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Invariants in Separated Variables: Yang–Baxter, Entwining and Transfer Maps

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Abstract. We present the explicit form of a family of Liouville integrable maps in 3 variables, the so-called *triad family of maps* and we propose a multi-field generalisation of the latter. We show that by imposing separability of variables to the invariants of this family of maps, the H_{I}, H_{II} and H^A_{III} Yang–Baxter maps in general position of singularities emerge. Two different methods to obtain entwining Yang–Baxter maps are also presented. The outcomes of the first method are entwining maps associated with the H_{I}, H_{II} and H^A_{III} Yang–Baxter maps, whereas by the second method we obtain non-periodic entwining maps associated with the whole F and H-list of quadrirational Yang–Baxter maps. Finally, we show how the transfer maps associated with the H-list of Yang–Baxter maps can be considered as the $(k-1)$-iteration of some maps of simpler form. We refer to these maps as *extended transfer maps* and in turn they lead to k-point alternating recurrences which can be considered as alternating versions of some hierarchies of discrete Painlevé equations.

Key words: discrete integrable systems; Yang–Baxter maps; entwining maps; transfer maps

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1 Introduction

The quantum Yang–Baxter equation originates from the theory of exactly solvable models in statistical mechanics [11, 73]. It reads

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.1}$$

where $R\colon V\otimes V \mapsto V\otimes V$ a linear operator and R_{lm}, $l\neq m\in\{1,2,3\}$ the operators that acts as R on the l-th and m-th factors of the tensor product $V\otimes V\otimes V$. For the history of the latter and for the early developments on the theory see [36]. Replacing the vector space V with any set X and the tensor product with the cartesian product, Drinfeld [21] introduced the *set theoretical version* of (1.1). Solutions of the latter appeared under the name of *set theoretical solutions of the quantum Yang–Baxter equation*. The first instance of such solutions, appeared in [24, 65]. The term *Yang–Baxter maps* was proposed by Veselov [69] as an alternative name to the Drinfeld's one. Early results on the context of Yang–Baxter maps were provided in [1, 40, 57]. In the recent years, many results arose in the interplay between studies on Yang–Baxter maps and the theory of discrete integrable systems [8, 9, 10, 12, 18, 19, 20, 31].

In [23] it was considered a special type of set theoretical solutions of the quantum Yang–Baxter equation, the so called *non degenerate rational maps*. Nowadays, this type of solutions is referred to as *quadrirational Yang–Baxter maps*. Note that the notion of quadrirational maps, was extended in [46] to the notion of 2^n*-rational maps*, where highly symmetric higher dimensional maps were considered. Under the assumption of quadrirationality and modulo conjugation (see Definition 3.1), in [5, 59] a list of ten families of maps was obtained. Five of them were given in [5], which constitute the so-called F-list of quadrirational Yang–Baxter maps and five more in [59], which constitute the so-called H-list of quadrirational Yang–Baxter maps. For their explicit form see Appendix A. The Yang–Baxter maps of the F-list and the H-list can also be obtained from some of the integrable lattice equations in the classification scheme of [4], by using the invariants of the generators of the Lie point symmetry group of the latter [60]. In the series of papers [44, 45, 56], from the Yang–Baxter maps of the F-list and of the H-list, integrable lattice equations and correspondences (relations) were systematically constructed. Invariant, under the maps, functions where the variables appeared in separated form, played an important role to this construction. The cornerstone of this manuscript are invariant functions where the variables appear in separated form.

In [3], it was introduced a family rational of maps in 3 variables that preserves two rational functions the so-called *the triad map*. The triad map serves as a generalisation of the QRT map [61] (cf. [22]). In Section 2 we present an explicit formula for Adler's triad map as well as we prove the Liouville integrability of the latter. We also propose an extension of the triad map in $k \geq 3$ number of variables. If one imposes separability to the variables of the invariants of the triad map, the $H_{\rm I}$, the $H_{\rm II}$ and the $H_{\rm III}^A$ Yang–Baxter maps in general positions of singularities, emerge. This is presented in Section 3 together with the explicit formulae for these maps.

In Section 4, we develop two methods to obtain non-equivalent *entwining maps* [51], i.e., maps R, S, T that satisfy the relation

$$R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}.$$

The first method gives us entwining maps associated with the $H_{\rm I}$, $H_{\rm II}$ and the $H_{\rm III}^A$ members of the H-list of Yang–Baxter maps. The second one produces entwining maps for the whole F-list and the H-list. In this manuscript we present the entwining maps associated with the H-list of quadrirational Yang–Baxter maps only.

In Section 5, we re-factorise the *transfer maps* [69] associated with the H-list of Yang–Baxter maps. We show that the transfer maps coincide with the $(k-1)$-iteration of some maps of simpler form that we refer to as *extended transfer maps*. Moreover, we show that the extended transfer maps, after an integration followed by a change of variables, are written as k-point recurrences, which some of them can be considered as alternating versions of discrete Painlevé hierarchies [16, 32, 57]. In Section 6 we end this manuscript with some conclusions and perspectives.

2 The Adler's triad family of maps

In [3], Adler proposed the so-called *triad family of maps*. The triad map is a family of maps in 3 variables that consists of the composition of involutions which preserve two rational invariants of a specific form. In what follows we present the explicit form of the latter in terms of its invariants.

Consider the polynomials

$$n^i = \sum_{j,k,l=0}^{1} \alpha^i_{j,k,l} x_1^{1-j} x_2^{1-k} x_3^{1-l}, \qquad d^i = \sum_{j,k,l=0}^{1} \beta^i_{j,k,l} x_1^{1-j} x_2^{1-k} x_3^{1-l}, \qquad i = 1, 2,$$

where x_1, x_2, x_3 are considered as variables and $\alpha^i_{j,k,l}$, $\beta^i_{j,k,l}$ as parameters. We consider also 3 maps R_{ij}, $i < j$, $i, j \in \{1, 2, 3\}$. These maps can be build out of the polynomials n^i, d^i and they read $R_{ij}\colon (x_1, x_2, x_3) \mapsto (X_1(x_1, x_2, x_3), X_2(x_1, x_2, x_3), X_3(x_1, x_2, x_3))$, where

$$X_i = x_i - 2\frac{\begin{vmatrix} \mathbf{D}_{x_i} n^1 \cdot d^1 & \mathbf{D}_{x_i} n^2 \cdot d^2 \\ \mathbf{D}_{x_j} n^1 \cdot d^1 & \mathbf{D}_{x_j} n^2 \cdot d^2 \end{vmatrix}}{\begin{vmatrix} \mathbf{D}_{x_i} n^1 \cdot d^1 & \mathbf{D}_{x_i} n^2 \cdot d^2 \\ \partial_{x_i}\mathbf{D}_{x_j} n^1 \cdot d^1 + \partial_{x_j}\mathbf{D}_{x_i} n^1 \cdot d^1 & \partial_{x_i}\mathbf{D}_{x_j} n^2 \cdot d^2 + \partial_{x_j}\mathbf{D}_{x_i} n^2 \cdot d^2 \end{vmatrix}},$$

$$X_j = x_j + 2\frac{\begin{vmatrix} \mathbf{D}_{x_i} n^1 \cdot d^1 & \mathbf{D}_{x_i} n^2 \cdot d^2 \\ \mathbf{D}_{x_j} n^1 \cdot d^1 & \mathbf{D}_{x_j} n^2 \cdot d^2 \end{vmatrix}}{\begin{vmatrix} \mathbf{D}_{x_j} n^1 \cdot d^1 & \mathbf{D}_{x_j} n^2 \cdot d^2 \\ \partial_{x_i}\mathbf{D}_{x_j} n^1 \cdot d^1 + \partial_{x_j}\mathbf{D}_{x_i} n^1 \cdot d^1 & \partial_{x_i}\mathbf{D}_{x_j} n^2 \cdot d^2 + \partial_{x_j}\mathbf{D}_{x_i} n^2 \cdot d^2 \end{vmatrix}},$$
$$X_k = x_k \qquad \text{for} \quad k \neq i, j, \tag{2.1}$$

with ∂_z we denote the partial derivative operator w.r.t. to z, i.e., $\partial_z h = \frac{\partial h}{\partial z}$. $\mathbf{D}_z$ is the Hirota's bilinear operator, i.e., $\mathbf{D}_z h \cdot k = (\partial_z h)k - h\partial_z k$.

Proposition 2.1. *The following holds:*

1. *Mappings R_{ij} depend on 32 parameters $\alpha^i_{j,k,l}$, $\beta^i_{j,k,l}$, $i = 1, 2$, $j, k, l \in \{0, 1\}$. Only 15 of them are essential.*
2. *The functions $H_1 = n^1/d^1$, $H_2 = n^2/d^2$ are invariant under the action of R_{ij}, i.e., $H_l \circ R_{ij} = H_l$, $l = 1, 2$.*
3. *Mappings R_{ij} are involutions, i.e., $R^2_{ij} = \mathrm{id}$.*
4. *Mappings R_{ij} are anti-measure preserving*[1] *with densities $m_1 = n^1 d^2$, $m_2 = n^2 d^1$.*
5. *Mappings R_{ij} satisfy the relation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.*

Proof. 1. The invariants H_1, H_2 depend on 3 variables and they include 32 parameters. Acting with a different Möbius transformation to each of the variables, 9 parameters can be removed. A Möbius transformation of an invariant remains an invariant, since we have 2 invariants, 6 more parameters can be removed. Finally, since any multiple of an invariant remains an invariant, 2 more parameters can be removed. That leaves us with $32 - 9 - 6 - 2 = 15$ essential parameters for the invariants H_1, H_2 and hence for the maps R_{ij}.

2. The functions $H_1 = n^1/d^1$, $H_2 = n^2/d^2$, reads

$$H_1(x_1, x_2, x_3) = \frac{a x_1 x_2 + b x_1 + c x_2 + d}{a_1 x_1 x_2 + b_1 x_1 + c_1 x_2 + d_1},$$
$$H_2(x_1, x_2, x_3) = \frac{k x_1 x_2 + l x_1 + m x_2 + n}{k_1 x_1 x_2 + l_1 x_1 + m_1 x_2 + n_1},$$

where $a, a_1, b, b_1, k, k_1, \ldots$ are linear functions of x_3 (note we have suppressed the dependency on x_3 of the functions H_1, H_2). From the set of equations

$$H_1(X_1, X_2, x_3) = H_1(x_1, x_2, x_3), \qquad H_2(X_1, X_2, x_3) = H_2(x_1, x_2, x_3), \tag{2.2}$$

by eliminating X_2 or by eliminating X_1 the resulting equations respectively factorize as

$$(X_1 - x_1)A = 0, \qquad (X_2 - x_2)B = 0.$$

The factor A is linear in X_1 and the factor B is linear in X_2. By solving these equations (we omit the trivial solution $X_1 = x_1$, $X_2 = x_2$) we obtain

$$X_1 = \frac{\gamma^{34}_{13} x_2^2 + (\gamma^{34}_{23} + \gamma^{34}_{14}) x_2 + \gamma^{34}_{24} + (\gamma^{14}_{13} x_2^2 + (\gamma^{24}_{13} + \gamma^{14}_{23}) x_2 + \gamma^{24}_{23}) x_1}{\gamma^{23}_{13} x_2^2 + (\gamma^{24}_{13} + \gamma^{23}_{14}) x_2 + \gamma^{24}_{14} + (\gamma^{13}_{12} x_2^2 + (\gamma^{23}_{12} + \gamma^{14}_{12}) x_2 + \gamma^{24}_{12}) x_1},$$
$$X_2 = \frac{\gamma^{24}_{12} x_1^2 + (\gamma^{23}_{24} + \gamma^{24}_{14}) x_1 + \gamma^{24}_{34} + (\gamma^{14}_{12} x_1^2 + (\gamma^{34}_{12} + \gamma^{23}_{14}) x_1 + \gamma^{23}_{34}) x_2}{\gamma^{12}_{23} x_1^2 + (\gamma^{34}_{12} + \gamma^{14}_{23}) x_1 + \gamma^{34}_{14} + (\gamma^{12}_{13} x_1^2 + (\gamma^{13}_{23} + \gamma^{14}_{13}) x_1 + \gamma^{34}_{13}) x_2}, \tag{2.3}$$

[1]A map $\phi\colon (x, y) \mapsto (X, Y)$ is called *measure preserving map* with density $m(x, y)$, if its Jacobian determinant $\frac{\partial(X,Y)}{\partial(x,y)}$ equals to $\frac{m(X,Y)}{m(x,y)}$. If the Jacobian determinant of the map ϕ equals to $-\frac{m(X,Y)}{m(x,y)}$, then the map ϕ is called *anti-measure preserving map* with density $m(x, y)$.

where $\gamma_{ij}^{kl} := \left|\begin{smallmatrix} u_{ij} & u_{kl} \\ v_{ij} & v_{kl}\end{smallmatrix}\right|$, with u_{ij} the determinants of a matrix generated by the i^{th} and j^{th} column of the matrix

$$u = \begin{pmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \end{pmatrix}$$

and v_{kl} the determinants of a matrix generated by the k^{th} and l^{th} column of the matrix

$$v = \begin{pmatrix} k & l & m & n \\ k_1 & l_1 & m_1 & n_1 \end{pmatrix}.$$

Now it is a matter of long and tedious calculation to prove that the map $\phi\colon (x_1, x_2, x_3) \mapsto (X_1, X_2, x_3)$, where X_1, X_2 are given by (2.3) coincides with the map R_{12} of (2.1). Similarly we can work on R_{13} and R_{23}.

3. Since the map $R_{12}\colon (x_1, x_2, x_3) \mapsto (X_1, X_2, x_3)$ satisfies (2.2), the proof of involutivity follows.

4. It is enough to prove that the map R_{12} anti-preserves the measure with density $m_1 = n^1 d^2$, i.e., the Jacobian determinant

$$\frac{\partial(X_1, X_2)}{\partial(x_1, x_2)} := \begin{vmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} \\ \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} \end{vmatrix}$$

equals

$$\frac{\partial(X_1, X_2)}{\partial(x_1, x_2)} = -\frac{n^1(X_1, X_2, x_3) d^2(X_1, X_2, x_3)}{n^1(x_1, x_2, x_3) d^2(x_1, x_2, x_3)}.$$

Since the functions $H_i = n^i/d^i$, $i = 1, 2$ are invariant under the action of the map R_{12}, it holds

$$\begin{aligned} n^1(X_1, X_2, x_3) &= \kappa(x_1, x_2, x_3) n^1(x_1, x_2, x_3), \\ d^1(X_1, X_2, x_3) &= \kappa(x_1, x_2, x_3) d^1(x_1, x_2, x_3), \\ n^2(X_1, X_2, x_3) &= \lambda(x_1, x_2, x_3) n^2(x_1, x_2, x_3), \\ d^2(X_1, X_2, x_3) &= \lambda(x_1, x_2, x_3) d^2(x_1, x_2, x_3), \end{aligned} \tag{2.4}$$

where κ, λ are rational functions of x_1, x_2, x_3. So,

$$\frac{n^1(X_1, X_2, x_3) d^2(X_1, X_2, x_3)}{n^1(x_1, x_2, x_3) d^2(x_1, x_2, x_3)} = \kappa(x_1, x_2, x_3)\lambda(x_1, x_2, x_3). \tag{2.5}$$

We differentiate equations (2.4) with respect to x_1 and we eliminate $\frac{\partial\kappa(x_1,x_2,x_3)}{\partial x_1}$ and $\frac{\partial\lambda(x_1,x_2,x_3)}{\partial x_1}$ to obtain

$$\begin{aligned} \frac{1}{n^1}\left(\frac{\partial \tilde{n}^1}{\partial x_1} - \kappa\frac{\partial n^1}{\partial x_1}\right) &= \frac{1}{d^1}\left(\frac{\partial \tilde{d}^1}{\partial x_1} - \kappa\frac{\partial d^1}{\partial x_1}\right), \\ \frac{1}{n^2}\left(\frac{\partial \tilde{n}^2}{\partial x_1} - \lambda\frac{\partial n^2}{\partial x_1}\right) &= \frac{1}{d^2}\left(\frac{\partial \tilde{d}^2}{\partial x_1} - \lambda\frac{\partial d^2}{\partial x_1}\right), \end{aligned} \tag{2.6}$$

here we have suppressed the dependency of κ, λ, n^i, d^i on x_1, x_2, x_3. By $\tilde{n}^i$ we denote $\tilde{n}^i := n^i(X_1, X_2, x_3)$, $i = 1, 2$, and similarly for $\tilde{d}^i$. Also if we differentiate the equations (2.4) with respect to x_2 and eliminate $\frac{\partial\kappa}{\partial x_2}$ and $\frac{\partial\lambda}{\partial x_2}$ we obtain

$$\frac{1}{n^1}\left(\frac{\partial \tilde{n}^1}{\partial x_2} - \kappa\frac{\partial n^1}{\partial x_2}\right) = \frac{1}{d^1}\left(\frac{\partial \tilde{d}^1}{\partial x_2} - \kappa\frac{\partial d^1}{\partial x_2}\right),$$

$$\frac{1}{n^2}\left(\frac{\partial \tilde{n}^2}{\partial x_2}-\lambda\frac{\partial n^2}{\partial x_2}\right)=\frac{1}{d^2}\left(\frac{\partial \tilde{d}^2}{\partial x_2}-\lambda\frac{\partial d^2}{\partial x_2}\right). \tag{2.7}$$

Due to the form of n^i, d^i, $i=1,2$, equations (2.6), (2.7) are linear in $\frac{\partial X_1}{\partial x_i}$, $\frac{\partial X_2}{\partial x_i}$, $i=1,2$. Hence we obtain $\frac{\partial X_1}{\partial x_i}$, $\frac{\partial X_2}{\partial x_i}$, $i=1,2$, in terms of X_1, X_2, x_1, x_2, x_3, κ, λ and by using (2.3), the Jacobian determinant reads $\frac{\partial(X_1,X_2)}{\partial(x_1,x_2)}=-\kappa\lambda$. Using (2.5) we have

$$\frac{\partial(X_1,X_2)}{\partial(x_1,x_2)}=-\kappa\lambda=-\frac{\tilde{n}^1\tilde{d}^2}{n^1d^2},$$

that completes the proof. Note that the same holds true for the remaining maps R_{ij}.

5. In [3] Adler presented a computational proof based on the fact that the maps R_{ij} map points that lie on the invariant curve

$$n^1(x_1,x_2,x_3)-C_1d^1(x_1,x_2,x_3)=0, \qquad n^2(x_1,x_2,x_3)-C_2d^2(x_1,x_2,x_3)=0, \tag{2.8}$$

that is the intersection of two surfaces of the form $A\colon N(x_1,x_2,x_3)=0$, where N is polynomial with degree at most one on each variable x_1, x_2 and x_3. In [3], it was proven that any surface of the form A that passes through the following five points

$$\left(\hat{x}_1,\tilde{x}_2,\hat{\tilde{x}}_3\right)\xleftarrow{R_{13}}\left(x_1,\tilde{x}_2,\tilde{x}_3\right)\xleftarrow{R_{23}}\left(x_1,x_2,x_3\right)\xrightarrow{R_{12}}\left(\bar{x}_1,\bar{x}_2,x_3\right)\xrightarrow{R_{13}}\left(\hat{\bar{x}}_1,\bar{x}_2,\hat{x}_3\right)$$

passes as well through the point $\left(\hat{\bar{x}}_1,Y,\hat{\tilde{x}}_3\right)$, that is the point of intersection of the straight line $L\colon (X,Z)=\left(\hat{\bar{x}}_1,\hat{\tilde{x}}_3\right)$ and the surface A, i.e., $L\cap A=\left(\hat{\bar{x}}_1,Y,\hat{\tilde{x}}_3\right)$. Since the invariant curve (2.8) is the intersection of two surfaces of the form A, it also passes through the point $\left(\hat{\bar{x}}_1,Y,\hat{\tilde{x}}_3\right)$ and there is $\tilde{\bar{x}}_2=Y$. So the values of $\tilde{\bar{x}}_2$ obtained in two different ways coincide and this is sufficient for the proof.

Alternatively, one can show by direct computation that the maps $T_1=R_{13}R_{12}$ and $T_2=R_{12}R_{23}$, commute, i.e., $T_1T_2=T_2T_1$. So there is

$$R_{13}R_{23}=R_{12}R_{23}R_{13}R_{12}$$

and due to the fact that the maps R_{ij} are involutions, $R_{ij}^2=\mathrm{id}$, from the equation above we obtain

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \qquad \blacksquare$$

Among all the maps that can be constructed by the involutions R_{ij}, the following maps

$$T_1=R_{13}R_{12}, \qquad T_2=R_{12}R_{23}, \qquad T_3=R_{23}R_{13}$$

are of special interest since they are not periodic and moreover they satisfy [3]

$$T_1T_2T_3=\mathrm{id}, \qquad T_iT_j=T_jT_i, \qquad i,j\in\{1,2,3\}.$$

Proposition 2.2. *For the maps T_i, $i=1,2,3$ it holds:*

1) *they preserve the functions H_1, H_2,*
2) *they are measure-preserving with densities m_1, m_2,*
3) *they preserve the following degenerate Poisson tensors,*

$$\Omega_i^j=m_j\left(\frac{\partial H_i}{\partial x_3}\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}-\frac{\partial H_i}{\partial x_2}\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}+\frac{\partial H_i}{\partial x_1}\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}\right), \qquad i,j\in\{1,2\},$$

where it holds

$$0=\Omega_1^j\nabla H_1, \qquad \Omega_1^j\nabla H_2=-\Omega_2^j\nabla H_1, \qquad \Omega_2^j\nabla H_2=0, \qquad j=1,2,$$

4) *they are Liouville integrable maps.*

Proof. The statements (1), (2) follows from Proposition 2.1. To prove the statement (3), (4), first note that since the maps T_i are measure preserving, they preserve the following polyvector fields

$$V^j = m_j \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

Hence, the contractions $V^j \rfloor dH_i$, $i,j \in \{1,2\}$ (see [29, 55]) are degenerate Poisson tensors. Namely,

$$\begin{aligned}\Omega_i^j &= \left(m_j \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}\right) \rfloor dH_i \\ &= m_j \left(\frac{\partial H_i}{\partial x_3}\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{\partial H_i}{\partial x_2}\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + \frac{\partial H_i}{\partial x_1}\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}\right),\end{aligned}$$

where $i,j \in \{1,2\}$.

(5) The maps T_i preserve the Poisson tensors Ω_i^j and the 2 invariants H_1, H_2, so they are Liouville integrable maps [14, 52, 68]. ■

Note that on the level surfaces $H_2(x_1,x_2,x_3) = c$, maps T_1, T_2, T_3 reduce to pair-wise commuting maps on the plane which preserve the function $\hat{H}_1(x_1,x_2;c)$. One of these reduced maps is the associated with the invariant $\hat{H}_1(x_1,x_2;c)$ QRT map. Examples of commuting maps with specific members of the QRT family of maps were also constructed in [30].

The involution R_{12} under the reduction $x_2 = x_1$, $H_2 = H_1 = H$, so $H = \frac{n}{d} = \frac{ax_1^2+bx_1+c}{kx_1^2+lx_1+m}$, reads

$$R_{12}\colon\ (x_1,x_3) \mapsto \left(x_1 - 2\frac{D_{x_1} n \cdot d}{\partial_{x_1} D_{x_1} n \cdot d}, x_3\right),$$

that coincides with the QRT involution i_x that preserves the invariant H. This formulae for the QRT involution i_x was firstly given in [37], where an elegant presentation of the QRT map was considered.

2.1 A generalisation of the triad family of maps

Following the same generalisation procedures introduced for the QRT family of maps [15, 29, 35, 62, 67], the triad family of maps can be generalised in similar manners. Here, in order to generalise the triad family of maps, we mimic the generalisation of the QRT family of maps introduced in [67].

Consider the following polynomials

$$\begin{aligned} n^i &= \sum_{j_1,j_2,\dots,j_k=0}^{1} \alpha^i_{j_1,j_2,\dots,j_k} x_1^{1-j_1} x_2^{1-j_2} \cdots x_k^{1-j_k}, \\ d^i &= \sum_{j_1,j_2,\dots,j_k=0}^{1} \beta^i_{j_1,j_2,\dots,j_k} x_1^{1-j_1} x_2^{1-j_2} \cdots x_k^{1-j_k}, \qquad i = 1,2 k \geq 3, \end{aligned} \tag{2.9}$$

where $x_1,x_2,\dots,x_k$ are considered as variables and $\alpha^i_{j_1,j_2,\dots,j_k}$, $\beta^i_{j_1,j_2,\dots,j_k}$ as parameters. We consider the $\binom{k}{2}$ maps R_{ij}, $i<j$, $i,j \in \{1,2,\dots,k\}$. These maps can be build out of the polynomials n^i, d^i and they read: $R_{ij}\colon (x_1,x_2,\dots,x_k) \mapsto (X_1,X_2,\dots,X_k)$, where $X_l = x_l$ $\forall\, l \neq i,j$ and X_i, X_j are given by the formulae (2.1), where n^i, d^i, $i=1,2$ are given by (2.9).

Proposition 2.1 is straight forward extended to the k-variables case.

Proposition 2.3. *The following holds:*

1. *Mappings* R_{ij} *depend on* $4\cdot 2^k$ *parameters* $\alpha^i_{j_1,j_2,\dots,j_k}$, $\beta^i_{j_1,j_2,\dots,j_k}$, $i=1,2$, $j_1,j_2,\dots,j_k\in\{0,1\}$. *Only* $4\cdot 2^k-3k-8$ *of them are essential.*
2. *The functions* $H_1=n^1/d^1$, $H_2=n^2/d^2$ *are invariant under the action of* R_{ij}, *i.e.,* $H_l\circ R_{ij}=H_l$, $l=1,2$.
3. *Mappings* R_{ij} *are involutions, i.e.,* $R_{ij}^2=\mathrm{id}$.
4. *Mappings* R_{ij} *are anti-measure preserving with densities* $m_1=n^1d^2$, $m_2=n^2d^1$.
5. *Mappings* R_{mn}, $m<n$, $m,n\in\{1,2,\dots,k\}$ *satisfy the relations* $R_{ij}R_{il}R_{jl}=R_{jl}R_{il}R_{ij}$.

Proof. 1. The invariants H_1, H_2 depend on $k\geq 3$ variables and they include $4\cdot 2^k$ parameters. Acting with a different Möbius transformation to each of the variables, $3k$ parameters can be removed. A Möbius transformation of an invariant remains an invariant, since we have 2 invariants, 6 more parameters can be removed. Finally, since any multiple of an invariant remains an invariant, 2 more parameters can be removed. That leaves us with $4\cdot 2^k-3k-6-2=4\cdot 2^k-3k-8$ essential parameters for the invariants H_1, H_2 and hence for the maps R_{ij}.

The proof of the remaining statements of this Proposition follows directly from the fact that for any 3 indices $p<q<r\in\{1,2,\dots,k\}$, the maps R_{pq}, R_{pr} and R_{qr}, coincide with the maps R_{12}, R_{13} and R_{23} respectively of Proposition 2.1. ■

We take a stand here to comment that for $k=3$ the construction above coincides with the Adler's triad family of maps hence we have Liouville integrability. For $k>3$ we have a generalisation of the latter and since always we will have maps in k variables with 2 invariants, Liouville integrability is not expected for generic choice of the parameters $\alpha^i_{j_1,j_2,\dots,j_k}$, $\beta^i_{j_1,j_2,\dots,j_k}$. For a specific but quite general choice of the parameters though, one can associate a Lax pair to these maps and recover the additional integrals which are required for the Liouville integrability to emerge.

We also have to note that the case $k=4$ was firstly introduced in [43]. Although for $k=4$ we have mappings in 4 variables with 2 invariants, Liouville integrability is not apparent unless we specify the parameters. A specific choice of the parameters which leads to integrability is presented to the following example.

Example 2.4 (the Adler–Yamilov map [7])**.** Consider the following special form of the functions n^i, d^i

$$d^1=d^2=1,\qquad n^1=x_1x_2+x_3x_4,\qquad n^2=x_1x_2x_3x_4+x_1x_4+x_2x_3+ax_1x_2+bx_3x_4.$$

Then the functions $H_i=n^i/d^i$, $i=1,2$ are preserved by construction by the maps R_{ij} as well as by the following elementary involutions

$$i\colon\ (x_1,x_2,x_3,x_4)\mapsto(x_2,x_1,x_4,x_3),\qquad \phi\colon\ (x_1,x_2,x_3,x_4)\mapsto(x_1x_2/x_3,x_3,x_2,x_3x_4/x_2).$$

The Adler–Yamilov map (ξ) is considered by the following composition

$$\xi:=R_{14}\phi i\colon\ (x_1,x_2,x_3,x_4)\mapsto\left(x_3-\frac{(a-b)x_1}{1+x_1x_4},x_4,x_1,x_2+\frac{(a-b)x_4}{1+x_1x_4}\right).$$

The Adler–Yamilov map is Liouville integrable since it preserves, and the invariants H_1, H_2 are in involution with respect to the canonical Poisson bracket. For further discussions on the Adler–Yamilov map see [30, 48].

3 Invariants in separated variables and Yang–Baxter maps

Mappings R_{mn}, $m < n \in \{1, 2, \dots, k\}$, presented in Section 2.1, satisfy the identities $R_{ij}R_{il}R_{jl} = R_{jl}R_{il}R_{ij}$, nevertheless as they stand they are not Yang–Baxter. Take for example the map $R_{12}\colon (x_1, x_2, x_3, \dots, x_k) \mapsto (X_1, X_2, x_3, \dots, x_k)$. The formulae for X_1 is fraction linear in x_1 with coefficients that depend on all the remaining variables and X_2 is fraction linear in x_2 with coefficients that depend on all the remaining variables. In order for R_{12} to be a Yang–Baxter map the coefficients of x_1 in the formulae of X_1 should depend only on x_2 and the coefficients of x_2 in the formulae of X_2 should depend only on x_1. This "separability" requirement can be easily achieved by requiring separability of variables on the level of the invariants of the map R_{12}. We have two invariants $H_1 = n^1/d^1$, $H_2 = n^2/d^2$, so we can have three different kinds of separability. (I) Both H_1 and H_2 to be multiplicative separable on the variables x_1 and x_2. (II) H_1 to be multiplicative and H_2 to be additive separable and finally (III) both H_1 and H_2 to be additive separable on the variables x_1 and x_2. In what follows we explicitly present these three different kinds of separability in all variables of the invariants H_1 and H_2.

(I) Multiplicative/multiplicative separability of variables:

$$H_1 = \prod_{i=1}^{k} \frac{a_i - b_i x_i}{c_i - d_i x_i}, \qquad H_2 = \prod_{i=1}^{k} \frac{A_i - B_i x_i}{C_i - D_i x_i}. \tag{3.1}$$

(II) Multiplicative/additive separability of variables:

$$H_1 = \prod_{i=1}^{k} \frac{a_i - b_i x_i}{c_i - d_i x_i}, \qquad H_2 = \sum_{i=1}^{k} \frac{A_i - B_i x_i}{C_i - D_i x_i}. \tag{3.2}$$

(III) Additive/additive separability of variables:

$$H_1 = \sum_{i=1}^{k} \frac{a_i - b_i x_i}{c_i - d_i x_i}, \qquad H_2 = \sum_{i=1}^{k} \frac{A_i - B_i x_i}{C_i - D_i x_i}. \tag{3.3}$$

In the formulas above, a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i, $i = 1, \dots, k$ are parameters, $8k$ in total. In all three cases above, the number of essential parameters is $3k - 6$. This argument can be proven by the following reasoning. Since the invariants H_1, H_2 depends on k variables, by a Möbius transformation on each of the k variables $3k$ parameters can be removed. Also any Möbius transformation of an invariant remains an invariant so since we have two invariants 2×3 more parameters can be removed. Finally, for each one of the $2k$ functions $\frac{a_i - b_i x_i}{c_i - d_i x_i}$, $\frac{A_i - B_i x_i}{C_i - D_i x_i}$, $i = 1, \dots, k$, one non-zero parameter can be absorbed simply by dividing with it (and reparametrise), so $2k$ more parameters can be removed. In total we have $8k - 3k - 2 \times 3 - 2k = 3k - 6$ essential parameters.

3.1 Multiplicative/multiplicative separability of variables

Let us first introduce some definitions.

Definition 3.1. The maps $R, \tilde{R}\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \mapsto \mathbb{CP}^1 \times \mathbb{CP}^1$ are $(\text{Möb})^2$ *equivalent* if there exists bijections $\phi, \psi\colon \mathbb{CP}^1 \mapsto \mathbb{CP}^1$ such that the following conjugation relation holds

$$\tilde{R} = \phi^{-1} \times \psi^{-1} R \phi \times \psi.$$

Definition 3.2. The map $R\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u,v) \mapsto (U,V) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, where

$$U = \frac{a_1 + a_2 u}{a_3 + a_4 u}, \qquad V = \frac{b_1 + b_2 v}{b_3 + b_4 v},$$

with a_i, b_i, $i = 1, \dots, 4$ known polynomials of v and u respectively, will be said to be of *subclass* $[\gamma : \delta]$, if the highest degree that appears in the polynomials a_i is γ and the higher degree that appears in the polynomials b_i is δ.

Clearly, maps that belong to different subclasses are not $(\text{Möb})^2$ equivalent.

Proposition 3.3. *Consider the multiplicative/multiplicative separability of variables of the invariants H_1 and H_2 (see (3.1)). Consider also the following sets of parameters*

$$\mathbf{p}_{ij} := \mathbf{p}_i \cup \mathbf{p}_j \qquad \text{where} \qquad \mathbf{p}_i := \{a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i\}, \qquad i < j \in \{1, 2, \dots, k\}$$

and the functions

$$f_i := \frac{a_i - b_i x_i}{c_i - d_i x_i}, \qquad g_i := \frac{A_i - B_i x_i}{C_i - D_i x_i}, \qquad i = 1, \dots, k.$$

The following holds:

1. *The invariants $H_1 = \prod_{i=1}^{k} f_i$, $H_2 = \prod_{i=1}^{k} g_i$ depend on $8k$ parameters. Only $3k - 6$ of them are essential.*
2. *Mappings R_{ij} explicitly read*

$$R_{ij}\colon \ (x_1, x_2, \dots, x_k) \mapsto (X_1, X_2, \dots, X_k),$$

where $X_l = x_l \ \forall\, l \neq i, j$ and X_i, X_j are given by the formulae

$$X_i = x_i - 2\frac{\begin{vmatrix} f_i' f_j & f_i f_j' \\ g_i' g_j & g_i g_j' \end{vmatrix}}{g_i' g_j \left(\frac{f_i'}{f_j'} \begin{vmatrix} f_j & f_j' \\ f_j' & f_j'' \end{vmatrix} + \frac{f_j'}{f_i'} \begin{vmatrix} f_i & f_i' \\ f_i' & f_i'' \end{vmatrix} \right) - f_i' f_j \left(\frac{g_i'}{g_j'} \begin{vmatrix} g_j & g_j' \\ g_j' & g_j'' \end{vmatrix} + \frac{g_j'}{g_i'} \begin{vmatrix} g_i & g_i' \\ g_i' & g_i'' \end{vmatrix} \right)},$$

$$X_j = x_j + 2\frac{\begin{vmatrix} f_i' f_j & f_i f_j' \\ g_i' g_j & g_i g_j' \end{vmatrix}}{g_j' g_i \left(\frac{f_i'}{f_j'} \begin{vmatrix} f_j & f_j' \\ f_j' & f_j'' \end{vmatrix} + \frac{f_j'}{f_i'} \begin{vmatrix} f_i & f_i' \\ f_i' & f_i'' \end{vmatrix} \right) - f_j' f_i \left(\frac{g_i'}{g_j'} \begin{vmatrix} g_j & g_j' \\ g_j' & g_j'' \end{vmatrix} + \frac{g_j'}{g_i'} \begin{vmatrix} g_i & g_i' \\ g_i' & g_i'' \end{vmatrix} \right)},$$

where $f_l' \equiv \frac{\partial f_l}{\partial x_l}$, $g_l' \equiv \frac{\partial g_l}{\partial x_l}$, $g_l'' \equiv \frac{\partial^2 g_l}{\partial x_l^2}$, etc. Note that in the expressions of X_i, X_j appears only the coordinates x_i, x_j and the parameters $\mathbf{p}_{ij}$. From further on we denote the maps R_{ij} as $R_{ij}^{\mathbf{p}_{ij}}$, in order to stress this separability feature.

3. *Mappings $R_{ij}^{\mathbf{p}_{ij}}$ are anti-measure preserving with densities $m_1 = n^1 d^2$, $m_2 = n^2 d^1$, where n^i, d^i the numerators and the denominators respectively, of the invariants H_i, $i = 1, 2$.*
4. *Mappings $R_{ij}^{\mathbf{p}_{ij}}$ satisfy the Yang–Baxter identity*

$$R_{ij}^{\mathbf{p}_{ij}} R_{ik}^{\mathbf{p}_{ik}} R_{jk}^{\mathbf{p}_{jk}} = R_{jk}^{\mathbf{p}_{jk}} R_{ij}^{\mathbf{p}_{ij}} R_{ij}^{\mathbf{p}_{ij}}.$$

5. *Mappings $R_{ij}^{\mathbf{p}_{ij}}$ are involutions with the sets of singularities*

$$\Sigma_{ij} = \{P_{ij}^1, P_{ij}^2, P_{ij}^3, P_{ij}^4\} = \left\{ \left(\frac{a_i}{b_i}, \frac{c_j}{d_j}\right), \left(\frac{c_i}{d_i}, \frac{a_j}{b_j}\right), \left(\frac{A_i}{B_i}, \frac{C_j}{D_j}\right), \left(\frac{C_i}{D_i}, \frac{A_j}{B_j}\right) \right\},$$

and the sets of fixed points

$$\Phi_{ij} = \{Q_{ij}^1, Q_{ij}^2, Q_{ij}^3, Q_{ij}^4\} = \left\{ \left(\frac{a_i}{b_i}, \frac{a_j}{b_j}\right), \left(\frac{c_i}{d_i}, \frac{c_j}{d_j}\right), \left(\frac{A_i}{B_i}, \frac{A_j}{B_j}\right), \left(\frac{C_i}{D_i}, \frac{C_j}{D_j}\right) \right\},$$

where in the formulae for P_{ij}^m and Q_{ij}^m, $m = 1, \dots, 4$, we have suppressed the dependency on the remaining variables. For example, with $P_{ij}^1 = \left(\frac{a_i}{b_i}, \frac{c_j}{d_j}\right)$ we denote $\left(x_1, \dots, x_{i-1}, \frac{a_i}{b_i}, x_{i+1}, \dots, x_{j-1}, \frac{c_j}{d_j}, x_{j+1}, \dots, x_k\right)$ and similarly for the remaining P_{ij}^m and Q_{ij}^m.

6. *Each one of the maps $R_{ij}^{\mathbf{p}_{ij}}$ is $(\text{Möb})^2$ equivalent to the H_{I} Yang–Baxter map.*

Proof. (1) See at the end of the previous subsection.

(2) Mappings (2.1) written in terms of the functions f_i, g_i get exactly the desired form.

(3) See Proposition 2.1.

(4) See Proposition 2.1.

(5) Because mappings $R_{ij}^{\mathbf{p}_{ij}}$, for generic parameter sets $\mathbf{p}_{ij}$, belong to the $[2:2]$ subclass, we expect at most 8 singular points, 4 singular points from the first fraction of the map and 4 from the second. By direct calculation we show that the singular points of the first and the second fraction of $R_{ij}^{\mathbf{p}_{ij}}$ coincide. Moreover, P_{ij}^m, $m = 1, \dots, 4$ are the singular points of the maps $R_{ij}^{\mathbf{p}_{ij}}$, i.e.,

$$R_{ij}^{\mathbf{p}_{ij}}\colon\ P_{ij}^m \mapsto \left(x_1, \dots, x_{i-1}, \frac{0}{0}, x_{i+1}, \dots, x_{j-1}, \frac{0}{0}, x_{j+1}, \dots, x_k\right).$$

Note that the values of the invariants H_i at the singular points P_{ij}^m are undetermined, i.e., $H_1(P_{ij}^m) = \frac{0}{0}$, $m = 1, 2$, $H_2(P_{ij}^m) = \frac{0}{0}$, $m = 3, 4$. For the fixed points Q_{ij}^m, $m = 1, \dots, 4$ it holds $R_{ij}^{\mathbf{p}_{ij}} : Q_{ij}^m \mapsto Q_{ij}^m$. Note also that $H_1(Q_{ij}^1) = 0$, $H_1(Q_{ij}^2) = \infty$, $H_2(Q_{ij}^3) = 0$, $H_2(Q_{ij}^4) = \infty$.

(6) Introducing the new variables y_i, y_j, $i \neq j = 1, \dots, k$ though

$$\begin{aligned} &\mathrm{CR}[x_i, a_i/b_i, c_i/d_i, A_i/B_i] = \mathrm{CR}[y_i, 0, 1, \infty], \\ &\mathrm{CR}[x_j, c_j/d_j, a_j/b_j, C_j/D_j] = \mathrm{CR}[y_j, \infty, 1, 0], \end{aligned}$$

after a re-parametrization mappings R_{ij} gets exactly the form of the H_{I} map. Here, with $\mathrm{CR}[a, b, c, d]$ we denote the cross-ratio of 4 points, namely

$$\mathrm{CR}[a, b, c, d] := \frac{(a-c)(b-d)}{(a-d)(b-c)}. \qquad \blacksquare$$

Each one of the maps R_{ij} has a set of singularities which consists of 4 distinct points. With appropriate limits we are allowed to merge some of the singularities and obtain Yang–Baxter maps which are not $(\text{Möb})^2$ equivalent with the original one.

By setting $C_i = \epsilon A_i$, $D_i = \epsilon B_i$, $A_j = \epsilon C_j$, $B_j = \epsilon D_j$ and letting $\epsilon \to 0$ the singular points P_{ij}^4 and P_{ij}^3 merge. The resulting maps, under a re-parametrization, coincide with the ones obtained in the multiplicative/additive case (see Section 3.2), hence are $(\text{Möb})^2$ equivalent with the H_{II} Yang–Baxter map. The same result can be obtained by merging P_{ij}^2 and P_{ij}^1. Note that merging P_{ij}^4 with P_{ij}^2 or P_{ij}^4 with P_{ij}^1 is not of interest since the resulting maps are trivial.

By further setting $c_i = \epsilon a_i$, $d_i = \epsilon b_i$, $a_j = \epsilon c_j$, $b_j = \epsilon d_j$ and letting $\epsilon \to 0$ the singular points P_{ij}^2 and P_{ij}^1 merge as well. The resulting maps, under a re-parametrization, coincide with the ones obtained in the additive/additive case (see Section 3.3), hence are $(\text{Möb})^2$ equivalent with the H_{III}^A Yang–Baxter map. Any further merging of singularities leads to trivial maps.

Remark 3.4. An interesting observation is that if we impose that the fixed points Q_{ij}^4 of the maps R_{ij} coincide with the singular points P_{ij}^2 or the fixed points Q_{ij}^4 coincide with P_{ij}^1, we obtain maps which belong to the $[1:1]$ subclass of maps. The same is true if we demand that the fixed points Q_{ij}^1 coincide with the singular points P_{ij}^3 or if the fixed points Q_{ij}^1 coincide with the singular points P_{ij}^4,

Remark 3.5. For generic sets of parameters $\mathbf{p}_{ij}$, each one of the $\binom{k}{2}$ maps $R_{ij}^{\mathbf{p}_{ij}}$, is $(\text{Möb})^2$ equivalent to the H_{I} Yang–Baxter map. For degenerate choices of the sets $\mathbf{p}_{ij}$, this is no longer the case. Hence, in that respect, mappings $R_{ij}^{\mathbf{p}_{ij}}$ are more general than the H_{I} map since they include degenerate cases as well. In the same respect Q_{V} [72], the rational version of the discrete Krichever–Novikov equation Q_4 [2], is more general.

Example 3.6 ($k=3$)**.** For $k=3$, the invariants $H_1 = f_1f_2f_3$, $H_2 = g_1g_2g_3$ are functions of 3 variables with 24 parameters, 3 of them are essential. Without loss of generality, after removing the redundancy of the parameters, the invariants H_1, H_2 can be cast into the form

$$H_1 = x_1x_2x_3, \qquad H_2 = \frac{x_1-p_1}{x_1-1}\frac{x_2-p_2}{x_2-1}\frac{x_3-p_3}{x_3-1}.$$

Then each of the mappings R_{ij}, $i \neq j \in \{1,2,3\}$ is exactly the H_{I} Yang–Baxter map. The H_{I} Yang–Baxter map explicitly reads $H_{\text{I}}\colon (u,v)\mapsto(U,V)$ where

$$U = vQ, \qquad V = uQ^{-1}, \qquad Q = \frac{(\alpha-1)uv + (\beta-\alpha)u + \alpha(1-\beta)}{(\beta-1)uv + (\alpha-\beta)v + \beta(1-\alpha)}. \tag{3.4}$$

By the identifications $u \equiv x_i$, $u \equiv x_j$, $\alpha \equiv p_i$ and $\beta \equiv p_j$, from (3.4) we recover the maps R_{ij}.

The maps $\phi_i\colon (x_1,x_2,x_3)\mapsto(X_1,X_2,X_3)$ where $X_l = x_l\ \forall\, l \neq i$ and $X_i = \frac{p_i}{x_i}$, $i=1,2,3$ and the maps $\psi_i\colon (x_1,x_2,x_3)\mapsto(X_1,X_2,X_3)$ where $X_l = x_l\ \forall\, l\neq i$ and $X_i = \frac{x_i-p_i}{x_i-1}$, $i=1,2,3$ satisfy

$$H_1\phi_1\phi_2\phi_3 = \frac{p_1p_2p_3}{H_1}, \qquad H_2\phi_1\phi_2\phi_3 = \frac{p_1p_2p_3}{H_2}, \qquad H_1\psi_1\psi_2\psi_3 = H_2, \qquad H_2\psi_1\psi_2\psi_3 = H_1.$$

The maps ϕ_i and ψ_i have a special role in [59] since though them the H_{I} map was derived out of the F_{I} Yang–Baxter map. We will discuss more about these maps in the next Section. We just quickly recall that $\phi_1R_{12}\phi_2$ is exactly the F_{I} Yang–Baxter map.

Remark 3.7. We have to remark that with loss of generality, mappings R_{ij} can belong on a different subclasses than the $[2:2]$ subclass of maps that the H_{I} map belongs to. For example, for

$$H_1 = (x_1-p_1)(x_2-p_2)(x_3-p_3), \qquad H_2 = \frac{x_1-p_1}{x_1}\frac{x_2}{x_2-p_2}\frac{x_3}{x_3-1},$$

R_{12} is the Hirota's KdV map (see [44]) that belongs on the subclass $[1:1]$ and R_{13}, R_{23} are maps which belong to the subclass $[2:1]$. Explicitly the maps read

$$\begin{aligned}
&R_{12}\colon\ (x_1,x_2,x_3)\mapsto\left(\frac{p_1(p_2x_1+p_1x_2-x_1x_2)}{p_2x_1}, \frac{p_2(p_2x_1+p_1x_2-x_1x_2)}{p_1x_2}, x_3\right),\\
&R_{13}\equiv S_{13}\colon\ (x_1,x_2,x_3)\mapsto\left(\frac{p_1(-1+x_3)(p_3x_1+p_1x_3-x_1x_3)}{-p_3x_1-p_1x_3+p_1p_3x_3+x_1x_3}, x_2, \frac{p_3x_1+p_1x_3-x_1x_3}{p_1x_3}\right),\\
&R_{23}\equiv T_{23}\colon\ (x_1,x_2,x_3)\mapsto\left(x_1, \frac{p_2x_3(-p_2+p_3x_2+p_2x_3-x_2x_3)}{-p_2p_3+p_3x_2+p_2p_3x_3-x_2x_3}, \frac{x_2(-p_3+x_3)}{p_2(-1+x_3)}\right).
\end{aligned}$$

The Hirota's KdV map entwines with S_{13} and T_{23}, since $R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}$ holds.

Example 3.8 ($k \geq 4$)**.** For $k = 4$ the invariants depend on 32 parameters and only 6 of them are essential. Without loss of generality they can be cast into the form

$$H_1 = x_1x_2x_3x_4, \qquad H_2 = \frac{x_1 - p_1}{x_1 - 1}\frac{x_2 - p_2}{x_2 - 1}\frac{x_3 - p_3}{x_3 - 1}\frac{\alpha_4 - \beta_4 x_4}{\beta_4 - \gamma_4 x_4}.$$

For $k > 4$ the invariants depend on $8k$ parameters and only $3k-6$ of them are essential. Without loss of generality they can be cast into the form

$$H_1 = \prod_{i=1}^{k} x_i, \qquad H_2 = \frac{x_1 - p_1}{x_1 - 1}\frac{x_2 - p_2}{x_2 - 1}\frac{x_3 - p_3}{x_3 - 1}\prod_{i=4}^{k}\frac{\alpha_i - \beta_i x_i}{\beta_i - \gamma_i x_i}.$$

3.2 Multiplicative/additive separability of variables

Proposition 3.9. *Consider the multiplicative/additive separability of variables of the invariants H_1 and H_2 (see (3.2)). Consider also the following sets of parameters*

$$\mathbf{p}_{ij} := \mathbf{p}_i \cup \mathbf{p}_j, \qquad \textit{where} \qquad \mathbf{p}_i := \{a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i\}, \qquad i < j \in \{1, 2, \ldots, k\}$$

and the functions

$$f_i := \frac{a_i - b_i x_i}{c_i - d_i x_i}, \qquad g_i := \frac{A_i - B_i x_i}{C_i - D_i x_i}, \qquad i = 1, \ldots, k.$$

The following holds:

1. *The invariants $H_1 = \prod_{i=1}^{k} f_i$, $H_2 = \sum_{i=1}^{k} g_i$ depend on $8k$ parameters. Only $3k-6$ of them are essential.*
2. *Mappings R_{ij} explicitly read*

$$R_{ij}\colon\ (x_1, x_2, \ldots, x_k) \mapsto (X_1, X_2, \ldots, X_k),$$

where $X_l = x_l\ \forall\, l \neq i, j$ and X_i, X_j are given by the formulae

$$X_i = x_i - 2\frac{\begin{vmatrix} f_i f_j' & f_i' f_j \\ g_j' & g_i' \end{vmatrix}}{\begin{vmatrix} f_i' f_j & g_i' \\ \frac{f_j'}{f_i'} f_i f_i'' + \frac{f_i'}{f_j'} f_j f_j'' - 2 f_i' f_j' & \frac{g_j'}{g_i'} g_i'' + \frac{g_i'}{g_j'} g_j'' \end{vmatrix}},$$

$$X_j = x_j + 2\frac{\begin{vmatrix} f_i f_j' & f_i' f_j \\ g_j' & g_i' \end{vmatrix}}{\begin{vmatrix} f_j' f_i & g_j' \\ \frac{f_j'}{f_i'} f_i f_i'' + \frac{f_i'}{f_j'} f_j f_j'' - 2 f_i' f_j' & \frac{g_j'}{g_i'} g_i'' + \frac{g_i'}{g_j'} g_j'' \end{vmatrix}},$$

where $f_l' \equiv \frac{\partial f_l}{\partial x_l}$, $g_l' \equiv \frac{\partial g_l}{\partial x_l}$, $g_l'' \equiv \frac{\partial^2 g_l}{\partial x_l^2}$, etc. Note that in the expressions of X_i, X_j appears only the coordinates x_i, x_j and the parameters $\mathbf{p}_{ij}$. From further on we denote the maps R_{ij} as $R_{ij}^{\mathbf{p}_{ij}}$, in order to stress this separability feature.

3. *Mappings $R_{ij}^{\mathbf{p}_{ij}}$ are anti-measure preserving with densities $m_1 = n^1 d^2$, $m_2 = n^2 d^1$, where n^i, d^i the numerators and the denominators respectively, of the invariants H_i, $i = 1, 2$.*

4. *Mappings $R_{ij}^{\mathbf{p}_{ij}}$ satisfy the Yang–Baxter identity*

$$R_{ij}^{\mathbf{p}_{ij}} R_{ik}^{\mathbf{p}_{ik}} R_{jk}^{\mathbf{p}_{jk}} = R_{jk}^{\mathbf{p}_{jk}} R_{ij}^{\mathbf{p}_{ij}} R_{ij}^{\mathbf{p}_{ij}}.$$

5. *Mappings $R_{ij}^{\mathbf{p}_{ij}}$ are involutions with the sets of singularities*

$$\Sigma_{ij} = \left\{P_{ij}^1, P_{ij}^2, P_{ij}^3\right\} = \left\{\left(\frac{a_i}{b_i}, \frac{c_j}{d_j}\right), \left(\frac{c_i}{d_i}, \frac{a_j}{b_j}\right), \left(\frac{C_i}{D_i}, \frac{C_j}{D_j}\right)^2\right\},$$

where the superscript 2 *in P_{ij}^3 denotes that these singular points appears with multiplicity* 2. *In the formulae for P_{ij}^m, $m = 1, \dots, 3$, we have suppressed the dependency on the remaining variables. For example, with $P_{ij}^1 = \left(\frac{a_i}{b_i}, \frac{c_j}{d_j}\right)$ we denote $\left(x_1, \dots, x_{i-1}, \frac{a_i}{b_i}, x_{i+1}, \dots, x_{j-1}, \frac{c_j}{d_j},\right.$ $\left.x_{j+1}, \dots, x_k\right)$ and similarly for the remaining P_{ij}^m.*

6. *Each one of the maps $R_{ij}^{\mathbf{p}_{ij}}$ is* $(\text{Möb})^2$ *equivalent to the H_{II} Yang–Baxter map.*

Proof. The proof follows similarly to the proof of Proposition 3.3. ■

Example 3.10 ($k \geq 3$)**.** For $k = 3$, the invariants $H_1 = f_1 f_2 f_3$, $H_2 = g_1 + g_2 + g_3$ are functions of 3 variables with 24 parameters, 3 of them are essential. Without loss of generality, after removing the redundancy of the parameters, the invariants H_1, H_2 can be cast into the form

$$H_1 = \frac{x_1 - p_1}{x_1} \frac{x_2 - p_2}{x_2} \frac{x_3 - p_3}{x_3}, \qquad H_2 = x_1 + x_2 + x_3.$$

Then each of the mappings R_{ij}, $i \neq j \in \{1, 2, 3\}$ is exactly the H_{II} Yang–Baxter map.

For $k > 3$ the invariants depend on $8k$ parameters and only $3k - 6$ of them are essential. Without loss of generality they can be cast into the form

$$H_1 = \frac{x_1 - p_1}{x_1} \frac{x_2 - p_2}{x_2} \frac{x_3 - p_3}{x_3} \prod_{i=4}^{k} \frac{\alpha_i - \beta_i x_i}{\beta_i - \gamma_i x_i}, \qquad H_2 = \sum_{i=1}^{k} x_i.$$

3.3 Additive/additive separability of variables

Proposition 3.11. *Consider the additive/additive separability of variables of the invariants H_1 and H_2 (see (3.3)). Consider also the following sets of parameters*

$$\mathbf{p}_{ij} := \mathbf{p}_i \cup \mathbf{p}_j, \qquad \text{where} \qquad \mathbf{p}_i := \{a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i\}, \qquad i \neq j < j \in \{1, 2, \dots, k\}$$

and the functions

$$f_i := \frac{a_i - b_i x_i}{c_i - d_i x_i}, \qquad g_i := \frac{A_i - B_i x_i}{C_i - D_i x_i}, \qquad i = 1, \dots, k.$$

The following holds:

1. *The invariants $H_1 = \prod_{i=1}^{k} f_i$, $H_2 = \sum_{i=1}^{k} g_i$ depend on $8k$ parameters. Only $3k - 6$ of them are essential.*

2. *Mappings R_{ij} explicitly read*

$$R_{ij}\colon\ (x_1, x_2, \dots, x_k) \mapsto (X_1, X_2, \dots, X_k),$$

where $X_l = x_l\ \forall\, l \neq i,j$ *and* X_i, X_j *are given by the formulae*

$$X_i = x_i - 2\frac{\begin{vmatrix} f_j' & f_i' \\ g_j' & g_i' \end{vmatrix}}{\begin{vmatrix} f_i' & g_i' \\ \dfrac{f_j'}{f_i'}f_i'' + \dfrac{f_i'}{f_j'}f_j'' & \dfrac{g_j'}{g_i'}g_i'' + \dfrac{g_i'}{g_j'}g_j'' \end{vmatrix}},$$

$$X_j = x_j + 2\frac{\begin{vmatrix} f_j' & f_i' \\ g_j' & g_i' \end{vmatrix}}{\begin{vmatrix} f_j' & g_j' \\ \dfrac{f_j'}{f_i'}f_i'' + \dfrac{f_i'}{f_j'}f_j'' & \dfrac{g_j'}{g_i'}g_i'' + \dfrac{g_i'}{g_j'}g_j'' \end{vmatrix}},$$

where $f_l' \equiv \frac{\partial f_l}{\partial x_l}$, $g_l' \equiv \frac{\partial g_l}{\partial x_l}$, $g_l'' \equiv \frac{\partial^2 g_l}{\partial x_l^2}$, *etc. Note that in the expressions of* X_i, X_j *appears only the coordinates* x_i, x_j *and the parameters* $\mathbf{p}_{ij}$. *From further on we denote the maps* R_{ij} *as* $R_{ij}^{\mathbf{p}_{ij}}$, *in order to stress this separability feature.*

3. *Mappings* $R_{ij}^{\mathbf{p}_{ij}}$ *are anti-measure preserving with densities* $m_1 = n^1 d^2$, $m_2 = n^2 d^1$, *where* n^i, d^i *the numerators and the denominators respectively, of the invariants* H_i, $i = 1,2$.
4. *Mappings* $R_{ij}^{\mathbf{p}_{ij}}$ *satisfy the Yang–Baxter identity*

$$R_{ij}^{\mathbf{p}_{ij}} R_{ik}^{\mathbf{p}_{ik}} R_{jk}^{\mathbf{p}_{jk}} = R_{jk}^{\mathbf{p}_{jk}} R_{ij}^{\mathbf{p}_{ij}} R_{ij}^{\mathbf{p}_{ij}}.$$

5. *Mappings* $R_{ij}^{\mathbf{p}_{ij}}$ *are involutions with the sets of singularities*

$$\Sigma_{ij} = \{P_{ij}^1, P_{ij}^2\} = \left\{ \left(\frac{c_i}{d_i}, \frac{c_j}{d_j}\right)^2, \left(\frac{C_i}{D_i}, \frac{C_j}{D_j}\right)^2 \right\},$$

where the superscript 2 *in* P_{ij}^1 *and* P_{ij}^2 *denotes that these singular points appears with multiplicity* 2. *In the formulae for* P_{ij}^m, $m = 1,\dots,2$, *we have suppressed the dependency on the remaining variables. For example, with* $P_{ij}^1 = \left(\frac{c_i}{d_i}, \frac{c_j}{d_j}\right)$ *we denote* $\left(x_1,\dots,x_{i-1},\frac{c_i}{d_i},x_{i+1},\dots,x_{j-1},\frac{c_j}{d_j},x_{j+1},\dots,x_k\right)$ *and similarly for* P_{ij}^2.

6. *Each one of the maps* $R_{ij}^{\mathbf{p}_{ij}}$ *is* $(\text{Möb})^2$ *equivalent to the* H_{III}^A *Yang–Baxter map.*

Proof. The proof follows similarly to the proof of Proposition 3.3. ■

Example 3.12 ($k \geq 3$)**.** For $k = 3$, the invariants $H_1 = f_1 + f_2 + f_3$, $H_2 = g_1 + g_2 + g_3$ are functions of 3 variables with 24 parameters, 3 of them are essential. Without loss of generality, after removing the redundancy of the parameters, the invariants H_1, H_2 can be cast into the form:

$$H_1 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \qquad H_2 = p_1 x_1 + p_2 x_2 + p_3 x_3.$$

Then each of the mappings R_{ij}, $i \neq j \in \{1,2,3\}$ is exactly the H_{III}^A Yang–Baxter map.

For $k > 3$ the invariants depend on $8k$ parameters and only $3k - 6$ of them are essential. Without loss of generality they can be cast into the form

$$H_1 = \sum_{i=1}^{k} \frac{1}{x_i}, \qquad H_2 = p_1 x_1 + p_2 x_2 + p_3 x_3 + \sum_{i=4}^{k} \frac{\alpha_i - \beta_i x_i}{\beta_i - \gamma_i x_i}.$$

4 Entwining Yang–Baxter maps

Following [51], three different maps S, T, U are called *entwining Yang–Baxter maps* if they satisfy

$$S_{12}T_{13}U_{23} = U_{23}T_{13}S_{12}.$$

We consider two maps to be different if they are not $(\text{Möb})^2$ equivalent. Hence, in order to ensure that we have different maps we require that at least one of the maps S, T, U either belongs to a different subclass than the remaining ones or it has different singularity pattern (even if it belongs to the same subclass with the remaining ones) or it has different periodicity. In what follows we present two methods to obtain entwining maps. The first one is based on degeneracy, i.e., we construct maps which belong to different subclasses and we obtain entwining maps associated with the $H_{\rm I}$, $H_{\rm II}$ and $H_{\rm III}^A$ families of maps. The second one is based on the symmetries of the H-list of Yang–Baxter maps and we obtain entwining maps for all members of the H-list.

4.1 Degeneracy and entwining Yang–Baxter maps

In Section 3.1 it was shown that for $k = 3$ and for the multiplicative/multiplicative case, the invariants H_1, H_2 depend on 3 essential parameters. Without loss of generality they read

$$H_1 = x_1x_2x_3, \qquad H_2 = \frac{x_1 - p_1}{x_1 - 1}\frac{x_2 - p_2}{x_2 - 1}\frac{x_3 - p_3}{x_3 - 1}.$$

The associated maps R_{12}, R_{13} and R_{23} which preserve the invariants have exactly the form of the $H_{\rm I}$ map. In order to obtain entwining maps associated with the $H_{\rm I}$ map, we consider

$$H_1 = x_1x_2x_3, \qquad H_2 = \frac{x_1 - p_1}{x_1 - 1}\frac{x_2 - p_2}{x_2 - 1}\frac{\alpha_3 - \beta_3x_3}{\beta_3 - \gamma_3x_3}.$$

For these invariants, R_{12} is exactly the $H_{\rm I}$ map and for generic α_3, β_3, γ_3 mappings R_{13} and R_{23} are $(\text{Möb})^2$ equivalent to the $H_{\rm I}$. In order to obtain entwining maps we need to violate this $(\text{Möb})^2$ equivalency of the maps R_{13} and R_{23} with the $H_{\rm I}$ map. This is achieved by violating the generality, e.g., setting $\alpha_3 = 0$ or $\beta_3 = 0$, the maps R_{13} and R_{23}, belongs to different subclasses than the $H_{\rm I}$ map does. Working similarly for the $H_{\rm II}$ map we find 1 family of maps which entwine with the latter without being $(\text{Möb})^2$ equivalent. Finally, for $H_{\rm III}^A$ we find also 1 family of entwining maps which are not $(\text{Möb})^2$ equivalent with the latter. Our results are presented in Propositions 4.1–4.3.

Table 1. Entwining maps associated with the $H_{\rm I}$ Yang–Baxter map through degeneracy.

map	$(u,v)\mapsto(U,V)$	subclass
${\rm e}^aH_{\rm I}$	$U = \dfrac{\alpha(1-u)+\beta(\alpha-1)uv}{\alpha-u}$, $V = \dfrac{uv(\alpha-u)}{\alpha(1-u)+\beta(\alpha-1)uv}$	$[1:2]$
${\rm e}^bH_{\rm I}$	$U = \dfrac{u-\alpha}{u-1}$, $V = \dfrac{uv(u-1)}{u-\alpha}$	$[0:2]$

Proposition 4.1. *The $H_{\rm I}$ Yang–Baxter map entwines with the maps ${\rm e}^aH_{\rm I}$ and ${\rm e}^bH_{\rm I}$ of Table 1 according to the entwining relation*

$$S_{12}T_{13}T_{23} = T_{23}T_{13}S_{12},$$

where S_{12} is the $H_{\rm I}$ map acting on the $(1,2)$-coordinates, T_{13} and T_{23} are ${\rm e}^aH_{\rm I}$ acting on $(1,3)$ and $(2,3)$ coordinates respectively, or ${\rm e}^bH_{\rm I}$ acting on $(1,3)$ and $(2,3)$ coordinates respectively.

Proof. Starting with the invariants

$$H_1 = x_1x_2x_3, \qquad H_2 = \frac{x_1-p_1}{x_1-1}\frac{x_2-p_2}{x_2-1}\frac{a-bx_3}{b-cx_3},$$

the map R_{12} is exactly the $H_{\rm I}$ map. By setting $a=0$, R_{13} and R_{23} takes the form of $\mathrm{e}^aH_{\rm I}$ of Table 1 (where $\beta\equiv c/b$). The map $\mathrm{e}^aH_{\rm I}$ is of subclass $[1:2]$ so clearly non-$(\text{Möb})^2$ equivalent to $H_{\rm I}$. By setting $b=0$, R_{13} and R_{23} takes the form of $\mathrm{e}^bH_{\rm I}$ of Table 1 (where $\beta\equiv a/c$). The map $\mathrm{e}^bH_{\rm I}$ is of subclass $[0:1]$ so clearly non-$(\text{Möb})^2$ equivalent to $H_{\rm I}$ or to $\mathrm{e}^aH_{\rm I}$. Finally, by setting $c=0$, mappings R_{13} and R_{23} are $(\text{Möb})^2$ equivalent to $\mathrm{e}^aH_{\rm I}$. ■

Proposition 4.2. *The $H_{\rm II}$ Yang–Baxter map entwines with the map of Table 2 according to the entwining relation*

$$S_{12}T_{13}T_{23} = T_{23}T_{13}S_{12},$$

where S_{12} is the $H_{\rm II}$ map acting on the $(1,2)$-coordinates, T_{13} and T_{23} are $\mathrm{e}^bH_{\rm II}$ acting on $(1,3)$ and $(2,3)$ coordinates respectively.

Table 2. Entwining maps associated with the $H_{\rm II}$ Yang–Baxter map though degeneracy.

map	$(u,v)\mapsto(U,V)$	subclass
$\mathrm{e}^bH_{\rm II}$	$U=\dfrac{\alpha v}{\alpha-u},\ V=u\dfrac{\alpha-u-v}{\alpha-u}$	$[1:1]$

Proof. Starting with the invariants

$$H_1 = x_1+x_2+x_3, \qquad H_2 = \frac{x_1-p_1}{x_1}\frac{x_2-p_2}{x_2}\frac{a-bx_3}{b-cx_3},$$

the map R_{12} is exactly the $H_{\rm II}$ map. By setting $a=0$, R_{13} and R_{23} are $(\text{Möb})^2$ equivalent to the $H_{\rm II}$ map. By setting $b=0$, R_{13} and R_{23} takes the form of $\mathrm{e}^bH_{\rm II}$ of Table 2. The map $\mathrm{e}^bH_{\rm II}$ is of subclass $[1:1]$ so clearly non-$(\text{Möb})^2$ equivalent to the $H_{\rm II}$ map. Finally, by setting $c=0$, mappings R_{13} and R_{23} are $(\text{Möb})^2$ equivalent to $\mathrm{e}^bH_{\rm II}$. ■

Proposition 4.3. *The $H_{\rm III}^A$ Yang–Baxter map entwines with the map of Table 3 according to the entwining relation*

$$S_{12}T_{13}T_{23} = T_{23}T_{13}S_{12},$$

where S_{12} is the $H_{\rm III}^A$ map acting on the $(1,2)$-coordinates, T_{13} and T_{23} are $\mathrm{e}^bH_{\rm III}^A$ acting on $(1,3)$ and $(2,3)$ coordinates respectively.

Table 3. Entwining maps associated with the $H_{\rm III}^A$ Yang–Baxter map though degeneracy.

map	$(u,v)\mapsto(U,V)$	subclass
$\mathrm{e}^bH_{\rm III}^A$	$U=\dfrac{\beta}{\alpha}u,\ V=\dfrac{\beta uv}{\beta(u+v)-\alpha u^2v}$	$[0:2]$

Proof. Starting with the invariants

$$H_1 = x_1 + x_2 + x_3, \qquad H_2 = p_1x_1 + p_2x_2 + \frac{a - bx_3}{b - cx_3},$$

the map R_{12} is exactly the $H^A_{\rm III}$ map. By setting $a = 0$, R_{13} and R_{23} are $(\text{Möb})^2$ equivalent to the $H^A_{\rm III}$ map. By setting $b = 0$ and R_{13} and R_{23} takes the form of ${\rm e}^bH^A_{\rm III}$ of Table 3 (where $\beta = a/c$). The map ${\rm e}^bH^A_{\rm III}$ is of subclass $[0 : 2]$ so clearly non-$(\text{Möb})^2$ equivalent to the $H^A_{\rm III}$ map. Finally, by setting $c = 0$, mappings R_{13} and R_{23} are $(\text{Möb})^2$ equivalent to the $H^A_{\rm III}$ map. ■

In the following subsection we are using the notion of *symmetry of Yang–Baxter maps* in order to generate entwining maps

4.2 Symmetries of Yang–Baxter maps and the entwining property

The notion of symmetry in the context of Yang–Baxter maps was introduced in [59].

Definition 4.4. An involution $\phi\colon \mathbb{CP}^1 \mapsto \mathbb{CP}^1$ is a symmetry of the Yang–Baxter map $R\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \mapsto \mathbb{CP}^1 \times \mathbb{CP}^1$ if it holds

$$\phi_1\phi_2R_{12} = R_{12}\phi_1\phi_2,$$

where ϕ_1 is the involution that acts as ϕ to the first factor of the cartesian product $\mathbb{CP}^1 \times \mathbb{CP}^1$ and ϕ_2 is the involution that acts as ϕ to the second factor of the cartesian product.

Let $m < n \in \{1, \dots, k\}$, $k \geq 3$ fixed. A direct consequence of the previous definition is that if ϕ is a symmetry of the Yang–Baxter map R, then the map $\phi_mR_{mn}\phi_n$ is a new Yang–Baxter map since it is not $(\text{Möb})^2$ equivalent with R_{mn}. By finding the symmetries of the F-list of Yang–Baxter maps, the authors of [59] derived the H-list of Yang–Baxter maps. Clearly the symmetries of the F-list are symmetries of the H-list and vice versa.

Theorem 4.5. *Let ϕ a symmetry of a Yang–Baxter map R and let ϕ_0 the identity map, i.e., $\phi_0\colon (x_1, \dots, x_k) \mapsto (x_1, \dots, x_k)$. Out of the possible 4^3 entwining relations of the form*

$$R_{12}\phi_iR_{13}\phi_jR_{23}\phi_k = R_{23}\phi_kR_{13}\phi_jR_{12}\phi_i, \qquad i, j, k \in \{0, 1, 2, 3\}, \tag{4.1}$$

apart the Yang–Baxter relation that holds, only the following three entwining relations holds

$$R_{12}R_{13}\phi_1R_{23}\phi_2 = R_{23}\phi_2R_{13}\phi_1R_{12}, \tag{4.2}$$
$$R_{12}\phi_2R_{13}\phi_3R_{23} = R_{23}R_{13}\phi_3R_{12}\phi_2, \tag{4.3}$$
$$R_{12}\phi_2R_{13}\phi_2R_{23}\phi_2 = R_{23}\phi_2R_{13}\phi_2R_{12}\phi_2. \tag{4.4}$$

Proof. To show that only the entwining relations (4.2), (4.3), (4.4) holds, we start with

$$R_{12}\phi_iR_{13}\phi_jR_{23}\phi_k = R_{23}\phi_kR_{13}\phi_jR_{12}\phi_i, \qquad i, j, k \in \{0, 1, 2, 3\}.$$

By direct calculations, we prove that if the Yang–Baxter relation holds out of the 4^3 different relations (4.1), only (4.2), (4.3), (4.4) holds.

For example let us show that (4.2) holds. We have

$$R_{12}R_{13}\phi_1R_{23}\phi_2 = R_{12}R_{13}R_{23}\phi_1\phi_2 = R_{23}R_{13}R_{12}\phi_1\phi_2, \tag{4.5}$$

since ϕ_1 commutes with R_{23} and the Yang–Baxter relation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ holds. But due to the symmetry we have $R_{12}\phi_1\phi_2 = \phi_1\phi_2R_{12}$ so (4.5) reads

$$R_{23}R_{13}R_{12}\phi_1\phi_2 = R_{23}R_{13}\phi_1\phi_2R_{12} = R_{23}\phi_2R_{13}\phi_1R_{12}$$

and that completes the proof that (4.2) holds. For the remaining relations we work similarly for their proof. ■

Note that any of the entwining relations (4.2), (4.3) and (4.4), is uniquely described by the symmetries ϕ_i, ϕ_j, ϕ_k that take part in this relation. For example in (4.2) the symmetries ϕ_0, ϕ_1, ϕ_2 appear in this order, hence we refer to (4.2) as relation of *entwining type* (ϕ_0, ϕ_1, ϕ_2) or by using just the subscripts, relation of entwining type $(0, 1, 2)$.

In Table 4, we present the entwining maps S, T, U that correspond to the entwining relations (4.2)–(4.4), where R is any Yang–Baxter map. In what follows, we specify R to be any member of the H-list[2] of quadrirational Yang–Baxter maps.

Table 4. Entwining maps S, T, U associated with a Yang–Baxter map R.

entwining type	S_{12}	T_{13}	U_{23}
$(0, 1, 2)$	R_{12}	$R_{13}\phi_1$	$R_{23}\phi_2$
$(2, 3, 0)$	$R_{12}\phi_2$	$R_{13}\phi_3$	R_{23}
$(2, 2, 2)$	$R_{12}\phi_2$	R_{13}	$\phi_2 R_{23}\phi_2$

4.2.1 Entwining maps associated with the H_{I} Yang–Baxter map

The involutions ϕ, ψ

$$\phi\colon\ u \mapsto \frac{\alpha}{u}, \qquad \psi\colon\ u \mapsto \frac{u-\alpha}{u-1},$$

where α a complex parameter, are symmetries for the H_{I} map (see [59]), since it holds

$$\phi_1\phi_2 R_{12} = R_{12}\phi_1\phi_2, \qquad \psi_1\psi_2 R_{12} = R_{12}\psi_1\psi_2,$$

where R_{12} is the H_{I} map acting on the 12-coordinates and

$$\begin{aligned} &\phi_1\colon\ (x_1, x_2) \mapsto (p_1/x_1, x_2), && \phi_2\colon\ (x_1, x_2) \mapsto (x_1, p_2/x_2),\\ &\psi_1\colon\ (x_1, x_2) \mapsto ((x_1 - p_1)/(x_1 - 1), x_2), && \psi_2\colon\ (x_1, x_2) \mapsto (x_1, (x_2 - p_2)/(x_2 - 1)). \end{aligned}$$

Note that the symmetries ϕ and τ can be derived from our considerations (see Example 3.6) since for $k = 3$ it holds

$$\begin{aligned} &H_1\phi_1\phi_2\phi_3 = \frac{p_1p_2p_3}{H_1}, && H_2\phi_1\phi_2\phi_3 = \frac{1}{H_2},\\ &H_1\psi_1\psi_2\psi_3 = H_2, && H_2\psi_1\psi_2\psi_3 = H_1. \end{aligned}$$

Remark 4.6. By using similar arguments as in the proof of the Theorem 4.5, entwining relations where the symmetries ϕ and ψ of the H_{I} map interlace do not exist, i.e., it does not exists for example any relation of entwining type (ϕ_i, ϕ_j, ψ_k).

In Table 5 we present the entwining maps associated with the H_{I} map which are generated by using the symmetries ϕ and ψ. In Table 5 it appears the H_{I} map, the companion of the H_{I} map that is denoted as cH_{I}, as well as $\tilde{c}F_{\mathrm{I}}$ which is the companion map of the map $\tilde{F}_{\mathrm{I}}$ that was derived in [59]. We also have four novel maps which are not $(\text{Möb})^2$ equivalent to H_{I}, which we refer to as Φ_{I}^a, Φ_{I}^b, Ψ_{I}^a and Ψ_{I}^b. In the proposition that follows we present their explicit form.

[2]It is easy to show that the entwining maps associated with the F-list of quadrirational Yang–Baxter maps are $(\text{Möb})^2$ equivalent to the corresponding to the H-list entwining maps. This is the reason that we present the entwining maps associated with the H-list only.

Table 5. Left table: Entwining maps S, T, U associated with $H_{\rm I}$ Yang–Baxter map using the symmetry ϕ. Right table: Entwining maps S, T, U associated with $H_{\rm I}$ Yang–Baxter map using the symmetry ψ.

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H_{\rm I}$	$\Phi_{\rm I}^a$	$\Phi_{\rm I}^a$
$(2,3,0)$	$\Phi_{\rm I}^b$	$\Phi_{\rm I}^b$	$H_{\rm I}$
$(2,2,2)$	$\Phi_{\rm I}^b$	$H_{\rm I}$	$cH_{\rm I}$

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H_{\rm I}$	$\Psi_{\rm I}^a$	$\Psi_{\rm I}^a$
$(2,3,0)$	$\Psi_{\rm I}^b$	$\Psi_{\rm I}^b$	$H_{\rm I}$
$(2,2,2)$	$\Psi_{\rm I}^b$	$H_{\rm I}$	$c\tilde{F}_{\rm I}$

Proposition 4.7. *The following non-periodic*[3] *maps $(u,v)\mapsto(U,V)$, where*

$$U=\alpha vQ,\qquad V=\frac{1}{u}Q^{-1},\qquad Q=\frac{\beta-\alpha+u(1-\beta)+v(\alpha-1)}{\beta(1-\alpha)u-\alpha(1-\beta)v+(\alpha-\beta)uv},\tag{$\Phi_{\rm I}^a$}$$

$$U=\frac{1}{v}Q^{-1},\qquad V=\beta uQ,\qquad Q=\frac{\beta-\alpha+u(1-\beta)+v(\alpha-1)}{\beta(1-\alpha)u-\alpha(1-\beta)v+(\alpha-\beta)uv},\tag{$\Phi_{\rm I}^b$}$$

$$U=vQ,\qquad V=\frac{u-\alpha}{u-1}Q^{-1},\qquad Q=\frac{\alpha(1-v)-\beta u+uv}{\beta(1-u)-\beta v+uv},\tag{$\Psi_{\rm I}^a$}$$

$$U=\frac{v-\beta}{v-1}Q,\qquad V=uQ^{-1},\qquad Q=\frac{\alpha(1-u-v)+uv}{\beta(1-u)-\alpha v+uv},\tag{$\Psi_{\rm I}^b$}$$

entwine with the $H_{\rm I}$ Yang–Baxter map according to the entwining relations of Table 5.

4.2.2 Entwining maps associated with the $H_{\rm II}$ Yang–Baxter map

The invariants

$$H_1=x_1+x_2+x_3,\qquad H_2=\frac{x_1-p_1}{x_1}\frac{x_2-p_2}{x_2}\frac{x_3-p_3}{x_3},$$

generate the maps R_{ij}, $i<j\in\{1,2,3\}$ which are exactly the $H_{\rm II}$ map acting on the (ij)-coordinates. Explicitly the $H_{\rm II}$ map reads

$$U=v+\frac{(\alpha-\beta)uv}{\beta u+\alpha v-\alpha\beta},\qquad V=u-\frac{(\alpha-\beta)uv}{\beta u+\alpha v-\alpha\beta}.\tag{$H_{\rm II}$}$$

A symmetry of the $H_{\rm II}$ map is $\phi\colon u\mapsto\alpha-u$, since it holds $\phi_1\phi_2R_{12}=R_{12}\phi_1\phi_2$, where R_{12} is the $H_{\rm II}$ map acting on the (12)-coordinates and

$$\phi_1\colon\ (x_1,x_2)\mapsto(p_1-x_1,x_2),\qquad \phi_2\colon\ (x_1,x_2)\mapsto(x_1,p_2-x_2).$$

Table 6. Entwining maps S, T, U associated with $H_{\rm II}$ Yang–Baxter map using the symmetry ϕ.

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H_{\rm II}$	$\Phi_{\rm II}^a$	$\Phi_{\rm II}^a$
$(2,3,0)$	$\Phi_{\rm II}^b$	$\Phi_{\rm II}^b$	$H_{\rm II}$
$(2,2,2)$	$\Phi_{\rm II}^b$	$H_{\rm II}$	$cH_{\rm II}$

[3]A non-periodic map cannot be equivalent by conjugation ($(\text{Möb})^2$ equivalent) to a periodic map. Since the $H_{\rm I}$ map is involutive, the maps presented in this proposition are not $(\text{Möb})^2$ to the $H_{\rm I}$ map.

Proposition 4.8. *The following non-periodic maps $(u,v)\mapsto(U,V)$, where*

$$U=\alpha v\frac{u-v+\beta-\alpha}{\beta u-\alpha v},\qquad V=\beta\frac{(\alpha-u)(u-v)}{\beta u-\alpha v},\tag{$\Phi^a_{\rm II}$}$$

$$U=\alpha\frac{(\beta-v)(u-v)}{\beta u-\alpha v},\qquad V=\beta u\frac{u-v+\beta-\alpha}{\beta u-\alpha v},\tag{$\Phi^b_{\rm II}$}$$

entwine with the $H_{\rm II}$ Yang–Baxter map according to the entwining relations of Table 6.

The map $cH_{\rm II}$ denotes the companion map of the $H_{\rm II}$ map.

4.2.3 Entwining maps associated with the $H^A_{\rm III}$ Yang–Baxter map

The invariants

$$H_1=\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3},\qquad H_2=p_1x_1+p_2x_2+p_3x_3,$$

generate the maps R_{ij}, $i<j\in\{1,2,3\}$ which are exactly the $H^A_{\rm III}$ map acting on the (ij)-coordinates. Explicitly the $H^A_{\rm III}$ map reads

$$U=\frac{v}{\alpha}\frac{\alpha u+\beta v}{u+v},\qquad V=\frac{u}{\beta}\frac{\alpha u+\beta v}{u+v}.\tag{$H^A_{\rm III}$}$$

Two symmetries of the $H^A_{\rm III}$ map are

$$\phi\colon\ u\mapsto\frac{1}{\alpha u},\qquad \psi\colon\ u\mapsto -u$$

since it holds

$$\phi_1\phi_2R_{12}=R_{12}\phi_1\phi_2,\qquad \psi_1\psi_2R_{12}=R_{12}\psi_1\psi_2,$$

where R_{12} is the $H^A_{\rm III}$ map acting on the (12)-coordinates and

$$\phi_1\colon\ (x_1,x_2)\mapsto\left(\frac{1}{p_1x_1},x_2\right),\qquad \phi_2\colon\ (x_1,x_2)\mapsto\left(x_1,\frac{1}{p_2x_2}\right),$$
$$\psi_1\colon\ (x_1,x_2)\mapsto(-x_1,x_2),\qquad \psi_2\colon\ (x_1,x_2)\mapsto(x_1,-x_2).$$

Note that the map $\phi_1R_{12}\phi_2$ is exactly the $H^B_{\rm III}$ Yang–Baxter map.

Proposition 4.9. *The following non-periodic maps $(u,v)\mapsto(U,V)$ where*

$$U=v\frac{1+\beta uv}{1+\alpha uv},\qquad V=\frac{1}{\beta u}\frac{1+\beta uv}{1+\alpha uv},\tag{$\Phi^a_{{\rm III}^A}$}$$

$$U=\frac{1}{\alpha v}\frac{1+\alpha uv}{1+\beta uv},\qquad V=u\frac{1+\alpha uv}{1+\beta uv},\tag{$\Phi^b_{{\rm III}^A}$}$$

$$U=\frac{v}{\alpha}\frac{\alpha u-\beta v}{u-v},\qquad V=\frac{u}{\beta}\frac{\alpha u-\beta v}{v-u},\tag{$\Psi^a_{{\rm III}^A}$}$$

$$U=\frac{v}{\alpha}\frac{\alpha u-\beta v}{v-u},\qquad V=\frac{u}{\beta}\frac{\alpha u-\beta v}{u-v},\tag{$\Psi^b_{{\rm III}^A}$}$$

entwine with the $H^A_{\rm III}$ Yang–Baxter map according to the entwining relations of Table 7.

The map $cH^A_{\rm III}$ denotes the companion map of the $H^A_{\rm III}$ map and with $\hat{H}^A_{\rm III}$ we denote a $({\rm M\ddot{o}b})^2$ equivalent map to the $H^A_{\rm III}$.

Table 7. Left table: Entwining maps S, T, U associated with $H^A_{\rm III}$ Yang–Baxter map using the symmetry ϕ. Right table: Entwining maps S, T, U associated with $H^A_{\rm III}$ Yang–Baxter map using the symmetry ψ.

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H^A_{\rm III}$	$\Phi^a_{{\rm III}^A}$	$\Phi^a_{{\rm III}^A}$
$(2,3,0)$	$\Phi^b_{{\rm III}^A}$	$\Phi^b_{{\rm III}^A}$	$H^A_{\rm III}$
$(2,2,2)$	$\Phi^b_{{\rm III}^A}$	$H^A_{\rm III}$	$\hat{H}^A_{\rm III}$

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H^A_{\rm III}$	$\Psi^a_{{\rm III}^A}$	$\Psi^a_{{\rm III}^A}$
$(2,3,0)$	$\Psi^b_{{\rm III}^A}$	$\Psi^b_{{\rm III}^A}$	$H^A_{\rm III}$
$(2,2,2)$	$\Psi^b_{{\rm III}^A}$	$H^A_{\rm III}$	$cH^A_{\rm III}$

4.2.4 Entwining maps associated with the $H^B_{\rm III}$ Yang–Baxter map

The invariants that were derived in [44, 45, 47, 56],

$$H_1 = x_1x_2x_3, \qquad H_2 = p_1x_1 + p_2x_2 + p_3x_3 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3},$$

generate the maps R_{ij}, $i < j \in \{1, 2, 3\}$ which are exactly the $H^B_{\rm III}$ map acting on the (ij)-coordinates. Explicitly the $H^B_{\rm III}$ map reads

$$U = v\frac{1+\beta uv}{1+\alpha uv}, \qquad V = u\frac{1+\alpha uv}{1+\beta uv}, \tag{$H^B_{\rm III}$}$$

The symmetries ϕ, ψ of the $H^A_{\rm III}$ map are symmetries of $H^B_{\rm III}$ as well.

Proposition 4.10. *The following non-periodic maps $(u, v) \mapsto (U, V)$, where*

$$U = \frac{v}{\alpha}\frac{\alpha u+\beta v}{u+v}, \qquad V = \frac{1}{u}\frac{u+v}{\alpha u+\beta v}, \tag{$\Phi^a_{{\rm III}^B}$}$$

$$U = \frac{1}{v}\frac{u+v}{\alpha u+\beta v}, \qquad V = \frac{u}{\beta}\frac{\alpha u+\beta v}{u+v}, \tag{$\Phi^b_{{\rm III}^B}$}$$

$$U = v\frac{1-\beta uv}{1-\alpha uv}, \qquad V = u\frac{1-\alpha uv}{-1+\beta uv}, \tag{$\Psi^a_{{\rm III}^B}$}$$

$$U = v\frac{1-\beta uv}{-1+\alpha uv}, \qquad V = u\frac{1-\alpha uv}{1-\beta uv}, \tag{$\Psi^b_{{\rm III}^B}$}$$

entwine with the $H^B_{\rm III}$ Yang–Baxter map according to the entwining relations of Table 8.

Table 8. Left table: Entwining maps S, T, U associated with $H^B_{\rm III}$ Yang–Baxter map using the symmetry ϕ. Right table: Entwining maps S, T, U associated with $H^B_{\rm III}$ Yang–Baxter map using the symmetry ψ.

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H^B_{\rm III}$	$\Phi^a_{{\rm III}^B}$	$\Phi^a_{{\rm III}^B}$
$(2,3,0)$	$\Phi^b_{{\rm III}^B}$	$\Phi^b_{{\rm III}^B}$	$H^B_{\rm III}$
$(2,2,2)$	$\Phi^b_{{\rm III}^B}$	$H^B_{\rm III}$	$\hat{H}^B_{\rm III}$

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H^B_{\rm III}$	$\Psi^a_{{\rm III}^B}$	$\Psi^a_{{\rm III}^B}$
$(2,3,0)$	$\Psi^b_{{\rm III}^B}$	$\Psi^b_{{\rm III}^B}$	$H^B_{\rm III}$
$(2,2,2)$	$\Psi^b_{{\rm III}^B}$	$H^B_{\rm III}$	$\tilde{H}^B_{\rm III}$

The maps $\hat{H}^B_{\rm III}$, $\tilde{H}^B_{\rm III}$ that appear in Table 8, are $(\text{Möb})^2$ equivalent to the map $H^B_{\rm III}$. The map $cH^B_{\rm III}$ denotes the companion map of the $H^B_{\rm III}$ map.

4.2.5 Entwining maps associated with the $H_{\rm V}$ Yang–Baxter map

The invariants that were derived in [44, 45, 47, 56],

$$H_1 = x_1 + x_2 + x_3, \qquad H_2 = x_1^3 + 3p_1x_1 + x_2^3 + 3p_2x_2 + x_3^3 + 3p_3x_3,$$

generate the maps R_{ij}, $i<j\in\{1,2,3\}$ which are exactly the $H_{\rm V}$ map acting on the (ij)-coordinates. Explicitly the $H_{\rm V}$ map reads

$$U=v-\frac{\alpha-\beta}{u+v},\qquad V=u+\frac{\alpha-\beta}{u+v}. \tag{$H_{\rm V}$}$$

The involution $\psi\colon u\mapsto -u$ is a symmetry of the $H_{\rm V}$ map.

Proposition 4.11. *The following non-periodic maps $(u,v)\mapsto(U,V)$, where*

$$U=v+\frac{\alpha-\beta}{u-v},\qquad V=-u-\frac{\alpha-\beta}{u-v}, \tag{$\Psi_{\rm V}^a$}$$

$$U=-v-\frac{\alpha-\beta}{u-v},\qquad V=u+\frac{\alpha-\beta}{u-v}, \tag{$\Psi_{\rm V}^b$}$$

entwine with the $H_{\rm V}$ Yang–Baxter map according to the entwining relations of Table 9.

Table 9. Entwining maps S, T, U associated with $H_{\rm V}$ Yang–Baxter map using the symmetry ψ.

entwining type	S_{12}	T_{13}	U_{23}
$(0,1,2)$	$H_{\rm V}$	$\Psi_{\rm V}^a$	$\Psi_{\rm V}^a$
$(2,3,0)$	$\Psi_{\rm V}^b$	$\Psi_{\rm V}^b$	$H_{\rm V}$
$(2,2,2)$	$\Psi_{\rm V}^b$	$H_{\rm V}$	$cH_{\rm V}$

The map $cH_{\rm V}$ denotes the companion map of the $H_{\rm V}$ map.

5 Transfer maps

The notion of *transfer maps* associated with Yang–Baxter maps was introduced by Veselov in [69]. In [70] dynamical aspects of the latter were discussed. The transfer maps associated with any reversible Yang–Baxter map are defined as

$$T_i^{(k)}=R_{ii+k-1}R_{ii+k-2}\cdots R_{ii+1},\qquad i\in\{1,\dots,k\},$$

where the indices are considered modulo k. There is:

$$T_i^{(k)}T_j^{(k)}=T_j^{(k)}T_i^{(k)},\qquad T_1^{(k)}T_2^{(k)}\cdots T_k^{(k)}=\mathrm{id}.$$

For example for $k=4$ we have $T_1^{(4)}=R_{14}R_{13}R_{12}$, $T_2^{(4)}=R_{12}R_{24}R_{23}$, $T_3^{(4)}=R_{23}R_{13}R_{34}$ and $T_4^{(4)}=R_{34}R_{24}R_{14}$.

Proposition 5.1. *For the transfer maps $T_i^{(k)}$ associated with the maps $R_{ij}^{\mathbf{p}_{ij}}$ of the Propositions 3.3, 3.9, 3.11, it holds:*

1) *they preserve the invariants H_1, H_2, presented in the Propositions 3.3, 3.9, 3.11,*
2) *for $k=2n+1$ they preserve the measures given in the Propositions 3.3, 3.9, 3.11,*
3) *for $k=2n$ they anti-preserve the measures given in the Propositions 3.3, 3.9, 3.11,*
4) *they possess Lax pairs,*
5) *for generic values of the parameter sets $\mathbf{p}_{ij}$, are equivalent by conjugation to the transfer maps associated with $H_{\rm I}$, $H_{\rm II}$ and $H_{\rm III}^A$ Yang–Baxter maps respectively,*

6) *for non-generic values of the parameter sets $\mathbf{p}_{ij}$, we have novel transfer maps.*

Proof. The statements (1)–(3) have already been proven (see Propositions 2.1, 3.3, 3.9, 3.11). As for the statement (4), one can construct a Lax matrix for the Yang–Baxter map R following [66]. Then the Lax equations associated with the transfer maps $T_i^{(k)}$, correspond to certain factorizations of the monodromy matrix (see [69]).

We will show the statement (5) for the transfer maps associated with $R_{ij}^{\mathbf{p}_{ij}}$ of Proposition 3.3 and for $k = 4$. The proof for arbitrary k follows by induction. In Proposition 3.3 it was shown that these maps are $(\text{Möb})^2$ equivalent to the H_{I} map. Let us denote as ν_l the maps defined by the cross-ratios

$$\mathrm{CR}[x_l, a_l/b_l, c_l/d_l, A_l/B_l] = \mathrm{CR}[y_l, 0, 1, \infty], \qquad l = 1, \dots, 4,$$

and as μ_l the maps defined by

$$\mathrm{CR}[x_l, c_l/d_l, a_l/b_l, C_l/D_l] = \mathrm{CR}[y_l, \infty, 1, 0], \qquad l = 1, \dots, 4.$$

Then the maps $\tilde{R}_{ij}^{\mathbf{p}_{ij}}$, where $\tilde{R}_{ij}^{\mathbf{p}_{ij}} = \mu_j^{-1}\mu_i^{-1}R_{ij}^{\mathbf{p}_{ij}}\mu_i\mu_j$ are exactly the H_{I} map acting on the (ij)-coordinates (see Proposition 3.3). For the transfer map $\tilde{T}_1^{(4)}$ associated with $\tilde{R}_{ij}^{\mathbf{p}_{ij}}$, there is

$$\begin{aligned}\tilde{T}_1^{(4)} &= \tilde{R}_{14}\tilde{R}_{13}\tilde{R}_{12} = \left(\nu_1^{-1}\mu_4^{-1}R_{14}\nu_1\mu_4\right)\left(\nu_1^{-1}\mu_3^{-1}R_{13}\nu_1\mu_3\right)\left(\nu_1^{-1}\mu_3^{-1}R_{13}\nu_1\mu_2\right)\\ &= \mu_4^{-1}\mu_3^{-1}\mu_2^{-1}\nu_1^{-1}R_{14}R_{13}R_{12}\nu_1\mu_2\mu_3\mu_4 = \mu_4^{-1}\mu_3^{-1}\mu_2^{-1}\nu_1^{-1}T_1^{(4)}\nu_1\mu_2\mu_3\mu_4.\end{aligned} \tag{5.1}$$

Note that we have omitted the parameter sets $\mathbf{p}_{ij}$ that the maps depends on for simplicity.

(6). For non-generic choice of the parameter sets $\mathbf{p}_{ij}$, the conjugation equivalence (5.1) does not holds. ■

5.1 On a re-factorisation of the transfer maps

First, let us introduce some maps. With π_{ij} we denote the transpositions

$$\begin{aligned}&\pi_{ij}\colon\ (x_1, \dots, x_k; \mathbf{p}_1, \dots, \mathbf{p}_k) \mapsto (X_1, \dots, X_k; \mathbf{P}_1, \dots, \mathbf{P}_k),\\ &X_l = x_l, \qquad \mathbf{P}_l = \mathbf{p}_l \qquad \forall\, l \neq i, j, \qquad X_i = x_j, \qquad X_j = x_i, \qquad \mathbf{P}_i = \mathbf{p}_j, \qquad \mathbf{P}_j = \mathbf{p}_i.\end{aligned}$$

and with π_0 we denote the following k-periodic map

$$\begin{aligned}&\pi_0\colon\ (x_1, \dots, x_k; \mathbf{p}_1, \dots, \mathbf{p}_k) \mapsto (X_1, \dots, X_k; \mathbf{P}_1, \dots, \mathbf{P}_k),\\ &X_l = x_{l+1}, \qquad \mathbf{P}_l = \mathbf{p}_{l+1}, \qquad \forall\, l \in \{1, \dots, k\}, \quad \text{modulo } k.\end{aligned}$$

Remark 5.2. Note that $\pi_0 = \pi_{12}\pi_{13}\cdots\pi_{1k}$ and the maps π_0, π_{ij} $\forall\, i, j \in \{1, \dots, k\}$, preserve the invariants H_1, H_2 of the Propositions 3.3, 3.9, 3.11. Moreover, the maps $S_i := \pi_{ii+1}R_{ii+1}$, $i \in \{1, \dots, k\}$, also preserve the invariants H_1, H_2. The following relations holds

$$S_i^2 = (S_iS_{i+1})^3 = \pi_0^k = \mathrm{id}, \qquad (S_iS_j)^2 = \mathrm{id}, \qquad |i - j| > 1, \qquad S_i\pi_0 = \pi_0S_{i+1}.$$

The group $g = \langle\pi_0, S_1, S_2, \dots, S_k\rangle$ generated by these maps provides a bi-rational realization of the extended Weyl group of type $A_{k-1}^{(1)}$.

Proposition 5.3. *The transfer maps $T_i^{(k)}$ of a Yang–Baxter map R, coincide with the $(k-1)$-iteration of the maps*

$$t_i^{(k)} := \pi_0\pi_{ii+1}R_{ii+1}^{\mathbf{p}_{ii+1}} = \pi_0S_i.$$

We refer to the maps $t_i^{(k)}$ as the extended transfer maps associated with the Yang–Baxter map R.

Proof. It is enough to show that the $(k-1)$-iteration of the map $t_1^{(k)}$ coincides with $T_1^{(k)}$. For small values of k, this can be proven by direct calculation. In-order to complete the proof, it is enough to show that for arbitrary k the maps $T_1^{(k)}$ and $\left(t_1^{(k)}\right)^{k-1}$ share the same Lax equation.

Let $L(x,\mathbf{p};\boldsymbol{\lambda})$ the Lax matrix associated with the Yang–Baxter map R. The Lax equation associated with the transfer map $T_1^{(k)}=R_{1k}^{\mathbf{p}_{1k}}R_{1k-1}^{\mathbf{p}_{1k-1}}\cdots R_{12}^{\mathbf{p}_{12}}$ reads

$$\begin{aligned}&L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})L(x_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(x_2,\mathbf{p}_2;\boldsymbol{\lambda})L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})\\&\qquad=L(X_1,\mathbf{p}_1;\boldsymbol{\lambda})L(X_k,\mathbf{p}_k;\boldsymbol{\lambda})L(X_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(X_2,\mathbf{p}_2;\boldsymbol{\lambda}).\end{aligned}\tag{5.2}$$

Since

$$\begin{aligned}\pi_{12}R_{12}^{\mathbf{p}_{12}}\colon\ &L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})L(x_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(x_2,\mathbf{p}_2;\boldsymbol{\lambda})L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})\\&\mapsto L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})L(x_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(x_2,\mathbf{p}_2;\boldsymbol{\lambda})L(x_1,\mathbf{p}_1;\boldsymbol{\lambda}),\end{aligned}$$

and

$$\begin{aligned}\pi_0\colon\ &L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})L(x_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(x_2,\mathbf{p}_2;\boldsymbol{\lambda})L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})\\&\mapsto L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})\cdots L(x_3,\mathbf{p}_3;\boldsymbol{\lambda})L(x_2,\mathbf{p}_2;\boldsymbol{\lambda}),\end{aligned}$$

there is

$$\begin{aligned}t_1^{(k)}\colon\ &L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})L(x_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(x_2,\mathbf{p}_2;\boldsymbol{\lambda})L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})\\&\mapsto L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})\cdots L(x_3,\mathbf{p}_3;\boldsymbol{\lambda})L(x_2,\mathbf{p}_2;\boldsymbol{\lambda}).\end{aligned}$$

So the map $t_1^{(k)}$ has the following Lax equation

$$\begin{aligned}&L(x_k,\mathbf{p}_k;\boldsymbol{\lambda})L(x_{k-1},\mathbf{p}_{k-1};\boldsymbol{\lambda})\cdots L(x_2,\mathbf{p}_2;\boldsymbol{\lambda})L(x_1,\mathbf{p}_1;\boldsymbol{\lambda})\\&\qquad=L(X_1,\mathbf{P}_1;\boldsymbol{\lambda})L(X_k,\mathbf{P}_k;\boldsymbol{\lambda})L(X_{k-1},\mathbf{P}_{k-1};\boldsymbol{\lambda})\cdots L(X_2,\mathbf{P}_2;\boldsymbol{\lambda}).\end{aligned}$$

But the map $t_1^{(k)}$ acts on the parameter sets $\mathbf{p}_i$ as follows

$$t_1^{(k)}\colon\ (\mathbf{p}_1,\dots,\mathbf{p}_k)\mapsto(\mathbf{P}_1,\dots,\mathbf{P}_k),$$

where

$$\mathbf{P}_1=\mathbf{p}_1,\qquad \mathbf{P}_k=\mathbf{p}_2\qquad\text{and}\qquad\forall\, i\neq 1,k\qquad \mathbf{P}_i=\mathbf{p}_{i+1},$$

that is periodic with period $k-1$, so the Lax equation of the map $\left(t_1^{(k)}\right)^{k-1}$ is exactly (5.2), i.e., the Lax equation of $T_1^{(k)}$. ■

Theorem 5.4. *The maps $t_i^{(k)}$ satisfy the relations*

$$\begin{aligned}&\left(t_i^{(k)}t_{i+1}^{(k)}\right)^{k/2}=\mathrm{id},\qquad t_1^{(k)}t_2^{(k)}\cdots t_k^{(k)}=\mathrm{id},\qquad k\quad\textit{even},\\&\left(t_i^{(k)}t_{i+i}^{(k)}\right)^{k}=\mathrm{id},\qquad \left(t_1^{(k)}t_2^{(k)}\cdots t_k^{(k)}\right)^2=\mathrm{id},\qquad k\quad\textit{odd}.\end{aligned}$$

Proof. Let us first prove that $t_1^{(k)}t_2^{(k)}\cdots t_k^{(k)}=\mathrm{id}$ for $k=2m$ even. There is

$$t_1^{(2m)}t_2^{(2m)}\cdots t_{2m}^{(2m)}=\pi_0S_1\pi_0S_2\cdots\pi_0S_{2m},$$

where we have the composition of m expressions of the form $\pi_0 S_i \pi_0 S_{i+1}$, and for each one of them (using Remark 5.2) it holds $\pi_0 S_i \pi_0 S_{i+1} = \pi_0 S_i^2 \pi_0 = \pi_0^2$. So

$$t_1^{(2m)} t_2^{(2m)} \cdots t_{2m}^{(2m)} = \underbrace{\pi_0^2 \pi_0^2 \cdots \pi_0^2}_{m\text{-times}} = \pi_0^{2m} = \mathrm{id}.$$

Let us now prove that $\big(t_i^{(k)} t_{i+1}^{(k)}\big)^{k/2} = \mathrm{id}$. We have

$$\big(t_i^{(k)} t_{i+1}^{(k)}\big)^{k/2} = \big(t_i^{(k)} t_{i+1}^{(k)}\big)^{m} = \big(\pi_0 S_i \pi_0 S_{i+1}\big)^{m} = \big(\pi_0^2 S_{i+1}^2\big)^{m} = \pi_0^{2m} = \mathrm{id}.$$

For $k = 2m+1$ odd, we have

$$\big(t_i^{(k)} t_{i+1}^{(k)}\big)^{k} = (t_i^{(k)} t_{i+1}^{(k)})^{2m+1} = \big(\pi_0^2 S_{i+1}^2\big)^{2m+1} = \big(\pi_0^{2m+1}\big)^{2} = \mathrm{id}.$$

Also,

$$\begin{aligned}\big(t_1^{(2m+1)} t_2^{(2m+1)} \cdots t_{2m+1}^{(2m+1)}\big)^2 &= \big(t_1^{(2m+1)} t_2^{(2m+1)} \cdots t_{2m}^{(2m+1)} \pi_0 S_{2m+1}\big)^2 \\ &= \big(\pi_0^{2m+1} S_{2m+1}\big)^2 = S_{2m+1}^2 = \mathrm{id},\end{aligned}$$

where we have used the fact that

$$t_1^{(2m+1)} t_2^{(2m+1)} \cdots t_{2m}^{(2m+1)} = \underbrace{\pi_0^2 \pi_0^2 \cdots \pi_0^2}_{m\text{-times}} = \pi_0^{2m}. \qquad \blacksquare$$

Remark 5.5. Note that for k odd, it holds the more general condition

$$\big(t_i^{(k)} t_j^{(k)}\big)^k = \mathrm{id}, \qquad i \neq j.$$

5.2 k-point recurrences associated with the transfer maps of the H-list of quadrirational Yang–Baxter maps

We refer to the extended transfer maps $t_i^{(k)}$ that correspond to the H_{I}, H_{II}, H_{III}^A, H_{III}^B and H_{V} Yang–Baxter maps respectively as $t_i^{H_{\mathrm{I}}(k)}$, $t_i^{H_{\mathrm{II}}(k)}$, $t_i^{H_{\mathrm{III}}^A(k)}$, $t_i^{H_{\mathrm{III}}^B(k)}$ and $t_i^{H_{\mathrm{V}}(k)}$.

Here, we associate k-point recurrences with the maps $t_i^{H_{\mathrm{I}}(k)}$, $t_i^{H_{\mathrm{II}}(k)}$, $t_i^{H_{\mathrm{III}}^A(k)}$, $t_i^{H_{\mathrm{III}}^B(k)}$ and $t_i^{H_{\mathrm{V}}(k)}$. Let us first introduce the shift operator T as follows

$$\begin{aligned}&T^0\colon\ x(n) \mapsto x(n), \qquad T^1\colon\ x(n) \mapsto x(n+1), \qquad T^l\colon\ x(n) \mapsto x(n+l),\\ &T^{-l}\colon\ x(n) \mapsto x(n-l), \qquad n, l \in \mathbb{Z}.\end{aligned}$$

The maps $t_2^{H_{\mathrm{I}}(k)}$, $t_2^{H_{\mathrm{II}}(k)}$, $t_2^{H_{\mathrm{III}}^A(k)}$, $t_2^{H_{\mathrm{III}}^B(k)}$ and $t_2^{H_{\mathrm{V}}(k)}$, explicitly read

$$(x_1, \dots, x_k; p_1, \dots, p_k) \mapsto (Tx_1, \dots, Tx_k; Tp_1, \dots, Tp_k),$$

where

$$\begin{aligned}&Tx_1 = x_2 \frac{p_3(1-p_2) + (p_2-p_3)x_3 + (p_3-1)x_2x_3}{p_2(1-p_3) + (p_3-p_2)x_2 + (p_2-1)x_2x_3}, \qquad Tp_1 = p_3, \qquad Tx_i = x_{i+1},\\ &Tx_2 = x_3 \frac{p_2(1-p_3) + (p_3-p_2)x_2 + (p_2-1)x_2x_3}{p_3(1-p_2) + (p_2-p_3)x_3 + (p_3-1)x_2x_3}, \qquad Tp_2 = p_2, \qquad Tp_i = p_{i+1}, \quad \big(t_2^{H_{\mathrm{I}}(k)}\big)\end{aligned}$$

$$Tx_1 = p_3 x_2 \frac{x_2 + x_3 - p_2}{p_3x_2 + p_2x_3 - p_2p_3}, \qquad Tp_1 = p_3, \qquad Tx_i = x_{i+1},$$

$$Tx_2 = p_2x_3\frac{x_2+x_3-p_3}{p_3x_2+p_2x_3-p_2p_3}, \qquad Tp_2 = p_2, \qquad Tp_i = p_{i+1}, \tag{$t_2^{H_{\rm II}(k)}$}$$

$$Tx_1 = \frac{x_2}{p_3}\frac{p_2x_2+p_3x_3}{x_2+x_3}, \qquad Tp_1 = p_3, \qquad Tx_i = x_{i+1},$$
$$Tx_2 = \frac{x_3}{p_2}\frac{p_2x_2+p_3x_3}{x_2+x_3}, \qquad Tp_2 = p_2, \qquad Tp_i = p_{i+1}, \tag{$t_2^{H_{\rm III}^A(k)}$}$$

$$Tx_1 = x_2\frac{1+p_2x_2x_3}{1+p_3x_2x_3}, \qquad Tp_1 = p_3, \qquad Tx_i = x_{i+1},$$
$$Tx_2 = x_3\frac{1+p_3x_2x_3}{1+p_2x_2x_3}, \qquad Tp_2 = p_2, \qquad Tp_i = p_{i+1}, \tag{$t_2^{H_{\rm III}^B(k)}$}$$

$$Tx_1 = x_2 - \frac{p_3-p_2}{x_2+x_3}, \qquad Tp_1 = p_3, \qquad Tx_i = x_{i+1},$$
$$Tx_2 = x_3 + \frac{p_3-p_2}{x_2+x_3}, \qquad Tp_2 = p_2, \qquad Tp_i = p_{i+1}, \tag{$t_2^{H_{\rm V}(k)}$}$$

with $i = 3, 4, \ldots, k$ and $Tx_k = x_1$, $Tp_k = p_1$. Moreover, not just $t_2^{(k)}$, but all the maps $t_i^{(k)}$, $i = 1, 2, \ldots, k$, preserve the invariants in separated variables (see Table 10)[4] and they anti-preserve the measures $m_i = n^i d^{i+1}$ where n^i, d^i the numerator and the denominator respectively of the invariants H_i, $i = 1, 2$. Additional invariant can be constructed though the Lax formulation (see the proof of Proposition 5.3).

Table 10. Invariants in separated variables for the maps $t_i^{H_{\rm I}(k)}$, $t_i^{H_{\rm II}(k)}$, $t_i^{H_{\rm III}^A(k)}$, $t_i^{H_{\rm III}^B(k)}$ and $t_i^{H_{\rm V}(k)}$.

map	H_1	H_2
$t_i^{H_{\rm I}(k)}$	$\prod_{i=1}^k p_ix_i$	$\prod_{i=1}^k \frac{x_i-p_i}{x_i-1}\frac{1}{p_i-1}$
$t_i^{H_{\rm II}(k)}$	$\sum_{i=1}^k x_i+p_i$	$\prod_{i=1}^k \frac{x_i-p_i}{p_ix_i}$
$t_i^{H_{\rm III}^A(k)}$	$\sum_{i=1}^k \frac{1}{x_i}+\frac{1}{p_i}$	$\sum_{i=1}^k p_ix_i$
$t_i^{H_{\rm III}^B(k)}$	$\prod_{i=1}^k p_ix_i$	$\sum_{i=1}^k \frac{1}{x_i}+p_ix_i+\frac{1}{p_i}$
$t_i^{H_{\rm V}(k)}$	$\sum_{i=1}^k x_i+p_i$	$\sum_{i=1}^k x_i^3+3p_ix_i+p_i^3$

Now we show how a k-point recurrence can be associated with the map $t_2^{H_{\rm V}(k)}$. Recall that the map $t_2^{H_{\rm V}(k)}$ reads

$$t_2^{H_{\rm V}(k)}\colon\ (x_1,\ldots,x_k;p_1,\ldots,p_k)\mapsto(Tx_1,\ldots,Tx_k;Tp_1,\ldots,Tp_k),$$

where

$$Tx_1 = x_2 - \frac{p_3-p_2}{x_2+x_3}, \qquad Tx_2 = x_3 + \frac{p_3-p_2}{x_2+x_3}, \qquad Tx_i = x_{i+1},$$
$$Tp_1 = p_3, \qquad Tp_2 = p_2, \qquad Tp_i = p_{i+1}, \qquad i = 3, \ldots, k,$$

[4]The invariants in separated variables that appear in Table 10, were firstly introduced, in a different context, in [44, 45, 47, 56]. Note that the invariants H_1, H_2 for $t_i^{H_{\rm III}^A(k)}$ were also given in [50].

and the indices are considered modulo k. Clearly we have, $x_3 = T^{2-k}x_1$, $p_3 = T^{2-k}p_1$. So we obtain

$$\begin{aligned} &Tx_1 = x_2 - \frac{T^{2-k}p_1 - p_2}{x_2 + T^{2-k}x_1}, \qquad Tx_2 = T^{2-k}x_1 + \frac{T^{2-k}p_1 - p_2}{x_2 + T^{2-k}x_1}, \\ &T^{k-1}p_1 = p_1, \qquad Tp_2 = p_2. \end{aligned} \tag{5.3}$$

Adding the first two equations from above we get the following invariance condition[5]

$$\left(T^1 - T^{2-k}\right)x_1 = \left(T^0 - T^1\right)x_2. \tag{5.4}$$

So it is guaranteed the existence of a potential function f such that

$$x_1 = c + \left(T^0 - T^1\right)f, \qquad x_2 = c + \left(T^1 - T^{2-k}\right)f, \qquad \text{where} \qquad c = \text{const}.$$

In terms of f, (5.3) becomes the following $(k+1)$-point recurrence

$$\left(T^2 - T^{2-k}\right)f = \frac{-p_2 + T^{2-k}p_1}{2c + \left(T - T^{3-k}\right)f}, \qquad T^{k-1}p_1 = p_1, \qquad Tp_2 = p_2. \tag{5.5}$$

In terms of a new variable h defined as $h := \lambda + \left(T^1 - T^0\right)f$, there is,

$$\left(T^2 - T^{2-k}\right)f = -\lambda k + \sum_{i=2-k}^{1} T^i h, \qquad \left(T - T^{3-k}\right)f = \lambda(2-k) + \sum_{i=3-k}^{0} T^i h,$$

so (5.5) becomes the k-point recurrence

$$\frac{2ck}{2-k} + \sum_{i=2-k}^{1} T^i h = \frac{-p_2 + T^{2-k}p_1}{\sum\limits_{i=3-k}^{0} T^i h}, \qquad T^{k-1}p_1 = p_1, \qquad Tp_2 = p_2,$$

where we chose $\lambda = \frac{2c}{k-2}$ to simplify the formulae.

Table 11. The invariance conditions (5.4) and the potential functions f for the maps $t_2^{H_{\mathrm{I}}(k)}$, $t_2^{H_{\mathrm{II}}(k)}$, $t_2^{H^A_{\mathrm{III}}(k)}$, $t_2^{H^B_{\mathrm{III}}(k)}$ and $t_2^{H_{\mathrm{V}}(k)}$.

map	invariance condition	potential function f
$t_2^{H_{\mathrm{I}}(k)}$	$\frac{Tx_1}{T^{2-k}x_1} = \frac{T^0x_2}{Tx_2}$	$x_1 = c\frac{T^0f}{Tf}$, $x_2 = c\frac{Tf}{T^{2-k}f}$
$t_2^{H_{\mathrm{II}}(k)}$	$\left(T - T^{2-k}\right)x_1 = \left(T^0 - T\right)x_2$	$x_1 = c + \left(T^0 - T\right)f$, $x_2 = c + \left(T - T^{2-k}\right)f$
$t_2^{H^A_{\mathrm{III}}(k)}$	$\left(T - T^{2-k}\right)\frac{1}{x_1} = \left(T^0 - T\right)\frac{1}{x_2}$	$\frac{1}{x_1} = \frac{1}{c} + \left(T^0 - T\right)f$, $\frac{1}{x_2} = \frac{1}{c} + \left(T - T^{2-k}\right)f$
$t_2^{H^B_{\mathrm{III}}(k)}$	$\frac{Tx_1}{T^{2-k}x_1} = \frac{T^0x_2}{Tx_2}$	$x_1 = c\frac{T^0f}{Tf}$, $x_2 = c\frac{Tf}{T^{2-k}f}$
$t_2^{H_{\mathrm{V}}(k)}$	$\left(T - T^{2-k}\right)x_1 = \left(T^0 - T\right)x_2$	$x_1 = c + \left(T^0 - T\right)f$, $x_2 = c + \left(T - T^{2-k}\right)f$

[5]This condition is a consequence of the fact that the $t_i^{H_{\mathrm{V}}(k)}$ preserves the invariant $H_1 = \sum\limits_{i=1}^{k} x_i$. Such a condition exists for the remaining extended transfer maps associated with the Yang–Baxter maps of the H-list. The latter enable us to write $t_2^{(k)}$ maps as k-point recurrences.

Proposition 5.6. *The following $(k+1)$-point recurrences corresponds to the extended transfer map $t_2^{(k)}$ associated with $H_{\rm I}$, $H_{\rm II}$, $H_{\rm III}^A$, $H_{\rm III}^B$ and $H_{\rm V}$ Yang–Baxter maps respectively. We refer to these $(k+1)$-point recurrences respectively as $rt_2^{H_{\rm I}(k)}$, $rt_2^{H_{\rm I}(k)}$, $rt_2^{H_{\rm III}^A(k)}$, $rt_2^{H_{\rm III}^B(k)}$ and $rt_2^{H_{\rm V}(k)}$*

$$\frac{T^2 f}{T^{2-k}f}=\frac{p_2\big(-1+T^{2-k}p_1\big)+c\big(p_2-T^{2-k}p_1\big)\dfrac{T^1 f}{T^{2-k}f}+c^2(1-p_2)\dfrac{T^1 f}{T^{3-k}f}}{\big(T^{2-k}p_1\big)(p_2-1)+c\big(-p_2+T^{2-k}p_1\big)\dfrac{T^{2-k}f}{T^{3-k}f}+c^2\big(1-T^{2-k}p_1\big)\dfrac{T^1 f}{T^{3-k}f}}, \quad \big(rt_2^{H_{\rm I}(k)}\big)$$

$$\frac{c+\big(T^1-T^2\big)f}{c+\big(T^1-T^{2-k}\big)f}$$
$$=\frac{\big(2c-p_2+\big(T^1-T^{3-k}\big)f\big)T^{2-k}p_1}{-p_2T^{2-k}p_1+c\big(p_2+T^{2-k}p_1\big)+\big(T^{2-k}p_1\big)\big(T^1-T^{2-k}\big)f+p_2\big(T^{2-k}-T^{3-k}\big)f}, \big(rt_2^{H_{\rm II}(k)}\big)$$

$$\frac{c+\big(T^1-T^2\big)f}{c+\big(T^1-T^{2-k}\big)f}=\frac{2c+\big(T^1-T^{3-k}\big)f}{c+\big(T^1-T^{2-k}\big)f+\dfrac{p_2}{T^{2-k}p_1}\big(c+\big(T^{2-k}-T^{3-k}\big)f\big)}, \quad \big(rt_2^{H_{\rm III}^A(k)}\big)$$

$$\frac{T^2 f}{T^{2-k}f}=\frac{T^{3-k}f+c^2\big(T^{2-k}p_1\big)T^1 f}{T^{3-k}f+c^2p_2T^1 f}, \quad \big(rt_2^{H_{\rm III}^B(k)}\big)$$

$$(T^2-T^{2-k})f=\frac{-p_2+T^{2-k}p_1}{2c+\big(T-T^{3-k}f\big)}. \quad \big(rt_2^{H_{\rm V}(k)}\big)$$

For each recurrence presented above we have that the parameters vary as follows: $Tp_2=p_2$, $T^{k-1}p_1=p_1$. So p_2 is constant and p_1 is periodic with period $k-1$.

Note that the recurrences $rt_2^{H_{\rm I}(k)}$ and $rt_2^{H_{\rm III}^B(k)}$ are bilinear. Some members of $rt_2^{H_{\rm I}(k)}$ and $rt_2^{H_{\rm III}^B(k)}$, for specific choices of the parameters c, p_2 and of the function p_1, are expected to exhibit the *Laurent property* [26, 27, 28].

Table 12. Definition of the variables h associated with the recurrences of Proposition 5.6.

recurrence	variable h	a choice for λ
$rt_2^{H_{\rm I}(k)}$	$h:=\lambda\dfrac{Tf}{T^0f}$	$\lambda=\dfrac{1}{c}$
$rt_2^{H_{\rm II}(k)}$	$h:=\lambda+\big(T-T^0\big)f$	$\lambda=\dfrac{2c}{k-2}$
$rt_2^{H_{\rm III}^A(k)}$	$h:=\lambda+\big(T-T^0\big)f$	$\lambda=\dfrac{2c}{k-2}$
$rt_2^{H_{\rm III}^B(k)}$	$h:=\lambda\dfrac{Tf}{T^0f}$	$\lambda=\dfrac{1}{c}$
$rt_2^{H_{\rm V}(k)}$	$h:=\lambda+\big(T-T^0\big)f$	$\lambda=\dfrac{2c}{k-2}$

Corollary 5.7. *The $(k+1)$-point recurrences $rt_2^{H_{\rm I}(k)}$, $rt_2^{H_{\rm II}(k)}$, $rt_2^{H_{\rm III}^A(k)}$, $rt_2^{H_{\rm III}^B(k)}$ and $rt_2^{H_{\rm V}(k)}$, in terms of the corresponding variables h defined in Table 12, get the form of the following k-point recurrences*

$$\prod_{i=3-k}^{1}T^ih=\frac{c^{-k}p_2\big(T^{2-k}p_1-1\big)+\big(p_2-T^{2-k}p_1\big)\displaystyle\prod_{i=2-k}^{0}T^ih+(1-p_2)\prod_{i=3-k}^{0}T^ih}{T^{2-k}p_1-p_2+T^{2-k}p_1(p_2-1)T^{2-k}h+c^k\big(1-T^{2-k}p_1\big)\displaystyle\prod_{i=3-k}^{0}T^ih}, \quad \big(\hat{r}t_2^{H_{\rm I}(k)}\big)$$

$$p_2\left(\frac{ck}{k-2}-Th\right)\left(\frac{ck}{k-2}-T^{2-k}p_1-T^{2-k}h\right)$$
$$=T^{2-k}p_1\left(\frac{ck}{k-2}-\sum_{i=2-k}^{0}T^ih\right)\left(\frac{ck}{k-2}+p_2-\sum_{i=3-k}^{1}T^ih\right), \qquad (\hat{r}t_2^{H_{\rm II}(k)})$$

$$p_2\left(\frac{ck}{k-2}-Th\right)\left(\frac{ck}{k-2}-T^{2-k}h\right)$$
$$=T^{2-k}p_1\left(\frac{ck}{k-2}-\sum_{i=2-k}^{0}T^ih\right)\left(\frac{ck}{k-2}-\sum_{i=3-k}^{1}T^ih\right), \qquad (\hat{r}t_2^{H^A_{\rm III}(k)})$$

$$\prod_{i=2-k}^{1}T^ih=\frac{c^{-k}+T^{2-k}p_1\prod\limits_{i=3-k}^{0}T^ih}{1+c^kp_2\prod\limits_{i=3-k}^{0}T^ih}, \qquad (\hat{r}t_2^{H^B_{\rm III}(k)})$$

$$-\frac{2ck}{k-2}+\sum_{i=2-k}^{1}T^ih=\frac{-p_2+T^{2-k}p_1}{\sum\limits_{i=3-k}^{0}T^ih} \qquad (\hat{r}t_2^{H_{\rm V}(k)})$$

and for each recurrence presented above we have that the parameters vary as follows: $Tp_2=p_2$, $T^{k-1}p_1=p_1$*. So* p_2 *is constant and* p_1 *is periodic with period* $k-1$*.*

Note that the $(k+1)$-point recurrences of Proposition 5.6, as well as the corresponding k-point ones introduced in Corollary 5.7 are non-autonomous. This is due to the fact that p_1 varies periodically ($T^{k-1}p_1=p_1$). The non-autonomous terms that will be introduced by integrating the relation $T^{k-1}p_1=p_1$ are periodic though. Proper de-autonomization for the recurrences $\hat{r}t_2^{H_{\rm V}(k)}$ and $\hat{r}t_2^{H^B_{\rm III}(k)}$ will be introduced in what follows.

5.2.1 The recurrences $\hat{r}t_i^{H_{\rm V}(k)}$ and discrete Painlevé equations

The dressing chain for the KdV equation [71], reads

$$(g_{i+1}+g_i)_t=g_{i+1}^2-g_i^2+p_{i+1}-p_i. \qquad (5.6)$$

The recurrences $\hat{r}t_i^{H_{\rm V}(k)}$, serve as its discretisations. Actually they are exactly the $(k-1)$-roots of the discretisations presented in [1]. So, $\hat{r}t_i^{H_{\rm V}(k)}$ corresponds to Liouville integrable maps.

Since the dressing chain (5.6) leads to Painlevé equations $P_{\rm IV}$ and $P_{\rm V}$ and their higher order analogues [71], the recurrences $\hat{r}t_i^{H_{\rm V}(k)}$ (after proper de-autonomisation) can be considered as their discrete counter-parts and/or the Bäcklund transformations of the higher order $P_{\rm IV}$ and $P_{\rm V}$ Painlevé equations.

A proper de-autonomisation of $\hat{r}t_2^{H_{\rm V}(k)}$ is achieved by breaking the periodicity of the p_1 assuming that $T^{k-1}p_1=p_1+(k-1)a$, where a constant. This de-autonomisation is proper since the resulting non-autonomous discrete system preserves the same Poisson structure[6] as the autonomous one. So we obtain the following hierarchy of discrete Painlevé equations

$$-\frac{2ck}{k-2}+\sum_{i=2-k}^{1}T^ih=\frac{-p_2+T^{2-k}p_1}{\sum\limits_{i=3-k}^{0}T^ih}, \qquad Tp_2=p_2, \qquad T^{k-1}p_1=p_1+(k-1)a. \qquad (5.7)$$

[6]The Poisson structures associated with the dressing chain for the KdV equation were first derived in [71], see also [25].

For $k=3$, (5.7) reads

$$-6c+Th+h+T^{-1}h=\frac{-p_2+T^{-1}p_1}{h},\qquad Tp_2=p_2,\qquad T^2p_1=p_1+2a.$$

So p_2 is constant and $p_1=b_0+b_1(-1)^n+an$, with b_0, b_1, a constants. We can choose $-p_2+b_0=b$ constant, hence we obtain the following discrete Painlevé equation which serves as Bäcklund transformation of P_{IV} [57]

$$-6c+Th+h+T^{-1}h=\frac{b+b_1(-1)^n+an}{h},\qquad n\in\mathbb{Z}. \tag{5.8}$$

For $k=4$, (5.7) reads

$$-4c+T^{-2}h+h+T^{-1}h+h+Th=\frac{-p_2+T^{-2}p_1}{h+T^{-1}h},\qquad Tp_2=p_2,\qquad T^3p_1=p_1+3a.$$

If we define a new variable w as $w:=h+T^{-1}h$, then we obtain the following discrete Painlevé equation which serves as Bäcklund transformation of P_{V}

$$-4c+T^{-1}w+Tw=\frac{-p_2+T^{-2}p_1}{w},\qquad Tp_2=p_2,\qquad T^3p_1=p_1+3a.$$

So for k odd (5.7) serves as Bäcklund transformation for the higher order analogues of P_{IV} and for k even (5.7) serves as Bäcklund transformation for the higher order analogues of P_{V}. Note that in [57], Bäcklund transformation for the higher order analogues of P_{IV} and P_{V} were given in terms of continued fractions. We can recover the form of discrete Painlevé equations introduced in [57] by making use of the alternating terms that appear in (5.7). For example for $k=3$, the term $(1)^n$ that appears in (5.8), suggests the introduction of the variables $y(m):=h(2n)$, $z(m):=h(2n+1)$. Then (5.8) takes to form of the *second discrete Painlevé equation* dP_{II}

$$y+z+T^{-1}z=\frac{b_0+b_1+am}{y},\qquad Ty+y+z=\frac{b_0-b_1+am}{z},\qquad m\in\mathbb{Z}.$$

5.2.2 The recurrences $\hat{r}t_i^{H^B_{\mathrm{III}}(k)}$ and discrete Painlevé equations

As we plan to show in our future work, the recurrences $\hat{r}t_i^{H^B_{\mathrm{III}}(k)}$ serves as Liouville integrable discretisations of the following chain introduced in [6]

$$(g_i+g_{i+1})_t=2(p_i\cosh g_i-p_{i+1}\cosh g_{i+1}).$$

A proper de-autonomisation of $\hat{r}t_2^{H^B_{\mathrm{III}}(k)}$ is achieved by breaking the periodicity of the p_1 in a way that the non-autonomous system preserves the same Poisson structure as the autonomous one. This is achieved by imposing that $T^{k-1}p_1=p_1a^{k-1}$, where a constant. So we obtain the following hierarchy of discrete Painlevé equations

$$\prod_{i=2-k}^{1}T^ih=\frac{c^{-k}+T^{2-k}p_1\displaystyle\prod_{i=3-k}^{0}T^ih}{1+c^kp_2\displaystyle\prod_{i=3-k}^{0}T^ih},\qquad Tp_2=p_2,\qquad T^{k-1}p_1=p_1a^{k-1}. \tag{5.9}$$

For $k=3$, (5.9) reads

$$ThT^{-1}h=\frac{1}{h}\frac{c^{-3}+hT^{-1}p_1}{1+c^3p_2h},\qquad Tp_2=p_2,\qquad T^2p_1=p_1a^2.$$

So p_2 is constant and $p_1 = b_0 a^n + b_1(-a)^n$, with b_0, b_1, a constants. Hence we obtain the $q - P_{\rm I}\big(A_6^{(1)}\big)$ discrete Painlevé equation (see [63]). For $k = 4$, (5.9) reads

$$ThT^0hT^{-1}hT^{-2}h = \frac{c^{-4} + hT^{-1}hT^{-2}p_1}{1 + c^4 p_2 hT^{-1}h}, \qquad Tp_2 = p_2, \qquad T^3 p_1 = p_1 a^3.$$

If we define a new variable w as $w := hT^{-1}h$, then we obtain the $q - P_{\rm II}\big(A_5^{(1)}\big)$ discrete Painlevé equation (see [63])

$$TwT^{-1}w = \frac{c^{-4} + wT^{-2}p_1}{1 + c^4 p_2 w}, \qquad Tp_2 = p_2, \qquad T^3 p_1 = p_1 a^3.$$

The Lax pair associated with the hierarchy (5.9) first appeared in [32].

Remark 5.8. As for the recurrences $\hat{r}t_i^{H^A_{\rm III}(k)}$, $\hat{r}t_i^{H_{\rm II}(k)}$, one could consider $T^{k-1}p_1 = p_1 + (k-1)a$ and for $\hat{r}t_i^{H_{\rm I}(k)}$ $T^{k-1}p_1 = p_1 a^{k-1}$, in order to de-autonomise them. We anticipate that this is a proper de-autonomisation, although we have no proof yet. The finding of the Poisson structures that the latter recurrences we anticipate that preserve, will sort this issue out.

Remark 5.9. As a final remark, we note that the k-point recurrences associated with the extended transfer maps of the Yang–Baxter map $F_{\rm V}$, are exactly the same as the k-point recurrences associated with the extended transfer maps of the Yang–Baxter map $H_{\rm V}$ which (one of them) were presented in Corollary 5.7. Since the $(k-1)$-iteration of the extended transfer maps of any Yang–Baxter map coincides with its transfer maps, we conclude that the dynamics of the transfer maps of the Yang–Baxter maps $F_{\rm V}$ and $H_{\rm V}$, are the same. The same holds true for the transfer maps associated with the Yang–Baxter maps $F_{\rm III}$ and $H^A_{\rm III}$. As for the remaining members of the F and the H lists of Yang–Baxter maps, further investigation is required in order to prove the equivalence of their transfer dynamics.

6 Conclusions

In Section 2 we have presented a family of maps in k variables which preserve 2 rational invariants of a specific form. One could mimic the procedures introduced in [29] to obtain rational maps in k variables which preserve m rational invariants where $m < k$. For example, there are $\binom{2k}{k}$ rational maps $(x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto (X_1, \ldots, X_k, Y_1, \ldots, Y_k)$ which preserve k invariants of the form:

$$H_i = \frac{\alpha_i x_i x_{i+1} + \beta_i x_i + \gamma_i x_{i+1} + \delta_i}{\kappa_i x_i x_{i+1} + \lambda_i x_i + \mu_i x_{i+1} + \nu_i}, \qquad i = 1, 2, \ldots, k, \tag{6.1}$$

where the indices are considered modulo k and α_i, β_i, κ_i, λ_i, etc. are given functions of the variables y_i, y_{i+1}.

If separability of variables on the invariants is imposed, then higher rank analogues of the Yang–Baxter maps of Propositions 3.3, 3.9 and 3.11 are expected. Moreover, solutions of the functional tetrahedron equation [41, 42, 49, 64], or even of higher simplex equations [17, 53, 54] are anticipated. For example if we consider the following, different than (6.1), choice of invariants:

$$H_1 = \sum_{i=1}^{6} x_i, \qquad H_2 = \frac{x_1 x_4 x_6}{x_3}, \qquad H_3 = x_2 x_3 x_4 x_5,$$

then the involutions R_{123}, R_{145}, R_{246}, and R_{356}, preserve H_i, $i=1,2,3$ and satisfy the functional tetrahedron equation

$$R_{123}R_{145}R_{246}R_{356}=R_{356}R_{246}R_{145}R_{123}.$$

They are exactly the Hirota's map [41, 42, 64], i.e., the map $R\colon (u,v,w)\mapsto (U,V,W)$, where

$$U=\frac{uv}{u+w},\qquad V=u+w,\qquad W=\frac{vw}{u+w},$$

acting on (123), (145), (246) and (356) coordinates respectively. For the involution $\phi\colon u\mapsto -u$, it holds $\phi_1\phi_2\phi_3R_{123}=R_{123}\phi_1\phi_2\phi_3$. So ϕ is a symmetry of the Hirota's map R and it can be easily proven that the following entwining relation holds

$$R_{123}\phi_3R_{145}\phi_5R_{246}\phi_6R_{356}=R_{356}R_{246}\phi_6R_{145}\phi_5R_{123}\phi_3.$$

Hence we have obtained a solution of the following entwining functional tetrahedron relation

$$S_{123}S_{145}S_{246}T_{356}=T_{356}S_{246}S_{145}S_{123},$$

where T is the Hirota's map acting on the (356) coordinates and $S\colon (u,v,w)\mapsto (U,V,W)$ a non-periodic map where

$$U=\frac{uv}{u-w},\qquad V=u-w,\qquad W=-\frac{vw}{u-w}.$$

The complete set of entwining relations and maps associated with the Hirota's map as well as with the Hirota–Miwa's map, will be considered elsewhere.

In Section 4, we considered two methods to obtain entwining maps. The first method uses degeneracy arguments and produces entwining maps associated with the H_{I}, H_{II} and H_{III}^A Yang–Baxter maps. The entwining maps of this method belongs to different subclasses than the $[2:2]$ subclass of maps that the H_{I}, H_{II} and H_{III}^A Yang–Baxter maps belongs to so they are not $(\text{Möb})^2$ equivalent to the latter. The outcomes of the second method are non-periodic[7] entwining maps of subclass $[2:2]$ associated with the whole H-list. The fact that the entwining maps which were presented in this Section preserve two invariants in separated variables, enable us to introduce appropriate potentials (as shown in [44, 45, 56]) to obtain integrable lattice equations. Actually we obtain integrable triplets of lattice equations (in some cases even correspondences). Note that integrable triplets of lattice equations were systematically derived in [13] and more recently in [33]. We plan to consider the integrable triplets of lattice equations derived from entwining maps, elsewhere.

In Section 6, we have proved that the transfer maps associated with the H list of Yang–Baxter maps can be considered as the $(k-1)$-iteration of some maps of simpler form. As a consequence of this re-factorisation we have obtained $(k+1)$-point (see Proposition 5.6) and k-point (see Corollary 5.7) alternating recurrences which can be considered as alternating versions of some hierarchies of discrete Painlevé equations. Moreover, the autonomous versions of some of the k-point recurrences presented in Corollary 5.7, can be obtained by *periodic reductions* [58] (cf. [34]) of integrable lattice equations. Here we have obtained alternating k-point recurrences from Yang–Baxter maps without performing periodic reductions. Hence, our results might be compared/extended to the novel and independent frameworks introduced in [8, 10] and [38, 39], where by using symmetry arguments, integrable lattice equations and discrete Painlevé equations of 2nd order were linked.

[7]The non-periodicity assures that these entwining maps are not $(\text{Möb})^2$ equivalent with the corresponding maps of the H-list.

A The $\boldsymbol{F}$-list and the $\boldsymbol{H}$-list of quadrirational Yang–Baxter maps

The Yang–Baxter maps R of the F and the H-list, explicitly read

$$R\colon\ \mathbb{CP}^1\times\mathbb{CP}^1 \ni (u,v)\mapsto (U,V)\in\mathbb{CP}^1\times\mathbb{CP}^1,$$

$$U=\alpha vP,\qquad V=\beta uP,\qquad P=\frac{(1-\beta)u+\beta-\alpha+(\alpha-1)v}{\beta(1-\alpha)u+(\alpha-\beta)uv+\alpha(\beta-1)v},\tag{$F_{\rm I}$}$$

$$U=\alpha vP,\qquad V=\beta uP,\qquad P=\frac{u-v+\beta-\alpha}{\beta u-\alpha v},\tag{$F_{\rm II}$}$$

$$U=\frac{v}{\alpha}P,\qquad V=\frac{u}{\beta}P,\qquad P=\frac{\alpha u-\beta v}{u-v},\tag{$F_{\rm III}$}$$

$$U=vP,\qquad V=uP,\qquad P=1+\frac{\beta-\alpha}{u-v},\tag{$F_{\rm IV}$}$$

$$U=v+P,\qquad V=u+P,\qquad P=\frac{\alpha-\beta}{u-v},\tag{$F_{\rm V}$}$$

$$U=vQ,\qquad V=uQ^{-1},\qquad Q=\frac{(\alpha-1)uv+(\beta-\alpha)u+\alpha(1-\beta)}{(\beta-1)uv+(\alpha-\beta)v+\beta(1-\alpha)},\tag{$H_{\rm I}$}$$

$$U=v+Q,\qquad V=u-Q,\qquad Q=\frac{(\alpha-\beta)uv}{\beta u+\alpha v-\alpha\beta},\tag{$H_{\rm II}$}$$

$$U=\frac{v}{\alpha}Q,\qquad V=\frac{u}{\beta}Q,\qquad Q=\frac{\alpha u+\beta v}{u+v},\tag{$H_{\rm III}^A$}$$

$$U=vQ,\qquad V=uQ^{-1},\qquad Q=\frac{1+\beta uv}{1+\alpha uv},\tag{$H_{\rm III}^B$}$$

$$U=v-Q,\qquad V=u+Q,\qquad Q=\frac{\alpha-\beta}{u+v}.\tag{$H_{\rm V}$}$$

The maps above are depending on 2 complex parameters α, β. The parameter α is associated with the first factor of the cartesian product $\mathbb{CP}^1\times\mathbb{CP}^1$, whereas the parameter β with the second factor.

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