquely characterized by the conditions:

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu, \qquad D^{(A_{n-1})} P_\lambda = \sum_{i=1}^{n} q^{\lambda_i} t^{n-i} \cdot P_\lambda.$$

Let $s_1, s_2, \dots, s_n \in \mathbb{C}$ be complex variables. Let $\mathsf{M}^{(n)}$ be the set of strict upper triangular matrices with entries in $\mathbb{Z}_{\ge 0}$, namely for $\theta^{(n)} = \big(\theta^{(n)}_{ij}\big)_{i,j \in \mathbb{Z}_{\ge 0}} \in \mathsf{M}^{(n)}$ $i \ge j$ implies $\theta^{(n)}_{ij} = 0$.

Definition 9.1. For $n \geq 1$, define recursively the rational functions $c_n(\theta^{(n)}; s_1, \dots, s_n; q, t) \in \mathbb{Q}(q, t, s_1, \dots, s_n)$ by $c_1(-; s_1; q, t) = 1$, and

$$c_n\big(\theta^{(n)}; s_1, \dots, s_n; q, t\big) = c_{n-1}\big(\theta^{(n-1)}; q^{-\theta_{1,n}} s_1, \dots, q^{-\theta_{n-1,n}} s_{n-1}; q, t\big) \times \prod_{1 \leq i \leq j \leq n-1} \frac{(ts_{j+1}/s_i; q)_{\theta_{i,n}}}{(qs_{j+1}/s_i; q)_{\theta_{i,n}}} \frac{\big(q^{-\theta_{j,n}} qs_j/ts_i; q\big)_{\theta_{i,n}}}{\big(q^{-\theta_{j,n}} s_j/s_i; q\big)_{\theta_{i,n}}}.$$

Definition 9.2. Set

$$\varphi^{(A_{n-1})}(s \,|\, x) = \sum_{\theta^{(n)} \in \mathsf{M}^{(n)}} c_n\big(\theta^{(n)}; s_1, \dots, s_n; q, t\big) \prod_{1 \leq i < j \leq n} \left(\frac{x_j}{x_i}\right)^{\theta_{ij}}.$$

Theorem 9.3 ([3, 16])**.** *Write $s_i = t^{n-1} q^{\lambda_i}$, $1 \leq i \leq n$ for simplicity. We have*

$$D^{(A_{n-1})} x^\lambda \varphi^{(A_{n-1})}(s \,|\, x) = (s_1 + \cdots + s_n) x^\lambda \varphi^{(A_{n-1})}(s \,|\, x).$$

When λ is a partition with $\ell(\lambda) \leq n$, we have

$$x^\lambda \varphi^{(A_{n-1})}(s \,|\, x) = P_\lambda(x).$$

Remark 9.4 (branching formulas)**.** We have the decomposition of the series $\varphi^{(A_{n-1})}$ in terms of the $\varphi^{(A_{n-2})}$ series as

$$\begin{aligned} &\varphi^{(A_{n-1})}(s_1 \dots, s_n \,|\, x_1 \dots, x_n) \\ &\quad = \sum_{\theta_{1n}, \dots, \theta_{n-1,n} \geq 0} \varphi^{(A_{n-2})}\big(q^{-\theta_{1n}} s_1, \dots, q^{-\theta_{n-1,n}} s_{n-1} \,|\, x_1, \dots, x_{n-1}\big) \\ &\qquad \times \prod_{1 \leq i \leq j \leq n-1} \frac{(ts_{j+1}/s_i; q)_{\theta_{i,n}}}{(qs_{j+1}/s_i; q)_{\theta_{i,n}}} \frac{\big(q^{-\theta_{j,n}} qs_j/ts_i; q\big)_{\theta_{i,n}}}{\big(q^{-\theta_{j,n}} s_j/s_i; q\big)_{\theta_{i,n}}} \cdot \prod_{i=1}^{n-1} \left(\frac{x_n}{x_i}\right)^{\theta_{in}}. \end{aligned}$$

9.2 Asymptotically free eigenfunction of type C_n

Let $n \in \mathbb{Z}_{>0}$. Let $x = (x_1, \dots, x_n)$ and $(s_1, \dots, s_n)$ be a pair of variables. Let $\mathcal{D}_x^{(C_n)} = \mathcal{D}_x\big({-t^{1/2}}, t^{1/2} - q^{1/2} t^{1/2}, q^{1/2} t^{1/2} \,|\, q, t\big)$ be the BC_n Koornwinder operator degenerated to the C_n case.

Definition 9.5. Set

$$s_i = t^{n-i+1} q^{\lambda_i}, \qquad 1 \leq i \leq n.$$

We define the asymptotically free eigenfunction $x^\lambda \varphi^{(C_n)}(s \,|\, x)$ of type C_n by

$$\begin{aligned} &\varphi^{(C_n)}(s \,|\, x) = \varphi^{(C_n)}(s_1, \dots, s_n \,|\, x_1, \dots, x_n) \\ &\qquad = \sum_{k_1, \dots, k_n \geq 0} c_{k_1, \dots, k_n}(s_1, \dots, s_n; q, t) \left(\frac{x_2}{x_1}\right)^{k_1} \cdots \left(\frac{x_n}{x_{n-1}}\right)^{k_{n-1}} \left(\frac{1}{x_n^2}\right)^{k_n}, \\ &\mathcal{D}_x^{(C_n)} x^\lambda \varphi^{(C_n)}(s \,|\, x) = \varepsilon^{(C_n)}(s) x^\lambda \varphi^{(C_n)}(s \,|\, x), \\ &\varepsilon^{(C_n)}(s) = \sum_{i=1}^n \big(s_i + s_i^{-1} - t^i - t^{-i}\big). \end{aligned}$$

9.3 C_2 case

Definition 9.6. Let x_1, x_2, s_1, s_2 be variables. Set

$$\begin{aligned}&\psi^{(C_2)}(s_1,s_2\,|\,x_1,x_2)\\&=\sum_{\theta_{12},\mu_{12},\rho_1,\rho_2\geq 0}c^{(C_2)}(\theta_{12},\mu_{12},\rho_1,\rho_2;s_1,s_2;q,t)\left(\frac{x_2}{x_1}\right)^{\theta_{12}}\left(\frac{1}{x_1x_2}\right)^{\mu_{12}}\left(\frac{1}{x_1^2}\right)^{\rho_1}\left(\frac{1}{x_2^2}\right)^{\rho_2},\end{aligned}$$

where

$$\begin{aligned}c^{(C_2)}(\theta_{12},\mu_{12},\rho_1,\rho_2;s_1,s_2;q,t)&=\frac{(t)_{\theta_{12}}}{(q)_{\theta_{12}}}\frac{(ts_2/s_1)_{\theta_{12}}}{(qs_2/s_1)_{\theta_{12}}}(q/t)^{\theta_{12}}\frac{(t)_{\mu_{12}}}{(q)_{\mu_{12}}}\frac{(t/s_1s_2)_{\mu_{12}}}{(q/s_1s_2)_{\mu_{12}}}(q/t)^{\mu_{12}}\\&\times\frac{(t/s_2)_{\mu_{12}}}{(q/s_2)_{\mu_{12}}}\frac{\left(q^{-\theta_{12}}q/ts_2\right)_{\mu_{12}}}{\left(q^{-\theta_{12}}/s_2\right)_{\mu_{12}}}\frac{(t/s_1)_{\mu_{12}}}{(q/s_1)_{\mu_{12}}}\frac{\left(q^{\theta_{12}}q/s_1\right)_{\mu_{12}}}{\left(q^{\theta_{12}}t/s_1\right)_{\mu_{12}}}\\&\times\frac{(t)_{\rho_1}}{(q)_{\rho_1}}\frac{\left(q^{\theta_{12}+\mu_{12}}t^2/s_1\right)_{\rho_1}}{\left(q^{\theta_{12}+\mu_{12}}qt/s_1\right)_{\rho_1}}(q/t)^{\rho_1}\frac{(t)_{\rho_2}}{(q)_{\rho_2}}\frac{\left(q^{-\theta_{12}+\mu_{12}}t/s_2\right)_{\rho_2}}{\left(q^{-\theta_{12}+\mu_{12}}q/s_2\right)_{\rho_2}}(q/t)^{\rho_2}.\end{aligned}$$

Conjecture 9.7. *We have* $\psi^{(C_2)}(s_1,s_2\,|\,x_1,x_2)=\varphi^{(C_2)}(s_1,s_2\,|\,x_1,x_2)$. *Namely, setting* $s_1=t^2q^{\lambda_1}$, $s_2=tq^{\lambda_2}$, $x^\lambda=x_1^{\lambda_1}x_2^{\lambda_2}$, *we have*

$$\begin{aligned}&\mathcal{D}_x^{(C_2)}x^\lambda\psi^{(C_2)}(s\,|\,x)=\varepsilon^{(C_2)}(s)x^\lambda\psi^{(C_2)}(s\,|\,x),\\&\varepsilon^{(C_2)}(s)=s_1+s_2+s_2^{-1}+s_1^{-1}-t^2-t-t^{-1}-t^{-2}.\end{aligned}$$

When $\lambda=(\lambda_1,\lambda_2)$ *is a partition, we have*

$$x^\lambda\psi^{(C_2)}(s\,|\,x)=P_\lambda^{(C_2)}(x\,|\,t;q,t).$$

9.4 C_3 case with rectangular diagrams

We can study the decomposition of the C_3 Macdonald polynomials $P_\lambda^{(C_3)}(x\,|\,t;q,t)$ in terms of the C_2 Macdonald polynomials. It seems that such a decomposition becomes rather simple when we consider the case of a rectangular diagram consisting of three equal rows $\lambda=(\lambda_3,\lambda_3,\lambda_3)$,

Definition 9.8. Let $\lambda_3\in\mathbb{C}$ and set

$$s_1=t^2s_3,\qquad s_2=ts_3,\qquad s_3=tq^{\lambda_3}.\tag{9.1}$$

Define

$$\begin{aligned}\psi^{(C_3),\mathrm{rect}}(s_3\,|\,x_1,x_2,x_3)&=\sum_{\mu_{13},\rho_1\geq 0}\frac{(t)_{\mu_{13}}}{(q)_{\mu_{13}}}\frac{\left(1/s_3^2\right)_{\mu_{13}}}{\left(q/ts_3^2\right)_{\mu_{13}}}(q/t)^{\mu_{13}}\frac{(t/s_3)_{\mu_{13}}}{(q/s_3)_{\mu_{13}}}\frac{(q/ts_3)_{\mu_{13}}}{(1/s_3)_{\mu_{13}}}\\&\times\frac{(t)_{\rho_1}}{(q)_{\rho_1}}\frac{\left(q^{\mu_{13}}t/s_3\right)_{\rho_1}}{\left(q^{\mu_{13}}q/s_3\right)_{\rho_1}}(q/t)^{\rho_1}\left(\frac{1}{x_1x_3}\right)^{\mu_{13}}\left(\frac{1}{x_1^2}\right)^{\rho_1}\varphi^{(C_2)}\left(ts_3,q^{-\mu_{13}}s_3\,|\,x_2,x_3\right).\end{aligned}$$

Conjecture 9.9. *We have* $\psi^{(C_3),\mathrm{rect}}(s_3\,|\,x_1,x_2,x_3)=\varphi^{(C_3)}\left(t^2s_3,ts_3,s_3\,|\,x_1,x_2,x_3\right)$. *Namely, setting* $x^\lambda=(x_1x_2x_3)^{\lambda_3}$, *we have*

$$\begin{aligned}&\mathcal{D}_x^{(C_3)}x^\lambda\varphi^{(C_3),\mathrm{rect}}(s_3\,|\,x_1,x_2,x_3)=\varepsilon^{(C_3)}(s)x^\lambda\varphi^{(C_3),\mathrm{rect}}(s_3\,|\,x_1,x_2,x_3),\\&\varepsilon^{(C_3)}(s)=s_1+s_2+s_3+s_3^{-1}+s_2^{-1}+s_1^{-1}-t^3-t^2-t-t^{-1}-t^{-2}-t^{-3},\end{aligned}$$

where s_1, s_2, s_3 *are as given in* (9.1). *When* λ_3 *is a nonnegative integer, we have*

$$x^\lambda\varphi^{(C_3),\mathrm{rect}}(s_3\,|\,x_1,x_2,x_3)=P^{(C_3)}_{(\lambda_3,\lambda_3,\lambda_3)}(x\,|\,t;q,t).$$

9.5 Folding of A_{2n-1} eigenfunctions and decomposition with respect to C_n eigenfunctions

Definition 9.10. Let $n \in \mathbb{Z}_{>0}$. Let $x = (x_1, \dots, x_n)$ and

$$s = (s_1, \dots, s_n), \qquad s_i = t^{n-i+1} q^{\lambda_i}, \qquad 1 \le i \le n,$$

be a pair of variables. Define the folded series $\widetilde{\varphi}^{(A_{2n-1})}(s \,|\, x) = \widetilde{\varphi}^{(A_{2n-1})}(s_1, \dots, s_n \,|\, x_1, \dots, x_n)$ by

$$\widetilde{\varphi}^{(A_{2n-1})}(s \,|\, x) = \varphi^{(A_{2n-1})}\big(t^{n-1} s_1, \dots, t^{n-1} s_n, t^{n-1}, t^{n-2}, \dots, t, 1 \,|\, x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}\big).$$

Proposition 9.11. *When $n = 1$, we have*

$$\widetilde{\varphi}^{(A_1)}(s \,|\, x) = \varphi^{(A_1)}\big(s_1, 1 \,|\, x_1, x_1^{-1}\big) = \sum_{\theta \ge 0} \frac{(t;q)_\theta}{(q;q)_\theta} \frac{(ts_1;q)_\theta}{(qs_1;q)_\theta} (q/t)^\theta x_1^{2\theta}.$$

Hence we have $\widetilde{\varphi}^{(A_1)}(s \,|\, x) = \varphi^{(C_1)}(s_1 \,|\, x_1)$.

We calculated the decomposition of the folded eigenfunctions $\widetilde{\varphi}^{(A_{2n-1})}(s \,|\, x)$ with respect to the C_n series $\varphi^{(C_n)}(s \,|\, x)$ for the cases $n = 2$ and 3 using Mathematica.

Conjecture 9.12. *We have*

$$\widetilde{\varphi}^{(A_3)}(s_1, s_2 \,|\, x_1, x_2) = \sum_{\mu_{12} \ge 0} e_2(s_1, s_2; \mu_{12}) \left(\frac{1}{x_1 x_2}\right)^{\mu_{12}} \varphi^{(C_2)}\big(q^{-\mu_{12}} s_1, q^{-\mu_{12}} s_2 \,|\, x_1, x_2\big),$$

where

$$e_2(s_1, s_2; \mu_{12}) = \frac{(t/s_1)_{\mu_{12}}}{(q/s_1)_{\mu_{12}}} \frac{(t/s_2)_{\mu_{12}}}{(q/s_2)_{\mu_{12}}} \frac{(t)_{\mu_{12}}}{(q)_{\mu_{12}}} \frac{\big(q^{\mu_{12}} q/ts_1 s_2\big)_{\mu_{12}}}{\big(q^{\mu_{12}}/s_1 s_2\big)_{\mu_{12}}} (q/t)^{\mu_{12}}.$$

Conjecture 9.13. *We have*

$$\begin{aligned}
\widetilde{\varphi}^{(A_5)}(s_1, s_2, s_3 \,|\, x_1, x_2, x_3) = {} & \sum_{\mu_{12}, \mu_{13}, \mu_{23} \ge 0} e_3(s_1, s_2, s_3; \mu_{12}, \mu_{13}, \mu_{23}) \left(\frac{1}{x_1 x_2}\right)^{\mu_{12}} \\
& \times \left(\frac{1}{x_1 x_3}\right)^{\mu_{13}} \left(\frac{1}{x_2 x_3}\right)^{\mu_{23}} \varphi^{(C_3)}\big(q^{-\mu_{12}-\mu_{13}} s_1, q^{-\mu_{12}-\mu_{23}} s_2, q^{-\mu_{13}-\mu_{23}} s_3 \,|\, x_1, x_2, x_3\big),
\end{aligned}$$

where

$$\begin{aligned}
e_3(s_1, s_2, s_3; \mu_{12}, \mu_{13}, \mu_{23}) = {} & \frac{(t/s_1)_{\mu_{12}+\mu_{13}}}{(q/s_1)_{\mu_{12}+\mu_{13}}} \frac{(t/s_2)_{\mu_{12}+\mu_{23}}}{(q/s_2)_{\mu_{12}+\mu_{23}}} \frac{(t/s_3)_{\mu_{13}+\mu_{23}}}{(q/s_3)_{\mu_{13}+\mu_{23}}} \\
& \times \frac{(t)_{\mu_{12}}}{(q)_{\mu_{12}}} \frac{\big(q^{\mu_{12}+\mu_{13}+\mu_{23}} q/ts_1 s_2\big)_{\mu_{12}}}{\big(q^{\mu_{12}+\mu_{13}+\mu_{23}}/s_1 s_2\big)_{\mu_{12}}} (q/t)^{\mu_{12}} \\
& \times \frac{(ts_3/s_1)_{\mu_{12}}}{(qs_3/s_1)_{\mu_{12}}} \frac{\big(q^{-\mu_{23}} qs_3/ts_1\big)_{\mu_{12}}}{\big(q^{-\mu_{23}} s_3/s_1\big)_{\mu_{12}}} \frac{(ts_3/s_2)_{\mu_{12}}}{(qs_3/s_2)_{\mu_{12}}} \frac{\big(q^{-\mu_{13}} qs_3/ts_2\big)_{\mu_{12}}}{\big(q^{-\mu_{13}} s_3/s_2\big)_{\mu_{12}}} \\
& \times \frac{(t)_{\mu_{13}}}{(q)_{\mu_{13}}} \frac{\big(q^{\mu_{12}+\mu_{13}+\mu_{23}} q/ts_1 s_3\big)_{\mu_{13}}}{\big(q^{\mu_{12}+\mu_{13}+\mu_{23}}/s_1 s_3\big)_{\mu_{13}}} (q/t)^{\mu_{13}} \frac{(ts_2/s_1)_{\mu_{13}}}{(qs_2/s_1)_{\mu_{13}}} \frac{\big(q^{-\mu_{23}} qs_2/ts_1\big)_{\mu_{13}}}{\big(q^{-\mu_{23}} s_2/s_1\big)_{\mu_{13}}} \\
& \times \frac{(t)_{\mu_{23}}}{(q)_{\mu_{23}}} \frac{\big(q^{\mu_{12}+\mu_{13}+\mu_{23}} q/ts_2 s_3\big)_{\mu_{23}}}{\big(q^{\mu_{12}+\mu_{13}+\mu_{23}}/s_2 s_3\big)_{\mu_{23}}} (q/t)^{\mu_{23}}.
\end{aligned}$$

Acknowledgements

Research of A.H. is supported by JSPS KAKENHI (Grant Number 16K05186). Research of J.S. is supported by JSPS KAKENHI (Grant Numbers 15K04808 and 16K05186). The authors thank M. Noumi, B. Feigin and H. Awata for stimulating discussion. They thank the anonymous referees for their various constructive comments.

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