A Family of GL_r Multiplicative Higgs Bundles on Rational Base

Rouven FRASSEK and Vasily PESTUN

Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France
E-mail: *frassek@ihes.fr, pestun@ihes.fr*

Abstract. In this paper we study a restricted family of holomorphic symplectic leaves in the Poisson–Lie group $\mathrm{GL}_r(\mathcal{K}_{\mathbb{P}^1_x})$ with rational quadratic Sklyanin brackets induced by a one-form with a single quadratic pole at $\infty \in \mathbb{P}_1$. The restriction of the family is that the matrix elements in the defining representation are linear functions of x. We study how the symplectic leaves in this family are obtained by the fusion of certain fundamental symplectic leaves. These symplectic leaves arise as minimal examples of (i) moduli spaces of multiplicative Higgs bundles on $\mathbb{P}^1$ with prescribed singularities, (ii) moduli spaces of $U(r)$ monopoles on $\mathbb{R}^2 \times S^1$ with Dirac singularities, (iii) Coulomb branches of the moduli space of vacua of 4d $\mathcal{N} = 2$ supersymmetric A_{r-1} quiver gauge theories compactified on a circle. While degree 1 symplectic leaves regular at $\infty \in \mathbb{P}^1$ (Coulomb branches of the superconformal quiver gauge theories) are isomorphic to co-adjoint orbits in $\mathfrak{gl}_r$ and their Darboux parametrization and quantization is well known, the case irregular at infinity (asymptotically free quiver gauge theories) is novel. We also explicitly quantize the algebra of functions on these moduli spaces by presenting the corresponding solutions to the quantum Yang–Baxter equation valued in Heisenberg algebra (free field realization).

Key words: symplectic leaves; Poisson–Lie group; Yang–Baxter equation; Sklyanin brackets; Coulomb branch; multiplicative Higgs bundles

2010 Mathematics Subject Classification: 16T25; 53D30; 81R12

1 Introduction

Complex completely integrable Hamiltonian systems can be typically constructed starting from a locus $\mathcal{M}$ in the moduli space $\mathrm{Bun}_G(\Sigma)$ of holomorphic G-bundles or sheaves of certain type on a complex holomorphic symplectic surface Σ with a structure of Lagrangian elliptic fibration $\Sigma \to X$, where the fibers Σ_x are possibly degenerate elliptic curves, and X is an algebraic curve typically called the base curve, see for example Section 0.3.6 in [27] and Section 3.8.2 in Donagi's lectures in [50] and [24, 25, 26, 27] for more complete details.

Indeed, the symplectic structure on Σ induces the symplectic structure on the space $\mathcal{M}$ which becomes the phase space of integrable system, the structure of Lagrangian fibration $\Sigma \to X$ induces the structure of Lagrangian fibration on $\mathcal{M}$, and the fact that the fibers Σ_x are abelian varieties (possibly degenerate elliptic curves) induces the structure of abelian varieties on the Lagrangian fibers in $\mathcal{M}$.

There are three cases to consider depending on whether the elliptic fibers are generically cusped elliptic, nodal elliptic or smooth elliptic.

(1) Fibers are cusped elliptic. If X is an algebraic curve, and $\Sigma \to X$ is a cotangent bundle whose fibers are compactified to cusped elliptic curves, this construction produces algebraic integrable system called Hitchin system on the curve X [28, 41, 49]. Hitchin system is an example of an abstract Higgs bundle on X valued in an abelian group K over X for the case when the group K is the canonical line bundle on X endowed with natural linear additive group structure

in the fiber direction. The Higgs field $\phi(x)$ is a holomorphic 1-form valued in the Lie algebra adjoint bundle ad $\mathfrak{g}$. The respective integrable system is of additive type in the fiber direction.

(2) Fibers are nodal elliptic. If $\Sigma \to X$ is a fibration whose fibers are nodal elliptic curves, then $\mathrm{Bun}_G(\Sigma)$ is equivalently described as a moduli space of multiplicative Higgs bundles $\mathrm{mHiggs}_G(X)$, that is moduli space of pairs (P, g) where P is a principal G-bundle on X, and Higgs field $g(x)$ is a section of Lie group adjoint bundle ad G. The respective integrable system is of multiplicative type in vertical direction. In Donagi's lectures in [26, Section 3.9] one finds a remark on three types of integrable system in the fiber direction corresponding to the three types of connected 1-dimensional complex groups: an elliptic curve, the multiplicative group $G_m = \mathbb{C}^\times$ and the additive group $G_a = \mathbb{C}$, and that the latter two can be considered as groups of non-singular points in the elliptic case in the nodal and cuspidal limit respectively. He goes on to clarify that Hitchin systems are associated to the cuspidal type and principal bundles on smooth elliptic fibrations to smooth elliptic type, and then asks "Is there an interesting geometric interpretation of the remaining "trigonometric" case, where the values are taken in multiplicative group G_m?" We believe so and we refer to several geometrical perspectives on the multiplicative case further in the introduction. For the basic definitions see [17, 30, 33, 42, 55].

(3) Fibers are smooth elliptic. If $\Sigma \to X$ is an elliptically fibered complex surface with generically smooth fibers, the corresponding case was studied in [25, 35]. Using Loojienga description of moduli space of G-bundles on a smooth elliptic fiber as a space conjugacy classes in the affine Kac–Moody Lie group $\hat{G}$ [47], we can also interpret $\mathrm{Bun}_G(\Sigma)$ as a moduli space $\mathrm{mHiggs}_{\hat{G}}(X)$ of multiplicative Higgs bundles for the affine Kac–Moody group $\hat{G}$. The respective integrable system is of elliptic type in vertical direction.

The case (1) of additive Higgs bundles (Hitchin systems) received large amount of attention in the mathematical literature in the context of geometrical Langlands correspondence and in the physical literature in the context of $6d$ $(2,0)$ superconformal self-dual tensor theory compactified on algebraic complex curve X for G of ADE type [2, 7, 45, 57]. Quantization of additive Higgs bundles on the curve X relates to the theory of Kac–Moody current algebras on X, conformal blocks of W-algebra on X with punctures, D-modules on $\mathrm{Bun}_G(X)$, and monodromy problems for various related differential equations.

The case (2) of multiplicative Higgs bundles on a complex curve X appeared first in the context of current Poisson–Lie groups $G(x)$ with spectral parameter $x \in X$. A Poisson–Lie group is a Lie group equipped with Poisson structure compatible with the group multiplication law. There is a standard way to equip $G(x)$ with Poisson structure called quadratic Skylanin bracket given a holomorphic no-where vanishing differential 1-form on X (possibly with poles). Quantization of this Poisson structure leads to the theory of quantum groups [29, 43] which have been discovered in the context of the inverse scattering method, quantum integrable spin chains, Yang–Baxter equation and R-matrix with spectral parameter. The standard horizontal trichotomy of the rational, trigonometric or elliptic R-matrix corresponds to taking the base X to be the $\mathbb{P}^1$ with 1-form with a single quadratic pole (rational type), the $\mathbb{P}^1$ with 1-form with two simple poles (trigonometric type), or smooth elliptic curve (elliptic type).

For the smooth elliptic base curve X the multiplicative Higgs bundle was studied in [42], following [11, 12]. Independently, the definition of multiplicative Higgs bundles was given in [33] where they were called G-pairs. On another hand, multiplicative Higgs bundles on X have been studied as periodic monopoles on real three-dimensional Riemannian manifold $X \times S^1$ via the monodromy map [15, 16, 17, 18, 37, 38, 55]. The relation between quantization of the moduli space of monopoles on $\mathbb{R}^3$ and Yangian has been proposed in [36] and further work in this direction has been in [44]. Recently a quantization of the holomorphic symplectic phase space of the moduli space of monopoles on $X \times S^1$ by a formal semi-holomorphic Chern–Simons functional on $X \times S^1 \times \mathbb{R}_t$, where $\mathbb{R}_t$ is the time direction, has been studied in [20, 21]. For simple Lie groups G of the ADE type these moduli spaces appear as Coulomb branches of the moduli

space of vacua of the $\mathcal{N} = 2$ supersymmetric ADE quiver gauge theory on $\mathbb{R}^3 \times S^1$ [13, 14, 53, 55]. Some constructions from the world of additive Higgs bundles have their versions in the world of multiplicative Higgs bundles [30] leading to difference equations and their monodromy problems [9, 58], q-geometric Langlands correspondence [1], $q-W$ algebras [46, 54, 56].

The goal of this paper is to present very concretely a Darboux coordinate system on a moduli space GL_r multiplicative Higgs bundles of degree 1 on the rational base $X = \mathbb{P}^1_x$. The base curve X is equipped with a holomorphic one-form $\mathrm{d}x$ that has the quadratic pole at $x_\infty = \infty$. The holomorphic one-form $\mathrm{d}x$ together with the Killing form on the Lie algebra induces the quadratic Sklyanin Poisson structure with the classical $\mathfrak{r}$-matrix of rational type in the spectral parameter x. Equivalently, we are studying degree 1 symplectic leaves in the rational Poisson–Lie group $\mathrm{GL}_r(\mathcal{K}_{\mathbb{P}^1})$, where $\mathcal{K}_{\mathbb{P}^1}$ denotes the field of rational functions on $\mathbb{P}^1$, and degree 1 means that all matrix elements $(L_{ij}(x))_{1 \leq i,j \leq r}$ of the multiplicative Higgs field $g(x)$ in the defining representation of GL_r by $r \times r$ matrices $L_{ij}(x)$ are degree 1 polynomials of x, i.e., linear functions of x.

By concrete presentation we mean introduction of explicit Darboux coordinates (canonically conjugated set of $(\underline{p}, \underline{q}) = (p_I, q^I)$ variables with $\{p_I, q^J\} = \delta_I^J$) and presentation of explicit formulae for the matrix elements $L_{ij}(x)$ in terms of (p_I, q^J). The complete set of commuting Hamiltonian functions is obtained from the coefficients of the spectral determinant of $L(x)$. The matrix $L(x)$ valued in functions on the phase space is called Lax matrix and its matrix elements satisfy quadratic Sklyanin Poisson brackets, see in particular [3, 31] but also the recent lecture notes [64].

The quadratic Sklyanin Poisson brackets can be also defined as semi-classical limit of the quantum Yang–Baxter equation [61, 62]. In this paper we find all rational solutions of degree 1 in the spectral parameter x associated to the classical Yang–Baxter equation defined by the rational $\mathfrak{gl}(r)$-invariant $\mathfrak{r}$-matrix, cf. [8]. In another note we plan to consider the trigonometric case associated to the base curve being a punctured nodal elliptic curve $X = \mathbb{C}^\times_x$ equipped with the holomorphic one-form $\frac{\mathrm{d}x}{x}$ that has simple pole at $x = 0$ and $x = \infty$, a related work appears in [32].

The case of $G = \mathrm{GL}_2$ is well studied. Here the Sklyanin relation admits three different elementary types of non-trivial solutions with matrix elements linear in the spectral parameter x that yield integrable models. These solutions are called the 2×2 elementary Lax matrices for the Heisenberg magnet, the DST chain and the Toda chain. For an overview we refer the reader to lecture notes of Sklyanin [63].

For higher rank r, to the best knowledge of the authors the explicit presentation of all linear solutions is missing in the literature. The case regular at the infinity $x_\infty \in X$ has been described in [60] and many other places. Some partial cases of Toda like solutions for irregular case have been described in [37, 38, 51]. The classifying labels appeared in [39]. In the quantum case some solutions to the Yang–Baxter equation were studied in connection to non-compact spin chains and Baxter Q-operators, in particular for the case of $\mathfrak{gl}_r$ we refer the reader to [5, 22]. The solutions relevant for non-compact spin chains can be obtained by realising the quantum R-matrix in terms of an infinite-dimensional oscillators algebra which is also known as free-field realisation, see, e.g., [23]. The Lax matrices relevant for Q-operators are certain degenerate solutions in the sense that the term proportional to the spectral parameter is not the identity matrix but a matrix of lower rank. These Lax matrices can be obtained from the non-degenerate case through a limiting procedure as discussed in [6] for $\mathfrak{gl}(2|1)$, [37, 38] for $\mathfrak{gl}_3$ or directly from the universal R-matrix as shown in [10] for $\mathfrak{gl}(3)$. Vice versa to the limiting procedure and as discussed in [5], one can also obtain the Lax matrices of non-compact spin chains by fusing the degenerate solutions relevant for Q-operators.

Here we follow the strategy of fusion in order to construct a family of GL_r Lax matrices $L(x)$ whose matrix elements are linear in spectral parameter x.

The discrete data of labels in our family is specified by two partitions λ and μ such that the total size is $|\lambda| + |\mu| = r$ and whose columns λ_i^t, μ_i^t are restricted by r. In addition to the discrete partition labels (λ, μ) we have a sequence of complex labels. There is a complex parameter x_i as-

signed to each column λ_i^t of the partition λ. Geometrically speaking, each pair $\left(\lambda_i^t, x_i\right)$ describes a type of singularity of the multiplicative Higgs field $g(x)$ at finite point $x_i \in \mathbb{C} = \mathbb{P}^1 \setminus \{x_\infty\}$ given by the conjugacy class of $(x - x_i)^{\check{\omega}_{\lambda_i^t}}$ where $\check{\omega}_k$ denotes k-th fundamental co-weight of GL_r: that is the highest weight of the k-th antisymmetric power of the fundamental representation for the Langlands dual group GL_r. Such highest weight is encoded by the column of height λ_i^t in the partition λ. Equivalently, in the neighborhood of the point x_i in the spectrum of the $r \times r$ Lax matrix $L(x)$ there are exactly λ_i^t eigenvalues which vanish linearly as x approaches x_i, and the remaining $r - \lambda_i^t$ eigenvalues are regular non-zero at x_i.

The partition μ specifies a dominant co-weight of singularity of the multiplicative Higgs field at the infinity point $x_\infty \in \mathbb{P}^1$, or equivalently the asymptotics of the eigenvalues of the Lax matrix $L(x)$ as $x \to \infty$: given r rows $(\mu_j)_{j \in [1,\dots,r]}$ of the partition μ, the j-th eigenvalue of the Lax matrix $L(x)$ has asymptotics $(x^{-1})^{\mu_j - 1}$ as $x \to \infty$.

We remark that the restriction on the total size of two partitions $|\mu| + \sum_i \lambda_i^t = r$ is a consequence of the restriction of the present paper to consider only Lax matrices whose matrix elements are linear functions of x. In the complete classification, if we allow higher degree of x in the matrix elements, which is not in the scope of the present paper, the label of a singularity at any finite point x_i is an arbitrary dominant GL_r co-weight described by an arbitrary partition λ_i, so that if rows of partition λ_i are denoted by $(\lambda_{ij})_{j \in [1,\dots,r]}$ then j-th eigenvalue of the Lax matrix $L(x)$ behaves as $(x - x_i)^{\lambda_{ij}}$ as $x \to x_i$. We leave for another note the presentation of explicit formulae for complete classification of the symplectic leaves of the degree d whose matrix elements are degree d polynomials of x for $|\mu| + \sum_i \lambda_i^t = dr$. (By looking at the determinant of $g(x)$ we see that the moduli space is non-empty only if the total size $|\mu| + \sum_i \lambda_i^t$ is integral multiple of rank r, cf., e.g., [42, 55]. This condition means that the total dominant co-weight summed over all singularities $\check{\omega}_{\mathrm{tot}}$ belongs to the lattice of co-roots.[1] To summarize, near every singularity on $\mathbb{CP}^1$ in a local coordinate w such that $w = 0$ is a position of singularity, we have asymptotics $[g(w)] \sim w^{\omega^\vee}$ where $\omega^\vee \colon \mathbb{C} \to T_G$ is a co-weight (either λ_i^t or μ) that characterizes the singularity. Normally, because the total degree ($U(1)$-charge) vanishes, the sum of degrees of all co-weights ω must vanish. We have chosen to shift the notational representation of the singularity co-weight at infinity by adding 1 to each row of the co-weight $\omega_\infty^\vee$ so that is described by a positive partition μ. In consequence, the sum over all partitions λ_i's and μ is r is no longer zero but r, since there are r rows in $\omega_\infty^\vee$, and each has been increased by 1 in our notations: $\mu_j = \omega_{\infty,j} + 1$ for each row $j = 1, \dots, r$.

In our solutions we can obtain higher (non-fundamental co-weight) singularities at finite point x_* by collision of several fundamental singularities at $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ which are associated to some columns $\lambda_{i_1}^t, \dots, \lambda_{i_k}^t$ of the partition λ by sending all of them to the common point x_*. In this case, generically, the multiplicative Higgs field $g(x)$ develops the singularity at point x_* specified by a higher (non-fundamental) co-weight $\sum_{j=1}^k \check{\omega}_{\lambda_{i_j}^t}$.

As we will see, all Lax matrices regular at the infinity x_∞, that is $\mu = \varnothing$ in the current notations, and arbitrary λ can be obtained by the fusion procedure of the elementary Lax matrices used in the Q-operator construction [5]. Also the case regular at infinity has been described in [60], where it was shown that degree 1 rational symplectic leaves for $G = \mathrm{GL}_r$ correspond to the co-adjoint orbits in the dual Lie algebra $\mathfrak{gl}_r^*$. The parametrization by Darboux coordinates of the holomorpic symplectic co-adjoint orbits in $\mathfrak{gl}_r^*$ identical to the present paper has been proposed in [4].

Then we proceed to build Lax matrices irregular at infinity from the fusion of a certain set of elementary Lax matrices whose irregularity at infinity is of the simplest type.

---

[1]In the monopole picture, the topological degree of gauge bundle induced on a surface enclosing all singularities is trivial. The topological degree is an element in $\pi_1(G) \simeq \check{\Lambda}/\check{Q}$, where $\check{\Lambda}$ and $\check{Q}$ denote the lattice of co-weights and co-roots.

Let us clarify the geometrical meaning of fusion. A Lax matrix $L_{\underline{\lambda},\underline{x},\mu}(x; \underline{p}, \underline{q})$ with a certain prescribed type of singularities at $\underline{x}, x_\infty$ parametrizes by a system of Darboux coordinates $(\underline{p}, \underline{q})$ a finite-dimensional symplectic leaf in the infinite-dimensional Poisson–Lie group $\mathcal{G} = \mathrm{GL}_r(\mathcal{K}_{\mathbb{P}^1})$ where $\mathcal{K}_{\mathbb{P}^1}$ denotes the field of rational functions on $\mathbb{P}^1$. More geometrically, a Lax matrix $L(x, \underline{p}, \underline{q})$ is a universal group valued (multiplicative) Higgs field on a Darboux chart in the second factor of $\mathbb{C}_x \times \mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ represented in $r \times r$ matrices, where $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ is a moduli space of multiplicative Higgs fields of a certain type $(\underline{\lambda}, \underline{x}, \mu)$, and complex spectral plane $\mathbb{C}_x$ is the domain of the Higgs field $g(x)$. So for us a Lax matrix $L_{\underline{\lambda},\underline{x},\mu}$ is a composition of Darboux chart parametrization

$$\mathbb{C}^{2d_{\underline{\lambda},\mu}} \to \mathcal{M}_{\underline{\lambda},\underline{x},\mu}$$

with a universal Higgs field map

$$\mathbb{C}_x \times \mathcal{M}_{\underline{\lambda},\underline{x},\mu} \to \mathrm{Mat}_{r \times r}.$$

Suppose we are given a symplectic leaf $\mathcal{M}_{\underline{\lambda},\underline{x},\mu} \subset \mathcal{G}$ described by a Lax matrix $L_{\underline{\lambda},\underline{x},\mu}(x; \underline{p}, \underline{q})$ and a symplectic leaf $\mathcal{M}'_{\underline{\lambda}',\underline{x}',\mu'} \subset \mathcal{G}$ described by a Lax matrix $L_{\underline{\lambda}',\underline{x}',\mu'}(x; \underline{p}', \underline{q}')$. By definition of Poisson–Lie group structure on $\mathcal{G}$ the group multiplication map

$$m \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G} \tag{1.1}$$

is a Poisson map, i.e., the pushforward of the product Poisson structure on $\mathcal{G} \times \mathcal{G}$ coincides with the Poisson structure on $\mathcal{G}$. The symplectic leaves $\mathcal{M}$, $\mathcal{M}'$ are, in particular, co-isotropic submanifolds of $\mathcal{G}$, hence $\mathcal{M} \times \mathcal{M}'$ is a co-isotropic submanifold of $\mathcal{G} \times \mathcal{G}$. Now, since the group multiplication map m in (1.1) is a Poisson map, and since the Poisson map preserves the co-isotropic property of the submanifolds, the image $m(\mathcal{M} \times \mathcal{M}') \subset \mathcal{G}$ is a co-isotropic subspace.

The $\mathcal{G}$-elements in the co-isotropic subspace $m(\mathcal{M} \times \mathcal{M}') \subset \mathcal{G}$ are represented by Lax matrices

$$L_{\underline{\lambda},\underline{x},\mu}(x; \underline{p}, \underline{q}) L_{\underline{\lambda}',\underline{x}',\mu'}(x; \underline{p}', \underline{q}') \tag{1.2}$$

and their type of singularities is typically a combination of the types of singularities of $(\underline{\lambda}, \underline{x}, \mu)$ and $(\underline{\lambda}', \underline{x}', \mu')$. However, $m(\mathcal{M} \times \mathcal{M}') \subset \mathcal{G}$ is not in general a symplectic leaf but a co-isotropic submanifold, and we can further slice it into symplectic leaves by determining the set of Casimir functions $\tilde{q}'$ on $m(\mathcal{M} \times \mathcal{M}')$ and a set of new conjugated coordinates $\tilde{p}, \tilde{q}$. We find that

$$L_{\underline{\lambda},\underline{x},\mu}(x; \underline{p}, \underline{q}) L_{\underline{\lambda}',\underline{x}',\mu'}(x; \underline{p}', \underline{q}') = \tilde{C}(\tilde{q}') \tilde{L}_{\tilde{\underline{\lambda}},\tilde{\underline{x}},\tilde{\mu}}(x; \tilde{p}, \tilde{q})$$

with the canonical transformation

$$\mathrm{d}\underline{p} \wedge \mathrm{d}\underline{q} + \mathrm{d}\underline{p}' \wedge \mathrm{d}\underline{q}' = \mathrm{d}\tilde{p} \wedge \mathrm{d}\tilde{q} + \mathrm{d}\tilde{p}' \wedge \mathrm{d}\tilde{q}'.$$

Notice that the conjugate variables $\tilde{p}'$ to the Casimir functions $\tilde{q}'$ on $\tilde{S}$ do not appear on the right side of (1.2). The Lax matrices $\tilde{L}_{\tilde{\underline{\lambda}},\tilde{\underline{x}},\tilde{\mu}}(x; \tilde{p}, \tilde{q})$ represent elements of $\mathcal{G}$ in a new symplectic leaf $\mathcal{M}_{\tilde{\underline{\lambda}},\tilde{\underline{x}},\tilde{\mu}}$ covered by Darboux coordinates $\tilde{p}, \tilde{q}$.

The symplectic leaves $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ arise as moduli spaces of multiplicative Higgs bundles of certain type [30], and like additive Higgs bundles (Hitchin system), the symplectic leaves $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ support the structure of an algebraic completely integrable system. In fact, the moduli spaces $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ can be also interpreted as moduli spaces of $U(r)$ monopoles on 3-dimensional Riemannian space $\mathbb{R}^2 \times S^1$ where $\mathbb{R}^2 \simeq \mathbb{C} = \mathbb{P}^1 \setminus \{x_\infty\}$, and consequently [16, 17, 18, 55] as moduli spaces of vacua of certain $\mathcal{N} = 2$ supersymmetric quiver gauge theories on $\mathbb{R}^3 \times S^1$ of quiver type A_{r-1}. The complex parameters $x_i \in \mathbb{C}$ which specify the position of singularities of the Lax matrix $L_{\underline{\lambda},\underline{x},\mu}$ play the role of the masses of the fundamental multiplets attached to the quiver node λ_i^t in

the A_{r-1} quiver diagram (i.e., the node associated to a simple root dual to the fundamental co-weight λ_i^t), and at the same time they play the role of the complex part of the coordinates of the positions of the Dirac singularities of the $U(r)$ monopoles on $\mathbb{R}^2 \times S^1$ under identification $\mathbb{R}^2 \simeq \mathbb{C}$. For polynomial Lax matrices that we consider in this paper the eigenvalues of $L(x)$ at the singularities $\underline{x}$ can have only zeros and no poles, thus the corresponding periodic monopoles can have only negatively charged Dirac singularities.

If the partition μ is empty, then the corresponding A_{r-1} quiver gauge theory is $\mathcal{N} = 2$ superconformal theory, and corresponding monopoles on $\mathbb{R}^2 \times S^1$ are regular at infinity. Non-empty partition μ corresponds to monopoles on $\mathbb{R}^2 \times S^1$ with non-trivial growth (or charge) at infinity controlled by μ, or to the Coulomb branches of asymptotically-free quiver gauge theories with β-function controlled by μ.

Consequently, the integrable system supported on a symplectic leaf $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ is identical to Seiberg–Witten integrable system for a certain A_{r-1} quiver gauge theory.

The complete set of commuting Hamiltonians functions H_{ij} can be extracted from the spectral determinant of the associated Lax matrix

$$\det \left(y - g_\infty L_{\underline{\lambda},\underline{x},\mu}(x; \underline{p}, \underline{q}) \right) = \sum_{i,j} H_{ij} x^i y^j \tag{1.3}$$

by taking coefficients at the monomials $x^i y^j$ where the appearing pairs of indices (i, j) can be described by certain profiles like Newton diagrams. The spectral curves (1.3) coincide with the spectral curves of the integrable systems studied in [55, 56]. Equivalently, since the determinant can be expanded in terms of the characters tr_{R_k} of the k-th external powers of the fundamental representation, the commuting Hamiltonians are expressed as coefficients at powers of x in the characters $\mathrm{tr}_{R^k} L(x)$.

We remark that by switching the role of variables $x \in \mathbb{C}$ and $y \in \mathbb{C}^\times$ (fiber-base duality) the spectral curve (1.3) of multiplicative Higgs bundle on X can be also interpreted as the spectral curve of additive Higgs bundle (Hitchin system) on $Y = \mathbb{C}^\times = \mathbb{P}^1_{0,\infty}$. This is a peculiarity related to the fact that we are considering the rational case of the base $X = \mathbb{P}^1$ corresponding to the monopoles on $\mathbb{R}^2 \times S^1$ and $4d$ quiver gauge theories rather than the trigonometric or elliptic base X corresponding to the monopoles on $\mathbb{R} \times S^1 \times S^1$ or $S^1 \times S^1 \times S^1$ that relate to $5d$ or $6d$ quiver gauge theories compactified on S^1 or $S^1 \times S^1$, and also that we take the gauge group to be of type GL_r. In this situation, the moduli space of $U(r)$ monopoles on $\mathbb{R}^2 \times S^1$ with several singularities has alternative presentation (Nahm duality) as GL_n Hitchin moduli space on $\mathbb{C}^\times$ with r singularities where n depends on the number and type of the singularities of the multiplicative Higgs bundle on X [16, 17, 18, 55].

Anyways, the fusion method of this paper allows us to analyze the multiplicative Higgs bundles in more general cases, which we leave for a future work, when Nahm duality of multiplicative Higgs bundle to a Hitchin system is not known. In particular, in the future one can study classification of symplectic leaves with matrix elements of higher degree in x, one can analyze trigonometric case with the base curve X is $\mathbb{C}^\times = \mathbb{P}^1 \setminus \{0, \infty\}$ or elliptic case when the base curve X is a smooth elliptic curve like [42] and consider arbitrary complex reductive Lie groups G.

The article is organised as follows. In Section 2 we remind and set notations about Poisson Lie groups, Sklyanin brackets and Lax matrices. In Section 3 we build the Lax matrices for arbitrary partitions $\underline{\lambda}$ and empty $\mu = \varnothing$ from certain elementary building blocks by fusion. Similarly, in Section 4 we build Lax matrices for arbitrary partitions μ with $\underline{\lambda} = \varnothing$ again employing certain elementary solutions using a slightly modified fusion procedure. In Section 5 we combine the solutions of Sections 3 and 4 to write down the Lax matrices for arbitrary $\underline{\lambda}$ and $\underline{\mu}$. In Section 6 we study the spectral determinant of the derived Lax matrices and compare our results with [55]. In Section 7 we say a few words on higher degree symplectic leaves. In Section 8 we consider the quantization of the algebra of functions and the integrable system.

2   Rational Poisson–Lie group and Sklyanin brackets

Let $X = \mathbb{P}^1$ be the base curve equipped with the differential holomorphic volume form dx that has a single quadratic pole at $x_\infty \in \mathbb{P}^1$. Fix a Killing form tr on $\mathfrak{g}$. Then the residue pairing

$$\mathrm{tr} \oint_{x=0} f(x)g(x)\mathrm{d}x$$

induces the metric on $\mathfrak{g}_D = \mathfrak{g}((x))$ with respect to which $\mathfrak{g}[[x]]$ and $x^{-1}\mathfrak{g}[x^{-1}]$ are isotropic subspaces and we have $\mathfrak{g}_D = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. This splitting induces the structure of the Lie bi-algebra on $\mathfrak{g}_+$, which means that the space of functions on $\mathfrak{g}_+$ is equipped with the Poisson bracket (induced from the Lie bracket on $\mathfrak{g}_-$). The data $(\mathfrak{g}_D, \mathfrak{g}_+, \mathfrak{g}_-)$ is called Manin triple. The Poisson bracket on the functions on $\mathfrak{g}_+$ can be extended to the Poisson bracket on the functions on the Lie group G_+ with the Lie algebra $\mathfrak{g}_+$, and the resulting bracket is called Sklyanin quadratic bracket with the rational $\mathfrak{r}$-matrix.

The space of rational multiplicative Higgs fields on $X = \mathbb{P}^1$ with a fixed framing of the gauge bundle at x_∞ forms a Poisson–Lie group [30].

In the following we consider gauge group $G = \mathrm{GL}_r$ and for a Higgs field $g(x)$ we call $L(x)$ the representation of $g(x)$ by $r \times r$ matrix valued functions $L(x)$ called Lax matrices. The space of Lax matrices $L(x)$ carries the quadratic Poisson bracket of rational Sklyanin type

$$\{L(x) \otimes I, I \otimes L(y)\} = [L(x) \otimes L(y), \mathfrak{r}(x - y)], \tag{2.1}$$

the quantization of which gives quantum Yang–Baxter equation [62]. Here the I denotes the $r \times r$ identity matrix, and the classical rational $\mathfrak{r}$-matrix of $\mathfrak{gl}(r)$ is

$$\mathfrak{r}(x) = x^{-1}\mathbb{P}, \qquad \text{with} \quad \mathbb{P} = \sum_{a,b=1}^{r} e_{ab} \otimes e_{ba}. \tag{2.2}$$

The bracket on the right-hand-side of (2.1) denotes the commutator $[X, Y] = XY - YX$. In a system of Darboux coordinates $(\underline{p}, \underline{q}) = (p_I, q^I)$, the Poisson bracket is

$$\{X, Y\} = \sum_I \left(\frac{\partial X}{\partial p_I} \frac{\partial Y}{\partial q^I} - \frac{\partial X}{\partial q^I} \frac{\partial Y}{\partial p_I} \right),$$

where we sum over all conjugate variables $(\underline{p}, \underline{q})$ in the Lax matrices L. In index notations, the Poisson bracket of matrix elements (2.2) reads as follows

$$\{L_{ij}(x), L_{kl}(y)\} = -\frac{1}{x - y}\left(L_{kj}(x)L_{il}(y) - L_{kj}(y)L_{il}(x) \right).$$

The solutions to the Sklyanin relation (2.1) that appear in this paper are labelled by two partitions

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r), \qquad \mu = (\mu_1, \mu_2, \dots, \mu_r),$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$ where $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$. The total number $|\lambda|$ of elements in the partition λ combined with total number $|\mu|$ of elements in the partition μ is equal to r. We study solutions $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ whose matrix elements are no higher than of degree 1 in the spectral parameter x. We can assume that

$$L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = x \times \mathrm{diag}(\underbrace{0, \dots, 0}_{\mu_1^t}, \underbrace{1, \dots, 1}_{r-\mu_1^t}) + M_{\lambda,\underline{x},\mu}(\underline{p}, \underline{q}).$$

Here μ_1^t denotes the first column, i.e., the first element in the transposed partition $\mu^t = \left(\mu_1^t, \mu_2^t, \ldots, \mu_r^t\right)$ and $M_{\lambda,\underline{x},\mu}(\underline{p},\underline{q})$ denotes an $r \times r$ matrix which is independent of the spectral parameter x. In total the matrix $M_{\lambda,\underline{x},\mu}(\underline{p},\underline{q})$ contains

$$d_{\lambda,\mu} = \frac{1}{2}\left(r^2 - \sum_{i=1}^{\lambda_1}\left(\lambda_i^t\right)^2 - \sum_{i=1}^{\mu_1}\left(\mu_i^t\right)^2\right), \tag{2.3}$$

pairs of variables $\left(p_I, q^I\right)$, i.e., $I = 1, 2, \ldots, d_{\lambda,\mu}$. Again the transposed partition is denoted as $\lambda^t = \left(\lambda_1^t, \lambda_2^t, \ldots, \lambda_r^t\right)$, and λ_i^t are called columns. The dimension of the corresponding symplectic leaf or the moduli space of multiplicative Higgs bundles will be given by

$$\dim_{\mathbb{C}} \mathcal{M}_{\lambda,\underline{x},\mu} = 2d_{\underline{\lambda},\mu}.$$

We fix the singularity of the $L(x)$ at points $x \to x_i$ to be of the form $[g(x)] \sim (x - x_i)^{\check{\omega}_{\lambda_i^t}}$, up to a regular factor, where $\check{\omega}_k$ is the k-th fundamental co-weight associated in Young notations to a column of height k, and at infinity $x \to x_\infty$ we take the singularity to be $[g(x)] \sim (x^{-1})^{\sum_i \check{\omega}_{\mu_i^t} - \check{\omega}_r}$. Here $\check{\omega}_r$ is a co-weight associated to the column of height r and denoting the diagonal homomorphism $\mathrm{GL}_1 \to T_{\mathrm{GL}_r}$ where T stands for the maximal torus, that is a co-weight dual to the weight of the determinant line representation.

The determinant of $L(x)$ determined by the partition λ is a polynomial of degree $|\lambda|$ with roots x_i of degeneracy λ_i^t:

$$\det L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = \prod_{i=1}^{\lambda_1}(x - x_i)^{\lambda_i^t}. \tag{2.4}$$

The explicit form of the matrices $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ is given in Section 3 for $\mu = \varnothing$, in Section 4 for $\lambda = \varnothing$ and for arbitrary λ and μ with $|\lambda| + |\mu| = r$ in Section 5.

As explained in the introduction, we can allow parameters x_i to collide in which case the dominant co-weight $\check{\omega}_*$ of the singularity at the collision point x_* is represented by a partition composed of several columns from λ, and is equal to the sum of the fundamental co-weights associated to each individual column in λ. In this way we get a symplectic leaf $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ whose singularity type at $x_* \in \underline{x}$ is described by a partition $\lambda_* \in \underline{\lambda}$. Bearing this in mind, in the following we assume that parameters x_i are assigned to individual columns λ_i^t of a single partition λ.

3   Degree 1 symplectic leaves regular
with fundamental singularity at infinity

In this section we focus on the GL_r Lax matrices that correspond to arbitrary partitions λ of size $|\lambda|$ and a single column μ-partition, $\mu = 1^{[r-|\lambda|]}$. In particular if $|\lambda| = r$ then μ is empty.

Since μ is a single column, the singularity at infinity is specified by a fundamental co-weight. The associated A_{r-1} quiver gauge theory with fundamental hypermultiplets [55] differs from the conformal class by absence of a single fundamental multiplet in the node μ_1^t.

We will assume that each element of the transposed partition λ^t, i.e., each column λ_i^t of the partition λ specifies a singularity of the Lax matrix $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ at point $x = x_i$ of the type $\check{\omega}_{\lambda_i^t}$. Here $\check{\omega}_k$ denotes a fundamental co-weight of GL_r of the form $(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{r-k})$ in the basis dual to the standard basis of weights of the defining representation.

This type of GL_r Lax matrices can be obtained by fusion of the fundamental solutions associated to a single column $\lambda = 1^{[|\lambda|]}$ and a single column $\mu = 1^{[r-|\lambda|]}$. The fundamental solutions are
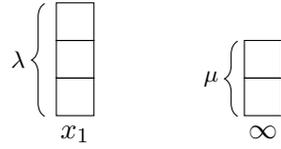
Figure 1. Single column partition for $r = 5$ with $\lambda = 1^{[3]}$ and $\mu = 1^{[2]}$.

given in Section 3.1, and the fusion is described in Section 3.3. The Lax matrices for arbitrary partitions λ are given in Section 3.2. We closely follow [5] where the elementary building blocks were derived, the factorisation was discussed on the quantum level and a closed formula for the Lax matrices was obtained for the case $\lambda = (r)$, see also [22].

3.1 Fundamental (λ, μ) orbits

The fundamental building blocks are $r \times r$ matrices that correspond to the partition

$$\mu = (\underbrace{1, \ldots, 1}_{|\mu|}), \qquad \lambda = (\underbrace{1, \ldots, 1}_{|\lambda|}),$$

with $r = |\lambda| + |\mu|$, see Fig. 1. They contain $|\lambda| \cdot |\mu|$ pairs of conjugate variables (p_{ij}, q_{ji}) where $1 \leq i \leq |\mu|$ and $|\mu| < j \leq r$ and can be written as

$$L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c} I & -P_{\mu,\lambda} \\ \hline Q_{\lambda,\mu} & (x - x_1)I - Q_{\lambda,\mu}P_{\mu,\lambda} \end{array} \right). \tag{3.1}$$

Here the upper diagonal block is of the size $|\mu| \times |\mu|$ and the lower one of size $|\lambda| \times |\lambda|$. The block matrices on the off-diagonal are parametrized as follows

$$(P_{\mu,\lambda})_{i,j} = p_{i,|\mu|+j}, \qquad (Q_{\lambda,\mu})_{i,j} = q_{|\mu|+i,j}.$$

The letter I denotes the identity matrix of appropriate size. In particular we have $L_{1^{[r]},\underline{x},\varnothing}(x) = (x - x_1)I$ and $L_{\varnothing,\varnothing,1^{[r]}} = I$.

The matrices $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ satisfy the Sklyanin relation (2.1) as can be verified by a direct computation using

$$\{(P_{\mu,\lambda})_{i,j}, (Q_{\lambda,\mu})_{k,l}\} = \delta_{i,l}\delta_{k,j}.$$

Consequently one finds

$$\{(Q_{\lambda,\mu}P_{\mu,\lambda})_{i,j}, (Q_{\lambda,\mu})_{k,l}\} = +(Q_{\lambda,\mu})_{i,l}\delta_{k,j}, \qquad \{(Q_{\lambda,\mu}P_{\mu,\lambda})_{i,j}, (P_{\mu,\lambda})_{k,l}\} = -(P_{\mu,\lambda})_{k,j}\delta_{i,l},$$

and

$$\{(Q_{\lambda,\mu}P_{\mu,\lambda})_{i,j}, (Q_{\lambda,\mu}P_{\mu,\lambda})_{k,l}\} = \delta_{k,j}(Q_{\lambda,\mu}P_{\mu,\lambda})_{i,l} - \delta_{i,l}(Q_{\lambda,\mu}P_{\mu,\lambda})_{k,j},$$

which is sufficient in order to check the Sklyanin Poisson bracket. It is instructive to see that $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ is factorized into a product of upper diagonal, diagonal and lower diagonal matrices:

$$L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c} I & 0 \\ \hline Q_{\lambda,\mu} & I \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & (x - x_1)I \end{array} \right) \left(\begin{array}{c|c} I & -P_{\mu,\lambda} \\ \hline 0 & I \end{array} \right).$$

The determinant is

$$\det L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = (x - x_1)^{|\lambda|}.$$
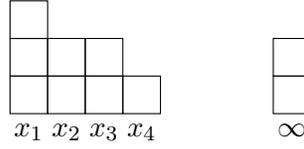
Figure 2. Regular partition with $\lambda = (4, 3, 1)$, $\mu = (1, 1)$ and $r = 10$.

3.2 Canonical coordinates on regular orbits

In this section we will construct solutions $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ for arbitrary partitions with λ composed of columns λ_i^t and a single column partition $\mu^t = \left(\mu_1^t\right)$.

The columns λ_i^t are associated to fundamental singularities at $x = x_i$ of type λ_i^t, which means that the singularity of $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ is in the conjugacy class of

$$\mathrm{diag}(\underbrace{(x - x_i), \ldots, (x - x_i)}_{\lambda_i^t}, 1, \ldots, 1), \qquad i = 1, \ldots, \lambda_1,$$

i.e., distinct λ_i^t eigenvalues of $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ are vanishing linearly at $x = x_i$.

The column μ_1^t describes a fundamental singularity at $x = \infty$ which means that the singularity of $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ at $x \to \infty$ is in conjugacy class of

$$x \, \mathrm{diag} \left(\underbrace{x^{-1}, \ldots, x^{-1}}_{\mu_1^t}, 1, \ldots, 1 \right).$$

We will prove recursively that regular Lax matrices $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ can be parametrized as a block matrix

$$L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c} I & -P_{\mu,\lambda} \\ \hline Q_{\lambda,\mu} & xI - J_{\lambda,\lambda} - Q_{\lambda,\mu}P_{\mu,\lambda} \end{array} \right), \tag{3.2}$$

where the upper-left block is of size $\mu_1^t \times \mu_1^t$ and bottom-right block is of size $|\lambda| \times |\lambda|$. The matrix elements of block $P_{\mu,\lambda}$ and block $Q_{\lambda,\mu}$ are canonically conjugated variables with

$$\{(P_{\mu,\lambda})_{ij}, (Q_{\lambda,\mu})_{kl}\} = \delta_{il}\delta_{jk}$$

and the matrix elements of $J_{\lambda,\lambda}$ satisfy the algebra of $\lambda \times \lambda$-matrices with respect to the Poisson brackets

$$\{J_{ij}, J_{kl}\} = \delta_{il}J_{kj} - \delta_{kj}J_{il}, \tag{3.3}$$

while Poisson commuting with matrix elements of $P_{\mu,\lambda}$ and $Q_{\lambda,\mu}$.

The matrix elements of the $|\lambda| \times |\lambda|$ matrix $J_{\lambda,\lambda}$ have an explicit parametrization in terms of the canonically conjugated coordinates as follows

$$J_{\lambda,\lambda} = Q_{\lambda,\lambda}(X_\lambda + [P_{\lambda,\lambda}Q_{\lambda,\lambda}]_+)Q_{\lambda,\lambda}^{-1}, \tag{3.4}$$

cf. [4, 60] and Appendix A. Here X_λ denotes the diagonal matrix

$$X_\lambda = \mathrm{diag}(\underbrace{x_1, \ldots, x_1}_{\lambda_1^t}, \underbrace{x_2, \ldots, x_2}_{\lambda_2^t}, \ldots, \underbrace{x_{\lambda_1}, \ldots, x_{\lambda_1}}_{\lambda_{\lambda_1}^t}). \tag{3.5}$$

The corresponding blocks on the diagonal are of the size $\lambda_1^t, \ldots, \lambda_{\lambda_1}^t$. The matrix $[P_{\lambda,\lambda} Q_{\lambda,\lambda}]_+$ is strictly upper block triangular and reads

$$
[P_{\lambda,\lambda}Q_{\lambda,\lambda}]_+ = \begin{pmatrix}
0 & \hat{P}_{1,2} & \hat{P}_{1,3} & \cdots & \hat{P}_{1,\lambda_1} \\
0 & 0 & \hat{P}_{2,3} & \cdots & \hat{P}_{2,\lambda_1} \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & 0 & \hat{P}_{\lambda_1-1,\lambda_1} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Here the matrices $\hat{P}_{ij}$ are of the size $\lambda_i^t \times \lambda_j^t$ and explicitly given by

$$
\hat{P}_{ij} = (P_{\lambda,\lambda})_{ij} + \sum_{k=j+1}^{\lambda_1} (P_{\lambda,\lambda})_{ik}(Q_{\lambda,\lambda})_{kj}. \tag{3.6}
$$

The matrix $Q_{\lambda,\lambda}$ is lower triangular and only depends on the variables $\underline{q}$ while $P_{\lambda,\lambda}$ is upper triangular and only depends on the variables $\underline{p}$. They read

$$
P_{\lambda,\lambda} = \begin{pmatrix}
0 & P_{1,2} & P_{1,3} & \cdots & P_{1,\lambda_1} \\
0 & 0 & P_{2,3} & \cdots & P_{2,\lambda_1} \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \ddots & P_{\lambda_1-1,\lambda_1} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
Q_{\lambda,\lambda} = \begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
Q_{2,1} & I & 0 & 0 & 0 \\
Q_{3,1} & Q_{3,2} & I & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
Q_{\lambda_1,1} & Q_{\lambda_1,2} & \cdots & Q_{\lambda_1,\lambda_1-1} & I
\end{pmatrix}, \tag{3.7}
$$

where Q_{ij} and P_{ij} denote $\lambda_i^t \times \lambda_j^t$ block matrices explicitly given by

$$
\begin{aligned}
(Q_{ij})_{kl} &= q_{\ell(i)+k,\ell(j)+l}, & k \in [1, \lambda_i^t], & \quad l \in [1, \lambda_j^t], \\
(P_{ij})_{kl} &= p_{\ell(i)+k,\ell(j)+l}, & k \in [1, \lambda_i^t], & \quad l \in [1, \lambda_j^t].
\end{aligned}
$$

Here we defined $\ell(i) = |\mu| + \sum_{k=1}^{i-1} \lambda_k^t$.

The realization (3.4) of the $\mathfrak{gl}(|\lambda|)$ algebra, also known as free field representation, can be constructed as algebra of twisted differential operators on the flag variety $G/P_{\lambda,+}$. Here $G = \mathrm{GL}(|\lambda|)$ and $P_{\lambda,+}$ denotes a parabolic subgroup of $\mathrm{GL}(|\lambda|)$ whose Levi is $\prod_i \mathrm{GL}(\lambda_i^t)$. The big cell of the flag variety $G/P_{\lambda,+}$ is identified with the λ^t-blocks unipotent subgroup $N_{\lambda,-}$ whose elements are represented by matrices $Q_{\lambda,\lambda}$ as in (3.7). In the classical limit twisted differential

operators in $J_{\lambda,\lambda}$ form a co-adjoint orbit O_{X_λ} in the dual Lie algebra $\mathfrak{g}^*$ for $\mathfrak{g} = \mathfrak{gl}(|\lambda|)$ of the semi-simple element X_λ (3.5). See details in Appendix A.

The number of pairs of conjugate variables in the Lax matrix (3.2) agrees with (2.3). There are $\mu_1^t \times |\lambda|$ pairs in $P_{\mu,\lambda}$, $Q_{\lambda,\mu}$ and $\sum_{i<j} \lambda_i^t \lambda_j^t$ in $J_{\lambda,\lambda}$. Further we verify that the determinant of (3.2) agrees with (2.4).

3.3   Regular orbits from fusion of fundamental orbits

We will construct the solution in the form of (3.2) associated to regular $(\tilde{\lambda}, \tilde{\mu})$ by fusion of two solutions associated to (λ, μ) and (λ', μ'). Here $(\tilde{\lambda}, \tilde{\mu})$ is defined such that

$$\tilde{\lambda}^t = \left(\lambda'^t, \lambda^t\right), \qquad |\tilde{\lambda}| = |\lambda| + |\lambda'|,$$

where (λ'^t, λ^t) denotes the partition given by the union of λ'^t and λ^t. The partitions μ, μ' and $\tilde{\mu}$ are single columns

$$\mu^t = (r - |\lambda|), \qquad \mu'^t = (r - |\lambda'|), \qquad \tilde{\mu}^t = \left(r - |\tilde{\lambda}|\right).$$

Then, by assumption of the recursion we represent $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ in the form

$$L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q}) = \left(\begin{array}{cc|c} I & 0 & -P_{\tilde{\mu},\lambda} \\ \hline 0 & I & -P_{\lambda',\lambda} \\ \hline Q_{\lambda,\tilde{\mu}} & Q_{\lambda,\lambda'} & xI - J_{\lambda,\lambda} - Q_{\lambda,\tilde{\mu}}P_{\tilde{\mu},\lambda} - Q_{\lambda,\lambda'}P_{\lambda',\lambda} \end{array}\right). \tag{3.8}$$

The blocks on the diagonal are of the size $|\tilde{\mu}|$, $|\lambda'|$ and $|\lambda|$ respectively, with $|\tilde{\mu}| + |\lambda'| + |\lambda| = r$. The matrix $L_{\lambda,\underline{x},\mu}(x; \underline{p}, \underline{q})$ explicitly depends on $|\lambda|(r - |\lambda|)$ pairs of conjugate variables arranged in the matrices $P_{\tilde{\mu},\lambda}$, $P_{\lambda',\lambda}$ and $Q_{\lambda,\tilde{\mu}}$, $Q_{\lambda,\lambda'}$ defined as

$$(P_{\tilde{\mu},\lambda})_{ij} = p_{i,|\tilde{\mu}|+|\lambda'|+j}, \qquad (P_{\lambda',\lambda})_{ij} = p_{|\tilde{\mu}|+i,|\tilde{\mu}|+|\lambda'|+j},$$
$$(Q_{\lambda,\tilde{\mu}})_{ij} = q_{|\tilde{\mu}|+|\lambda'|+i,j}, \qquad (Q_{\lambda,\lambda'})_{ij} = q_{|\tilde{\mu}|+|\lambda'|+i,|\tilde{\mu}|+j},$$

and the matrix $J_{\lambda,\lambda}$ of the size $|\lambda| \times |\lambda|$ defined in (3.4).

Similarly, we consider another Lax matrix

$$L'_{\lambda',\underline{x}',\mu'}(x; \underline{p}', \underline{q}') = \left(\begin{array}{c|c|c} I & -P'_{\tilde{\mu},\lambda'} & 0 \\ \hline Q'_{\lambda',\tilde{\mu}} & xI - J'_{\lambda',\lambda'} - Q'_{\lambda',\tilde{\mu}}P'_{\tilde{\mu},\lambda'} + P'_{\lambda',\lambda}Q'_{\lambda,\lambda'} & -P'_{\lambda',\lambda} \\ \hline 0 & -Q'_{\lambda,\lambda'} & I \end{array}\right), \tag{3.9}$$

with the same block structure as in (3.8). This matrix $L'_{\lambda',\underline{x}',\mu'}(x; \underline{p}', \underline{q}')$ explicitly depends on $|\lambda'|(r - |\lambda'|)$ pairs of conjugate variables

$$(Q'_{\lambda',\tilde{\mu}})_{ij} = q'_{|\tilde{\mu}|+i,j}, \qquad (Q'_{\lambda,\lambda'})_{ij} = q'_{|\tilde{\mu}|+|\lambda'|+i,|\tilde{\mu}|+j},$$
$$(P'_{\tilde{\mu},\lambda'})_{ij} = p'_{i,|\tilde{\mu}|+j}, \qquad (P'_{\lambda',\lambda})_{ij} = p'_{|\tilde{\mu}|+i,|\tilde{\mu}|+|\lambda'|+j},$$

and another set of variables appearing in the expression for $J'_{\lambda',\lambda'}$ like in (3.4). The matrix $L'_{\lambda',\underline{x}',\mu'}(x; \underline{p}', \underline{q}')$ that appears in (3.9) is obtained from the canonical form (3.2) by permutation, that is a conjugation by an element of the Weyl group of GL_r, and a canonical transformation in the variables $\underline{p}$ and $\underline{q}$.

In the next step we multiply the matrices (3.8) and (3.9). It was pointed out in [5] for the corresponding solutions of the quantum Yang–Baxter equation that the product can be written as

$$L_{\lambda,\underline{x},\mu}(x;\underline{p},\underline{q})L'_{\lambda',\underline{x}',\mu'}(x;\underline{p}',\underline{q}') = \tilde{Q}'\tilde{L}_{\tilde{\lambda},\tilde{\underline{x}},\tilde{\mu}}(x;\tilde{\underline{p}},\tilde{\underline{q}}). \tag{3.10}$$

Here $\tilde{L}_{\tilde{\lambda},\tilde{\underline{x}},\tilde{\mu}}(x;\tilde{\underline{p}},\tilde{\underline{q}})$ denotes a spectral parameter dependent Lax matrix and Casimir $\tilde{Q}'$ is a lower triangular matrix. They are of the form

$$\tilde{L}_{\tilde{\lambda},\tilde{\underline{x}},\tilde{\mu}}(x;\tilde{\underline{p}},\tilde{\underline{q}}) = WU\begin{pmatrix} I & 0 & 0 \\ 0 & xI - J'_{\lambda',\lambda'} & -\tilde{P}_{\lambda',\lambda} \\ 0 & 0 & xI - J_{\lambda,\lambda} \end{pmatrix}U^{-1}V^{-1}, \qquad \tilde{Q}' = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \tilde{Q}'_{\lambda,\lambda'} & I \end{pmatrix},$$

where

$$W = \begin{pmatrix} I & 0 & 0 \\ \tilde{Q}_{\lambda',\tilde{\mu}} & I & 0 \\ \tilde{Q}_{\lambda,\tilde{\mu}} & 0 & I \end{pmatrix}, \qquad U = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \tilde{Q}_{\lambda,\lambda'} & I \end{pmatrix}, \qquad V = \begin{pmatrix} I & \tilde{P}_{\tilde{\mu},\lambda'} & \tilde{P}_{\tilde{\mu},\lambda} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

expressed in terms of the new variables

$$\begin{aligned}
&\tilde{P}_{\lambda'\lambda} = P'_{\lambda'\lambda} + P_{\lambda'\lambda} - Q'_{\lambda'\tilde{\mu}}P_{\tilde{\mu}\lambda}, && \tilde{Q}_{\lambda\lambda'} = Q'_{\lambda\lambda'}, \\
&\tilde{P}_{\tilde{\mu}\lambda'} = P'_{\tilde{\mu}\lambda'} - P_{\tilde{\mu}\lambda}Q'_{\lambda\lambda'}, && \tilde{Q}_{\lambda'\tilde{\mu}} = Q'_{\lambda'\tilde{\mu}}, \\
&\tilde{P}_{\tilde{\mu}\lambda} = P_{\tilde{\mu}\lambda}, && \tilde{Q}_{\lambda\tilde{\mu}} = Q_{\lambda\tilde{\mu}} + Q'_{\lambda\lambda'}Q'_{\lambda'\tilde{\mu}}, \\
&\tilde{P}'_{\lambda'\lambda} = P_{\lambda'\lambda}, && \tilde{Q}'_{\lambda\lambda'} = Q_{\lambda\lambda'} - Q'_{\lambda\lambda'}.
\end{aligned} \tag{3.11}$$

The polynomial change of variables (3.11) is a symplectomorphism (i.e., canonical transformation) as we can directly verify. Indeed, computing the differentials we find

$$\begin{aligned}
\mathrm{d}\tilde{P}_{\lambda'\lambda} \wedge \mathrm{d}\tilde{Q}_{\lambda\lambda'} &= (\mathrm{d}P'_{\lambda'\lambda} + \mathrm{d}P_{\lambda'\lambda} - \mathrm{d}Q'_{\lambda'\tilde{\mu}}P_{\tilde{\mu}\lambda} - Q'_{\lambda'\tilde{\mu}}\mathrm{d}P_{\tilde{\mu}\lambda}) \wedge \mathrm{d}Q'_{\lambda\lambda'}, \\
\mathrm{d}\tilde{P}_{\tilde{\mu}\lambda'} \wedge \mathrm{d}\tilde{Q}_{\lambda'\tilde{\mu}} &= (\mathrm{d}P'_{\tilde{\mu}\lambda'} - \mathrm{d}P_{\tilde{\mu}\lambda}Q'_{\lambda\lambda'} - P_{\tilde{\mu}\lambda}\mathrm{d}Q'_{\lambda\lambda'}) \wedge \mathrm{d}Q'_{\lambda'\tilde{\mu}}, \\
\mathrm{d}\tilde{P}_{\tilde{\mu}\lambda} \wedge \mathrm{d}\tilde{Q}_{\lambda\tilde{\mu}} &= \mathrm{d}P_{\tilde{\mu}\lambda} \wedge (\mathrm{d}Q_{\lambda\tilde{\mu}} + \mathrm{d}Q'_{\lambda\lambda'}Q'_{\lambda'\tilde{\mu}} + Q'_{\lambda\lambda'}\mathrm{d}Q'_{\lambda'\tilde{\mu}}) \\
\mathrm{d}\tilde{P}'_{\lambda'\lambda} \wedge \mathrm{d}\tilde{Q}'_{\lambda\lambda'} &= \mathrm{d}P_{\lambda'\lambda} \wedge (\mathrm{d}Q_{\lambda\lambda'} - \mathrm{d}Q'_{\lambda\lambda'}),
\end{aligned}$$

and hence, after cancellations, we find that the canonical symplectic form is invariant

$$\begin{aligned}
\sum_{I \in \{\tilde{\mu}\lambda',\tilde{\mu}\lambda,\lambda'\lambda\}} \mathrm{d}\tilde{P}_I \wedge \mathrm{d}\tilde{Q}_{I^t} &+ \sum_{I \in \{\lambda'\lambda\}} \mathrm{d}\tilde{P}'_I \wedge \mathrm{d}\tilde{Q}'_i \\
= \sum_{I \in \{\tilde{\mu}\lambda,\lambda'\lambda\}} \mathrm{d}P_I \wedge \mathrm{d}Q_{I^t} &+ \sum_{I \in \{\tilde{\mu}\lambda',\lambda'\lambda\}} \mathrm{d}P'_I \wedge \mathrm{d}Q'_{I^t}.
\end{aligned}$$

In analogy to the Yang–Baxter equation, the product of two solutions to the Sklyanin relation (2.1) with different sets of conjugate variables $(\underline{p},\underline{q})$ is again a solution to the Sklyanin relation (2.1).

Therefore the matrix in (3.10) satisfies the Sklyanin bracket when taking the Poisson bracket with respect to the variables $(\tilde{\underline{p}},\tilde{\underline{q}})$ which denote the elements of the matrices defined in (3.11).

Finally, we note that the result is independent of $\tilde{P}'_{\lambda',\lambda}$ which allows us to strip off the matrix $\tilde{Q}'$ from (3.10). Thus we conclude that

$$\tilde{L}_{\tilde{\lambda},\tilde{x},\tilde{\mu}}(x;\tilde{p},\tilde{q}) = \left(\begin{array}{c|c} I & -\tilde{P}_{\tilde{\mu},\tilde{\lambda}} \\ \hline \tilde{Q}_{\tilde{\lambda},\tilde{\mu}} & xI - \tilde{J}_{\tilde{\lambda},\tilde{\lambda}} - \tilde{Q}_{\tilde{\lambda},\tilde{\mu}}\tilde{P}_{\tilde{\mu},\tilde{\lambda}} \end{array} \right),$$

with

$$\left(\tilde{P}_{\tilde{\mu},\tilde{\lambda}}\right)_{ij} = \tilde{p}_{i,|\tilde{\mu}|+j}, \qquad \left(\tilde{Q}_{\tilde{\lambda},\tilde{\mu}}\right)_{ij} = \tilde{q}_{|\tilde{\mu}|+i,j}$$

is a solution of the Sklyanin relation. Here the generators $\tilde{J}_{\tilde{\lambda},\tilde{\lambda}}$ of the $\mathfrak{gl}(|\lambda'| + |\lambda|)$ subalgebra are realised as

$$\tilde{J}_{\tilde{\lambda},\tilde{\lambda}} = \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}_{\lambda,\lambda'} & I \end{array} \right) \cdot \left(\begin{array}{c|c} J'_{\lambda',\lambda'} & \tilde{P}_{\lambda',\lambda} \\ \hline 0 & J_{\lambda,\lambda} \end{array} \right) \cdot \left(\begin{array}{c|c} I & 0 \\ \hline -\tilde{Q}_{\lambda,\lambda'} & I \end{array} \right),$$

where

$$\left(\tilde{P}_{\lambda',\lambda}\right)_{ij} = \tilde{p}_{|\tilde{\mu}|+i,|\tilde{\mu}|+|\lambda'|+j}, \qquad \left(\tilde{Q}_{\lambda,\lambda'}\right)_{ij} = \tilde{q}_{|\tilde{\mu}|+|\lambda'|+i,|\tilde{\mu}|+j}.$$

Let us remark that here we have chosen a certain order of fusion, but depending on the order we would get different parametrization related by a polynomial choice of variables, see, e.g., Appendix G. It would be interesting to explore the resulting cluster structure in more details.

3.3.1 Linear fusion

Now to demonstrate the particular parametrization (3.4) for $\tilde{J}_{\tilde{\lambda},\tilde{\lambda}}$ it is sufficient to assume that λ' is a single column partition $\lambda'^t = (\lambda'^t_1)$ while λ is an arbitrary collection of columns. In this case

$$J'_{\lambda',\lambda'} = X_{\lambda'}, \qquad J_{\lambda,\lambda} = Q_{\lambda,\lambda}(X_\lambda + [P_{\lambda,\lambda}Q_{\lambda,\lambda}]_+)Q_{\lambda,\lambda}^{-1}.$$

Then we find that $\tilde{J}_{\tilde{\lambda}\tilde{\lambda}}$ can be again represented in the form

$$\begin{aligned}
\tilde{J}_{\tilde{\lambda}\tilde{\lambda}} = & \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}_{\lambda\lambda'} & I \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & Q_{\lambda\lambda} \end{array} \right) \left(\begin{array}{c|c} X_{\lambda'} & \tilde{P}_{\lambda'\lambda}Q_{\lambda,\lambda} \\ \hline 0 & X_\lambda + [P_{\lambda,\lambda}Q_{\lambda,\lambda}]_+ \end{array} \right) \\
& \times \left(\begin{array}{c|c} I & 0 \\ \hline 0 & Q_{\lambda,\lambda} \end{array} \right)^{-1} \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}_{\lambda\lambda'} & I \end{array} \right)^{-1}
\end{aligned} \tag{3.12}$$

or

$$\tilde{J}_{\tilde{\lambda},\tilde{\lambda}} = Q_{\tilde{\lambda},\tilde{\lambda}}(X_{\tilde{\lambda}} + [P_{\tilde{\lambda},\tilde{\lambda}}Q_{\tilde{\lambda},\tilde{\lambda}}]_+)Q_{\tilde{\lambda},\tilde{\lambda}}^{-1}$$

with

$$Q_{\tilde{\lambda},\tilde{\lambda}} = \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}_{\lambda\lambda'} & Q_{\lambda,\lambda} \end{array} \right), \qquad P_{\tilde{\lambda},\tilde{\lambda}} = \left(\begin{array}{c|c} 0 & \tilde{P}_{\lambda'\lambda} \\ \hline 0 & P_{\lambda,\lambda} \end{array} \right).$$

As a consequence it follows that the Lax matrices (3.2) satisfy the Sklyanin relation (2.1).
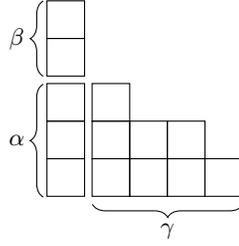
Figure 3. Example of the decomposition in (4.1) for $\mu = (5, 4, 2, 1, 1)$. We have $\alpha = (1, 1, 1)$, $\beta = (1, 1)$ and $\gamma = (4, 3, 1)$.

4 Degree 1 symplectic leaves singular only at infinity

In the following section we focus on the Lax matrices that correspond to $\lambda = \varnothing$ and arbitrary partition μ. Similar to the case labelled by pure λ partitions in Section 3 the present case can be obtained from fusion of the basic building blocks. These basic building blocks are generalisations of the well-known Lax matrix of the Toda chain [31] corresponding to the partition $\mu = (2)$. They are introduced in Sections 4.1 and 4.2. The Lax matrices for arbitrary partitions μ are presented in Section 4.3. As discussed in Section 4.4 we can apply a similar fusion procedure as in Section 3.2 to derive the general form of the Lax matrices.

To describe the Lax matrices it is convenient to introduce the partitions

$$\alpha = \big(\underbrace{1, \ldots, 1}_{\mu_2^t} \big), \qquad \beta = \big(\underbrace{1, \ldots, 1}_{\mu_1^t - \mu_2^t} \big), \qquad \gamma = \big(\mu_2^t, \ldots, \mu_{\mu_1}^t \big)^t, \tag{4.1}$$

as shown in Fig. 3. The partition μ is then written as $\mu^t = \big(|\alpha| + |\beta|, \gamma_1^t, \ldots, \gamma_{\gamma_1}^t \big)$. For simplicity we are only considering partitions with $\mu_i \geq \mu_j$ for $1 \leq i < j \leq \mu_1^t$.

4.1 Elementary μ partitions: $\alpha = \gamma$ and $\beta = 0$

First we introduce the Lax matrices that correspond to the partitions $\lambda = \varnothing$ and $\mu = 2^{\lceil \frac{r}{2} \rceil}$ with

$$\alpha = \big(\underbrace{1, \ldots, 1}_{\mu_1^t} \big), \qquad \beta = 0, \qquad \gamma = \big(\underbrace{1, \ldots, 1}_{\mu_2^t} \big),$$

where $\mu_1^t = \mu_2^t = \frac{r}{2}$ and $r \in 2\mathbb{N}$. The Lax matrices $L_\mu(x; \underline{p}, \underline{q}) = L_{\varnothing, \varnothing, \mu}(x; \underline{p}, \underline{q})$ are $r \times r$ matrices with $|\alpha| + |\gamma| = r$ whose determinant evaluates to unity. They contain $\left(\frac{r}{2} \right)^2$ pairs of conjugate variables (p_I, q^I) and can be written in the form

$$L_\mu(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c} 0 & K_{\alpha, \gamma} \\ \hline \bar{K}_{\gamma, \alpha} & xI - F_{\gamma, \gamma} \end{array} \right). \tag{4.2}$$

For later purposes we labeled the upper block by α and the lower block by γ such that the block on the diagonal are of equal size $|\alpha| \times |\alpha|$ and $|\gamma| \times |\gamma|$ respectively. Further we introduced the matrices

$$F_{\gamma, \gamma} = Q_- G Q_-^{-1}, \qquad \text{with} \quad G = P_0 + [P_+ Q_-]_+ + Q_0 [Q_+ P_-]_- Q_0^{-1}, \tag{4.3}$$

where $[\ ]_\pm$ denotes the projection on the upper and lower diagonal part respectively and

$$\bar{K}_{\gamma, \alpha} = Q_- Q_0 Q_+, \qquad K_{\alpha, \gamma} = -Q_+^{-1} Q_0^{-1} Q_-^{-1} = -\bar{K}_{\gamma, \alpha}^{-1}. \tag{4.4}$$

The matrices $Q_{\pm,0}$ are parametrized in terms of the conjugate variables $(\underline{p}, \underline{q})$ as follows

$$Q_- = I + \sum_{|\mu| \geq i > j > \mu_1^t} q_{ij} e_{ij}, \qquad Q_+ = I + \sum_{\mu_1^t < i < j \leq |\mu|} q_{ij} e_{ij}, \qquad Q_0 = \sum_{i=\mu_1^t+1}^{|\mu|} e^{q_{ii}} e_{ii},$$

and

$$P_- = \sum_{|\mu| \geq i > j > \mu_1^t} p_{ij} e_{ij}, \qquad P_+ = \sum_{\mu_1^t < i < j \leq |\mu|} q_{ij} e_{ij}, \qquad P_0 = \sum_{i=\mu_1+1}^{|\mu|} p_{ii} e_{ii}.$$

We note that Q_+ is an upper triangular matrix containing variables q_{ij} with $i > j$, while Q_- is lower triangular containing the variables q_{ij} with $i < j$. The diagonal matrix Q_0 only contains the exponential function of q_{ii}. All variables p_{ij} are contained in G which is decomposed as the sum of a diagonal, a lower diagonal and an upper diagonal matrix.

The Sklyanin relation is equivalent to the commutators

$$\{F_{ij}, K_{kl}\} = -K_{kj}\delta_{il}, \qquad \{F_{ij}, \bar{K}_{kl}\} = \bar{K}_{il}\delta_{kj}, \qquad \{K_{ij}, \bar{K}_{kl}\} = 0, \tag{4.5}$$

$$\{F_{ij}, F_{kl}\} = \delta_{kj} F_{il} - \delta_{il} F_{kj}. \tag{4.6}$$

Here the latter can be identified with commutators of the $\mathfrak{gl}(\frac{r}{2})$ algebra, while the parametrization of $K_{\alpha,\gamma}$ is given in terms of a Gauss decomposition of $\mathrm{GL}(\frac{r}{2})$. These relations are verified explicitly in Appendix C.

For $\mu = (2)$, i.e., $|\alpha| = |\gamma| = 1$, the Lax matrix in (4.2) reproduces the well known Lax matrix for the Toda chain

$$L_{(2)}(x; p, q) = \begin{pmatrix} 0 & -e^{-q} \\ e^q & x - p \end{pmatrix}.$$

4.2 Elementary μ partitions: $\alpha = \gamma$ and $\beta \neq 0$

We can extend the elementary Lax matrices to the case $\beta \neq 0$ which can be used to obtain the Lax matrices for arbitrary partitions μ. They correspond to the partitions

$$\mu = (\underbrace{2, \dots, 2}_{|\alpha|=|\gamma|}, \underbrace{1, \dots, 1}_{|\beta|}),$$

with $|\alpha| + |\beta| + |\gamma| = r$ and contain $|\gamma|(|\alpha| + |\beta|)$ pairs of conjugate variables. The Lax matrices can be defined from $L_\mu(x; \underline{p}, \underline{q})$ with $\alpha = \gamma$ and $\beta = 0$ given in (4.2) as

$$L_\mu(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c|c} 0 & 0 & K_{\alpha,\gamma} \\ \hline 0 & I & -P_{\beta,\gamma} \\ \hline \bar{K}_{\gamma,\alpha} & Q_{\gamma,\beta} & xI - F_{\gamma,\gamma} - Q_{\gamma,\beta} P_{\beta,\gamma} \end{array} \right). \tag{4.7}$$

Here $F_{\gamma,\gamma}$, $K_{\alpha,\gamma}$ and $\bar{K}_{\gamma,\alpha}$ are defined in (4.4) and do not depend on β. The diagonal block containing the identity matrix I is of the size $|\beta|$. The matrices $P_{\beta,\gamma}$ and $Q_{\gamma,\beta}$ read

$$(P_{\beta,\gamma})_{i,j} = p_{|\alpha|+i,|\alpha|+|\beta|+j}, \qquad (Q_{\gamma,\beta})_{i,j} = q_{|\alpha|+|\beta|+i,|\alpha|+j}. \tag{4.8}$$

The proof of Sklyanin relation is straightforward combining the proofs in Sections 3.1 and 4.1. The determinant can be obtained using (B.1) and yields unity. From here one may build all other Lax matrices corresponding to arbitrary μ partitions by factorisation. The result is presented in the next subsection.

4.3   Lax matrices for μ partitions

The Lax matrix for arbitrary μ partitions can be written in the form

$$L_\mu(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c|c} 0 & 0 & K_{\alpha,\gamma} \\ \hline 0 & I & -P_{\beta,\gamma} \\ \hline \bar{K}_{\gamma,\alpha} & Q_{\gamma,\beta} & xI - F_{\gamma,\gamma} - Q_{\gamma,\beta}P_{\beta,\gamma} \end{array} \right). \tag{4.9}$$

The blocks on the diagonal of this Lax matrix are of the size $|\alpha|$, $|\beta|$ and $|\gamma|$, respectively, with $|\alpha| + |\beta| + |\gamma| = r$. The matrices $P_{\beta,\gamma}$ and $Q_{\gamma,\beta}$ are defined as in (4.8) and contain $|\gamma| \cdot |\beta|$ pairs of conjugate variables. The remaining matrix elements can then be expressed in terms of γ_1 copies of the matrices defined in (4.3) and (4.4). We have

$$F_{\gamma,\gamma} = Q_{\gamma,\gamma} \left(\begin{array}{c|c|c|c|c} F_{1,1} & \hat{P}_{1,2} & \hat{P}_{1,3} & \cdots & \hat{P}_{1,\gamma_1} \\ \hline W_{2,1} & F_{2,2} & \hat{P}_{2,3} & \cdots & \hat{P}_{2,\gamma_1} \\ \hline W_{3,1} & W_{3,2} & F_{3,3} & \ddots & \vdots \\ \hline \vdots & \vdots & \ddots & \ddots & \hat{P}_{\gamma_1-1,\gamma_1} \\ \hline W_{\gamma_1,1} & W_{\gamma_1,2} & \cdots & W_{\gamma_1,\gamma_1-1} & F_{\gamma_1,\gamma_1} \end{array} \right) Q_{\gamma,\gamma}^{-1}, \tag{4.10}$$

$$K_{\alpha,\gamma} = \left(\begin{array}{c|c|c} D_{\alpha,\alpha}^{[\gamma_1]} \cdots D_{\alpha,\alpha}^{[2]} K_{\alpha,1} & \cdots & D_{\alpha,\alpha}^{[\gamma_1]} K_{\alpha,\gamma_1-1} & K_{\alpha,\gamma_1} \end{array} \right) Q_{\gamma,\gamma}^{-1} \tag{4.11}$$

and

$$\bar{K}_{\gamma,\alpha} = Q_{\gamma,\gamma} \left(\begin{array}{c} \bar{K}_{1,\alpha} \\ \hline \bar{K}_{2,\alpha} D_{\alpha,\alpha}^{[1]} \\ \hline \cdots \\ \hline \bar{K}_{\gamma_1,\alpha} D_{\alpha,\alpha}^{[\gamma_1-1]} \cdots D_{\alpha,\alpha}^{[1]} \end{array} \right). \tag{4.12}$$

Each block (i,j) in (4.10) is of the size $\gamma_i^t \times \gamma_j^t$. The matrices $\hat{P}_{ij}$ are defined as in (3.6) with

$$\hat{P}_{ij} = (P_{\gamma,\gamma})_{ij} + \sum_{k=j+1}^{\gamma_1} (P_{\gamma,\gamma})_{ik}(Q_{\gamma,\gamma})_{kj}.$$

The corresponding matrices $Q_{\gamma,\gamma}$ and $P_{\gamma,\gamma}$ as defined for the partition λ in (3.7) read

$$Q_{\gamma,\gamma} = \left(\begin{array}{c|c|c|c|c} I & 0 & 0 & 0 & 0 \\ \hline Q_{2,1} & I & 0 & 0 & 0 \\ \hline Q_{3,1} & Q_{3,2} & I & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \ddots & 0 \\ \hline Q_{\gamma_1,1} & Q_{\gamma_1,2} & \cdots & Q_{\gamma_1,\gamma_1-1} & I \end{array} \right), \quad P_{\gamma,\gamma} = \left(\begin{array}{c|c|c|c|c} 0 & P_{1,2} & P_{1,3} & \cdots & P_{1,\gamma_1} \\ \hline 0 & 0 & P_{2,3} & \cdots & P_{2,\gamma_1} \\ \hline 0 & 0 & 0 & \ddots & \vdots \\ \hline 0 & 0 & 0 & \ddots & P_{\gamma_1-1,\gamma_1} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

where Q_{ij} and P_{ij} denote block matrices explicitly given by

$$(Q_{ij})_{kl} = q_{\ell(s)+k,\ell(t)+l}, \qquad k \in [1,\gamma_i^t], \quad l \in [1,\gamma_j^t],$$
$$(P_{ij})_{kl} = p_{\ell(s)+k,\ell(t)+l}, \qquad k \in [1,\gamma_i^t], \quad l \in [1,\gamma_j^t].$$

Here we defined $\ell(i) = |\alpha| + |\beta| + \sum_{l=1}^{i-1} \gamma_l^t$.

The elements on the lower diagonal of the middle part of $F_{\gamma,\gamma}$ in (4.10) are defined as the product

$$W_{ij} = -\bar{K}_{i,\alpha} D_{\alpha,\alpha}^{[i-1]} \cdots D_{\alpha,\alpha}^{[j+1]} K_{\alpha,j},$$

which in particular yields $W_{i+1,i} = -\bar{K}_{i+1,\alpha} K_{\alpha,i}$. The remaining matrices are parametrized in terms of the matrices defined in (4.3) and (4.4) as

$$F_{k,k} = \left(Q_- G Q_-^{-1}\right)_{\gamma_k^t,\gamma_k^t} + Q_{\gamma_k^t,|\alpha|-\gamma_k^t} P_{|\alpha|-\gamma_k^t,\gamma_k^t}, \qquad K_{\alpha,k} = -\left(\begin{array}{c} (Q_- Q_0 Q_+)^{-1}_{\gamma_k^t,\gamma_k^t} \\ \hline P_{|\alpha|-\gamma_k^t,\gamma_k^t} \end{array}\right),$$

$$\bar{K}_{k,\alpha} = \left(\begin{array}{c|c} (Q_- Q_0 Q_+)_{\gamma_k^t,\gamma_k^t} & Q_{\gamma_k^t,|\alpha|-\gamma_k^t} \end{array}\right), \qquad D_{\alpha,\alpha}^{[k]} = \mathrm{diag}(\underbrace{0,\ldots,0}_{\gamma_k^t}, \underbrace{1,\ldots,1}_{|\alpha|-\gamma_k^t}). \quad (4.13)$$

Here the matrices $(Q_- Q_0 Q_+)_{\gamma_k^t,\gamma_k^t}$, $(Q_- Q_0 Q_+)^{-1}_{\gamma_k^t,\gamma_k^t}$ and $\left(Q_- G Q_-^{-1}\right)_{\gamma_k^t,\gamma_k^t}$ are built from the variables q_{ij} and p_{ji} where $\ell(k) < i,j \le \ell(k+1)$ with $\ell(k) = |\alpha| + |\beta| + \sum_{l=1}^{k-1} \gamma_l^t$. Further the matrices $Q_{\gamma_k^t,|\alpha|-|\gamma|_k^t}$ and $P_{|\alpha|-\gamma_k^t,|\gamma|_k^t}$ are of the form

$$\left(Q_{\gamma_k^t,|\alpha|-|\gamma|_k^t}\right)_{ij} = q_{\ell(k)+i,\gamma_k^t+j}, \qquad i \in [1,\gamma_k^t], \quad j \in [1,|\alpha|-\gamma_k^t],$$
$$\left(P_{|\alpha|-\gamma_k^t,|\gamma|_k^t}\right)_{ij} = p_{\gamma_k^t+i,\ell(k)+j}, \qquad i \in [1,|\alpha|-\gamma_k^t], \quad j \in [1,\gamma_k^t].$$

The total number of pairs (p_I, q_I) in the Lax matrix for general μ partitions is $\frac{1}{2}\left(r^2 - \sum_{i=1}^{\mu_1} \left(\mu_i^t\right)^2\right)$. Here $|\beta| \cdot |\gamma|$ pairs come from the elements P and Q in (4.9), the matrices $P_{\gamma,\gamma}$ and $Q_{\gamma,\gamma}$ contain $\sum_{i<j} \gamma_i^t \gamma_j^t$ pairs of conjugate variables and the matrices $F_{k,k}$, $K_{\alpha,k}$ and $\bar{K}_{k,\alpha}$ in (4.13) contain for $k = 1, \ldots, \gamma_1$ in total $|\alpha| \cdot |\gamma|$ pairs of variables.

The expression for the Lax matrix $L_\mu(x; \underline{p}, \underline{q})$ in (4.9) is in principle valid for any ordering of columns where $|\alpha| + |\beta|$ denotes the height of the biggest columns and γ the partition that remains after removing that column. If the partition is ordered, i.e., $\lambda_i \ge \lambda_j$ for $i < j$, we have that $D_{\alpha,\alpha}^{[i]} D_{\alpha,\alpha}^{[j]} = D_{\alpha,\alpha}^{[j]} D_{\alpha,\alpha}^{[i]} = D_{\alpha,\alpha}^{[i]}$ for $i < j$ and $D_{\alpha,\alpha}^{[1]} = 0$ which simplifies the expressions above.

4.4 Fusion procedure for μ partitions

The formula for the Lax matrices of the μ partitions can be shown in analogy to Section 3.2. We define three partitions μ, μ' and $\tilde{\mu}$ with $|\mu| = |\mu'| = |\tilde{\mu}| = r$. They are related by fusion via

$$|\tilde{\alpha}| = \max(|\alpha|, |\alpha'|), \qquad |\tilde{\beta}| = \min(|\beta|, |\beta'|), \qquad \tilde{\gamma}^t = \left(\gamma^t, \gamma'^t\right). \qquad (4.14)$$

Here we consider a solution to the Sklyanin relation of the form (4.7) written as a 4×4 block matrix

$$
L_\mu(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c|c|c}
D_{\tilde{\alpha}, \tilde{\alpha}} & 0 & 0 & K_{\tilde{\alpha}, \gamma} \\
\hline
0 & I & 0 & -P_{\tilde{\beta}, \gamma} \\
\hline
0 & 0 & I & -P_{\gamma', \gamma} \\
\hline
\bar{K}_{\gamma, \tilde{\alpha}} & Q_{\gamma, \tilde{\beta}} & Q_{\gamma, \gamma'} & xI - F_{\gamma, \gamma} - Q_{\gamma, \tilde{\beta}} P_{\tilde{\beta}, \gamma} - Q_{\gamma, \gamma'} P_{\gamma', \gamma}
\end{array}\right). \tag{4.15}
$$

The blocks on the diagonal are of the size $|\tilde{\alpha}|$, $|\tilde{\beta}|$, $|\gamma'|$ and $|\gamma|$, respectively with $|\tilde{\alpha}| + |\tilde{\beta}| + |\gamma'| + |\gamma| = r$. The matrices $Q_{\gamma, \tilde{\beta}}$, $Q_{\gamma, \gamma'}$ and $P_{\tilde{\beta}, \gamma}$, $P_{\gamma', \gamma}$ are explicitly given in terms of the conjugate variables. They read

$$
(P_{\tilde{\beta}, \gamma})_{i,j} = p_{|\tilde{\alpha}|+i, |\tilde{\alpha}|+|\tilde{\beta}|+|\gamma'|+j}, \qquad (P_{\gamma', \gamma})_{i,j} = p_{|\tilde{\alpha}|+|\tilde{\beta}|+i, |\tilde{\alpha}|+|\tilde{\beta}|+|\gamma'|+j},
$$
$$
(Q_{\gamma, \tilde{\beta}})_{i,j} = q_{|\tilde{\alpha}|+|\tilde{\beta}|+|\gamma'|+i, |\tilde{\alpha}|+j}, \qquad (Q_{\gamma, \gamma'})_{i,j} = q_{|\tilde{\alpha}|+|\tilde{\beta}|+|\gamma'|+i, |\tilde{\alpha}|+|\tilde{\beta}|+j}.
$$

Furthermore we define a second Lax matrix, cf. (3.9), which also is a solution of the Sklyanin relation. It has the same block structure as (4.15) and reads

$$
L'_{\mu'}(x; \underline{p}', \underline{q}') = \left(\begin{array}{c|c|c|c}
D'_{\tilde{\alpha}, \tilde{\alpha}} & 0 & K'_{\tilde{\alpha}, \gamma'} & 0 \\
\hline
0 & I & -P'_{\tilde{\beta}, \gamma'} & 0 \\
\hline
\bar{K}'_{\gamma', \tilde{\alpha}} & Q'_{\gamma', \tilde{\beta}} & xI - F'_{\gamma', \gamma'} - Q'_{\gamma', \tilde{\beta}} P'_{\tilde{\beta}, \gamma'} + P'_{\gamma', \gamma} Q'_{\gamma, \gamma'} & -P'_{\gamma', \gamma} \\
\hline
0 & 0 & -Q'_{\gamma, \gamma'} & I
\end{array}\right).
$$

We got

$$
(P'_{\tilde{\beta}, \gamma'})_{i,j} = p_{|\tilde{\alpha}|+i, |\tilde{\alpha}|+|\tilde{\beta}|+j}, \qquad (P'_{\gamma', \gamma})_{i,j} = p_{|\tilde{\alpha}|+|\tilde{\beta}|+i, |\tilde{\alpha}|+|\tilde{\beta}|+|\gamma'|+j},
$$
$$
(Q'_{\gamma', \tilde{\beta}})_{i,j} = q_{|\tilde{\alpha}|+|\tilde{\beta}|+i, |\tilde{\alpha}|+j}, \qquad (Q_{\gamma, \gamma'})_{i,j} = q_{|\tilde{\alpha}|+|\tilde{\beta}|+|\gamma'|+i, |\tilde{\alpha}|+|\tilde{\beta}|+j}.
$$

We proceed as in Section 3.2 and multiply the two solutions of the Sklyanin relation. The product can again be written as

$$
L_{\lambda, \underline{x}, \mu}(x; \underline{p}, \underline{q}) L'_{\mu'}(x; \underline{p}', \underline{q}') = \tilde{Q}' \tilde{L}_{\tilde{\mu}}(x, \tilde{\underline{p}}, \tilde{\underline{q}}),
$$

cf. (3.10). The spectral parameter dependent matrix $\tilde{L}_{\tilde{\mu}}(x, \tilde{\underline{p}}, \tilde{\underline{q}})$ and the matrix $\tilde{Q}'$ take the form

$$
\tilde{L}_{\tilde{\mu}}(x, \tilde{\underline{p}}, \tilde{\underline{q}}) = WU \left(\begin{array}{c|c|c|c}
D_{\tilde{\alpha}, \tilde{\alpha}} D'_{\tilde{\alpha}, \tilde{\alpha}} & 0 & D_{\tilde{\alpha}, \tilde{\alpha}} K_{\tilde{\alpha}, \gamma'}{}' & K_{\tilde{\alpha}, \gamma} \\
\hline
0 & I & 0 & 0 \\
\hline
\bar{K}'_{\gamma', \tilde{\alpha}} & 0 & xI - F'_{\gamma', \gamma'} & -\tilde{P}_{\gamma', \gamma} \\
\hline
\bar{K}_{\gamma, \tilde{\alpha}} D'_{\tilde{\alpha}, \tilde{\alpha}} & 0 & \bar{K}_{\gamma \tilde{\alpha}} K'_{\tilde{\alpha}, \gamma'} & xI - F_{\gamma, \gamma}
\end{array}\right) U^{-1} V^{-1},
$$

and

$$
\tilde{Q}' = \left(\begin{array}{c|c|c|c}
I & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 \\
\hline
0 & 0 & I & 0 \\
\hline
0 & 0 & \tilde{Q}'_{\gamma, \gamma'} & I
\end{array}\right).
$$

Here we have written $\tilde{L}_{\tilde{\mu}}(x, \tilde{\underline{p}}, \tilde{\underline{q}})$ in a factorised form and introduced the matrices

$$
W = \left(\begin{array}{c|c|c|c}
I & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 \\
\hline
0 & \tilde{Q}_{\gamma',\tilde{\beta}} & I & 0 \\
\hline
0 & \tilde{Q}_{\gamma,\tilde{\beta}} & 0 & I
\end{array}\right), \qquad
U = \left(\begin{array}{c|c|c|c}
I & 0 & 0 & 0 \\
\hline
0 & I & 0 & 0 \\
\hline
0 & 0 & I & 0 \\
\hline
0 & 0 & \tilde{Q}_{\gamma,\gamma'} & I
\end{array}\right),
$$

$$
V^{-1} = \left(\begin{array}{c|c|c|c}
I & 0 & 0 & 0 \\
\hline
0 & I & \tilde{P}_{\tilde{\beta},\gamma'} & \tilde{P}_{\tilde{\beta},\gamma} \\
\hline
0 & 0 & I & 0 \\
\hline
0 & 0 & 0 & I
\end{array}\right).
$$

They are expressed in terms of the new variables

$$
\tilde{P}_{\gamma'\gamma} = P'_{\gamma'\gamma} + P_{\gamma'\gamma} - Q'_{\gamma'\tilde{\beta}}P_{\tilde{\beta}\gamma}, \qquad \tilde{Q}_{\gamma\gamma'} = Q'_{\gamma\gamma'},
$$
$$
\tilde{P}_{\tilde{\beta}\gamma'} = P'_{\tilde{\beta}\gamma'} - P_{\tilde{\beta}\gamma}Q'_{\gamma\gamma'}, \qquad \tilde{Q}_{\gamma'\tilde{\beta}} = Q'_{\gamma'\tilde{\beta}},
$$
$$
\tilde{P}_{\tilde{\beta}\gamma} = P_{\tilde{\beta}\gamma}, \qquad \tilde{Q}_{\gamma\tilde{\beta}} = Q_{\gamma\tilde{\beta}} + Q'_{\gamma\gamma'}Q'_{\gamma'\tilde{\beta}},
$$
$$
\tilde{P}'_{\gamma'\gamma} = P_{\gamma'\gamma}, \qquad \tilde{Q}'_{\gamma\gamma'} = Q_{\gamma\gamma'} - Q'_{\gamma\gamma'}.
$$

This is the same change of variables as in (3.11) and therefore it is canonical. Following the same logic as in Section 3.2 we conclude that $\tilde{L}_{\tilde{\mu}}(x, \tilde{\underline{p}}, \tilde{\underline{q}})$ is a solution of the Sklyanin relation. For convenience we write it in the same form as $L_\mu(x; \underline{p}, q)$ such that

$$
\tilde{L}_{\tilde{\mu}}(x, \tilde{\underline{p}}, \tilde{\underline{q}}) = \left(\begin{array}{c|c|c}
\tilde{D}_{\tilde{\alpha},\tilde{\alpha}} & 0 & K_{\tilde{\alpha},\tilde{\gamma}} \\
\hline
0 & I & -\tilde{P}_{\tilde{\beta},\tilde{\gamma}} \\
\hline
\bar{K}_{\tilde{\gamma},\tilde{\alpha}} & \tilde{Q}_{\tilde{\gamma},\tilde{\beta}} & xI - \tilde{F}_{\tilde{\gamma},\tilde{\gamma}} - \tilde{Q}_{\tilde{\gamma},\tilde{\beta}}\tilde{P}_{\tilde{\beta},\tilde{\gamma}}
\end{array}\right). \tag{4.16}
$$

The size of the block matrices on the diagonal is $|\tilde{\alpha}|$, $|\tilde{\beta}|$ and $|\tilde{\gamma}|$. We defined the matrices

$$
\left(\tilde{P}_{\tilde{\beta},\tilde{\gamma}}\right)_{i,j} = \tilde{p}_{|\tilde{\alpha}|+i, |\tilde{\alpha}|+|\tilde{\beta}|+j}, \qquad \left(\tilde{Q}_{\tilde{\gamma},\tilde{\beta}}\right)_{i,j} = \tilde{p}_{|\tilde{\alpha}|+|\tilde{\beta}|+i, |\tilde{\alpha}|+j},
$$

while the remaining elements are given by

$$
\tilde{F}_{\tilde{\gamma},\tilde{\gamma}} = \tilde{Q}_- \left(\begin{array}{c|c}
F'_{\gamma',\gamma'} & \tilde{P}'_{\gamma',\gamma} \\
\hline
-\bar{K}_{\gamma,\tilde{\alpha}}K'_{\tilde{\alpha},\gamma'} & F_{\gamma,\gamma}
\end{array}\right) \tilde{Q}_-^{-1}, \qquad \text{with} \quad \tilde{Q}_- = \left(\begin{array}{c|c}
I & 0 \\
\hline
\tilde{Q}'_{\gamma,\gamma'} & I
\end{array}\right),
$$

$$
\bar{K}_{\tilde{\gamma},\tilde{\alpha}} = \tilde{Q}_- \left(\begin{array}{c}
\bar{K}'_{\gamma',\tilde{\alpha}} \\
\hline
\bar{K}_{\gamma,\tilde{\alpha}}D'_{\tilde{\alpha},\tilde{\alpha}}
\end{array}\right), \qquad K_{\tilde{\alpha},\tilde{\gamma}} = \left(\begin{array}{c|c}
D_{\tilde{\alpha},\tilde{\alpha}}K'_{\tilde{\alpha},\gamma'} & K_{\tilde{\alpha},\gamma}
\end{array}\right) \tilde{Q}_-^{-1},
$$

and $\tilde{D}_{\tilde{\alpha},\tilde{\alpha}} = D_{\tilde{\alpha},\tilde{\alpha}}D'_{\tilde{\alpha},\tilde{\alpha}}$.

4.4.1 Recursion

We specify the matrix elements in the fusion procedure to describe the fusion of one arbitrary partition μ as proposed in Section 4.3 and an elementary matrix (4.7) corresponding to the partition μ' with the restriction $\alpha' = \gamma'$. The resulting partition $\tilde{\mu}$ is then written in terms of μ and μ' as in (4.14).

This can be seen as follows. The primed letters correspond to elements of the Lax matrix corresponding to μ' and read

$$F'_{\gamma',\gamma'} = \left(Q_- G Q_-^{-1}\right)_{\gamma',\gamma'} + Q_{\gamma',\tilde{\alpha}-\alpha'} P_{\tilde{\alpha}-\alpha',\gamma'},$$

$$\bar{K}'_{\gamma',\tilde{\alpha}} = \left(\ (Q_- Q_0 Q_+)_{\gamma',\alpha'} \ \middle| \ Q_{\gamma',\tilde{\alpha}-\alpha'} \ \right), \qquad K'_{\tilde{\alpha},\gamma'} = -\left(\frac{(Q_- Q_0 Q_+)^{-1}_{\alpha',\gamma'}}{P_{\tilde{\alpha}-\alpha',\gamma'}} \right)$$

and

$$D'_{\tilde{\alpha},\tilde{\alpha}} = D^{[\gamma']}_{\tilde{\alpha},\tilde{\alpha}} = \mathrm{diag}(\underbrace{0,\ldots,0}_{\alpha'}, \underbrace{1,\ldots,1}_{\tilde{\alpha}-\alpha'}).$$

The unprimed letters correspond to the partition μ as given in (4.10), (4.11) and (4.12). We find that

$$F_{\tilde{\gamma},\tilde{\gamma}} = Q_{\tilde{\gamma},\tilde{\gamma}} \left(\begin{array}{c|cccc} F'_{\gamma',\gamma'} & & \multicolumn{3}{c}{\tilde{P}_{\gamma',\gamma} Q_{\gamma,\gamma}} \\ \hline & \begin{array}{c|c|c|c} F_{1,1} & \hat{P}_{1,2} & \cdots & \hat{P}_{1,\gamma_1} \\ \hline W_{2,1} & F_{2,2} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \hat{P}_{\gamma_1-1,\gamma_1} \\ \hline W_{\gamma_1,1} & \cdots & W_{\gamma_1,\gamma_1-1} & F_{\gamma_1,\gamma_1} \end{array} \\ -Q^{-1}_{\gamma,\gamma} \bar{K}_{\gamma,\tilde{\alpha}} K'_{\tilde{\alpha},\gamma'} & \end{array} \right) Q^{-1}_{\tilde{\gamma},\tilde{\gamma}},$$

where similar as for the case of λ-partitions in (3.12) we identify

$$\tilde{P}_{\gamma',\gamma} Q_{\gamma,\gamma} = \left(\ \hat{P}_{\gamma',1} \ \middle| \ \hat{P}_{\gamma',2} \ \middle| \ \cdots \ \middle| \ \hat{P}_{\gamma',\gamma_1} \ \right).$$

Furthermore we identify

$$-Q^{-1}_{\gamma,\gamma} \bar{K}_{\gamma,\tilde{\alpha}} K'_{\tilde{\alpha},\gamma'} = -\left(\frac{\bar{K}_{1,\tilde{\alpha}} K'_{\tilde{\alpha},\gamma'}}{\frac{\bar{K}_{2,\tilde{\alpha}} D^{[1]}_{\tilde{\alpha},\tilde{\alpha}} K'_{\tilde{\alpha},\gamma'}}{\frac{\vdots}{\bar{K}_{\gamma_1,\tilde{\alpha}} D^{[\gamma_1-1]}_{\tilde{\alpha},\tilde{\alpha}} \cdots D^{[1]}_{\tilde{\alpha},\tilde{\alpha}} K'_{\tilde{\alpha},\gamma'}}}} \right) = \left(\frac{W_{1,\gamma'}}{\frac{W_{2,\gamma'}}{\frac{\vdots}{W_{\gamma_1,\gamma'}}}} \right)$$

and obtain

$$\bar{K}_{\tilde{\gamma},\tilde{\alpha}} = Q_{\tilde{\gamma},\tilde{\gamma}} \begin{pmatrix} \bar{K}'_{\gamma',\tilde{\alpha}} \\ \hline \bar{K}_{1,\tilde{\alpha}} D^{[\gamma']}_{\tilde{\alpha},\tilde{\alpha}} \\ \hline \bar{K}_{2,\tilde{\alpha}} D^{[1]}_{\tilde{\alpha},\tilde{\alpha}} D^{[\gamma']}_{\tilde{\alpha},\tilde{\alpha}} \\ \hline \vdots \\ \hline \bar{K}_{\gamma_1,\tilde{\alpha}} D^{[\gamma_1-1]}_{\tilde{\alpha},\tilde{\alpha}} \cdots D^{[1]}_{\tilde{\alpha},\tilde{\alpha}} D^{[\gamma']}_{\tilde{\alpha},\tilde{\alpha}} \end{pmatrix}$$

and

$$K_{\tilde{\alpha},\tilde{\gamma}} = \left(\begin{array}{c|c|c|c|c} D^{[\gamma_1]}_{\tilde{\alpha},\tilde{\alpha}} \cdots D^{[1]}_{\tilde{\alpha},\tilde{\alpha}} K'_{\tilde{\alpha},\gamma'} & D^{[\gamma_1]}_{\tilde{\alpha},\tilde{\alpha}} \cdots D^{[2]}_{\tilde{\alpha},\tilde{\alpha}} K_{\tilde{\alpha},1} & \cdots & D^{[\gamma_1]}_{\alpha,\alpha} K_{\alpha,\gamma_1-1} & K_{\alpha,\gamma_1} \end{array} \right) Q^{-1}_{\tilde{\gamma},\tilde{\gamma}}.$$

Thus we conclude that (4.9) satisfies Sklyanin's quadratic Poisson bracket.

5   Generic degree 1 symplectic leaves

We will now define the Lax matrices $L_{\lambda,\underline{x},\mu}(x;\underline{p},\underline{q})$ for arbitrary partitions λ and μ. They can be obtained by fusing the Lax matrix for regular partitions (3.2) with the Lax matrix for μ partitions (4.9).

5.1   Lax matrix for λ, μ partitions

The Lax matrix for arbitrary partitions λ and μ can compactly be written as

$$L_{\lambda,\underline{x},\mu}(x;\underline{p},\underline{q}) = \left(\begin{array}{c|c|c} 0 & 0 & K_{\alpha,\gamma\lambda} \\ \hline 0 & I & -P_{\beta,\gamma\lambda} \\ \hline \bar{K}_{\gamma\lambda,\alpha} & Q_{\gamma\lambda,\beta} & xI - F_{\gamma\lambda,\gamma\lambda} - Q_{\gamma\lambda,\beta} P_{\beta,\gamma\lambda} \end{array} \right). \tag{5.1}$$

The blocks on the diagonal are of the size $|\alpha|$, $|\beta|$ and $|\gamma| + |\lambda|$ respectively. Here $Q_{\lambda,\gamma}$ and $P_{\gamma,\lambda}$ are defined as

$$(P_{\gamma,\lambda})_{i,j} = p_{|\alpha|+|\beta|+i,|\alpha|+|\beta|+|\gamma|+j}, \qquad (Q_{\lambda,\gamma})_{i,j} = q_{|\alpha|+|\beta|+|\gamma|+i,|\alpha|+|\beta|+j}.$$

The remaining matrix elements in (5.1) are given in terms of the components of the Lax matrix for regular partitions (3.2) and the Lax matrix for μ partitions (4.9). We have

$$F_{\gamma\lambda,\gamma\lambda} = \left(\begin{array}{c|c} I & 0 \\ \hline Q_{\lambda,\gamma} & I \end{array} \right) \cdot \left(\begin{array}{c|c} F_{\gamma,\gamma} & P_{\gamma,\lambda} \\ \hline -Q_{\lambda,\alpha} K_{\alpha,\gamma} & J_{\lambda,\lambda} + Q_{\lambda,\alpha} P_{\alpha,\lambda} \end{array} \right) \cdot \left(\begin{array}{c|c} I & 0 \\ \hline -Q_{\lambda,\gamma} & I \end{array} \right),$$

and

$$\bar{K}_{\gamma\lambda,\alpha} = \left(\begin{array}{c} \bar{K}_{\gamma,\alpha} \\ \hline Q_{\lambda,\gamma} \bar{K}_{\gamma,\alpha} \end{array} \right), \qquad K_{\alpha,\gamma\lambda} = \left(\begin{array}{c|c} K_{\alpha,\gamma} + P_{\alpha,\lambda} Q_{\lambda,\gamma} & -P_{\alpha,\lambda} \end{array} \right),$$

$$Q_{\gamma\lambda,\beta} = \begin{pmatrix} Q_{\gamma,\beta} \\ \hline Q_{\lambda,\beta} \end{pmatrix}, \qquad\qquad P_{\beta,\gamma\lambda} = \begin{pmatrix} P_{\beta,\gamma} & | & P_{\beta,\lambda} \end{pmatrix}.$$

Again we can check that the number of pairs of conjugate variables agrees with (2.3). First we note that F contains $\sum_{i<j} \gamma_i^t \gamma_j^t + |\alpha| \cdot |\gamma|$ and J contains $\sum_{i<j} \lambda_i^t \lambda_j^t$ pairs. The remaining variables are contained in $P_{\beta,\gamma\lambda}$, $Q_{\gamma\lambda,\beta}$, $P_{\alpha,\lambda}$, $Q_{\lambda,\alpha}$ and $P_{\gamma,\lambda}$, $Q_{\lambda,\gamma}$. By construction, cf. Section 5.2, the determinant of the Lax matrix in (5.1) satisfies (2.4).

The symplectic leaves that we found in the Poisson–Lie group $\mathcal{G}$ are orbits of certain representative elements under the dressing action of the dual Poisson–Lie group $\mathcal{G}^*$. These representative elements are easily seen as Lax matrices at $\underline{p} = \underline{q} = 0$. Here the Lax matrix (5.1) reduces to a block matrix of the form

$$L_{\lambda,\underline{x},\mu}(x; \varnothing, \varnothing) = \left(\begin{array}{c|c} xI_\mu - \Sigma_\mu & 0 \\ \hline 0 & xI - X_\lambda \end{array} \right),$$

where

$$I_\mu = \mathrm{diag}(\underbrace{0,\ldots,0}_{|\alpha|+|\beta|}, \underbrace{1\ldots,1}_{|\gamma|})$$

and X_λ denotes the diagonal matrix defined in (3.5). The matrix Σ_μ is a permutation matrix containing the elements ± 1. This matrix can be block diagonalized such that it contains $|\alpha|+|\beta|$ blocks of the size μ_i, $i = 1,\ldots,|\alpha|+|\beta|$, corresponding to the rows of the partitions μ. The diagonal of each block i reads $\mathrm{diag}(0, x, \ldots, x)$ and its remaining elements ± 1 correspond to a cyclic permutation of length μ_i. For example for a row of $\mu_i = 4$ we obtain

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ 1 & 0 & 0 & x \end{pmatrix}.$$

For $\mu_i = 1$ where $i = |\alpha|+1, \ldots, |\alpha|+|\beta|$, we obtain a 1×1 block containing only the element 1.

5.2 Fusion procedure

The Lax matrix (5.1) can be derived using the factorisation formula in Section 4.4 when substituting the γ block for a $\lambda = \tilde{\lambda}$ block as follows

$$F_{\gamma,\gamma} \to J_{\tilde{\lambda},\tilde{\lambda}} + Q_{\tilde{\lambda},\tilde{\alpha}} P_{\tilde{\alpha},\tilde{\lambda}}, \qquad \bar{K}_{\gamma,\tilde{\alpha}} \to Q_{\tilde{\lambda},\tilde{\alpha}}, \qquad K_{\tilde{\alpha},\gamma} \to -P_{\tilde{\alpha},\tilde{\lambda}}, \qquad D_{\tilde{\alpha},\tilde{\alpha}} \to I.$$

Here I is the $|\tilde{\alpha}| \times |\tilde{\alpha}|$ identity matrix. The primed elements in the second Lax matrix L' are taken to be as defined in (4.9) as

$$F'_{\gamma',\gamma'} = F_{\tilde{\gamma},\tilde{\gamma}}, \qquad \bar{K}'_{\gamma',\tilde{\alpha}} = \bar{K}_{\tilde{\gamma},\tilde{\alpha}}, \qquad K'_{\tilde{\alpha},\gamma'} = K_{\tilde{\alpha},\tilde{\gamma}}, \qquad D'_{\tilde{\alpha},\tilde{\alpha}} = 0.$$

Here D' is equal to the $|\tilde{\alpha}| \times |\tilde{\alpha}|$ zero matrix.

This factorisation corresponds to the fusion of the partitions λ, μ and λ', μ' expressed in terms of the resulting partition $\tilde{\lambda}$, $\tilde{\mu}$ via

$$\alpha = \varnothing, \qquad \beta = (\underbrace{1,\ldots,1}_{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|}), \qquad \gamma = \varnothing, \qquad \lambda = \tilde{\lambda},$$

$$\alpha' = \tilde{\alpha}, \qquad \beta' = (\underbrace{1,\ldots,1}_{|\tilde{\beta}|+|\tilde{\lambda}|}), \qquad \gamma' = \tilde{\gamma}, \qquad \lambda' = \varnothing.$$
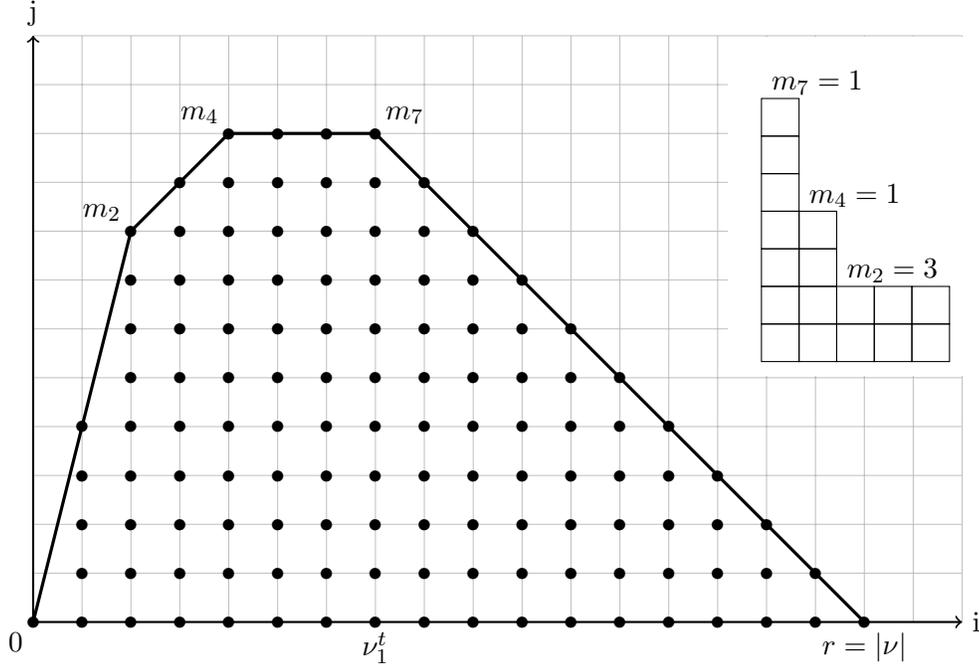
Figure 4. An example of non-vanishing commuting Hamiltonians $\mathcal{X}_{i,j}^{[\lambda,\mu]}$ for a GL_r Lax matrix $L_{\lambda,\underline{x},\mu}(x;\underline{p},\underline{q})$ corresponding to a partition $\nu = (5,5,2,2,1,1,1)$ with $r = 17$ and $d_\nu = 106$, with non-zero $m_2 = 3$ and $m_4 = m_7 = 1$. The horizontal axis i labels the nodes of the A_{r-1} quiver diagram for $i \in [1, r-1]$, and the vertical coordinate j of the enveloping profile denotes the color ranks n_i in the quiver diagram.

The final result of the factorisation can be directly read off from (4.16). We conclude that (5.1) is a solution to Skyanin's relation (2.1).

6 Algebraic completely integrable systems and Coulomb branches of A_{r-1} quiver gauge theory

The symplectic leaf $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$, i.e., the moduli space of multiplicative Higgs bundles with fixed singularities, supports fibration of an algebraic completely integrable system

$$H\colon \ \mathcal{M}_{\underline{\lambda},\underline{x},\mu} \to \mathcal{U}_{\underline{\lambda},\underline{x},\mu}. \tag{6.1}$$

Here H denotes a complete set of independent commuting Hamiltonian functions (also known as conserved charges or action variable or integrals of motion of an integrable Hamilonian dynamical system) and $\mathcal{U}_{\underline{\lambda},\underline{x},\mu}$ denotes the space where the complete set of independent Hamiltonians takes value. The fibers $\mathcal{A}_u = H^{-1}(u), u \in \mathcal{U}_{\underline{\lambda},\underline{x},\mu}$ of the map (6.1) are abelian varieties which are holomorphic Lagrangians with respect to the holomorphic symplectic structure on $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$, so that

$$\dim_{\mathbb{C}} \mathcal{U}_{\underline{\lambda},\underline{x},\mu} = \dim_{\mathbb{C}} \mathcal{A}_u = \tfrac{1}{2} \dim \mathcal{M}_{\underline{\lambda},\underline{x},\mu}$$

Let

$$d_{\underline{\lambda},\underline{x},\mu} = \tfrac{1}{2} \dim \mathcal{M}_{\underline{\lambda},\underline{x},\mu}$$

denote the half-dimension of the symplectic leaf (phase space) $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$. In the context of Seiberg–Witten integrable systems [17, 18, 55, 59] the holomorphic symplectic phase space $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ is the Coulomb branch of the hyperkähler moduli space of vacua of $\mathcal{N}=2$ supersymmetric quiver gauge theory on $\mathbb{R}^3 \times S^1$ viewed as a holomorphic symplectic manifold at a distinguished point $\zeta = 0$ on the twistor sphere of complex structures. The complex base space $\mathcal{U}_{\underline{\lambda},\underline{x},\mu}$ is the moduli space of vacua of the same $\mathcal{N}=2$ supersymmetric gauge theory on $\mathbb{R}^4$ called $\mathcal{U}$-plane in the respective context. In terms of action-angle variables, the complex action variables parametrize the base $\mathcal{U}$-plane, and the complex angle variables parametrize the abelian fibers $\mathcal{A}_u$.

To realize the structure of an algebraic completely integrable system (6.1) we need to construct $d_{\underline{\lambda},\underline{x},\mu}$ independent Poisson commuting Hamiltonian functions on $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$. Like in the case of additive Higgs bundles (Hitchin system), the commuting Hamiltonian functions on multiplicative Higgs bundles (or more general abstract Higgs bundles) can be realized by the abstract cameral cover construction [27]. In the case of additive Higgs bundles on X, the cameral cover construction generates Poisson commuting Hamiltonian functions as coefficients of $P(\phi(x))$ where the Higgs field $\phi(x)$ is a section of $\mathrm{ad}\,\mathfrak{g} \otimes K_X$ and P is an adjoint invariant function on the Lie algebra $\mathfrak{g}$. Similarly, in the case of multiplicative Higgs bundles on X, the cameral cover construction generates Poisson commuting Hamiltonian functions as coefficients of $\chi(g(x))$ where χ is an adjoint invariant function on the group G and multiplicative Higgs field $g(x)$ is a section of $\mathrm{Ad}\,G$ on X. The complete set of independent Poisson commuting Hamiltonians for simple G is spanned by the characters χ_{R_i} of the fundamental irreducible highest weight modules R_k whose highest weight is the fundamental weight ω_k for each k in the set of nodes of the Dynkin diagram of g.

If $G = \mathrm{GL}_r$, the irreducible highest weight module R_k with highest weight ω_k associated to the k-th node of the A_{r-1} Dynkin diagram of the simple factor $\mathrm{SL}_r \subset \mathrm{GL}_r$ is isomorphic to the k-th external power $R_k = \bigwedge^k R_1$, for $k = 1, \ldots, r-1$, of the defining r-dimensional representation R_1, and we set $R_r = \bigwedge^r R_1$ to be the determinant 1-dimensional representation. It is convenient to assemble the fundamental characters $\chi_k = \chi_{R_k}$ for $k = 1, \ldots, r$ into the spectral polynomial

$$\det(yI_{r\times r} - L(x))_{r\times r} = \sum_{k=0}^{r}(-1)^k y^{r-k}\chi_k(g(x)),$$

where $\chi_k(g(x)) = \mathrm{tr}_{R_k}\rho_k(g(x))$ is a character for a fundamental representation $\rho_k \colon G \to \mathrm{End}(R_k)$ evaluated on Higgs field $g(x)$.

Now we illustrate explicitly the construction of commuting Hamiltonians for the Lax matrices constructed in the previous sections that describe the symplectic leaves $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$.

First, for any symplectic leaf $\mathcal{M}_{\underline{\lambda},\underline{x},\mu}$ and its representing Higgs field $g_{\underline{\lambda},\underline{x},\mu}(x)$ we define its twisted version

$$g_{\underline{\lambda},\underline{x},\mu,g_L;g_R}(x) = g_L g_{\underline{\lambda},\underline{x},\mu}(x)g_R,$$

which represents a symplectic leaf $\mathcal{M}_{\underline{\lambda},\underline{x},\mu;g_L,g_R}$. Here $g_L \in G, g_R \in G$ are arbitrary constant (x-independent) Higgs fields. We remark that the symplectic leaves $\mathcal{M}_{\underline{\lambda},\underline{x},\mu;g_L,g_R}$ are isomorphic for various g_L, g_R, and for certain relation between g_L and g_R they in fact coincide, in this sense the labeling by both g_L and g_R are redundant.[2] For the following, it is sufficient to take, $g_L \equiv g_\infty$, $g_R \equiv 1$, and define the Lax matrix

$$L_{\underline{\lambda},\underline{x},\mu,g_\infty}(x) = \rho_1(g_\infty)L_{\underline{\lambda},\underline{x},\mu}(x), \tag{6.2}$$

---

[2] What is exactly the degree of redundancy? We can see that for the case regular at infinity $\mu = \varnothing$, when $L(x)$ is an x-shifted co-adjoint orbit in $\mathfrak{g}$, the non-redundant label is the product $g_R g_L$. In any case, the resulting completely integrable system depends only on the product $g_R g_L$ as we see from the spectral polynomial (6.3).

where $\rho_1(g_\infty)$ is $r \times r$ matrix representing $g_\infty \in G$. Note that due to the symmetries of the $\mathfrak{r}$-matrix in (2.1) the product $\rho_1(g_\infty)L_{\underline{\lambda},\underline{x},\mu}(x)$ is a solution to the Sklyanin relation if $L_{\underline{\lambda},\underline{x},\mu}$ is.

Now define the spectral determinant to be a polynomial of two variables x and y

$$W_{\underline{\lambda},\underline{x},\mu,g_\infty}(x,y) = \det(y - L_{\underline{\lambda},\underline{x},\mu,g_\infty}(x)) = \sum_{k=0}^{r}(-1)^k y^{r-k}\chi_k(x). \tag{6.3}$$

The commuting Hamiltonians are coefficients of the monomials $x^j y^i$. With

$$\chi_k(x) = \mathcal{Q}^{[\lambda]}_{r-k}\mathcal{X}^{[\lambda,\mu]}_k(x), \tag{6.4}$$

we find that the spectral determinant can be written as

$$W_{\underline{\lambda},\underline{x},\mu,g_\infty}(x,y) = y^r + \sum_{i=1}^{r-1}(-1)^{r-i}\mathcal{Q}^{[\lambda]}_i(x)\mathcal{X}^{[\lambda,\mu]}_{r-i}(x)y^i + (-1)^r\mathcal{Q}^{[\lambda]}_0(x), \tag{6.5}$$

where $\mathcal{Q}^{[\lambda]}_i$ is a polynomial in x, cf. (E.2), which is independent of the conjugate variables (p,q) of the Lax matrix and which takes the form

$$\mathcal{Q}^{[\lambda]}_i(x) = \prod_{j=1}^{\lambda_{i+1}}(x - x_j)^{\lambda^t_j - i}. \tag{6.6}$$

All commuting Hamiltonians are thus contained in $\mathcal{X}^{[\lambda,\mu]}_i(x)$. More precisely $\mathcal{X}^{[\lambda,\mu]}_{r-k}(x)$ is a polynomial in x of degree

$$n^{[\lambda,\mu]}_k = \sum_{j=1}^{k}(\nu_j - 1) \qquad \text{with} \quad \nu_i = \lambda_i + \mu_i, \quad k \in [1, r-1]. \tag{6.7}$$

We note that the number of independent commuting Hamiltonians only depends on the total dominant co-weight represented by the partition obtained by the union of columns of the partitions λ and μ minus the shift by the diagonal co-representation (see below (6.10). The charges are obtained as the coefficients of the expansion

$$\mathcal{X}^{[\lambda,\mu]}_{r-i}(x) = \sum_{j=0}^{n^{[\lambda,\mu]}_i} \mathcal{X}^{[\lambda,\mu]}_{r-i,j}x^j,$$

cf. Fig. 4. The highest coefficients do not depend on the conjugate variables (p_I, q_I) but all other coefficients in the expansion do. The total number of linearly independent charges is equal to the number of conjugate pairs in the corresponding Lax matrix

$$d_{\underline{\lambda},\mu} = \sum_{k=1}^{r-1} n^{\lambda,\mu}_k, \tag{6.8}$$

cf. (2.3). This relation is shown using Frobenius-like coordinates for the partitions in Appendix D.

For given partitions we can plot the non vanishing coefficients of the spectral determinant in a Newton diagram as done in Fig. 4. Here we introduce the parameters

$$m^{[\lambda,\mu]}_k = \left(n^{[\lambda,\mu]}_k - n^{[\lambda,\mu]}_{k-1}\right) - \left(n^{[\lambda,\mu]}_{k+1} - n^{[\lambda,\mu]}_k\right) = \nu_k - \nu_{k+1} \qquad \text{for} \quad k \in [1, r-1] \tag{6.9}$$

to label the partition and the corresponding Newton diagram.

The representation theoretical meaning of the equations (6.7), (6.8) and (6.9) is the following. The partitions $\underline{\lambda}$ and μ encode the GL_r co-weights of the respective singularity of the multiplicative Higgs field at finite points $\underline{x}$ and x_∞. The encoding is in the r-dimensional basis of the dual to the weights of the defining representation that we call $\check{e}_k$ with $k = 1, \ldots, r$. In terms of e_i define the simple co-roots α_i of SL_r to be

$$\check{\alpha}_k = \check{e}_k - \check{e}_{k-1}, \qquad k \in [1, r-1]$$

and define the fundamental weights to be

$$\check{\omega}_k = \sum_{j=1}^{k} \check{e}_k - \frac{k}{r} \sum_{j=1}^{r} \check{e}_j, \qquad k \in [1, r-1].$$

The dominant co-weight associated to each singularity $x_* \in \underline{x}$ with associated partition $\lambda_* = \lambda_{*,1} \geq \lambda_{*,2} \geq \cdots \geq \lambda_{*,r} = 0$ is given by

$$\check{\omega}_* = \sum_{k=1}^{r} \lambda_{*,i} \check{e}_i$$

and the dominant co-weight associated to the singularity at $x_\infty = \infty$ with associated partition μ is given by

$$\check{\omega}_\infty = \sum_{k=1}^{r} (\mu_i - 1) \check{e}_i, \tag{6.10}$$

so that the GL_r multiplicative Higgs field behaves up to a multiplication by a regular function as $(x - x_*)^{\check{\omega}_*}$ as $x \to x_*$ and as $(1/x)^{\check{\omega}_\infty}$ as $x \to \infty$.

Let ρ be the Weyl vector

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{k=1}^{r-1} \omega_k = \sum_{k=1}^{r} \frac{r - (2k+1)}{2} e_k$$

and let

$$\check{\omega}_{\mathrm{tot}} = \check{\omega}_\infty + \sum_{x_* \in \underline{x}} \check{\omega}_{x_*}$$

be the sum of the co-weights of all singularities in $\underline{x}$ and x_∞. Then we see that the dimension formula (6.8) is equivalent to

$$\dim \mathcal{M}_{\underline{\lambda}, \underline{x}, \mu; g_L, g_R} = 2 d_{\underline{\lambda}, \underline{x}, \mu} = 2(\rho, \check{\omega}_{\mathrm{tot}})$$

in agreement with the general formula for the dimension of the moduli space of monopoles with Dirac singularities encoded by the total co-weight $\check{\omega}_{\mathrm{tot}}$.

The numbers n_k and m_k for $k \in [1, r-1]$ are the number of colors in the node k and the number of fundamental flavours attached to the node k of the Dynkin quiver [55], including the "deficit" fundamental flavours multiplets of asymptotically free theory which would make it conformal if added. We have

$$\check{\omega}_{\mathrm{tot}} = \sum_{k=1}^{r-1} \check{\alpha}_k n_k = \sum_{k=1}^{r-1} \check{\omega}_k m_k$$
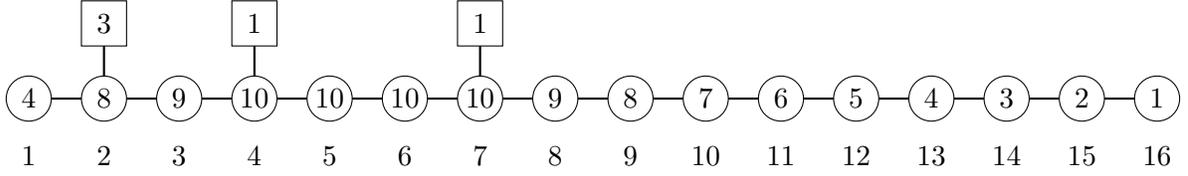
Figure 5. Representation of the Newton polygon in Fig. 4 corresponding to the partition $\nu = (5, 5, 2, 2, 1, 1, 1)$ with $r = 17$ as quiver diagram. Here the integers in the circles denote the number of charges for a given index i indicated below. The parameters $m_i^{[\lambda, \mu]}$ are given in the squared boxes.
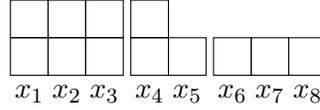
Figure 6. Decomposition of the partition $\lambda = (8, 4)$ for $r = 3$ into three partitions $(3, 3)$, $(2, 1)$ and (3).

in agreement with (6.7) and (6.9). The position of each singularity $x_* \in \underline{x}$ to which we have associated a column λ_*^t encoding a fundamental co-weight $\breve{\omega}_{\lambda_*^t}$ is the mass of the fundamental flavour multiplet at the node λ_*^t.

Now the spectral curve can be compared with [55] where a slightly different notation is used. To do so we note that the polynomials $\mathcal{Q}_i^{[\lambda]}$ in (6.6) can be written in terms of the parameters $m_i^{[\lambda, \varnothing]}$ introduced in (6.9) as

$$\mathcal{Q}_k^{[\lambda]}(x) = \prod_{i=k+1}^{|\lambda|} \mathcal{P}_{r-i}^{i-k}(x), \qquad \text{with} \quad \mathcal{P}_{r-i}(x) = \prod_{j=1}^{m_i^{[\lambda, \varnothing]}} (x - x_{\lambda_i - j + 1}). \tag{6.11}$$

This relation is shown in Appendix E. Setting $m_i^{[\lambda, \varnothing]} = 0$ for $i > |\lambda|$ we can now write the spectral determinant (6.5) in the notation used in (7.5) of [55]. We find

$$W_{\underline{\lambda}, \underline{x}, \mu, g_\infty}(x, y) = y^r + \sum_{i=1}^{r-1} (-\zeta(x))^i \prod_{j=1}^{i-1} \mathcal{P}_j^{i-j}(x) \mathcal{X}_i(x) y^{r-i} + (-\zeta(x))^r \prod_{j=1}^{r-1} \mathcal{P}_j^{r-j}(x),$$

where we defined

$$\zeta(x) = \mathcal{P}_0(x) \qquad \mathcal{X}_i(x) = \mathcal{X}_i^{[\lambda, \mu]}(x),$$

with $i = 1, \ldots, r - 1$. We note that here the so-called matter polynomials $\mathcal{P}$ only depend on the partition λ and not on μ.

For singularities associated to the partition $\nu = (5, 5, 2, 2, 1, 1, 1)$ as plotted in Fig. 4 the quiver diagram is depicted in Fig. 5. Further examples are discussed in Appendix F.

7 Higher degree symplectic leaves

In this section we discuss some symplectic leaves of higher degree n in the spectral parameter x corresponding to the partitions of the total size nr for $G = \mathrm{GL}_r$.

In the case $r = 2$ we can factorize the higher degree Lax matrices for partitions $\lambda = (2n)$ as a product of degree 1 Lax matrices labelled by partitions $\lambda = (2)$

$$L_{(2n), \underline{x}, \varnothing}(x; \underline{p}, \underline{q}) = L_{(2), (x_1, x_2), \varnothing}(x; p_1, q_1) \cdots L_{(2), (x_{2n-1}, x_{2n}), \varnothing}(x; p_n, q_n),$$

cf. [60].

For $r = 3$ the partitions are of the form $\lambda = (\lambda_1, 3n - \lambda_1)$ with $\lambda_1 \leq 3n$. The Lax matrices for these partitions can be factorized as a product of Lax matrices of partitions $\lambda = (3)$, $\lambda = (2,1)$ and their conjugates. The conjugates correspond to the partitions $\lambda = (3,3)$ and $\lambda = (2,1)$ respectively and are obtained via

$$\bar{L}_{\lambda,\underline{x},\mu}(x;\underline{p},\underline{q}) = \det L_{\lambda,\underline{x},\mu}(x;\underline{p},\underline{q}) L_{\lambda,\underline{x},\mu}^{-1}(-x;\underline{p},\underline{q}).$$

This can be seen as follows. If $3n - \lambda_1 = 0$ we can build the partition from copies of $\lambda = (3)$ as for the case $r = 2$. Any such partition is extended to the case where $3n - \lambda_1 = 1$ by adding a partition $\lambda = (2,1)$ which again extends to $3n - \lambda_1 = 2$ by adding another partition $\lambda = (2,1)$. Now we note that any $\lambda = (\lambda_1, 3n - \lambda_1)$ can be reduced to the cases discussed when stripping off multiples of the partition $\lambda = (3,3)$. An example is shown in Fig. 6.

A similar factorization of higher degree leaves to the product of the degree 1 leaves applies to the case of GL_4. However in the case of GL_5 and higher rank such factorization fails, first time for the $n = 2$ and the partition $\lambda = (4,3,3)$, i.e., $\lambda^t = (3,3,3,1)$, of the total size $|\lambda| = 10$. The Lax matrix associated to this partition is not factorized into a product of degree 1 Lax matrices. However, we can compute $L_{\lambda=(4,3,3)}$ using the fusion method.

7.1   Fusion of degree 2

In this subsection we present the degree 2 fusion of two elementary Lax matrices

$$L(x) = \left(\begin{array}{c|c|c} x - x_1 + P_{12}Q_{21} & -P_{12} & P_{12}P_{23} \\ \hline -Q_{21} & I & -P_{23} \\ \hline -Q_{32}Q_{21} & Q_{32} & x - x_1 - Q_{32}P_{23} \end{array} \right)$$

and

$$L'(x) = \left(\begin{array}{c|c|c} x - x_2 + P'_{13}Q'_{31} & P'_{13}Q'_{32} & -P'_{13} \\ \hline P'_{23}Q'_{31} & x - x_2 + P'_{23}Q'_{32} & -P'_{23} \\ \hline -Q'_{31} & -Q'_{32} & I \end{array} \right)$$

as introduced in (3.1). Here the blocks on the diagonal are of the size $k_1 \times k_1$, $k_2 \times k_2$ and $k_3 \times k_3$ and the Lax matrices contain $k_2(k_1 + k_3)$ and $k_3(k_1 + k_2)$ pairs of conjugate variables respectively.

We find that their product can be decomposed as

$$L(x)L'(x) = \tilde{Q}\tilde{L}(x),$$

where

$$\tilde{Q} = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & \tilde{Q}_{32} & I \end{array} \right)$$

and

$$\tilde{L}(x) = \left(\begin{array}{c|c} (x - x_1)(x - x_2)I + \tilde{P}_{1\tilde{2}}(xI - \tilde{J})\tilde{Q}_{\tilde{2}1} & -\tilde{P}_{1\tilde{2}}(xI - \tilde{J}) \\ \hline -(xI - \tilde{J})\tilde{Q}_{\tilde{2}1} & xI - \tilde{J} \end{array} \right). \tag{7.1}$$

Here we defined

$$\tilde{P}_{1\tilde{2}} = \left(\begin{array}{c|c} \tilde{P}_{12} & \tilde{P}'_{13} \end{array} \right), \qquad \tilde{Q}_{\tilde{2}1} = \left(\begin{array}{c} \tilde{Q}_{21} \\ \hline -\tilde{Q}'_{31} \end{array} \right),$$

and

$$\tilde{J} = \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}'_{32} & I \end{array} \right) \left(\begin{array}{c|c} x_2 I & \tilde{P}'_{23} \\ \hline 0 & x_1 I \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \hline -\tilde{Q}'_{32} & I \end{array} \right).$$

As in the linear fusion we introduced the new canonical variables

$$\begin{aligned}
\tilde{P}_{12} &= P_{12} - P'_{13}Q'_{32}, & \tilde{Q}_{21} &= Q_{21}, \\
\tilde{P}'_{13} &= P'_{13}, & \tilde{Q}'_{31} &= Q'_{31} + Q'_{32}Q_{21}, \\
\tilde{P}'_{23} &= P'_{23} + P_{23} - Q_{21}P'_{13}, & \tilde{Q}'_{32} &= Q'_{32}, \\
\tilde{P}_{23} &= P_{23}, & \tilde{Q}_{32} &= Q_{32} - Q'_{32}.
\end{aligned}$$

The final Lax matrix has $k_1k_2 + k_1k_3 + k_2k_3$ pairs $(\underline{p}, \underline{q})$ and $\det \tilde{L}(x) = (x-x_1)^{k_1+k_3}(x-x_2)^{k_1+k_2}$. It corresponds to the partition $\tilde{\lambda}^t = (k_1+k_3, k_1+k_2)$. This can be seen when setting all p and q equal to zero in (7.1). One obtains

$$\tilde{L}(x)\Big|_{\underline{p},\underline{q}=\varnothing} = \mathrm{diag}\left(\begin{array}{c|c|c} (x-x_1)(x-x_2)I_{k_1 \times k_1} & (x-x_2)I_{k_2 \times k_2} & (x-x_1)I_{k_3 \times k_3} \end{array} \right).$$

7.2   Full fusion of degree 2

We can further multiply the resulting Lax matrix in (7.1)

$$L'(x) = \left(\begin{array}{c|c} (x-x_1)(x-x_2)I + P'_{1\tilde{2}}(xI - J')Q'_{\tilde{2}1} & -P'_{1\tilde{2}}(xI - J') \\ \hline -(xI - J')Q'_{\tilde{2}1} & xI - J' \end{array} \right),$$

with

$$L(x) = \left(\begin{array}{c|c} I & -P_{1\tilde{2}} \\ \hline Q_{\tilde{2}1} & xI - J - Q_{\tilde{2}1}P_{1\tilde{2}} \end{array} \right),$$

which corresponds to a general regular partition (3.2) with arbitrary λ and $\mu = 1^{[k_1]}$. The blocks on the diagonal are of the size $k_1 \times k_1$ and $\tilde{k}_2 = k_2 + k_3$ as defined in Section 7.1. One finds

$$L(x)L'(x) = \tilde{Q}\tilde{L}(x)$$

with

$$\tilde{Q} = \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}_{\tilde{2}1} & I \end{array} \right),$$

and

$$\tilde{L}(x) = \left(\begin{array}{c|c} I & 0 \\ \hline \tilde{Q}'_{\tilde{2}1} & I \end{array} \right) \left(\begin{array}{c|c} (x-x_1)(x-x_2)I & -\tilde{P}'_{1\tilde{2}}(xI - J') \\ \hline 0 & (xI - J)(xI - J') \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \hline -\tilde{Q}'_{\tilde{2}1} & I \end{array} \right).$$

Here we defined

$$\tilde{P}'_{1\tilde{2}} = P'_{1\tilde{2}} + P_{1\tilde{2}}, \qquad \tilde{P}_{1\tilde{2}} = P_{1\tilde{2}}, \qquad \tilde{Q}'_{\tilde{2}1} = Q'_{\tilde{2}1}, \qquad \tilde{Q}_{\tilde{2}1} = Q_{\tilde{2}1} - Q'_{\tilde{2}1}.$$

The final Lax matrix contains $k_1 \tilde{k}_2 + k_J + k_{J'}$ pairs of conjugate variables where k_J and $k_{J'}$ denote the number of pairs in J and J' respectively. It corresponds to the partition $\tilde{\lambda}^t = (k_1 + k_3, k_1 + k_2, \lambda^t)$. Setting $p, q = 0$ yields the block matrix

$$\tilde{L}(x)\Big|_{\underline{p},\underline{q}=\varnothing}(x) = \mathrm{diag}\left((x - x_1)(x - x_2)I_{k_1 \times k_1} \ \Big| \ (x - X_\lambda)(x - X'_{\lambda'})I_{\tilde{k}_2 \times \tilde{k}_2} \right). \tag{7.2}$$

Here X_λ is defined in (3.5) corresponding to an arbitrary partition λ and $X'_{\lambda'}$ follows from (7.1) and reads

$$X'_{\lambda'} = \mathrm{diag}\left(((x - x_2)I_{k_2 \times k_2} \ \Big| \ (x - x_1)I_{k_3 \times k_3} \right).$$

7.3 Example $\tilde{\lambda} = (4, 3, 3)$

The case GL_5 and $\tilde{\lambda} = (4, 3, 3)$ we discussed at the beginning of this section corresponds to setting $k_1 = 1$ and $\tilde{k}_2 = 4$ in Section 7.2 while setting $\lambda = (2, 1, 1)$ such that $n_J = 3$. The Lax matrix L' is obtained in the case $k_1 = 1$, $k_2 = 2$ and $k_3 = 2$ from the fusion in Section 7.1 and thus yields $n_{J'} = 4$. The half-dimension is $\frac{1}{2} \dim_{\mathbb{C}} \mathcal{M}_{4,3,3} = 11$. For $p, q = 0$ the diagonal of the Lax matrix follows from (7.2) when taking $X_\lambda = \mathrm{diag}(x_3, x_3, x_3, x_4)$.

8 Quantization

Notice that the classical Yang–Baxter equation (2.1) is a limit of quantum Yang–Baxter equation

$$R_{12}(x - y)\hat{L}_{13}(x)\hat{L}_{23}(y) = \hat{L}_{23}(y)\hat{L}_{13}(x)R_{12}(x - y)$$

in $\mathrm{End}(V) \otimes \mathrm{End}(V) \otimes A$ where $V \simeq \mathbb{C}^r$ is the fundamental representation of $\mathfrak{gl}_r$ and A is the quantized algebra of functions on the classical phase space parametrized locally by (p_I, q^I). Here the quantum R-matrix is $R \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ and the quantum L-operator is $\hat{L} \in \mathrm{End}(V) \otimes A$, that is an $r \times r$ matrix valued in operators in A. The quantum R-matrix is

$$R(x) = 1 + \frac{\epsilon \mathbb{P}}{x},$$

where $\epsilon = -i\hbar$ is the quantization parameter and $\mathbb{P}$ is the permutation operator (2.2).

In terms of matrix elements $\hat{L}_{ij}$ we have the quantum Yang Baxter equation in A

$$\left[\hat{L}_{ij}(x), \hat{L}_{kl}(y)\right] = -\frac{\epsilon}{x - y}\left(\hat{L}_{kj}(x)\hat{L}_{il}(y) - \hat{L}_{kj}(y)\hat{L}_{il}(x)\right)$$

and its classical limit is

$$\{L_{ij}(x), L_{kl}(y)\} = -\frac{1}{x - y}\left(L_{kj}(x)L_{il}(y) - L_{kj}(y)L_{il}(x)\right)$$

with the standard convention

$$\left[\hat{\phi}, \hat{\psi}\right] = \epsilon\{\phi, \psi\} + O(\epsilon^2), \qquad \epsilon \to 0,$$

where $[\hat{\phi}, \hat{\psi}]$ denotes the commutator of the elements $\hat{\phi}$, $\hat{\psi}$ of the algebra A that correspond to the quantization of the functions ϕ, ψ on the classical phase space with Poisson brackets $\{\phi, \psi\}$. In particular the canonical coordinates p_I, q^I have Poisson bracket

$$\{p_I, q^J\} = \delta_I^J$$

and the respective operators have commutation relations

$$[\hat{p}_I, \hat{q}^J] = \epsilon \delta_I^J$$

that can be represented in the algebra of differential operators acting on Hilbert space of states represented by function of q^I as

$$\hat{q}^I \mapsto q^I, \qquad \hat{p}_I \mapsto \epsilon \frac{\partial}{\partial q^I}.$$

For a polynomial function $f(\hat{q}, \hat{p})$ the normal ordering notation $:f(\hat{q}, \hat{p}):$ means placing all operators $\hat{p}_I$ to the right of the operators $\hat{q}^I$ in each monomial.

The quantum version $\hat{L}_{\underline{\lambda}, \underline{x}, \mu}(x)$ of all our classical solutions $L_{\underline{\lambda}, \underline{x}, \mu}(x)$ is obtained by replacing all variables (p, q) by the operators $\hat{p}$, $\hat{q}$ and assuming normal ordering convention. One can check that such operator valued matrix $\hat{L}_{\underline{\lambda}, \underline{x}, \mu}(x)$ satisfies quantum Yang–Baxter equation. The commuting Hamiltonians are obtained from the expansion of the quantum spectral determinant (quantum spectral curve) as in [19]

$$\hat{W}_{x,y} = \operatorname{tr} A_r \big(y - \mathrm{e}^{\epsilon \partial_x} \hat{L}_1'(x)\big)\big(y - \mathrm{e}^{\epsilon \partial_x} \hat{L}_2'(x)\big) \cdots \big(y - \mathrm{e}^{\epsilon \partial_x} \hat{L}_r'(x)\big)$$

$$= \sum_{k=0}^{r} (-1)^k y^{r-k} \hat{\chi}_k(x + \epsilon) \mathrm{e}^{\epsilon k \partial_x}, \tag{8.1}$$

where $\hat{L}'(x) = \rho_1(g_\infty)\hat{L}_{\underline{\lambda}, \underline{x}, \mu}(x)$, cf. (6.2), and A_k is the normalised antisymmetrizer acting on the k-fold tensor product of $\mathbb{C}^r$. The quantum characters whose coefficients generate the algebra of quantum commuting Hamiltonians (Bethe subalgebra) are

$$\hat{\chi}_k(x) = \operatorname{tr} A_k \hat{L}_1'(x) \cdots \hat{L}_k'(x + \epsilon(k-1)) \tag{8.2}$$

see also [52]. The definition of the quantum spectral determinant (8.1) is a quantum version of the classical spectral curve (6.3), and there is a quantum version of the factorization (6.4)

$$\hat{\chi}_k(x) = \hat{\mathcal{Q}}_{r-k}^{[\lambda]}(x + \epsilon|\mu|)\hat{\mathcal{X}}_k^{[\lambda, \mu]}(x),$$

where the c-valued polynomials are

$$\hat{\mathcal{Q}}_i^{[\lambda]}(x) = \prod_{j=1}^{\lambda_{i+1}} \prod_{k=1}^{\lambda_j^t - i} \left(x - x_j + \epsilon \left(\sum_{l=1}^{j-1} \lambda_l^t + k - 1\right)\right).$$

The quantization of the corresponding integrable systems in the context of the $\mathcal{N} = 2$ supersymmetric quiver gauge theories has been considered in [56], in particular the q-character functions appearing in [56] after [34] stand for the eigenvalue of the quantum commuting Hamiltonians (8.2).

The quantized symplectic leaves $\hat{\mathcal{M}}_{\underline{\lambda}, \underline{x}, \mu}$ are modules, typically infinite-dimensional, for the dual Yangian algebra $\mathbf{Y}(\mathfrak{gl}_r)^*$ which is a quantum deformation algebra of the space of functions on the Poisson–Lie group $\mathrm{GL}_r(\mathcal{K}_{\mathbb{P}^1_x})$. This representation theory relates to the 'pre-fundamental' modules of Hernandez–Jimbo [40] associated to the individual singularities at points x_i labeled by a fundamental co-weight $\breve{\omega}_{\lambda_i^t}$.

A   Twisted cotangent bundles of generalized flag varieties

Let $\mathfrak{g}$ be a reductive Lie algebra and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a decomposition of $\mathfrak{g}$ into the Cartan subalgebra $\mathfrak{h}$, the negative nilpotent subspace $\mathfrak{n}_- = \oplus_{\alpha<0}\mathfrak{g}_\alpha$ and positive nilpotent subspace $\mathfrak{n}_+ = \oplus_{\alpha>0}\mathfrak{g}_\alpha$. Here α denote a root of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ the α-root subspace of $\mathfrak{g}$. Let $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathfrak{h} + \mathfrak{n}_-$ be the respective Borel subalgebras. If $\mathfrak{g} = \mathfrak{gl}_r$ then $\mathfrak{b}_+$ (or $\mathfrak{b}_-$) is represented by upper (or lower) triangular matrices including the diagonal, and $\mathfrak{n}_+$ (or $\mathfrak{n}_-$) is represented by strictly upper (or lower) triangular matrices excluding the diagonal.

Let G, H, $N_\pm$, $B_\pm$ the respective Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}_\pm$, $\mathfrak{b}_\pm$, and let $x \in \mathfrak{h}^*$ be a weight. Here we record explicit formulas for representation of $U\mathfrak{g}$ in x-twisted differential operators on the complete flag manifold G/B_+ following the approach of Harish-Chandra, Springer, Kostant, Beilinson–Bernstein. We identify the big cell of G/B_+ with N_- and denote elements of N_- by Q.

We compute the vector field L_X associated to the action of Lie algebra element X on G/B_+ from the left. Let ε be infinitesimal parameter, and let $1+\varepsilon X$ be a group element corresponding to Lie algebra element $X \in \mathfrak{g}$. Let $\tilde{Q} = Q + \varepsilon\delta_X Q$ denotes a coset representative in G/B_+ obtained from the action of $1 + \varepsilon X$ on Q from the left:

$$(1 + \varepsilon X)Q = (Q + \varepsilon\delta_X Q)(1 + \varepsilon n_+ + \varepsilon h), \qquad n_+ \in \mathfrak{n}_+, \quad h \in \mathfrak{h}, \tag{A.1}$$

where $(1 + \varepsilon n_+ + \varepsilon h)$ is an element of B_+ that gauges the deformation of Q. We find

$$XQ = Q(n_+ + h) + \delta_X Q, \qquad n_+ \in \mathfrak{n}_+, \quad h \in \mathfrak{h}$$

and thus

$$\delta_X Q = Q\big[Q^{-1}XQ\big]_-,$$

where $[\ ]_-$ denotes the projection $\mathfrak{g} \to \mathfrak{n}_-$. The corresponding vector field and the differential operator on scalar functions on N_- is $L_X = -\delta_X Q \frac{\partial}{\partial Q}$ that is

$$L_X = -Q\big[Q^{-1}XQ\big]_- \frac{\partial}{\partial Q},$$

where the minus sign comes from the standard convention of defining the vector fields associated to the group actions on manifolds in such a way as to preserve the Lie algebra bracket.

We are actually interested in a more general situation, when the differential operator L_X acts not on functions on G/B_+ but on sections of line bundle induced from the H-bundle $G/N_+ \to G/B_+$ by a semi-simple co-weight $x \in \mathfrak{h}^*$, e.g.,

$$x = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix}.$$

The additional connection term is $-x(h_X)$ for the diagonal variation h_X in the coset computation (A.1)

$$L_{X,x} = L_X - x(h)$$

and since

$$h_X = \big[Q^{-1}XQ\big]_0$$

where $[\ ]_0$ denotes the projection to the diagonal part $\mathfrak{g} \to \mathfrak{h}$, we find the differential operator

$$L_{X,x} = -Q[Q^{-1}XQ]_- \frac{\partial}{\partial Q} - \langle x, [Q^{-1}XQ]_0 \rangle.$$

acting on sections of the line bundle on G/B_+.

Now we fix $\mathfrak{g} = \mathfrak{gl}_r$. Let $(e_{ij})_{i,j \in [1,r]}$ denote the standard basis elements of $\mathfrak{gl}_r$ represented by matrices whose (i,j)-entry is equal to 1 and the rest is 0. The upper-triangular Borel subgroup B_+ preserves the standard full flag

$$0 \subset \mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \cdots \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \cdots \oplus \mathbb{C}e_r.$$

Further we define the coordinates $(q_{i,j})$ with $1 \leq j < i \leq r$ on N_- taking the matrix elements of $Q \in N_-$ in the defining representation of $\mathfrak{gl}_r$

$$q_{ij} := Q_{ij}, \qquad 1 \leq j < i \leq r$$

for example, for $\mathfrak{gl}_3$ we have

$$Q = \begin{pmatrix} 1 & & \\ q_{2,1} & 1 & \\ q_{3,1} & q_{3,2} & 1 \end{pmatrix}.$$

We evaluate in coordinates $(q)_{ij}$ the differential operator $L_{x,x}$ associated to each basis element $X = e_{ij}$ in $\mathfrak{g}$, and we assemble $r \times r$ matrix $\hat{L}$ valued in twisted differential operators

$$\hat{L}_{ij,x} = L_{e_{ji},x}.$$

Let us denote

$$p_{ij} = \frac{\partial}{\partial q_{ij}}, \qquad i > j$$

with

$$[p_{ij}, q_{kl}] = \delta_{ik}\delta_{jl},$$

and assemble the upper triangular matrix with only non-zero entries $(P)_{ij} = p_{ji}$ for $i > j$

$$P = \begin{pmatrix} 0 & p_{21} & p_{31} \\ 0 & 0 & p_{32} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$L_{e_{ij},x} = -\operatorname{tr} Q[Q^{-1}e_{ij}Q]_- P - \operatorname{tr} Q^{-1}e_{ij}Qx = -: \operatorname{tr} e_{ij}Q(x + [PQ]_+)Q^{-1}:$$
$$= -:\left(Q(x + [PQ]_+)Q^{-1}\right)_{ji}:$$

where normal ordering notation $:\ :$ means that all symbols of the operators p_{ij} are kept to the right, and consequently we find

$$\hat{L}_{ij} = -:\left(Q(x + [PQ]_+)Q^{-1}\right)_{ij}:.$$

B   Determinant formula

The determinant of a block matrix can be written as

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det\left(A - BD^{-1}C\right)\det D. \tag{B.1}$$

C   Sklyanin relation for elementary μ partitions

In order to show that the Lax matrix (4.2) satisfies the Sklyanin relation (2.1) we verify (4.5) and (4.6) in the following.

Starting with (4.6) we first note that $F_{\gamma,\gamma}$ in (4.3) can be written as

$$F_{\gamma,\gamma} = J_{\gamma,\gamma} + Q_- G' Q_-^{-1} \qquad \text{with} \quad G' = P_0 + Q_0[Q_+ P_-]_- Q_0^{-1},$$

where $J_{\gamma,\gamma}$ has been defined in (3.4) and satisfies the $\mathfrak{gl}\left(\frac{r}{2}\right)$ commutation relations (3.3).

It follows that (4.6) is equivalent to

$$\left[\left\{ ([P_+ Q_-]_+)_2, \left(Q_-^{-1}\right)_1 \right\} (Q_-)_1, G_1' \right] - \left[\left\{ ([P_+ Q_-]_+)_1, \left(Q_-^{-1}\right)_2 \right\} (Q_-)_2, G_2' \right]$$
$$+ \{G_1', G_2'\} = (G_1' - G_2')\mathbb{P}, \tag{C.1}$$

where $[X, Y] = XY - YX$ denotes the anticommutator. Further we use the notation $X_1 = X \otimes I$ and $X_2 = I \otimes X$ and $\mathbb{P}$ act as a permutation such that $\mathbb{P} X_1 = X_2 \mathbb{P}$. It is convenient to consider different cases. Writing (C.1) in components and taking into account that G' and Q_- are lower diagonal while $[P_+ Q_-]_+$ is upper diagonal results in the conditions

$$\{G_{ab}', G_{cd}'\} = \delta_{cb} G_{ad}' - \delta_{ad} G_{cb}' \qquad \text{for} \quad a \geq b \wedge c \geq d,$$
$$\left[\left\{ ([P_+ Q_-]_+)_{cd}, Q_-^{-1} \right\} Q_-, G' \right]_{ab} = \delta_{cb} G_{ad}' - \delta_{ad} G_{cb}' \qquad \text{for} \quad c < d.$$

The first relation can be verified straightforwardly. The second relation follows when noting that $\left\{ ([P_+ Q_-]_+)_{ab}, Q_-^{-1} \right\} Q_- = -e_{ba}$ for $a < b$.

To show (4.5) we note that $\bar{K}_{\gamma,\alpha}$ and $K_{\alpha,\gamma}$ are independent of the variables p_{ij}. Thus the brackets reduces to

$$\sum_{s,t=1}^{\frac{r}{2}} \frac{\partial F_{ij}}{\partial p_{st}} \frac{\partial \bar{K}_{kl}}{\partial q_{ts}} = +\bar{K}_{il}\delta_{kj}, \qquad \sum_{s,t=1}^{\frac{r}{2}} \frac{\partial F_{ij}}{\partial p_{st}} \frac{\partial K_{kl}}{\partial q_{ts}} = -K_{kj}\delta_{il}. \tag{C.2}$$

Here we suppressed the subindeces α and γ. The Latin indices take values $i, j, k, l = 1, \ldots, \frac{r}{2}$.

Using the relation $\partial_q K = K\left(\partial_q \bar{K}\right)K$ which follows from (4.4), one finds that the two equations in (C.2) are equivalent. They can be written as

$$\sum_{s,t} \frac{\partial G_{ij}}{\partial p_{st}} \frac{\partial \bar{K}_{kl}}{\partial q_{ts}} = (Q_0 Q_+)_{il}(Q_-)_{kj}. \tag{C.3}$$

In order to show this relation it is again convenient to consider different cases and take into account the dependence of G on p. For a particular choice of the indices i and j we see that (C.3) is equivalent to:

Case $i < j$

$$\sum_{s<t} \frac{\partial (P_+ Q_-)_{ij}}{\partial p_{st}} \frac{\partial (Q_-)_{kl}}{\partial q_{ts}} = \delta_{il}(Q_-)_{kj}.$$

Case $i > j$

$$\sum_{s>t} \frac{\partial (Q_+ P_-)_{ij}}{\partial p_{st}} \frac{\partial (Q_+)_{kl}}{\partial q_{ts}} = (Q_+)_{il}\delta_{kj}.$$

Case $i = j$

$$\sum_{s=t} \frac{\partial (P_0)_{ij}}{\partial p_{st}} \frac{\partial (Q_0)_{kl}}{\partial q_{ts}} = (Q_0)_{il}\delta_{kj}.$$

These equations can be checked explicitly. The derivatives with respect to q_{ij} for $i \neq j$ essentially yield delta functions while the derivatives with respect to p_{ij} give the q-dependence.
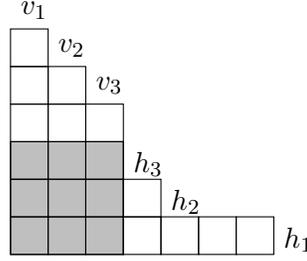
Figure 7. Frobenius-like coordinates of the partition $(7, 4, 3, 3, 2, 1)$: $f = 3$, $h_1 = 4$, $h_2 = 1$, $h_3 = 0$, $v_1 = 3$, $v_2 = 2$, $v_3 = 1$.

D The number of independent commuting Hamiltonians

In order to show the relation (6.8) it is convenient to introduce Frobenius-like coordinates to label the partitions, compare, e.g., [48].

Let us focus on the case $\mu = \varnothing$ which is sufficient as we will argue at the end of this section. Using Frobenius-like coordinates any partition λ can be written as

$$\lambda = (h_1 + f, \ldots, h_f + f, \underbrace{f, \ldots, f}_{v_f}, \underbrace{f - 1, \ldots, f - 1}_{v_{f-1} - v_f}, \ldots, \underbrace{1, \ldots, 1}_{v_1 - v_2}).$$

Here f denotes the length of the sides of the maximal square which fits into the lower left corner of the corresponding Young diagram as shown in Fig. 7. the variables h_i denote the number of boxes on the right of the square in the ith row while the variables v_k denote the number of boxes above the square in the kth row. The coordinates introduced in this way have the advantage that the transpose can be obtained by interchanging h_i and v_i where $i = 1, \ldots, f$ such that

$$\lambda^t = \lambda\big|_{v_i \leftrightarrow h_i} = (v_1 + f, \ldots, v_f + f, \underbrace{f, \ldots, f}_{h_f}, \underbrace{f - 1, \ldots, f - 1}_{h_{f-1} - h_f}, \ldots, \underbrace{1, \ldots, 1}_{h_1 - h_2}).$$

To introduce Frobenius-like coordinates in (6.8) we decompose the sum over the elements of the partition λ as

$$\sum_{k=0}^{r} n_k^{[\lambda, \varnothing]} = \sum_{k=1}^{r} \sum_{j=1}^{k} \lambda_j - \frac{r(r + 1)}{2}$$

$$= \sum_{k=1}^{f} (r - k + 1)\lambda_k + \sum_{k=1}^{r-f} (r - f - k + 1)\lambda_{k+f} - \frac{r(r + 1)}{2}. \tag{D.1}$$

Now the first term on the right-hand-side of (D.1) can be written in terms of the variables h_i as

$$\sum_{k=1}^{f} (r - k + 1)\lambda_k = \sum_{k=1}^{f} (r - k + 1)(h_k + f),$$

while the second one can be written in terms of the variables v_i as

$$\sum_{k=1}^{r-f} (r - f - k + 1)\lambda_{k+f} = f \sum_{k=1}^{v_f} (r - f - k + 1) + \sum_{l=1}^{f-1} l \sum_{k=v_{l+1}+1}^{v_l} (r - f - k + 1)$$

$$= -\frac{1}{2} \left(\sum_{l=1}^{f} v_l^2 + (2f - 2r - 1) \sum_{l=1}^{f} v_l \right).$$

Finally, using the relation $r = \sum\limits_{i=1}^{f} h_i + f^2 + \sum\limits_{i=1}^{f} v_i$ we find that

$$\sum_{k=0}^{r} n_k^{[\lambda,\varnothing]} = \frac{1}{2}\left(r^2 - \sum_{l=1}^{f}(v_l + f)^2 - \sum_{l=1}^{f} h_l(2l-1)\right) = \frac{1}{2}\left(r^2 - \sum_{i=1}^{\lambda_1}\left(\lambda_i^t\right)^2\right) = d_{[\lambda,\varnothing]}.$$

As λ is arbitrary it follows that the same relation holds for partitions μ with $\lambda = \varnothing$. Combining the two relations we find that

$$\sum_{k=0}^{r} n_k^{[\lambda,\mu]} = \sum_{k=0}^{r}\left(n_k^{[\lambda,\varnothing]} + n_k^{[\varnothing,\mu]}\right) + \frac{r(r+1) - |\lambda|(|\lambda|+1) - |\mu|(|\mu|+1)}{2}$$

$$= d_{[\lambda,\varnothing]} + d_{[\varnothing,\mu]} + \frac{r^2 - |\lambda|^2 - |\mu|^2}{2} = d_{[\lambda,\mu]},$$

where we used that $r = |\lambda| + |\mu|$. Another way to show (6.8) is to use the relation (E.1).

E Rewritten polynomials

The relation in (6.11) can be shown using the relation

$$\lambda^t = (\underbrace{r,\ldots,r}_{\lambda_r}, \underbrace{r-1,\ldots,r-1}_{\lambda_{r-1}-\lambda_r}, \ldots, \underbrace{1,\ldots,1}_{\lambda_1-\lambda_2}). \tag{E.1}$$

First it follows that $\mathcal{Q}_i^{[\lambda]}$ in (6.6) can be written as

$$\mathcal{Q}_i^{[\lambda]}(x) = \prod_{k=1}^{\lambda_{i+1}}(x-x_k)^{\lambda_k^t - i} = \prod_{l=i+1}^{r}\prod_{k=\lambda_{l+1}+1}^{\lambda_l}(x-x_k)^{\lambda_k^t - i} = \prod_{l=i+1}^{r}\prod_{k=\lambda_{l+1}+1}^{\lambda_l}(x-x_k)^{l-i}, \tag{E.2}$$

with $\lambda_{r+1} = 0$ and using that $\lambda_k^t = l$ for $k = \lambda_{l+1}+1,\ldots,\lambda_l$. Note that the polynomiality of $\mathcal{Q}_i^{[\lambda]}$ is now manifest. Further we find

$$\mathcal{Q}_i^{[\lambda]}(x) = \prod_{l=i+1}^{r}\prod_{k=\lambda_{l+1}+1}^{\lambda_l}(x-x_k)^{l-i} = \prod_{l=i+1}^{r}\prod_{k=1}^{\lambda_l-\lambda_{l+1}}(x-x_{\lambda_l-k+1})^{l-i}.$$

Identifying $m_l = \lambda_l - \lambda_{l+1}$ we recover (6.11). We remark that we can write the polynomial simply as

$$\mathcal{Q}_i^{[\lambda]}(x) = \prod_{l=i+1}^{r}\prod_{k=\lambda_{l+1}+1}^{\lambda_l}(x-x_k)^{l-i} = \prod_{l=i+1}^{r}\prod_{k=1}^{\lambda_l}(x-x_k),$$

after rearranging the product.

F Examples

We consider a couple of explicit examples associated to the diagrams as shown on Figs. 9 and 8. The ranks in the gauged (circled) nodes on the right correspond to the heights of the profile on the left. The ranks of the framed (boxed) nodes on the right counting the number of fundamentals correspond to the (minus) change of the slope of the profile on the left in the corresponding corner (negative second difference), cf. Section 6 and [55].
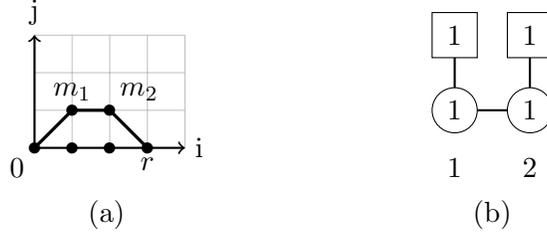
Figure 8. (a) shows the Newton polygon and (b) shows the quiver diagram for $\mu = (2,1)$, $\lambda = \varnothing$ and $r = 3$. The ranks in the gauged (circle) nodes on the right correspond to the heights of the profile on the left. The ranks of the framed (boxed) nodes on the right counting the number of fundamentals correspond to the change of the slope of the profile on the left.

F.1 $\mu = (2, 1)$ and $\lambda = \varnothing$

The Lax matrix corresponding to the partitions $\mu = (2,1)$ and $\lambda = \varnothing$ can be obtained from (4.7). It contains two conjugate pairs of variables and reads

$$L_{(2,1)}(x; \underline{p}, \underline{q}) = \begin{pmatrix} 0 & 0 & -\mathrm{e}^{-q_{3,3}} \\ 0 & 1 & -p_{2,3} \\ \mathrm{e}^{q_{3,3}} & q_{3,2} & x - p_{3,3} - q_{3,2}p_{2,3} \end{pmatrix}.$$

It resembles the structure of the DST and Toda chain. The Hamiltonians can be computed explicitly. The emerging Newton polygon and quiver diagram are depicted in Fig. 8, cf. Section 6.

F.2 $\mu = (2, 2)$ and $\lambda = \varnothing$

The Lax matrix corresponding to the partitions $\mu = (2,2)$ and $\lambda = \varnothing$ can be obtained from (4.2). They contain four conjugate pairs of variables and can be written as

$$L_{(2,2)}(x; \underline{p}, \underline{q}) = \left(\begin{array}{c|c} 0 & K_{(2,2)} \\ \hline \bar{K}_{(2,2)} & xI - F_{(2,2)} \end{array} \right),$$

where the 2×2 block matrices are given by

$$F_{(2,2)} = \begin{pmatrix} p_{3,3} - q_{4,3}p_{3,4} & p_{3,4} \\ \mathrm{e}^{q_{4,4}-q_{3,3}}p_{4,3} - q_{4,3}(p_{4,4} - p_{3,3} + q_{4,3}p_{3,4}) & p_{4,4} + q_{4,3}p_{3,4} \end{pmatrix},$$

$$K_{(2,2)} = \begin{pmatrix} -\mathrm{e}^{-q_{3,3}} - \mathrm{e}^{-q_{4,4}}q_{3,4}q_{4,3} & \mathrm{e}^{-q_{4,4}}q_{3,4} \\ \mathrm{e}^{-q_{4,4}}q_{4,3} & -\mathrm{e}^{-q_{4,4}} \end{pmatrix},$$

$$\bar{K}_{(2,2)} = \begin{pmatrix} \mathrm{e}^{q_{3,3}} & \mathrm{e}^{q_{3,3}}q_{3,4} \\ \mathrm{e}^{q_{3,3}}q_{4,3} & \mathrm{e}^{q_{4,4}} + \mathrm{e}^{q_{3,3}}q_{3,4}q_{4,3} \end{pmatrix}.$$

G Cluster structures

In this section we elaborate on the cluster structure of the fusion procedure in Section 3.3. As an example we study the GL_3 case and introduce the Lax matrices

$$L_1(x) = \begin{pmatrix} x + p''_{12}q''_{21} + p''_{13}q''_{31} & -p''_{12} & -p''_{13} \\ -q''_{21} & 1 & 0 \\ -q''_{31} & 0 & 1 \end{pmatrix},$$
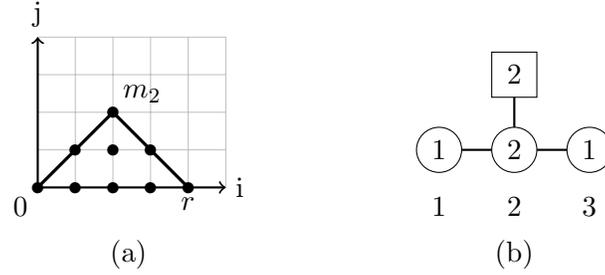
Figure 9. (a) shows the Newton polygon and (b) shows the quiver diagram for $\mu = (2, 2)$, $\lambda = \varnothing$ and $r = 4$.

$$L_2(x) = \begin{pmatrix} 1 & -p'_{12} & 0 \\ q'_{21} & x - q'_{21}p'_{12} + p'_{23}q'_{32} & -p'_{23} \\ 0 & -q'_{32} & 1 \end{pmatrix},$$

$$L_3(x) = \begin{pmatrix} 1 & 0 & -p_{13} \\ 0 & 1 & -p_{23} \\ q_{31} & q_{32} & x - q_{31}p_{13} - q_{32}p_{23} \end{pmatrix}.$$

Looking at the product

$$L_3(x - x_3)L_2(x - x_2)L_1(x - x_1)$$

there are two ways to proceed with the fusion. The standard way used in Section 3.3 first computes the Lax matrix resulting from the fusion $L_3(x - x_3)L_2(x - x_2)$ and then computes the final result by fusing the result with $L_1(x - x_1)$ from the right. It yields the matrix

$$L_{(32)1}(x) = Q(x - X - [PQ]_+)Q^{-1}$$

as introduced in Section 3.3. Alternatively we can first fuse $L_2(x - x_2)L_1(x - x_1)$, bring it to the canonical form, and then multiply by $L_3(x - x_3)$ from the left. This procedure results in

$$L_{3(21)} = \tilde{Q}^{-1}(x - X - [\tilde{Q}\tilde{P}]_+)\tilde{Q}$$

with

$$\tilde{Q} = \begin{pmatrix} 1 & 0 & 0 \\ -\tilde{q}_{21} & 1 & 0 \\ -\tilde{q}_{31} & -\tilde{q}_{32} & 1 \end{pmatrix}, \qquad \tilde{P} = \begin{pmatrix} 0 & \tilde{p}_{12} & \tilde{p}_{13} \\ 0 & 0 & \tilde{p}_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

The formulas are related by the canonical transformation

$$\tilde{p}_{12} = p_{12} - p_{13}q_{32}, \qquad \tilde{p}_{23} = p_{23} - q_{21}p_{13}, \qquad \tilde{q}_{31} = q_{31} + q_{32}q_{21}. \tag{G.1}$$

In other words, the fusion procedure to obtain the Darboux coordinates is not associative, and the failure of the associativity is described by the symplectomorphism (G.1).

Acknowledgements

References

[1] Aganagic M., Frenkel E., Okounkov A., Quantum q-Langlands correspondence, *Trans. Moscow Math. Soc.* **79** (2018), 1–83, arXiv:1701.03146.

[2] Alday L.F., Gaiotto D., Tachikawa Y., Liouville correlation functions from four-dimensional gauge theories, *Lett. Math. Phys.* **91** (2010), 167–197.

[3] Babelon O., Bernard D., Talon M., Introduction to classical integrable systems, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 2003.

[4] Babich M.V., Birational Darboux coordinates on (co)adjoint orbits of GL(N, $\mathbb{C}$), *Funct. Anal. Appl.* **50** (2016), 17–30.

[5] Bazhanov V.V., Frassek R., Łukowski T., Meneghelli C., Staudacher M., Baxter Q-operators and representations of Yangians, *Nuclear Phys. B* **850** (2011), 148–174, arXiv:1010.3699.

[6] Bazhanov V.V., Tsuboi Z., Baxter's Q-operators for supersymmetric spin chains, *Nuclear Phys. B* **805** (2008), 451–516, arXiv:0805.4274.

[7] Beilinson A.A., Drinfel'd V.G., Quantization of Hitchin's fibration and Langlands' program, in Algebraic and Geometric Methods in Mathematical Physics (Kaciveli, 1993), *Math. Phys. Stud.*, Vol. 19, Kluwer Acad. Publ., Dordrecht, 1996, 3–7.

[8] Belavin A.A., Drinfel'd V.G., Solutions of the classical Yang–Baxter equation for simple Lie algebras, *Funct. Anal. Appl.* **16** (1982), 159–180.

[9] Birkhoff G.D., The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc. Amer. Acad. Arts Sci.* **49** (1913), 521–568.

[10] Boos H., Göhmann F., Klümper A., Nirov K.S., Razumov A.V., Exercises with the universal R-matrix, *J. Phys. A: Math. Theor.* **43** (2010), 415208, 35 pages, arXiv:1004.5342.

[11] Bottacin F., Poisson structures on moduli spaces of sheaves over Poisson surfaces, *Invent. Math.* **121** (1995), 421–436.

[12] Bottacin F., Symplectic geometry on moduli spaces of stable pairs, *Ann. Sci. École Norm. Sup. (4)* **28** (1995), 391–433.

[13] Braverman A., Finkelberg M., Nakajima H., Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian, arXiv:1604.03625.

[14] Braverman A., Finkelberg M., Nakajima H., Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II, arXiv:1601.03586.

[15] Charbonneau B., Hurtubise J., Singular Hermitian–Einstein monopoles on the product of a circle and a Riemann surface, *Int. Math. Res. Not.* **2011** (2011), 175–216, arXiv:0812.0221.

[16] Cherkis S., Kapustin A., Nahm transform for periodic monopoles and $\mathcal{N} = 2$ super Yang–Mills theory, *Comm. Math. Phys.* **218** (2001), 333–371, arXiv:hep-th/0006050.

[17] Cherkis S.A., Kapustin A., Hyper-Kähler metrics from periodic monopoles, *Phys. Rev. D* **65** (2002), 084015, 10 pages, arXiv:hep-th/0109141.

[18] Cherkis S.A., Kapustin A., Periodic monopoles with singularities and $\mathcal{N} = 2$ super-QCD, *Comm. Math. Phys.* **234** (2003), 1–35, arXiv:hep-th/0011081.

[19] Chervov A., Talalaev D., Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence, arXiv:hep-th/0604128.

[20] Costello K., Supersymmetric gauge theory and the Yangian, arXiv:1303.2632.

[21] Costello K., Witten E., Yamazaki M., Gauge theory and integrability, I, *ICCM Not.* **6** (2018), 46–119, arXiv:1709.09993.

[22] Derkachov S.E., Manashov A.N., $\mathcal{R}$-matrix and Baxter $\mathcal{Q}$-operators for the noncompact SL(N, $\mathbb{C}$) invariant spin chain, *SIGMA* **2** (2006), 084, 20 pages, arXiv:nlin.SI/0612003.

[23] Di Francesco P., Mathieu P., Sénéchal D., Conformal field theory, *Graduate Texts in Contemporary Physics*, Springer-Verlag, New York, 1997.

[24] Donagi R.Y., Spectral covers, in Current Topics in Complex Algebraic Geometry (Berkeley, CA, 1992/93), *Math. Sci. Res. Inst. Publ.*, Vol. 28, Cambridge University Press, Cambridge, 1995, 65–86.

[25] Donagi R.Y., Principal bundles on elliptic fibrations, *Asian J. Math.* **1** (1997), 214–223, arXiv:alg-geom/9702002.

[26] Donagi R.Y., Geometry and integrability, in Geometry and Integrability, *London Math. Soc. Lecture Note Ser.*, Vol. 295, Cambridge Univerity Press, Cambridge, 2003, 21–59.

[27] Donagi R.Y., Gaitsgory D., The gerbe of Higgs bundles, *Transform. Groups* **7** (2002), 109–153, arXiv:math.AG/0005132.

[28] Donagi R.Y., Markman E., Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles, in Integrable Systems and Quantum Groups (Montecatini Terme, 1993), *Lecture Notes in Math.*, Vol. 1620, Springer, Berlin, 1996, 1–119, arXiv:alg-geom/9507017.

[29] Drinfel'd V.G., Quantum groups, in Proceedings of the International Congress of Mathematicians, Vols. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.

[30] Elliott C., Pestun V., Multiplicative Hitchin systems and supersymmetric gauge theory, arXiv:1812.05516.

[31] Faddeev L.D., Takhtajan L.A., Hamiltonian methods in the theory of solitons, *Springer Series in Soviet Mathematics*, Springer-Verlag, Berlin, 1987.

[32] Finkelberg M., Tsymbaliuk A., Multiplicative slices, relativistic Toda and shifted quantum affine algebras, arXiv:1708.01795.

[33] Frenkel E., Ngô B.C., Geometrization of trace formulas, *Bull. Math. Sci.* **1** (2011), 129–199, arXiv:1004.5323.

[34] Frenkel E., Reshetikhin N., The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras, in Recent Developments in Quantum Affine Algebras and related Topics (Raleigh, NC, 1998), *Contemp. Math.*, Vol. 248, Amer. Math. Soc., Providence, RI, 1999, 163–205, arXiv:math.QA/9810055.

[35] Friedman R., Morgan J., Witten E., Vector bundles and F theory, *Comm. Math. Phys.* **187** (1997), 679–743, arXiv:hep-th/9701162.

[36] Gerasimov A., Kharchev S., Lebedev D., Oblezin S., On a class of representations of the Yangian and moduli space of monopoles, *Comm. Math. Phys.* **260** (2005), 511–525, arXiv:math.AG/0409031.

[37] Gorsky A., Gukov S., Mironov A., Multiscale $N = 2$ SUSY field theories, integrable systems and their stringy/brane origin, *Nuclear Phys. B* **517** (1998), 409–461, arXiv:hep-th/9707120.

[38] Gorsky A., Gukov S., Mironov A., SUSY field theories in higher dimensions and integrable spin chains, *Nuclear Phys. B* **518** (1998), 689–713, arXiv:hep-th/9710239.

[39] Haouzi N., Schmid C., Little string origin of surface defects, *J. High Energy Phys.* **2017** (2017), no. 5, 082, 53 pages, arXiv:1608.07279.

[40] Hernandez D., Jimbo M., Asymptotic representations and Drinfeld rational fractions, *Compos. Math.* **148** (2012), 1593–1623, arXiv:1104.1891.

[41] Hitchin N., Stable bundles and integrable systems, *Duke Math. J.* **54** (1987), 91–114.

[42] Hurtubise J.C., Markman E., Elliptic Sklyanin integrable systems for arbitrary reductive groups, *Adv. Theor. Math. Phys.* **6** (2002), 873–978, arXiv:math.AG/0203031.

[43] Jimbo M., Introduction to the Yang–Baxter equation, in Braid group, knot theory and statistical mechanics, *Adv. Ser. Math. Phys.*, Vol. 9, World Sci. Publ., Teaneck, NJ, 1989, 111–134.

[44] Kamnitzer J., Webster B., Weekes A., Yacobi O., Yangians and quantizations of slices in the affine Grassmannian, *Algebra Number Theory* **8** (2014), 857–893, arXiv:1209.0349.

[45] Kapustin A., Witten E., Electric-magnetic duality and the geometric Langlands program, arXiv:hep-th/0604151.

[46] Kimura T., Pestun V., Quiver W-algebras, *Lett. Math. Phys.* **108** (2018), 1351–1381, arXiv:1512.08533.

[47] Looijenga E., Root systems and elliptic curves, *Invent. Math.* **38** (1976), 17–32.

[48] Macdonald I.G., Symmetric functions and Hall polynomials, *Oxford Mathematical Monographs*, The Clarendon Press, Oxford University Press, New York, 1979.

[49] Markman E., Spectral curves and integrable systems, *Compositio Math.* **93** (1994), 255–290.

[50] Mason L., Nutku Y. (Editors), Geometry and integrability, *London Mathematical Society Lecture Note Series*, Vol. 295, Cambridge University Press, Cambridge, 2003.

[51] Meneghelli C., Superconformal gauge theory, Yangian symmetry and Baxter's Q-operator, Ph.D. Thesis, Humboldt University Berlin, 2011.

[52] Molev A., Yangians and classical Lie algebras, *Mathematical Surveys and Monographs*, Vol. 143, Amer. Math. Soc., Providence, RI, 2007.

[53] Nakajima H., Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I, *Adv. Theor. Math. Phys.* **20** (2016), 595–669, arXiv:1503.03676.

[54] Nekrasov N., BPS/CFT correspondence: non-perturbative Dyson–Schwinger equations and qq-characters, *J. High Energy Phys.* **2016** (2016), no. 3, 181, 69 pages, arXiv:1512.05388.

[55] Nekrasov N., Pestun V., Seiberg–Witten geometry of four dimensional $\mathcal{N} = 2$ quiver gauge theories, arXiv:1211.2240.

[56] Nekrasov N., Pestun V., Shatashvili S., Quantum geometry and quiver gauge theories, *Comm. Math. Phys.* **357** (2018), 519–567, arXiv:1312.6689.

[57] Nekrasov N., Witten E., The omega deformation, branes, integrability and Liouville theory, *J. High Energy Phys.* **2010** (2010), no. 9, 092, 83 pages, arXiv:1002.0888.

[58] Sauloy J., Isomonodromy for complex linear q-difference equations, in Théories asymptotiques et équations de Painlevé, *Sémin. Congr.*, Vol. 14, Soc. Math. France, Paris, 2006, 249–280.

[59] Seiberg N., Witten E., Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ super-symmetric Yang–Mills theory, *Nuclear Phys. B* **426** (1994), 19–52, arXiv:hep-th/9407087.

[60] Shapiro A., Grothendieck resolution, affine Grassmannian, and Yangian, Ph.D. Thesis, University of California, Berkeley, 2016.

[61] Sklyanin E.K., On complete integrability of the Landau–Lifshitz equation, Tech. Rep. LOMI-E-79-3, 1979.

[62] Sklyanin E.K., Some algebraic structures connected with the Yang–Baxter equation, *Funct. Anal. Appl.* **16** (1982), 263–270.

[63] Sklyanin E.K., Bäcklund transformations and Baxter's Q-operator, in Integrable Systems: from Classical to Quantum (Montréal, QC, 1999), *CRM Proc. Lecture Notes*, Vol. 26, Amer. Math. Soc., Providence, RI, 2000, 227–250, arXiv:nlin.SI/0009009.

[64] Torrielli A., Classical integrability, *J. Phys. A: Math. Theor.* **49** (2016), 323001, 31 pages, arXiv:1606.02946.