Simple Vectorial Lie Algebras in Characteristic 2 and their Superizations

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Abstract. We overview the classifications of simple finite-dimensional modular Lie algebras. In characteristic 2, their list is wider than that in other characteristics; e.g., it contains desuperizations of modular analogs of complex simple vectorial Lie superalgebras. We consider odd parameters of deformations. For all 15 Weisfeiler gradings of the 5 exceptional families, and one Weisfeiler grading for each of 2 serial simple complex Lie superalgebras (with 2 exceptional subseries), we describe their characteristic-2 analogs – new simple Lie algebras. Descriptions of several of these analogs, and of their desuperizations, are far from obvious. One of the exceptional simple vectorial Lie algebras is a previously unknown deform (the result of a deformation) of the characteristic-2 version of the Lie algebra of divergence-free vector fields; this is a new simple Lie algebra with no analogs in characteristics distinct from 2. In characteristic 2, every simple Lie superalgebra can be obtained from a simple Lie algebra by one of the two methods described in arXiv:1407.1695. Most of the simple Lie superalgebras thus obtained from simple Lie algebras we describe here are new.

Key words: modular vectorial Lie algebra; modular vectorial Lie superalgebra

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To Dmitry Borisovich Fuchs with admiration

Contents

Index

1 Notation and background

To make the text understandable for the uninitiated, we place the most basic facts before the Introduction which we have divided into two parts to make it more readable; for the same reason we divided the background to accommodate both students and experts. Statements proved directly, or by means of *Mathematica*-based *SuperLie* package [26], are called Claims.

We recall the basics and show how to modify familiar formulas in order to pass from $\mathbb{C}$ to fields of positive characteristic, especially characteristic 2. In some formulas given for $p = 2$, we retain notation convenient for comparison with the cases where $p \neq 2$.

1.1 Main points

1. We give an overview of the classification of simple finite-dimensional modular Lie algebras and Lie superalgebras over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$. We update the conjectures for various values of $p > 0$.

2. We use our results on classification of Lie superalgebras of vector fields with polynomial coefficients over $\mathbb{C}$ to describe their characteristic-p versions, especially, their desuperizations, for all 15 Weisfeiler gradings of all 5 exceptional simple vectorial Lie superalgebras, and for several serial ones, also exceptional in a sense.

3. One of the deforms[1] of the divergence-free Lie algebra $\mathfrak{svect}(5; \underline{N})$ which exists only if $p = 2$. It is one of the exceptional simple vectorial Lie algebras – a desuperization of an exceptional simple vectorial Lie superalgebra. *This is the most unexpected result of this paper.*

1.2 Generalities

As is now customary, we denote the elements of $\mathbb{Z}/2$ by $\bar{0}$ and $\bar{1}$, to distinguish them from integers. For us, $\mathbb{N} := \{1, 2, \dots\}$, as it used to be in the past, and still is in some countries; we set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. The parity p of a non-zero element v of a $\mathbb{Z}/2$-graded space V, called a *superspace*, is equal to i if and only if $v \in V_i$. Any $\mathbb{Z}/2$-graded algebra is called a *superalgebra*.

Hereafter $\mathbb{K}$ is an algebraically closed field of characteristic $p > 0$; usually, $p = 2$ and all Lie (super)algebras are finite-dimensional unless otherwise stated. Mostly (exceptions indicated), Π denotes the change of parity functor, i.e., tensoring by $\Pi(\mathbb{Z})$.

The superization of most formulas of algebra is achieved via the following *sign rule*

"If something of parity a is moved past something of parity b, the sign $(-1)^{ab}$ accrues. Formulas defined only on homogeneous elements are extended to arbitrary elements via linearity."

Note that only even homomorphisms are considered as *morphisms of superalgebras*.

Observe that sometimes applying the Sign Rule requires some dexterity. For example, we have to distinguish between two versions both of which turn in the non-super case into one, called skew- or anti-commutativity, which are synonyms only in the *non-super* case; for two elements a and b of a superalgebra we call the following conditions

$$ba = (-1)^{p(b)p(a)} ab \qquad \text{super commutativity,}$$
$$ba = -(-1)^{p(b)p(a)} ab \qquad \text{super anti-commutativity,}$$
$$ba = (-1)^{(p(b)+1)(p(a)+1)} ab \qquad \text{super skew-commutativity,}$$
$$ba = -(-1)^{(p(b)+1)(p(a)+1)} ab \qquad \text{super antiskew-commutativity.}$$

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[1] *Deform* is the result of a deformation, like transform is the result of a transformation.

Examples: the bracket in any Lie superalgebra is *super anti-commutative*; the anti-bracket $\{-,-\}_{\text{B.b.}}$, see (2.5), being anti-commutative relative the parity in the Lie superalgebra is, however, *super antiskew-commutative* relative to the *natural parity of generating functions*.

1.2.1 Conventions and notation often used

In what follows, we assume that every *supercommutative* superalgebra is associative with 1; their morphisms send 1 to 1.

We denote by $\mathfrak{c}$ the center of a given Lie (super)algebra; both $\mathfrak{c}(\mathfrak{g})$ and $\mathfrak{cg} := \mathfrak{c} \oplus \mathfrak{g}$ denote a trivial central extension of $\mathfrak{g}$.

Let $\mathfrak{a} \ltimes \mathfrak{b}$ or $\mathfrak{b} \rtimes \mathfrak{a}$ denote the semi-direct sum of modules (algebras) in which $\mathfrak{a}$ is a submodule (ideal).

Let $\mathfrak{d}(\mathfrak{g}) := \mathfrak{g} \ltimes \mathbb{K}D$, where D is an outer derivation of $\mathfrak{g}$; unless specified otherwise, D is the grading operator of $\mathfrak{g}$. For example, for $\mathfrak{d}(\mathfrak{o}_\Pi(2k))$, we take $D = \text{diag}(I_k, 0_k)$.

Let $\mathfrak{g}^{\cdot k} := [\mathfrak{g}, [\mathfrak{g}, \ldots [\mathfrak{g}, \mathfrak{g}] \ldots]]$, the k-fold bracket.

We denote the functor of raising to the nth symmetric (resp. exterior) power by S^n (resp. E^n, often denoted also by Λ^n); sometimes we denote the exterior (Grassmann) algebra by $\Lambda[\theta]$ or $\Lambda(r)$ in generators $\theta = (\theta_1, \ldots, \theta_r)$ satisfying anti-commutativity relations (and, additionally, $\theta_i^2 = 0$ for all i if $p = 2$).

The symbol id (sometimes id_n, $\text{id}_{a|b}$) denotes not only the identity operator (in the space of dimension n, resp. superspace of superdimension $a|b$), but also the tautological module V over the linear Lie superalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$; sometimes we write $\text{id}_\mathfrak{g} := V$ for clarity.

L_D is the Lie derivative along the vector field D.

1.2.2 Definition of Lie superalgebras for $p \neq 2, 3$

The "naive" definition of Lie superalgebras for $p \neq 2, 3$ is obtained by applying the Sign Rule to anti-commutativity and Jacobi identities. To understand deformations with odd parameter, we need a more sophisticated approach using the functor of points. The multiplication in the Lie superalgebra will be called *super-bracket* or just *bracket*.

1.2.3 Definition of Lie superalgebras for $p = 2$

If $p = 2$, the *antisymmetry* condition for Lie algebra $\mathfrak{g}_{\bar{0}}$ should be replaced by an equivalent for $p \neq 2$, but otherwise stronger, *alternating* or *antisymmetry* condition

$[x, x] = 0$ for any $x \in \mathfrak{g}_{\bar{0}}$.

If $p = 2$, a *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module (made two-sided by symmetry), with a *squaring* $x \mapsto x^2$ and a *bracket* of odd elements, which are defined via a linear map $s \colon S^2(\mathfrak{g}_{\bar{1}}) \longrightarrow \mathfrak{g}_{\bar{0}}$, where S^2 denotes the operator of raising to symmetric square, as follows:

$$x^2 := s(x \otimes x),$$
$$[x, y] := s(x \otimes y + y \otimes x) \text{ for any } x, y \in \mathfrak{g}_{\bar{1}}.$$

The linearity of the $\mathfrak{g}_{\bar{0}}$-valued function s implies that

$(ax)^2 = a^2 x^2$ for any $x \in \mathfrak{g}_{\bar{1}}$ and $a \in \mathbb{K}$, and

$[x, y]$ is a bilinear form on $\mathfrak{g}_{\bar{1}}$ with values in $\mathfrak{g}_{\bar{0}}$.

The *Jacobi identity* involving odd elements takes the form of the following two conditions:

$$[x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_{\bar{1}}, y \in \mathfrak{g}_{\bar{0}}, \quad [x^2, x] = 0 \text{ for any } x \in \mathfrak{g}_{\bar{1}}. \tag{1.1}$$

The (super)algebra satisfying only Jacobi identity, without any symmetry conditions, is called a *Leibniz (super)algebra*.

Over $\mathbb{Z}/2$, the condition (1.1) must (for a reason, see [44]) be replaced with a more general one:

$$[x^2, y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } y \in \mathfrak{g}. \tag{1.2}$$

For any other ground field, this condition is equivalent to condition (1.1).

More generally, for any Lie superalgebra $\mathfrak{g}$, since we want the Lie superalgebra $\mathfrak{der}\,\mathfrak{g}$ of all derivations of $\mathfrak{g}$ to be a Lie superalgebra, we have to add (to the Leibniz rule) the following condition on derivations (it becomes (1.2) for $D = \mathrm{ad}_y$)

$$D(x^2) = [D(x), x] \text{ for odd elements } x \in \mathfrak{g}_{\bar{1}} \text{ and any } D \in \mathfrak{der}\,\mathfrak{g}.$$

By an *ideal* $\mathfrak{i}$ of a Lie superalgebra $\mathfrak{g}$, one always means $\mathfrak{i}$ *homogeneous* with respect to parity, i.e., equal to $\mathfrak{i} \cap \mathfrak{g}_{\bar{0}} \bigoplus \mathfrak{i} \cap \mathfrak{g}_{\bar{1}}$; for $p = 2$, the ideal should be closed with respect to squaring.

Recall that a given Lie (super)algebra $\mathfrak{g}$ is said to be *simple* if $\dim \mathfrak{g} > 1$ and $\mathfrak{g}$ has no proper ideals; $\mathfrak{g}$ is *semisimple* if its radical is zero; $\mathfrak{g}$ is *almost simple* if it can be sandwiched (non-strictly) between a simple Lie superalgebra $\mathfrak{s}$ and the Lie superalgebra $\mathfrak{der}(\mathfrak{s})$ of derivations of $\mathfrak{s}$, i.e., $\mathfrak{s} \subset \mathfrak{g} \subset \mathfrak{der}(\mathfrak{s})$.

The definition of the *derived of the Lie superalgebra* $\mathfrak{g}$ changes when $p = 2$: let $\mathfrak{g}^{(0)} := \mathfrak{g}$ and for any $i \geq 0$, set

$$\mathfrak{g}^{(i+1)} = (\mathfrak{g}^{(i)})^{(1)} := \begin{cases} [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] & \text{if } p \neq 2, \\ [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\left\{g^2 \mid g \in \mathfrak{g}_{\bar{1}}^{(i)}\right\} & \text{if } p = 2. \end{cases}$$

An even linear mapping $r \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is said to be a *representation of the Lie superalgebra* $\mathfrak{g}$, and V is said to be a $\mathfrak{g}$-*module* if

$$r([x, y]) = [r(x), r(y)] \quad \text{for any } x, y \in \mathfrak{g},$$
$$r(x^2) = (r(x))^2 \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}.$$

1.2.4 Definition of Lie superalgebras for $p = 3$

Since we are giving a review of the context for any $p > 0$, we have to note a peculiarity of $p = 3$, where the *Jacobi identity* for *Lie superalgebras* entails, additionally, that

$$[x, [x, x]] = 0 \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \tag{1.3}$$

The super anti-commutative algebra satisfying the Jacobi identity, but not (1.3) is called a *pre-Lie superalgebra*. For interesting examples of pre-Lie superalgebras, see [2].

1.3 Basics on the functor of points

In this subsection, we follow [59]; we advise the reader interested in subtleties that we, like most authors, do not dwell on, to read the Appendices to [60].

For a fixed object M and any object X of a category C, the association $X \longmapsto \mathrm{Hom}_{\mathsf{C}}(X, M)$ defines a functor $F \colon \mathsf{C} \rightsquigarrow \mathsf{Sets}$. The idea is

(A) To consider $\mathrm{Hom}_{\mathsf{C}}(X, M)$ as the set of points of M, which is indeed the case for any set M if X is a point of M, and $\mathsf{C} = \mathsf{Sets}$;

(B) Considering objects of the category C of sets endowed with a structure (of a group, algebra, module over a fixed algebra, topological space, etc.), and the morphisms in C being the

maps of these sets preserving (exactly, or up to an equivalence, see [59, Section 1.16.3]) a certain structure (that of a group, or of an associative algebra, or of Lie algebra, etc.), as a model, we'd like to imitate these sets-with-a-structure by objects of another category.

For example, a *Lie supergroup* is any group in the category of supermanifolds, see [59].

Likewise, a *Lie superalgebra* is any Lie algebra in the category of linear supervarieties, see [44]. There we reformulate the naive definition of Lie superalgebras, which are $\mathbb{Z}/2$-graded linear spaces with multiplication satisfying certain identities, in terms of supervarieties.

1.3.1 PBW-theorem for Lie superalgebras

In [23], an interesting description of conditions when the Poincaré–Birkhoff–Witt theorem for Lie superalgebras holds (or not) is offered for $p > 0$. Note, although we will not use this in this paper, that for Lie superalgebras understood "naively", the PBW theorem holds.

1.3.2 Deformations of the brackets

Let C be a supercommutative superalgebra, let $\operatorname{Spec} C$ be the affine super scheme.

Recall, see [62], where the non-super case is considered, that a *deformation* of a Lie superalgebra $\mathfrak{g}$ over $\operatorname{Spec} C$, is a Lie algebra $\mathfrak{G}$ such that $\mathfrak{G} \simeq \mathfrak{g} \otimes C$, as superspaces. The deformation is *trivial* if $\mathfrak{G} \simeq \mathfrak{g} \otimes C$, as Lie superalgebras, not just as superspaces, and *non-trivial* otherwise.

Generally, the *deforms* of a Lie superalgebra $\mathfrak{g}$ over $\mathbb{K}$ are Lie superalgebras $\mathfrak{G} \otimes_I \mathbb{K}$, where I is any closed point in $\operatorname{Spec} C$.

In particular, consider a deformation with an odd parameter τ. This is a Lie superalgebra $\mathfrak{G}$ isomorphic to $\mathfrak{g} \otimes \mathbb{K}[\tau]$ as a *super space*; if, moreover, $\mathfrak{G} \simeq \mathfrak{g} \otimes \mathbb{K}[\tau]$ as a *Lie superalgebra*, i.e.,

$$[a \otimes f, b \otimes g] = (-1)^{p(f)p(b)}[a,b] \otimes fg \quad \text{for all } a, b \in \mathfrak{g} \text{ and } f, g \in \mathbb{K}[\tau],$$

then the deformation is considered *trivial* (and *non-trivial* otherwise). Observe that $\mathfrak{g} \otimes \tau$ is not an ideal of $\mathfrak{G}$: the ideal should be a free $\mathbb{K}[\tau]$-module.

Comment. Consider formal deformations over $\mathbb{K}[[\tau]]$. If the formal series in τ converges in a domain D, we can evaluate τ for any $\tau \in D$ and – if $\dim \mathfrak{g} < \infty$ – consider copies $\mathfrak{g}_\tau$, where $\tau \in D$, of the same dimension as $\mathfrak{g}$. If the parameter is formal or odd, such an evaluation is possible only trivially: $\tau \mapsto 0$.

1.4 Linear (matrix) Lie superalgebras

Certain basics of linear superalgebra are not well-known, or given wrongly in the literature; no harm in recalling about them.

The *general linear* Lie superalgebra of all supermatrices of size Size corresponding to linear operators in the superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over the ground field $\mathbb{K}$ is denoted by $\mathfrak{gl}(\text{Size})$, where $\text{Size} = (p_1, \ldots, p_{|\text{Size}|})$ is an ordered collection of parities of the basis vectors of V for which we take only vectors *homogeneous with respect to parity* and $|\text{Size}| := \dim V$; usually, for the *standard* (simplest from a certain point of view) format, $\mathfrak{gl}(\bar{0}, \ldots, \bar{0}, \bar{1}, \ldots, \bar{1})$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar{0}} | \dim V_{\bar{1}})$. Any supermatrix from $\mathfrak{gl}(\text{Size})$ can be uniquely expressed as the sum of its even and odd parts; in the standard format this is the following block expression; on non-zero summands the parity is defined:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \qquad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar{0}, \qquad p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar{1}.$$

The *supertrace* is the map $\mathfrak{gl}(\text{Size}) \longrightarrow \mathbb{K}$, $(X_{ij}) \longmapsto \sum (-1)^{p_i(p(X)+1)} X_{ii}$. Thus, in the standard format, $\text{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{tr}\, A - \text{tr}\, D$. Observe that for the Lie superalgebra $\mathfrak{gl}_{\mathcal{C}}(p|q)$ over a super-commutative superalgebra $\mathcal{C}$, i.e., for supermatrices with elements in $\mathcal{C}$, we have

$$\text{str}\, X = \text{tr}\, A - (-1)^{p(X)} \text{tr}\, D \quad \text{for any } X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$\text{where } p(X) = p(A_{ij}) = p(D_{kl}) = p(B_{il}) + \bar{1} = p(C_{kj}) + \bar{1}.$$

So if $\mathcal{C}_{\bar{1}} \neq 0$, then on odd supermatrices the supertrace coincides with the trace.

Since $\text{str}\, [x, y] = 0$, the subsuperspace of supertraceless matrices constitutes a Lie subsuperalgebra of $\mathfrak{gl}(\text{Size})$ called *special linear* and denoted $\mathfrak{sl}(\text{Size})$.

1.4.1 The queer version of $\mathfrak{gl}(n)$

There are at least two super versions of $\mathfrak{gl}(n)$, not one; for reasons, see [52, Chapters 1 and 7]. The other version – $\mathfrak{q}(n)$ – is called the *queer* Lie superalgebra and is defined as the one that preserves – if $p \neq 2$ – the complex structure given by an *odd* operator J, i.e., $\mathfrak{q}(n)$ is the centralizer $C(J)$ of J:

$$\mathfrak{q}(n) = C(J) = \{X \in \mathfrak{gl}(n|n) \,|\, [X, J] = 0\}, \quad \text{where } J^2 = -\,\text{id}\,.$$

It is clear that by a change of basis we can reduce J to the form (shape) J_{2n} in the standard format, and then $\mathfrak{q}(n)$ takes the form

$$\mathfrak{q}(n) = \left\{(A, B) := \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \text{ where } A, B \in \mathfrak{gl}(n) \text{ and } J_{2n} := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}\right\}.$$

(Over any algebraically closed field $\mathbb{K}$, instead of J we can take any odd operator K such that $K^2 = a\,\text{id}_{n|n}$, where $a \in \mathbb{K}^{\times}$; and the Lie superalgebras $C(K)$ are isomorphic for distinct K; if $p = 2$, it is natural to select $K^2 = \text{id}$, and hence $\Pi_{n|n} := \Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ can serve as the normal shape of K.)

On $\mathfrak{q}(n)$, the supertrace vanishes, but the *queertrace* is defined: $\text{qtr}: (A, B) \longmapsto \text{tr}\, B$. Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices; set $\mathfrak{psq}(n) := \mathfrak{sq}(n)/\mathbb{K}1_{2n}$.

If $p = 2$, on $\mathfrak{q}(n)$ there is another (even) trace htr: $(A, B) \mapsto \text{tr}(A)$, see [44].

1.4.2 The supermatrix of the dual operator (after [52])

Let $F \in \text{End}_{\mathcal{C}}(V)$. The passage from the matrix of F in a basis of V to the matrix of F^* in the dual basis of V^* is performed by means of the *supertransposition* which in the standard format is of the shape

$$X = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \mapsto X^{\text{st}} := \begin{cases} \begin{pmatrix} R^{\text{t}} & T^{\text{t}} \\ -S^{\text{t}} & U^{\text{t}} \end{pmatrix} & \text{if } p(X) = \bar{0}, \\ \begin{pmatrix} R^{\text{t}} & -T^{\text{t}} \\ S^{\text{t}} & U^{\text{t}} \end{pmatrix} & \text{if } p(X) = \bar{1}, \end{cases}$$

where M^{t} is the transposed of the matrix M.

1.4.3   Lie superalgebras preserving bilinear forms

The supermatrices $X \in \mathfrak{gl}(\text{Size})$ such that

$$X^{\text{st}}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\text{Size})$$

constitute the Lie superalgebra $\mathfrak{aut}(\mathcal{B})$ that preserves the bilinear form $\mathcal{B}$ on V whose Gram matrix $B = (B_{ij})$ is given by the formula

$$B_{ij} = (-1)^{p(B)p(v_i)}\mathcal{B}(v_i, v_j) \quad \text{for the basis vectors } v_i \in V. \tag{1.4}$$

In order to identify a bilinear form $B(V, W)$ with an operator, an element of $\text{Hom}(V, W^*)$, the matrix B of the bilinear form $\mathcal{B}$ is defined in [52, Chapter 1] by equation (1.4), not by *seemingly natural* – but inappropriate for such an identification – formula

$$B_{ij} = \mathcal{B}(v_i, v_j) \quad \text{for the basis vectors } v_i \in V. \tag{1.5}$$

Moreover, the would-be definition (1.5) contradicts the manifest symmetry of the odd bilinear form qtr on $\mathfrak{q}(n)$. To correctly define symmetry of bilinear forms, consider the *upsetting* of bilinear forms $u \colon \text{Bil}(V, W) \longrightarrow \text{Bil}(W, V)$, see [52, Chapter 1], given by the formula

$$u(\mathcal{B})(w, v) = (-1)^{p(v)p(w)}\mathcal{B}(v, w) \quad \text{for any } v \in V \text{ and } w \in W.$$

If $V = W$, we say that $\mathcal{B}$ is *symmetric* if

$$u(B) = B, \quad \text{where } u(B) = \begin{pmatrix} R^{\text{t}} & (-1)^{p(B)}T^{\text{t}} \\ (-1)^{p(B)}S^{\text{t}} & -U^{\text{t}} \end{pmatrix} \text{ for } B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}.$$

Similarly, $\mathcal{B}$ is *anti-symmetric* if $u(B) = -B$.

1.4.4   Notational convention

By abuse of notation we will often denote the bilinear form $\mathcal{B}$ by its Gram matrix B in a normal shape.

1.5   Analogs of polynomials for $p > 0$

Let $\mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_a]$ denote the supercommutative superalgebra of polynomials in indeterminates x in their *standard order*; i.e., let the first m indeterminates be even and the other n be odd $(m + n = a)$. Among the bases of $\mathbb{C}[x]$ in which the structure constants are integers, the two bases are usually considered: the monomial basis and the basis of *divided powers* constructed as follows.

For any multi-index $\underline{r} = (r_1, \dots, r_a)$, where $r_1, \dots, r_m \in \mathbb{Z}_+$ and $r_{m+1}, \dots, r_a \in \{0, 1\}$, we set

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{1 \leq i \leq a} u_i^{(r_i)}. \tag{1.6}$$

Clearly, we have

$$u^{(\underline{r})} \cdot u^{(\underline{s})} = \left(\prod_{m+1 \leq i \leq a} \min(1, 2 - r_i - s_i) \cdot (-1)^{\sum\limits_{m < i < j \leq a} r_j s_i} \right) \cdot \binom{\underline{r} + \underline{s}}{\underline{r}} u^{(\underline{r} + \underline{s})}, \tag{1.7}$$

$$\text{where} \quad \binom{\underline{r} + \underline{s}}{\underline{r}} := \prod_{1 \leq i \leq m} \binom{r_i + s_i}{r_i}.$$

These $u_i^{(r_i)}$ form an "integer basis" (i.e., a basis in which all structure constants with respect to the product (1.7) are integers) of $\mathbb{C}[x]$.

1.5.1 Notational convention

In what follows, for clarity, we will write exponents of divided powers in parentheses.

Over any field $\mathbb{K}$ of characteristic $p > 0$, we consider the supercommutative superalgebra (now we do not have any elements x, only the $u_i^{(r_i)}$)

$$\mathcal{O}(m; \underline{N}|n) := \mathbb{K}[u; \underline{N}] := \mathrm{Span}_{\mathbb{K}} \left(u^{(\underline{r})} \,|\, r_i \begin{cases} < p^{N_i} & \text{for } i \leq m \\ = 0 \text{ or } 1 & \text{for } i > m \end{cases} \right),$$

with multiplication given by formula (1.7) where $\underline{N} = (N_1, \ldots, N_m)$ is the *shearing vector* with $N_i \in \mathbb{Z}_+ \cup \infty$ (we assume that $p^\infty = \infty$). Important particular cases of shearing vectors:

$$\mathbb{1} := (1, \ldots, 1) \quad \text{and} \quad \underline{N}_\infty := (\infty, \ldots, \infty); \quad \text{we set } \widehat{\mathcal{O}}(m) := \mathcal{O}(m; \underline{N}_\infty). \tag{1.8}$$

The algebra

$$\mathcal{F} := \mathcal{O}(m; \underline{N}|n) = \mathbb{K}[u; \underline{N}] \quad \text{and its completion } \widehat{\mathcal{O}}(m; \underline{N}_\infty|n) \tag{1.9}$$

are called the *algebras of divided powers*. We will sometimes need completion $\widehat{\mathcal{O}}(\widehat{\underline{N}}_i)$ with respect to only one indeterminate, where $(\widehat{\underline{N}}_i)_j = \underline{N}_j$ except for $(\widehat{\underline{N}}_i)_i = \infty$.

Clearly, $\mathcal{O}(a; \mathbb{1}) = \mathbb{K}[u; \mathbb{1}]$ is the algebra of truncated polynomials. Only $\mathbb{K}[u; \mathbb{1}]$ is indeed generated by the declared indeterminates whereas the list of generators of $\mathbb{K}[u; \underline{N}]$ consists of $u_i^{(p^{k_i})}$ for all i and all k_i such that $0 \leq k_i < N_i$ if u_i is even.

1.6 The (generalized) Cartan prolongation

Let $\mathfrak{g}_- = \bigoplus_{-d \leq i \leq -1} \mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie (super)algebra and $\mathfrak{g}_0$ a Lie sub(super)algebra of the Lie (super)algebra $\mathfrak{der}_0(\mathfrak{g}_-)$ of degree 0 derivations of $\mathfrak{g}_-$. Recall that the graded Lie superalgebra $\mathfrak{b} = \oplus_{p \geqslant -d} \mathfrak{b}_p$ is said to be *transitive* if for all $p \geq 0$ we have

$$\{x \in \mathfrak{b}_p \,|\, [x, \mathfrak{b}_-] = 0\} = 0, \quad \text{where } \mathfrak{b}_- := \oplus_{i<0} \mathfrak{b}_i.$$

The maximal transitive $\mathbb{Z}$-graded Lie (super)algebra whose non-positive part is $\mathfrak{g}_- \oplus \mathfrak{g}_0$ is called the (generalized) *Cartan prolong* of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ and is denoted by $(\mathfrak{g}_-, \mathfrak{g}_0)_*$.

If $p = 0$, we can realize $\mathfrak{g}_-$ by elements of negative degree of $\mathfrak{vect}(n|m; \vec{r})$ and $\mathfrak{g}_0$ by elements of 0th degree of $\mathfrak{vect}(n|m; \vec{r})$ in a non-standard (see Section 2.4.1) grading of $\mathfrak{vect}(n|m)$, where $n|m = \mathrm{sdim}\mathfrak{g}_-$. Then the Cartan prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_* := \bigoplus_{k \geq -d} \mathfrak{g}_k$ of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ is obtained for any $k > 0$ by

$$\mathfrak{g}_k := \{D \in \mathfrak{vect}(n|m; \vec{r})_k \,|\, [D, \mathfrak{g}_i] \subset \mathfrak{g}_{k+i} \text{ for any } i < 0\}.$$

The above-described procedure is called *generalized* prolongation because the initial Cartan prolongation was defined for $d = 1$ only.

1.6.1 Partial Cartan prolongation involving positive components

Let $\mathfrak{h}_1 \subset \mathfrak{g}_1$ be a proper $\mathfrak{g}_0$-submodule such that $[\mathfrak{g}_{-1}, \mathfrak{h}_1] = \mathfrak{g}_0$. If such $\mathfrak{h}_1$ exists (usually, $[\mathfrak{g}_{-1}, \mathfrak{h}_1] \subset \mathfrak{g}_0$), define the 2nd prolongation of $(\bigoplus_{i \leq 0} \mathfrak{g}_i) \oplus \mathfrak{h}_1$ to be

$$\mathfrak{h}_2 := \{D \in \mathfrak{g}_2 \,|\, [D, \mathfrak{g}_{-1}] \subset \mathfrak{h}_1\}.$$

The terms $\mathfrak{h}_i$, where $i > 2$, are similarly defined. Set $\mathfrak{h}_i := \mathfrak{g}_i$ for $i \leq 0$ and call $\mathfrak{h}_* := \oplus\mathfrak{h}_i$ the *partial Cartan prolong involving positive components*.

Examples. The Lie superalgebra $\mathfrak{vect}(1|n; n)$ is a subalgebra of $\mathfrak{k}(1|2n; n)$. The former is obtained as the Cartan prolong of the same nonpositive part as $\mathfrak{k}(1|2n; n)$ and a submodule of $\mathfrak{k}(1|2n; n)_1$. The simple exceptional superalgebra $\mathfrak{kas}$ discovered in [64, 65] is another example.

1.7 Vectorial Lie algebras and algebras of divided powers

The Cartan prolong of $(\mathfrak{g}_-, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ acts faithfully on $\mathfrak{g}_-$ and $\operatorname{sdim}\mathfrak{g}_- = m|n$, can be embedded into the superalgebra of polynomial vector fields of m even and n odd indeterminates, i.e., into $\mathfrak{der}\,\mathbb{C}[x_1, \ldots, x_a]$ (where $a = m + n$, the first m indeterminates are even, and the rest are odd), see [66].

Over a field $\mathbb{K}$ of characteristic $p > 0$, if one tries to follow the recipe of Section 1.6 naively and use derivations of usual polynomials, instead of divided powers, it would not work. For example, let us consider the prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, where $\mathfrak{g}_- = \mathfrak{g}_{-1}$, $\operatorname{sdim}\mathfrak{g}_{-1} = \operatorname{sdim}\mathfrak{g}_0 = 1|0$, and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is non-trivial. It has the form $\bigoplus\limits_{i=-1}^{\infty} \mathfrak{g}_i$ such that $\operatorname{sdim}\mathfrak{g}_i = 1|0$ and $\mathfrak{g}_i = [\mathfrak{g}_{-1}, \mathfrak{g}_{i+1}]$ for all $i \geq -1$.

The corresponding prolong over $\mathbb{C}$ would be embedded isomorphically into $\mathfrak{der}\,\mathbb{C}[x]$ so that $\mathfrak{g}_i$ would be mapped into $\operatorname{Span}\left(x^{i+1}\partial_x\right)$. Over $\mathbb{K}$, the construction of embedding would fail for $\mathfrak{g}_{p-1}$, because $[\partial_x, x^p\partial_x] = 0$, and there is no element X such that $[\partial_x, X] = x^{p-1}\partial_x$.

However, over $\mathbb{K}$, Cartan prolongs can be embedded into the superalgebra of derivations of the algebra of divided powers. Let us first say a bit about these derivations.

Over $\mathbb{C}$, consider the action of derivation ∂_{x_i} of $\mathbb{C}[x_1, \ldots, x_a]$ in the basis of divided powers. It is given by (recall the definition (1.6) of $u^{(r)}$)

$$
\partial_{x_i} u^{(r)} = \begin{cases} 0 & \text{if } r_i = 0, \\ (-1)^{\max(0, i-m-1)} u^{((r_1, \ldots, r_{i-1}, r_i-1, r_{i+1}, \ldots, r_a))} & \text{otherwise.} \end{cases} \tag{1.10}
$$

Since all the coefficients are integer, the map given by this formula is a derivation of $\mathbb{K}[m; \underline{N}|n]$. We will denote this map $\partial_i := \partial_{x_i}$ and call the maps $\partial_1, \ldots, \partial_a$ *distinguished* partial derivatives.

The *general Lie algebra of vector fields* consists of the following derivations expressed in terms of *distinguished* partial derivatives

$$
\mathfrak{vect}(m; \underline{N}|n) = \left\{ \sum_{1 \leq i \leq a} f_i \partial_i \mid f_i \in \mathcal{O}(m; \underline{N}|n) \right\}.
$$

Note that if $\underline{N} \neq \mathbb{1}$, then $\mathfrak{vect}(m; \underline{N}|n)$ is not the whole $\mathfrak{der}\,\mathbb{K}[m; \underline{N}|n]$. Maps $\partial_i^{p^k}$, where $1 \leq i \leq m$ and $1 \leq k < N_i$, are also derivations of $\mathbb{K}[m; \underline{N}|n]$, and a general derivation of $\mathbb{K}[m; \underline{N}|n]$ has the form[2]

$$
\sum_{1 \leq i \leq m} \sum_{0 \leq k \leq N_i - 1} f_{i,k} \partial_i^{p^k} + \sum_{m+1 \leq i \leq a} f_i \partial_i, \quad \text{where } f_{i,k}, f_i \in \mathbb{K}[m; \underline{N}|n].
$$

The Lie superalgebra $\mathfrak{vect}(m; \underline{N}|n)$ and its subalgebras, are called *vectorial* Lie superalgebras (cf. with *matrix* or *linear* Lie superalgebras). Cartan prolongs can be embedded into $\mathfrak{vect}(m; \underline{N}|n)$; in particular, the above Cartan prolong would be isomorphic to $\mathfrak{vect}(1; \underline{N}_\infty|0)$, with $\mathfrak{g}_i$ corresponding to $\operatorname{Span}\left(u_1^{(i+1)}\partial_1\right)$.

1.7.1 Notation, again

Hereafter, the symbol $\mathfrak{g}(a|b)$ or $\mathfrak{g}(a; \underline{N}|b)$ will designate the vectorial Lie superalgebra with given name $\mathfrak{g}$ realized by vector fields on the linear supermanifold $\mathcal{K}^{a|b}$ (the one corresponding to the superspace $\mathbb{K}^{a|b}$, see [44]), and endowed with a W-grading, see Table (2.18). The *standard*

---

[2]It is easy to see that $\mathbb{K}[m; \underline{N}|n]$ is isomorphic to $\mathbb{K}[\sum N_i; \mathbb{1}|n]$, so their algebras of derivations are isomorphic as well, so it is not surprising that a general element of $\mathfrak{der}\,\mathbb{K}[m; \underline{N}|n]$ has $\sum N_i + n$ functional parameters.

grading is taken as a point of reference for regradings governed by the vector $\vec{r}$ of degrees, which often can be described by one number r that usually (for details, see [56, 65]) is equal to the number of odd indeterminates of degree 0. The regraded Lie superalgebra is denoted by $\mathfrak{g}(a|b;r)$. In the standard grading the parameter r is usually omitted, see Table (2.18) and tables in Section 25.4.

The module $\mathcal{F}$ of "functions" over $\mathfrak{vect}(m;\underline{N}|n)$ and its subalgebras (usually with the same negative part) is an analog of the *tautological* module V over $\mathfrak{gl}(V)$ and its subalgebras.

1.7.2 Names

The Lie algebra $\mathfrak{vect}(1;\underline{N})$ is called a *Zassenhaus algebra*. For $p = 2$ it is not simple. Observe that $\mathfrak{vect}(1;\underline{N}) \simeq \mathfrak{k}(1;\underline{N})$ (indeed, $f\partial_x \longleftrightarrow K_f$, see definition (2.2); clearly, ∂_x is the distinguished derivative with respect to the only indeterminate). The simple derived algebra $\mathfrak{vect}^{(1)}(1;\underline{N}) \simeq \mathfrak{k}^{(1)}(1;\underline{N})$ is also called a *Zassenhaus algebra* causing confusion, while $\mathfrak{vect}(1;\underline{1})$ is lately called (even for $p = 0$) the *Witt algebra* in honor of Witt who was the first to study one of its modular incarnations, see Introduction to the first volume of [73].

In the old literature, $\mathfrak{vect}(m;\underline{N})$, like its version for $p = 0$, was called *the general Lie algebra of Cartan type*; lately, it is called the *Jacobson–Witt algebra*, whereas the name *Witt algebra* is reserved for the particular case $\mathfrak{vect}(1;\underline{1})$ for $p > 2$.

1.8 Traces and divergencies on vectorial Lie superalgebras

On any Lie (super)algebra $\mathfrak{g}$ over a supercommutative superalgebra $\mathcal{C}$, e.g., over a field $\mathcal{C} = \mathbb{K}$, a *trace* is any linear mapping $\mathrm{tr}\colon \mathfrak{g} \longrightarrow \mathcal{C}$ such that

$$\mathrm{tr}\left(\mathfrak{g}^{(1)}\right) = 0. \tag{1.11}$$

Now let $\mathfrak{g}$ be a $\mathbb{Z}$-graded vectorial Lie (super)algebra with $\mathfrak{g}_- := \bigoplus_{i<0} \mathfrak{g}_i$ generated by $\mathfrak{g}_{-1}$, and let tr be a trace on $\mathfrak{g}_0$. Recall that any $\mathbb{Z}$-grading of a given vectorial Lie (super)algebra is given by degrees of the indeterminates, so the space of functions $\mathcal{F}$, see equation (1.9), is also $\mathbb{Z}$-graded.

The *divergence* $\mathrm{div}\colon \mathfrak{g} \longrightarrow \mathcal{F}$ is a degree-preserving $\mathrm{ad}_{\mathfrak{g}_{-1}}$-invariant extension of the trace to the Cartan prolong; this extension should satisfy the following conditions, so $\mathrm{div} \in Z^1(\mathfrak{g};\mathcal{F})$, i.e., is a cocycle:

$$X_i(\mathrm{div}\, D) = \mathrm{div}([X_i, D]) \ \text{ for all elements } X_i \text{ that span } \mathfrak{g}_{-1},$$
$$\mathrm{div}\,|_{\mathfrak{g}_0} = \mathrm{tr},$$
$$\mathrm{div}\,|_{\mathfrak{g}_-} = 0.$$

We denote by $\mathrm{Vol}(u;\underline{N})$ or simply $\mathrm{Vol}_u := \mathcal{F}^*$ the $\mathfrak{vect}(m;\underline{N}|n)$-module of *volume forms* dual to $\mathcal{F}$ over $\mathcal{F}$. As an $\mathcal{F}$-module, Vol_u is generated by the *volume element* $\mathrm{vol}_u = 1^*$ with fixed indeterminates ("coordinates") u which we often do not indicate. On the rank-1 $\mathcal{F}$-module of *weighted* λ-densities $\mathrm{Vol}^\lambda(m;\underline{N}|n)$ with generator vol_u^λ over $\mathcal{F}$, the $\mathfrak{vect}(m;\underline{N}|n)$-action is given for any $f \in \mathcal{F}$ and $D \in \mathfrak{vect}(m;\underline{N}|n)$ by the *Lie derivative*

$$L_D\left(f\,\mathrm{vol}_u^\lambda\right) = \left(D(f) + (-1)^{p(f)p(D)}\lambda f\,\mathrm{div}(D)\right)\mathrm{vol}_u^\lambda. \tag{1.12}$$

The *special* Lie algebra $\mathfrak{sg} := \mathrm{Ker}\,\mathrm{div}$ of *divergence-free* elements of $\mathfrak{g}$ is the Cartan prolong of $(\mathfrak{g}_-, \mathrm{Ker}\,\mathrm{tr}\,|_{\mathfrak{g}_0})$. For example, $\mathfrak{svect}(m;\underline{N}|n) = (\mathrm{id}_{\mathfrak{sl}(m|n)}, \mathfrak{sl}(m|n))_{*,\underline{N}}$.

The $\mathfrak{vect}(0|n)$-module $\mathrm{Vol}(0|n)$ contains a submodule $\mathrm{Vol}_0(0|n)$ of codimension 1:

$$\mathrm{Vol}_0(0|n) := \left\{ f\,\mathrm{vol} \mid \int f\,\mathrm{vol} = 0, \text{ where } f \in \mathcal{F} = \Lambda(n) \right\}, \tag{1.13}$$

where the *Berezin integral* $\int f \, \mathrm{vol}_\theta$ is equal to the coefficient of the monomial of the highest in θs degree.

Over $\mathfrak{svect}(0|n)$, we often identify $\mathrm{Vol}_0(0|n)$ with a submodule of $\mathcal{F}$ and omit $(0|n)$; set

$$T_0^0(0|n) := \mathrm{Vol}_0(0|n)/\mathbb{K} \cdot \mathrm{vol}\,.$$

1.8.1   Examples of several divergences

On $\mathfrak{vect}(m; \underline{N}|n)$, the explicit expression of the *standard* divergence is as follows

$$\mathrm{div}\colon \ \sum f_i \partial_{x_i} \longrightarrow \sum (-1)^{p(x_i)p(f_i)} \partial_{x_i} f_i. \tag{1.14}$$

The supertrace restricted from $\mathfrak{gl}$ vanishes on $\mathfrak{q}$, but there is an "indigenous" queer trace on $\mathfrak{q}$; analogously, the standard divergence (1.14) vanishes on certain Lie subsuperalgebras of $\mathfrak{vect}(m; \underline{N}|n)$ on which there might be defined an "indigenous" divergence. This happens, e.g., with $\mathfrak{k}(2n+1|2n+2)$ and $\mathfrak{m}(n)$ as will be shown later on.

If there are several traces on $\mathfrak{g}_0$, and hence divergences on $\mathfrak{g} = (\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}}$, there are several types of special subalgebras, and we need an individual name for each.

If $\mathfrak{g}$ is a Lie *super*algebra, then the linear functional tr satisfying condition (1.11) is often called, for emphasis, *super*trace and denoted by str. If we were consistent, we should, accordingly, use the term *super*divergence but instead we drop the preface "super" in both cases.

1.9   Critical coordinates and unconstrained shearing vectors

The coordinate of the shearing vector $\underline{N}$ corresponding to an even indeterminate of the $\mathbb{Z}$-graded vectorial Lie (super)algebra $\mathfrak{g}$ is said to be *critical* if it cannot take an arbitrarily big value.

The shearing vector without any imposed restrictions on its coordinates is said to be *unconstrained*; we denote it by $\underline{N}^u$. Let

$$\dim \underline{N} \text{ be the number of coordinates of } \underline{N},$$
$$\mathrm{Par}\, \underline{N}^u := \dim \underline{N}^u - \mathrm{card}(\{\text{critical coordinates}\})$$
$$\text{be the number of parameters } \underline{N}^u \text{ depends on.}$$

We established the (non)critical coordinates of the shearing vectors of the $\mathbb{Z}$-graded vectorial Lie (super)algebra $\mathfrak{g}$ with a computer's aid by explicitly computing the bases of the first several terms $\mathfrak{g}_i$ for $i \geq 0$ without imposing any constraints on $\underline{N}$.

Conjecture 1.1. *If the value of the coordinate $\underline{N}_i$ (of the $\mathbb{Z}$-graded vectorial Lie (super)algebra $\mathfrak{g}$) can be > 1, then it can take any value.*

2   Background continued. Subtleties

2.1   The serial simple vectorial Lie superalgebras over $\mathbb{C}$ as prolongs

When we only need the vectorial Lie superalgebras considered as abstract, not realized by vector fields, we may consider their simplest filtrations with the smallest codimension of their maximal subalgebras, and gradings associated with such filtrations.

2.1.1   Convention: on central element $z \in \mathfrak{g}_0$

We chose the central element $z \in \mathfrak{g}_0$ so that it acts on $\mathfrak{g}_i$ as $i \cdot \mathrm{id}$. The irreducible 1-dimensional module over the commutative Lie algebra spanned by z which acts as $i \cdot \mathrm{id}$ is denoted by $\mathbb{K}[i]$.

2.2 The two types of superizations of the contact series over $\mathbb{C}$

The type $\mathfrak{k}$: Define the Lie superalgebra $\mathfrak{hei}(2n|m)$ on the direct sum of a $(2n, m)$-dimensional superspace W, endowed with a non-degenerate antisymmetric bilinear form B, and a $(1,0)$-dimensional space spanned by z. Clearly, we have

$$\mathfrak{k}(2n+1|m) = (\mathfrak{hei}(2n|m), \mathfrak{cosp}(m|2n))_*$$

and, given $\mathfrak{hei}(2n|m)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cosp}(m|2n)$, we call $(\mathfrak{hei}(2n|m), \mathfrak{g})_*$ the *k-prolong* of $(W, \mathfrak{g})$, where W is the tautological $\mathfrak{osp}(m|2n)$-module.

The type $\mathfrak{m}$: The "odd" analog of $\mathfrak{k}$ is associated with the following "odd" analog of $\mathfrak{hei}(2n|m)$. Denote by $\mathfrak{ba}(n)$ the *antibracket* Lie superalgebra ($\mathfrak{ba}$ is Anti-Bracket read backwards). Its space is $W \oplus \mathbb{C} \cdot z$, where W is an $n|n$-dimensional superspace endowed with a non-degenerate antisymmetric odd bilinear form B; the bracket in $\mathfrak{ba}(n)$ is given by the following relations:

z is odd and lies in the center; $[v, w] = B(v, w) \cdot z$ for any $v, w \in W$.

Given $\mathfrak{ba}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cpe}(n)$, we call $(\mathfrak{ba}(n), \mathfrak{g})_*$ the *m-prolong* of $(W, \mathfrak{g})$, where W is the tautological $\mathfrak{pe}(n)$-module.

2.3 Generating functions over $\mathbb{C}$

A laconic way of describing $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via generating functions.

On the $2n+1|m$-dimensional superspace with even coordinates t, and $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, and odd indeterminates ("coordinates") (ξ, η, θ), where $\xi = (\xi_1, \ldots, \xi_k)$, $\eta = (\eta_1, \ldots, \eta_k)$ and $\theta = (\theta_1, \ldots, \theta_s)$, the *odd contact form* α_1 is defined to be

$$\alpha_1 = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq k} (\xi_j d\eta_j + \eta_j d\xi_j) + \begin{cases} 0 & \text{if } m = 2k, \\ \sum_{1 \leq \ell \leq s} \theta_\ell d\theta_\ell & \text{if } m = 2k + s. \end{cases} \quad (2.1)$$

For any $f \in \mathbb{C}[t, p, q, \xi, \eta, \theta]$, set

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} + H_f + \frac{\partial f}{\partial t}E, \quad (2.2)$$

where $E = \sum_i y_i \frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator*, and

$$H_f = \sum_{i \leq n} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \left(\sum_{\ell \leq s} \frac{\partial f}{\partial \theta_\ell}\frac{\partial}{\partial \theta_\ell} + \sum_{j \leq k} \left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j} \right) \right).$$

The *Hamiltonian vector field* H_f with Hamiltonian f preserves the symplectic form $\omega_0 := d\alpha_1$.

On the $(n|n+1)$-dimensional superspace with even coordinates $q = (q_1, \ldots, q_n)$, and odd indeterminates ("coordinates") $\xi = (\xi_1, \ldots, \xi_n)$, and τ, the *even contact form* α_0, is defined to be

$$\alpha_0 = d\tau + \sum_{i \leq n} (\xi_i dq_i + q_i d\xi_i). \quad (2.3)$$

For any $f \in \mathbb{C}[q, \xi, \tau]$, set

$$M_f = (2 - E)(f)\partial_\tau - \text{Le}_f - (-1)^{p(f)}\partial_\tau(f)E, \quad \text{where } E = \sum_i y_i \partial_{y_i} \text{ and where } y = (q, \xi),$$

$$\mathrm{Le}_f = \sum_{i \leq n} \left(\frac{\partial f}{\partial q_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial q_i} \right).$$

Since

$$L_{K_f}(\alpha_1) = 2 \frac{\partial f}{\partial t} \alpha_1 = K_1(f) \alpha_1,$$

$$L_{M_f}(\alpha_0) = -(-1)^{p(f)} 2 \frac{\partial f}{\partial \tau} \alpha_0 = -(-1)^{p(f)} M_1(f) \alpha_0. \tag{2.4}$$

Let $\mathfrak{k}(2n+1|m)$ be the (contact) Lie superalgebra preserving the distribution given by the Pfaff equation with the 1-form α_1; let $\mathfrak{m}(n) := \mathfrak{m}(n+1|n)$ be the pericontact ("odd" contact) Lie superalgebra preserving the distribution given by the Pfaff equation with the 1-form α_0.

Equation (2.4) implies that $K_f \in \mathfrak{k}(2n+1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$p(\mathrm{Le}_f) = p(M_f) = p(f) + \bar{1}.$$

Contact brackets, Poisson bracket, antibracket a.k.a. Buttin (Schouten) bracket. To the (super)commutators $[K_f, K_g]$ and $[M_f, M_g]$ there correspond *contact brackets* of the generating functions:

$$[K_f, K_g] = K_{\{f,\,g\}_{\mathrm{k.b.}}},$$

$$[M_f, M_g] = M_{\{f,\,g\}_{\mathrm{m.b.}}}.$$

The explicit expressions for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on t (resp. τ).

The *Poisson bracket* $\{-,-\}_{\mathrm{P.b.}}$ (in the realization with the form $\omega_0 := d\alpha_1$ for $m = 2k+1$ it is given by the formula

$$\{f, g\}_{\mathrm{P.b.}} = \sum_{i \leq n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

$$- (-1)^{p(f)} \left(\sum_{\ell \leq s} \frac{\partial f}{\partial \theta_\ell} \frac{\partial}{\partial \theta_\ell} + \sum_{j \leq k} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} \right) \right) \quad \text{for any } f, g \in \mathbb{C}[p, q, \xi, \eta, \theta].$$

The *Buttin*[3] *bracket* $\{-,-\}_{\mathrm{B.b.}}$, discovered by Schouten and initially known as the *Schouten bracket*, is very popular in physics under the name *antibracket*, see [24]. It is given by the formula

$$\{f, g\}_{\mathrm{B.b.}} = \sum_{i \leq n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial q_i} \right) \quad \text{for any } f, g \in \mathbb{C}[q, \xi]. \tag{2.5}$$

In terms of the Poisson and Buttin brackets, respectively, the contact brackets are as follows:

$$\{f, g\}_{\mathrm{k.b.}} = (2 - E)(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} (2 - E)(g) - \{f, g\}_{\mathrm{P.b.}},$$

$$\{f, g\}_{\mathrm{m.b.}} = (2 - E)(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (2 - E)(g) - \{f, g\}_{\mathrm{B.b.}}. \tag{2.6}$$

The Lie superalgebras of *Hamiltonian vector fields* (or *Hamiltonian superalgebras*) and their special subalgebras (defined only if $n = 0$) are

$$\mathfrak{h}(2n|m) = \{ D \in \mathfrak{vect}(2n|m) \,|\, L_D \omega_0 = 0, \quad \text{where } \omega_0 := d\alpha_1 \},$$

$$\mathfrak{h}^{(1)}(0|m) = \left\{ H_f \in \mathfrak{h}(0|m) \,|\, \int f \,\mathrm{vol} = 0 \right\}.$$

---

[3]C. Buttin was the first to publish that the Schouten bracket satisfies super Jacobi identity.

The "odd" analogues of the Lie superalgebra of Hamiltonian fields are the Lie superalgebra of vector fields $\mathfrak{Le}_f$ introduced in [50], and its special subalgebra:

$$\mathfrak{le}(n) = \{D \in \mathfrak{vect}(n|n) \,|\, L_D\omega_1 = 0, \text{ where } \omega_1 := d\alpha_0\},$$
$$\mathfrak{sle}(n) = \{D \in \mathfrak{le}(n) \,|\, \operatorname{div} D = 0\}. \tag{2.7}$$

It is not difficult to prove the following isomorphisms as Lie superalgebras with the brackets on the right-hand sides given by the above-described brackets k.b. (where K_f and H_f are involved) and m.b. (where M_f and Le_f are involved)

$$\mathfrak{k}(2n+1|m) = \{K_f \,|\, f \in \mathbb{C}[t, p, q, \xi]\} \cong \mathbb{C}[t, p, q, \xi],$$

$$\mathfrak{po}(2n|m) := \left\{ K_f \,|\, f \in \mathbb{C}[t, p, q, \xi] \text{ such that } \frac{\partial f}{\partial t} = 0 \right\} \cong \mathbb{C}[p, q, \xi],$$

$$\mathfrak{h}(2n|m) = \{H_f \,|\, f \in \mathbb{C}[p, q, \xi]\} \simeq \mathbb{C}[p, q, \xi]/\mathbb{C} \cdot 1,$$
$$\mathfrak{m}(n) = \{M_f \,|\, f \in \mathbb{C}[\tau, q, \xi]\} \cong \mathbb{C}[\tau, q, \xi],$$

$$\mathfrak{b}(n) := \left\{ M_f \,|\, f \in \mathbb{C}[\tau, q, \xi), \text{ such that } \frac{\partial f}{\partial \tau} = 0 \right\} \cong \Pi(\mathbb{C}[q, \xi]),$$

$$\mathfrak{le}(n) = \{\mathrm{Le}_f \,|\, f \in \mathbb{C}[q, \xi]\} \cong \Pi(\mathbb{C}[q, \xi]/\mathbb{C} \cdot 1).$$

We have

$$\mathfrak{po}^{(1)}(0|m) = \left\{ K_f \in \mathfrak{po}(0|m) \,\Big|\, \int f \operatorname{vol} = 0 \right\},$$

$$\mathfrak{h}^{(1)}(0|m) = \mathfrak{po}^{(1)}(0|m)/\mathbb{C} \cdot K_1.$$

2.3.1 Generating functions over $\mathbb{K}$

Recall that the contact Lie superalgebra $\mathfrak{k}(n_{\bar 0} + 1; \underline{N}|n_{\bar 1})$ consists of the vector fields D that preserve the contact structure (a non-integrable distribution given by a contact form α_1, cf. (2.1)) on the supervariety $\mathcal{M}$ associated with the superspace $\mathbb{K}^{n_{\bar 0}+1|n_{\bar 1}}$:

$$L_D(\alpha_1) = F_D\alpha_1 \text{ for some } F_D \in \mathcal{F}, \text{ where } \mathcal{F} \text{ is the space of functions on } \mathcal{M}. \tag{2.8}$$

Consider the form (here $n_{\bar 0} = 2k_{\bar 0}$; if $n_{\bar 0}$ is odd, no contact form exists)

$$\alpha_1 = dx_0 + \sum_{1 \le i \le k} x_i dx_{k+i}$$

$$+ \begin{cases} 0 & \text{if } n = (n_{\bar 0} + 1) + n_{\bar 1} = (2k_{\bar 0} + 1) + 2k_{\bar 1} = 2k + 1, \\ x_{2k+1}dx_{2k+1} & \text{if } n = (n_{\bar 0} + 1) + n_{\bar 1} = (2k_{\bar 0} + 1) + (2k_{\bar 1} + 1) = 2k + 2, \end{cases} \tag{2.9}$$

where in order to make expressions for brackets simpler, we consider the following nonstandard order of indeterminates the constituents of dual pairs one above/under the other:

$$\text{even: } x_0, \quad \text{even: } x_1, \qquad \dots, x_{k_{\bar 0}}, \qquad \text{odd: } x_{k_{\bar 0}+1}, \dots, x_{k_{\bar 0}+k_{\bar 1}},$$
$$\text{even: } x_{k_{\bar 0}+k_{\bar 1}+1}, \dots, x_{2k_{\bar 0}+k_{\bar 1}}, \quad \text{odd: } x_{2k_{\bar 0}+k_{\bar 1}+1}, \dots, x_{2k}; \quad \text{odd: } x_{2k+1}.$$

The vector fields D satisfying (2.8) for some function F_D look differently for different characteristics:

For $p \ne 2$, and also if $p = 2$ and $n = 2k + 1$, the fields D satisfying (2.8) have, for any $f \in \mathcal{F}$, the following form (compare with (2.2)):

$$K_f = (1 - E')(f)\frac{\partial}{\partial x_0} + \frac{\partial f}{\partial x_0}E' + \sum_{1 \le i \le k_{\bar 0}} \left(\frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}} \right) \tag{2.10}$$

$$-(-1)^{\Pi(f)}\left(\sum_{k_{\bar 0}+1\leq i\leq k}\left(\frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i}+\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}}\right)+\begin{cases}0 & \text{if } n=2k+1\\ \frac{1}{2}\frac{\partial f}{\partial x_{2k+1}}\frac{\partial}{\partial x_{2k+1}} & \text{if } n=2k+2\end{cases}\right),$$

where

$$E':=\sum_{1\leq i\leq k}x_i\partial_{x_i}+\begin{cases}0 & \text{if } n=2k+1,\\ \frac{1}{2}x_{2k+1}\partial_{x_{2k+1}} & \text{if } n=2k+2.\end{cases}$$

For $p=2$ and $n=2k+2$, we cannot use formula (2.10) *anymore (at least, not for arbitrary f)* since it contains $\frac{1}{2}$. In this case, the elements of the contact algebra are of the following three types, and their linear combinations, where $k=k_{\bar 0}+k_{\bar 1}$:

a) For any $f\in\mathcal{F}$ such that $\frac{\partial f}{\partial x_0}=\frac{\partial f}{\partial x_{2k+1}}=0$, we have

$$K_f=(1+E')(f)\frac{\partial}{\partial x_0}+\sum_{1\leq i\leq k}\left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_{k+i}}+\frac{\partial f}{\partial x_{k+i}}\frac{\partial}{\partial x_i}\right),\tag{2.11}$$

where $E':=\sum_{1\leq i\leq k}x_i\frac{\partial}{\partial x_i}$.

Observe that in (2.10) and (2.11) we can also take $E':=\sum_{k+1\leq i\leq 2k}x_i\frac{\partial}{\partial x_i}$.

b) Set

$$\widehat{\mathcal{F}}:=\mathcal{O}(x_0,x_1,\ldots,x_{k_{\bar 0}},x_{k+1},\ldots,x_{k+k_{\bar 0}};\underline{N}|x_{k_{\bar 0}+1},\ldots,x_k,x_{k+k_{\bar 0}+1},\ldots,x_{2k}).\tag{2.12}$$

For any $g\in\widehat{\mathcal{F}}$, or equivalently, for any $g\in\mathcal{F}$ such that $\frac{\partial g}{\partial x_{2k+1}}=0$, we set

b1) $A_g:=g\left(x_{2k+1}\partial_{x_0}+\partial_{x_{2k+1}}\right)$,

b2) $B_g:=gx_{2k+1}\partial_{x_{2k+1}}$.

For the pericontact Lie superalgebra $\mathfrak{m}(n;\underline{N}|n)$ the analog of the formula (2.9) takes the form

$$\alpha_0=dx_0+\sum_{1\leq i\leq k}x_i dx_{k+i},$$

where the parities of indeterminates are such that $p(x_i)=p(x_{i+k})+\bar 1$; e.g., they are as follows

even: $x_1,\ldots,x_k$;   odd: $x_{k+1},\ldots,x_{2k}$, and x_0.

Theorem 2.1 (on explicit squaring and contact brackets for $p=2$). *Brackets and squares of contact vector fields, and the corresponding contact brackets of generating functions, are given by formulas* (2.14). *Both the contact brackets $\{-,-\}_{\text{k.b.}}$ and $\{-,-\}_{\text{m.b.}}$ are of the shape*

$$\{f,f_1\}=\frac{\partial f}{\partial x_0}(1+E')(f_1)+(1+E')(f)\frac{\partial f_1}{\partial x_0}+\sum_{1\leq i\leq k}\left(\frac{\partial f}{\partial x_i}\frac{\partial f_1}{\partial x_{k+i}}+\frac{\partial f}{\partial x_{k+i}}\frac{\partial f_1}{\partial x_i}\right),$$

where $E':=\sum_{1\leq i\leq k}x_i\frac{\partial}{\partial x_i}$ or $E':=\sum_{k+1\leq i\leq 2k}x_i\frac{\partial}{\partial x_i}$ for any $f,f_1\in\widehat{\mathcal{F}}$. (2.13)

Then, for any $f,f_1,g,g_1\in\widehat{\mathcal{F}}$, we deduce

$$[K_f,K_{f_1}]=K_{\{f,f_1\}_{\text{k.b.}}},\qquad [M_f,M_{f_1}]=M_{\{f,f_1\}_{\text{m.b.}}},$$
$$(K_f)^2=K_{\sum_{1\leq i\leq k}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_{k+i}}},\qquad (M_f)^2=M_{\sum_{1\leq i\leq k}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_{k+i}}},$$

$$[K_f, A_g] = A_{\{f,g\}_{\text{k.b.}}}, \qquad (A_g)^2 = B_{g \frac{\partial g}{\partial x_0}} + g^2 K_1,$$

$$[K_f, B_g] = B_{\{f,g\}_{\text{k.b.}}}, \qquad [A_g, B_{g_1}] = A_{gg_1},$$

$$[A_g, A_{g_1}] = B_{\frac{\partial(gg_1)}{\partial x_0}}, \qquad [B_g, B_{g_1}] = (B_g)^2 = 0. \tag{2.14}$$

Proof is based on direct computations, see formulas (2.14). ∎

In particular, for $\mathfrak{k}(1; \underline{N}|0)$, we have (unless $a = b = 0$)

$$\{x^{(a)}, x^{(b)}\} = \left(\binom{a+b-1}{a-1} + \binom{a+b-1}{b-1} \right) x^{(a+b-1)} = \binom{a+b}{a} x^{(a+b-1)}.$$

Lemma 2.2 (A helpful lemma). *For any $g \in \widehat{\mathcal{F}}$, see (2.12), we have $g^2 \in \mathbb{K}$, see the expressions for $(A_g)^2$ and $[A_g, B_g]$ in equation (2.14).*

Proof. Indeed, $g = \sum_r g_r x^{(r)}$, where r is a $(2k+1)$-tuple of non-negative numbers and the sum runs over a set of such tuples, and $g_r \in \mathbb{K}$ for all r. Then,

$$g \cdot g = \sum_{r,s} g_r g_s x^{(r)} \cdot x^{(s)},$$

i.e., the terms with $r \neq s$, are encountered 2 times, so what remains is

$$\sum_r g_r^2 x^{(r)} \cdot x^{(r)} = \sum_r g_r^2 \binom{2r}{r} x^{(2r)} = \sum_r g_r^2 \left(\prod_{i=0}^{2k} \binom{2r_i}{r_i} \right) x^{(2r)}. \tag{2.15}$$

If $a > 0$, then $\binom{2a}{a} = 0 \pmod 2$. By definition, $\binom{0}{1} := 0$; hence the product in equation (2.15) vanishes even if there are only even indeterminates involved. Therefore, only the summand with $r = (0, \ldots, 0)$, i.e., $(g_{(0,\ldots,0)})^2$, survives. ∎

Claim 2.3 (Grading operators in $\mathfrak{k}(2n_{\bar{0}} + 1; \underline{N}|m)$ and $\mathfrak{m}(n; \underline{N}|n + 1)$). *For $p \neq 2$ and the bracket (2.13), the x_0-action gives a grading of $\mathfrak{g}$ by the formula $\operatorname{ad}_{x_0}|_{\mathfrak{g}_i} = i\,\mathrm{id}$; this action also defines the 1-dimensional $\mathfrak{g}_0$-module we denote by $\mathbb{K}[i]$.*

For $p = 2$, let

$$\mathbb{K}[*] \quad \text{denote the } \mathbb{K}x_0\text{-module analogous to } \mathbb{K}[i]; \quad \Phi := \sum_{1 \leq i \leq k} x_i x_{k+i}. \tag{2.16}$$

For $p = 2$, the element x_0 annihilates a subspace $ann(x_0)|_{\mathfrak{g}_{-1}}$ of $\mathfrak{g}_{-1}$ and acts as multiplication by $\overline{1}$ on both $\mathfrak{g}_{-1}/(ann(x_0)|_{\mathfrak{g}_{-1}})$, and $\mathfrak{g}_{-2}$.

For $p = 2$, the operators $\operatorname{ad}_{x_0}|_{\mathfrak{g}_0}$ and $\operatorname{ad}_\Phi|_{\mathfrak{g}_0}$ interchange their roles as compared with $p \neq 2$: now ad_Φ commutes with $\mathfrak{g}_0$.

Claim 2.4 (brackets in $\mathfrak{k}(2k_{\bar{0}} + 1; \underline{N}|2k_{\bar{1}} + 1)$). *In items 1)–3) the isomophisms are between the LHS, defined as the Lie superalgebras of vector fields that multiply the contact form α by a function, and the RHS, the brackets and squarings in which is given by formulas (2.14).*

1) $\mathfrak{k}(1; \underline{N}|0) \simeq \mathfrak{vect}(1; \underline{N}|0)$.

2) $\mathfrak{k}(1; \underline{N}|1) \simeq (\mathbb{K}K_1 \oplus \mathcal{O}(x_0; \underline{N})) \bigoplus \Pi(\mathcal{O}(x_0; \underline{N}))$. *The even part of the simple Lie superalgebra $\mathfrak{k}^{(1)}(1; \underline{N}|1)$ is solvable.* (For other examples of the same phenomenon indigenous to $p = 2$, see [10, Section 16.2].) *This Lie superalgebra is the superization of $\mathfrak{vect}^{(1)}(1; \underline{N} + 1)$ by "method 2", see [12] and [43], and the desuperization $\boldsymbol{F}(\mathfrak{k}^{(1)}(1; \underline{N}))$ is simple Lie algebra $\mathfrak{vect}^{(1)}(1; \underline{N} + 1)$.*

3) $\mathfrak{k}(2k_{\bar{0}} + 1; \underline{N}|2k_{\bar{1}} + 1) \simeq \left(\mathfrak{po}(2k_{\bar{0}}; \hat{\underline{N}}|2k_{\bar{1}}) \oplus \widehat{\mathcal{F}} \right) \bigoplus \Pi(\widehat{\mathcal{F}})$, *where $\underline{N} = (N_0, \hat{\underline{N}})$ and at least one of $k_{\bar{0}}$ and $k_{\bar{1}}$ is non-zero. These Lie superalgebras and their desuperizations are not simple, the ideal $\mathfrak{i}$ is generated by the A_g and B_g; recall (2.12). We have*

$$\mathfrak{k}(2k_{\bar{0}} + 1; \underline{N}|2k_{\bar{1}} + 1)/\mathfrak{i} \simeq \mathfrak{h}(2k_{\bar{0}}; \underline{N}|2k_{\bar{1}}).$$

2.4   Weisfeiler gradings

For vectorial Lie superalgebras, the invariant notion is filtration, not grading. In characteristic 0, the *Weisfeiler* filtrations were used in the description of the infinite-dimensional Lie (super)algebras $\mathcal{L}$ by selecting a maximal subalgebra $\mathcal{L}_0$ of finite codimension; for the simple vectorial Lie algebra, there is only one such $\mathcal{L}_0$. (Dealing with finite-dimensional algebras for $p > 0$, we can confine ourselves to maximal subalgebras of *least* codimension, or "almost least".)

Let $\mathcal{L}_{-1}$ be a minimal $\mathcal{L}_0$-invariant subspace strictly containing $\mathcal{L}_0$; for $i \geq 1$, set:

$$\mathcal{L}_{-i-1} = \begin{cases} [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} & \text{unless } p = 2 \text{ and } i = 1, \\ [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} + \text{Span}\left(X^2 \,|\, X \in \mathcal{L}_{-1}\right) & \text{for } p = 2 \text{ and } i = 1, \end{cases}$$
$$\mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \,|\, [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}.$$

We thus get a filtration:

$$\mathcal{L} = \mathcal{L}_{-d} \supset \mathcal{L}_{-d+1} \supset \cdots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots. \tag{2.17}$$

The d in (2.17) is called the *depth* of $\mathcal{L}$, and of the associated *Weisfeiler-graded* Lie superalgebra $\mathfrak{g} = \bigoplus_{-d \leq i} \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathcal{L}_i/\mathcal{L}_{i+1}$. We will for brevity say *W-graded* and *W-filtered*.

For the list of simple *W-graded* vectorial Lie superalgebras $\mathfrak{g} = \bigoplus_{-d \leq i} \mathfrak{g}_i$ over $\mathbb{C}$, see [55] reproduced in Tables 25.2 and 25.3.

2.4.1   The $\mathbb{Z}$-gradings of vectorial Lie superalgebras

These gradings are defined by the vector $\vec{r}$ of degrees of the indeterminates, but this vector can be shortened *for W-gradings* to a number r, or a symbol, which we do not indicate for $r = 0$. Let the indeterminates t, p_i, q_j, and u_ℓ be even, while τ, ξ_i, η_j, and θ_ℓ be odd. Let the contact Lie superalgebra $\mathfrak{k}(2n+1|m)$ preserve the distribution given by the Pfaff equation

$$\alpha_1(X) = 0 \quad \text{for } X \in \mathfrak{vect}(2n+1|m),$$

where the form α_1 is given by (2.1). For the $\mathfrak{k}$ series, let $u = (t; p, q)$ be even indeterminates, the odd indeterminates being the θ (resp. θ, ξ, η), see (2.1).

For the $\mathfrak{m}$ series, the indeterminates in Table (2.18) are denoted as in formula (2.3), i.e., the q_i even, the ξ_i, and τ odd.

In Table (2.18), the "standard" gradings correspond to $r = 0$, they are marked by an asterisk ($*$). For $r = 0$, the codimension of $\mathcal{L}_0$ is the smallest.

Lie superalgebra	its $\mathbb{Z}$-grading	
$\mathfrak{vect}(n\|m; r)$,	$\deg u_i = \deg \theta_j = 1$ for any i, j	$(*)$
where $0 \leq r \leq m$	$\deg \theta_j = 0$ for $1 \leq j \leq r$;	
	$\deg u_i = \deg \theta_{r+s} = 1$ for any i, s	
$\mathfrak{m}(n; r)$,	$\deg \tau = 2$, $\deg q_i = \deg \theta_i = 1$ for any i	$(*)$
where $0 \leq r < n - 1$	$\deg \tau = \deg q_i = 2$, $\deg \xi_i = 0$ for $1 \leq i \leq r$;	
and one more grading (next line):	$\deg q_{r+j} = \deg \xi_{r+j} = 1$ for any j	
$\mathfrak{m}(n; n)$	$\deg \tau = \deg q_i = 1$, $\deg \xi_i = 0$ for $1 \leq i \leq n$	
$\mathfrak{k}(2n+1\|m; r)$,	$\deg t = 2$, $\deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_\ell = 1$	
where $0 \leq r \leq [\frac{m}{2}]$,	for any i, j, ℓ	$(*)$
$r \neq k - 1$ if $(n, m) = (0, 2k)$	$\deg t = \deg \xi_i = 2$, $\deg \eta_i = 0$ for $1 \leq i \leq r$;	
and one more grading (next line):	$\deg p_i = \deg q_i = \deg \theta_\ell = 1$ for $\ell \geq 1$ and all i	
$\mathfrak{k}(1\|2m; m)$	$\deg t = \deg \xi_i = 1$, $\deg \eta_i = 0$ for $1 \leq i \leq m$	

$$\tag{2.18}$$

2.5 Divergence-free and traceless subalgebras

In this subsection, the ground field is any $\mathbb{K}$ for $p \neq 2$. The peculiarities of $p = 2$ are considered in Sections 2.9 and 2.10. Here we will not mention $\underline{N}$ if $p > 2$.

2.5.1 $\mathfrak{k}$ series

Since the restriction of the standard divergence (1.14) to the subalgebra of degree 0 is (super)trace, and since the space $\mathfrak{g}_0/[\mathfrak{g}_0, \mathfrak{g}_0]$, where $\mathfrak{g} := \mathfrak{k}(2n + 1|m)$, is spanned by K_t for $(n, m) \neq (0, 2)$, it is easy to calculate that

$$\operatorname{div} K_f = (2n + 2 - m)\partial_t(f) \quad \text{if } 2n + 2 - m \neq 0, \tag{2.19}$$

it follows that the divergence-free (relative the restriction of the divergence (1.14) to $\mathfrak{k}(2n+1|m)$) subalgebra of the contact Lie superalgebra either coincides with it for $m = 2n+2$ or is the Poisson superalgebra singled our by the condition $\partial_t(f) = 0$.

On $\mathfrak{k}(2n + 1|2n + 2)$ there is its own, "indigenous" divergence $K_f \mapsto \partial_t(f)$; it also singles out the Poisson superalgebra. This, however, is not the whole story: the case $\mathfrak{k}(1|2)$ is exceptional.

The case of $\mathfrak{k}(1|2)$. Let $\alpha_1 = dt + \xi d\eta + \eta d\xi$. Since $\mathfrak{k}(1|2)_0$ is commutative and 2-dimensional, there are 2 linearly independent traces on it: one – tr – is equal to 1 at t and vanishes at $\xi\eta$, the other one – call it $\operatorname{tr}_{(2)}$ – is equal to 1 at $\xi\eta$ and vanishes at t.

Clearly, the condition $K_1(f) = 0$ singles out the subalgebra $\mathfrak{k}_- \oplus \mathbb{K}\xi\eta$ of $\mathfrak{k}(1|2)$. In other words, the operator $\partial_t = \frac{1}{2}K_1$ in the adjoint representation is an analog of the divergence – the prolong of the trace on $\mathfrak{k}_0$; this analog is equal to 1 at t and vanishes at $\xi\eta$.

The divergence-free condition $\operatorname{div} D = 0$, where $D \in \mathfrak{g}$ for a $\mathbb{Z}$-graded vectorial Lie superalgebra $\mathfrak{g}$, should single out the complete prolong of $(\mathfrak{g}_-, \mathfrak{s})$, where $\mathfrak{s} = \{g \in \mathfrak{g}_0 \mid \operatorname{tr} g = 0\}$. Therefore, the condition that determines the divergence is

$$X(\operatorname{div} D) = \operatorname{div}([X, D]) \quad \text{for any } X \in \mathfrak{g}_-. \tag{2.20}$$

Since $\mathfrak{g}_{-1}$ generates the negative part, it suffices to require fulfillment of the condition (2.20) for any $X \in \mathfrak{g}_{-1}$.

Therefore, we have to express the divergence not in terms of partial derivatives, but in terms of the operators commuting (not supercommuting) with $\mathfrak{g}_-$ (recall that in [66], the operators that span $\mathfrak{g}_-$ are denoted by X_i, and the operators commuting with $\mathfrak{g}_-$ are denoted by Y_i).

To write the second divergence $\operatorname{div}_{(2)}$, which is the prolong of $\operatorname{tr}_{(2)}$, we need two operators commuting (not supercommuting!) with $\mathfrak{k}(1|2)_-$. In our case, the X-operators are

$$K_1 = 2\partial_t, \quad K_\xi = \partial_\eta + \xi\partial_t \quad \text{and} \quad K_\eta = \partial_\xi + \eta\partial_t,$$

then the needed Y-operators are

$$\tilde{K}_1 = \partial_t, \quad \tilde{K}_\xi(f) = (-1)^{p(f)}(\partial_\eta - \xi\partial_t)(f) \quad \text{and} \quad \tilde{K}_\eta(f) = (-1)^{p(f)}(\partial_\xi - \eta\partial_t)(f). \tag{2.21}$$

Let $\alpha_1 = dt + \xi d\eta$ which works for any characteristic. Then, the X-operators are

$$K_1 = \partial_t, \quad K_\xi = \partial_\eta \quad \text{and} \quad K_\eta = \partial_\xi + \eta\partial_t,$$

and the Y-operators are

$$\tilde{K}_1 = \partial_t, \quad \tilde{K}_\xi(f) = (-1)^{p(f)}(\partial_\eta - \xi\partial_t)(f) \quad \text{and} \quad \tilde{K}_\eta(f) = (-1)^{p(f)}\partial_\xi(f). \tag{2.22}$$

Claim 2.5 (the second divergence on $\mathfrak{k}(1|2)$). *The prolong of* $\mathrm{tr}_{(2)}$ *composed of the Y-operators* (2.21) *or* (2.22) *is the same*

$$\mathrm{div}_{(2)} := \tilde{K}_\eta \tilde{K}_\xi - \tilde{K}_1,$$

and $\mathrm{div}_{(2)}(\xi\eta) = 1$ *while* $\mathrm{div}_{(2)}(t) = 0$.

The $\mathfrak{k}(1|2)$-module of weighted densities. Over contact Lie superalgebras $\mathfrak{k}(2n+1|m)$ it is natural to express the spaces of weighted densities in terms of the conformally preserved form α_1. This recalculation is well-known for $m = 0$, where $\mathrm{vol} = \alpha_1 \wedge (d\alpha_1)^n$. The general case follows from equation (2.19): from the point of view of the $\mathfrak{k}(2n+1|m)$-action

$$\mathrm{vol} = \begin{cases} (\alpha_1)^{(2n+2-m)/2} & \text{if } (m,n) \neq (0,0), \\ \alpha_1 & \text{if } (m,n) = (0,0). \end{cases}$$

Since the center of $\mathfrak{k}(1|2)_0$ is of dimension 2, the weights of the spaces of weighted densities have 2 parameters, not one: $\mathcal{F}_{a,b} := \mathcal{F}\alpha_1^a \beta^b$, where $a,b \in \mathbb{K}$.

Let β be the symbol of the *class* of the differential form $d\xi$ (or, equivalently, $(d\eta)^{-1}$)in the quotient space $\Omega^1/\mathcal{F}\alpha_1$ of 1-forms. The Lie derivative acts as follows

$$\begin{aligned} L_{K_f}\left(\alpha_1^a \beta^b\right) &= \left(a\partial_t(f) + (-1)^{p(f)}b\,\mathrm{div}_2(K_f)\right)\left(\alpha_1^a \beta^b\right) \\ &= \left((a - (-1)^{p(f)}b)\partial_t(f) + (-1)^{p(f)}b\tilde{K}_\eta \tilde{K}_\xi(f)\right)\left(\alpha_1^a \beta^b\right). \end{aligned} \tag{2.23}$$

The space $\mathcal{F}_{a,b}$ of weighted densities over $\mathfrak{k}(1|2)$ is a rank-1 module generated by $\alpha_1^a \beta^b$ over the algebra of functions $\mathcal{F} = \mathcal{F}_{0,0}$.

2.5.2 $\mathfrak{m}$ series, its simple subalgebras, and weighted densities

For the pericontact series, the situation is more interesting than that for contact series: the divergence-free subalgebra is simple and new (only as compared with the above-described algebras; it is known since ca 1978, see [1]).

Let $p \neq 2$. Since

$$\mathrm{div}\,M_f = (-1)^{p(f)}2\left((1-E)\frac{\partial f}{\partial \tau} - \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}\right),$$

it follows that the divergence-free subalgebra of the pericontact superalgebra is

$$\mathfrak{sm}(n) = \mathrm{Span}\left(M_f \in \mathfrak{m}(n)\,|\,(1-E)\frac{\partial f}{\partial \tau} = \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}\right).$$

In particular,

$$\mathrm{div}\,\mathrm{Le}_f = (-1)^{p(f)}2\sum_{i\leq n}\frac{\partial^2 f}{\partial q_i \partial \xi_i} = (-1)^{p(f)}2\Delta(f), \quad \text{where } \Delta := \sum_{i\leq n}\frac{\partial^2}{\partial q_i \partial \xi_i}.$$

The divergence-free vector fields from $\mathfrak{sle}(n)$ are generated by *harmonic* functions, i.e., such that $\Delta(f) = 0$.

Rank 1 over the algebra $\mathcal{F}$ modules $\mathcal{F}_{a,b}^{\mathfrak{m}} := \mathcal{F}\alpha_0^a \gamma^b$, where $a,b \in \mathbb{K}$, are generated by $\alpha_0^a \gamma^b$, where γ is a symbol of the class of differential forms (whose explicit expression is irrelevant, same as that of β, see equation (2.23)). The Lie derivative acts as follows:

$$L_{M_f}\left(\alpha_0^a \gamma^b\right) = \left((-1)^{p(f)}b\partial_\tau(f) + a\,\Delta^{\mathfrak{m}}(f)\right)\left(\alpha_0^a \gamma^b\right).$$

The divergence-free relative the standard divergence Lie superalgebras $\mathfrak{sle}(n)$, $\mathfrak{sb}(n)$ and $\mathfrak{svect}(1|n)$ have traceless ideals $\mathfrak{sle}^{(1)}(n)$, $\mathfrak{sb}^{(1)}(n)$ and $\mathfrak{svect}^{(1)}(n)$ of codimension 1; they are defined from the exact sequences

$$0 \longrightarrow \mathfrak{sle}^{(1)}(n) \longrightarrow \mathfrak{sle}(n) \longrightarrow \mathbb{K} \cdot \mathrm{Le}_{\xi_1 \ldots \xi_n} \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{sb}^{(1)}(n) \longrightarrow \mathfrak{sb}(n) \longrightarrow \mathbb{K} \cdot M_{\xi_1 \ldots \xi_n} \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{svect}^{(1)}(1|n) \longrightarrow \mathfrak{svect}(1|n) \longrightarrow \mathbb{K} \cdot \xi_1 \cdots \xi_n \partial_t \longrightarrow 0.$$

2.5.3   A deform of the series $\mathfrak{b}$

Let $p \neq 2$. For an explicit form of M_f, see Section 2.3. Set

$$\mathfrak{b}_{a,b}(n) = \left\{ M_f \in \mathfrak{m}(n) \,|\, a \,\operatorname{div} M_f = (-1)^{p(f)} 2(aE - bn) \frac{\partial f}{\partial \tau} \right\}.$$

We denote the operator that singles out $\mathfrak{b}_\lambda(n)$ in $\mathfrak{m}(n)$ as follows, cf. (1.12):

$$\operatorname{div}_\lambda = (bn - aE) \frac{\partial}{\partial \tau} - a\Delta, \quad \text{for } \lambda = \frac{2a}{n(a-b)} \quad \text{and} \quad \Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}.$$

Taking the explicit form of the divergence of M_f into account, we get

$$\mathfrak{b}_{a,b}(n) = \left\{ M_f \in \mathfrak{m}(n) \,|\, (bn - aE) \frac{\partial f}{\partial \tau} = a\Delta f \right\}$$

$$= \left\{ D \in \mathfrak{vect}(n|n+1) \,|\, L_D \big(\operatorname{vol}_{q,\xi,\tau}^a \, \alpha_0^{a-bn} \big) = 0 \right\}. \tag{2.24}$$

It is subject to a direct verification that $\mathfrak{b}_{a,b}(n) \simeq \mathfrak{b}_\lambda(n)$ for $\lambda = \frac{2a}{n(a-b)} \in \mathbb{K}P^1$. Obviously, if $\lambda = 0, 1, \infty$ (where $\mathfrak{b}_0 := \mathfrak{b}$ and $\mathfrak{b}_\infty := \mathfrak{b}_{a,a}$) the structure of $\mathfrak{b}_\lambda(n)$ differs from the other members of the parametric family: the following exact sequences single out simple Lie superalgebras (the quotient $\mathfrak{le}(n)$ and ideals, the first derived subalgebras):

$$0 \longrightarrow \mathbb{K}M_1 \longrightarrow \mathfrak{b}(n) \longrightarrow \mathfrak{le}(n) \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{b}_1^{(1)}(n) \longrightarrow \mathfrak{b}_1(n) \longrightarrow \mathbb{K} \cdot M_{\xi_1 \ldots \xi_n} \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{b}_\infty^{(1)}(n) \longrightarrow \mathfrak{b}_\infty(n) \longrightarrow \mathbb{K} \cdot M_{\tau \xi_1 \ldots \xi_n} \longrightarrow 0. \tag{2.25}$$

Problem 2.6. *The Lie superalgebras $\mathfrak{b}_\lambda(n)$ can be further deformed at certain points λ, see* [58], *where $\mathbb{K} = \mathbb{C}$; the Lie superalgebras of series $\mathfrak{h}$ and $\mathfrak{le}$ also have extra deformations. Describe the deformations of $\mathfrak{b}_\lambda(n; \underline{N})$, as well as $\mathfrak{h}$ and $\mathfrak{le}$ for all $p > 0$.*

2.6   Passage from $p = 0$ to $p > 0$

Here we have collected answers to several questions that stunned us while we were writing this paper. We hope that even the simplest of these answers will help the reader familiar with representations of Lie algebra over $\mathbb{C}$, but with no experience of working with characteristic $p > 0$.

   For $p = 2$, several of our definitions are new, see Sections 2.9 and 2.10.

2.6.1   The Lie (super)algebras preserving symmetric non-degenerate bilinear forms $\mathcal{B}$

We often denote the Gram matrix of the bilinear form $\mathcal{B}$ also by $\mathcal{B}$, let $\mathfrak{aut}(\mathcal{B})$ be the Lie (super)algebra preserving $\mathcal{B}$. If $\mathcal{B}$ is odd and the superspace, on which it is defined, is of superdimension $n|n$, we write $\mathfrak{pe}_\mathcal{B}(n)$ instead of $\mathfrak{aut}(\mathcal{B})$.

Let $p \neq 2$ and $\mathfrak{g} = \mathfrak{pe}_{\mathcal{B}}(n)$. The Lie superalgebra $\mathfrak{g}$ consists of the supermatrices of the form

$$X = \begin{pmatrix} A & B \\ C & -A^{\mathrm{t}} \end{pmatrix}, \quad \text{where } B \text{ is symmetric and } C \text{ is antisymmetric}$$

if the form $\mathcal{B}$ is in its normal shape $\Pi_{n|n} := \Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$.

Clearly, $\operatorname{str} X = 2 \operatorname{tr} A$. We also have $\mathfrak{g}^{(1)} = \mathfrak{spe}(n)$, i.e., $\mathfrak{spe}(n)$ is of codimension 1; it is singled out by the condition $\operatorname{str} X = 0$, which is equivalent to $\operatorname{tr} A = 0$.

The Lie superalgebra $\mathfrak{le}(n; \underline{N}|n)$ is, by definition, the Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n))_{*,\underline{N}}$.

Over $\mathbb{C}$, there is no shearing vector, and $\mathfrak{le}(n) := \mathfrak{le}(n|n)$ is spanned by the elements Le_f, where $f \in \mathbb{C}[q, \xi]$.

If $p > 2$, the elements of $\mathcal{O}(q; \underline{N}|\xi) \oplus \mathrm{Span}\left(q_i^{(p^{N_i})} \mid N_i < \infty\right)$, or $\mathcal{O}(q; \underline{N}_\infty|\xi)$ for $\underline{N} = \underline{N}_\infty$, see (1.8), generate $\mathfrak{le}(n; \underline{N})$. If $N_i < \infty$ for at least one i, the additional part Irreg does not change while the regular part looks the same for any $p > 2$:

$$\begin{aligned} \mathrm{Reg} &= \mathrm{Span}(\mathrm{Le}_f \mid f \in \mathcal{O}(q; \underline{N}|\xi)) \oplus \mathrm{Span}\left(q_i^{(p^{N_i})} \mid N_i < \infty\right), \\ \mathrm{Irreg} &= \mathrm{Span}(\xi_i \partial_{q_i})_{i=1}^n. \end{aligned} \tag{2.26}$$

In other words: *there are vector fields corresponding to non-existing generating functions, like* $q_i^{(p^{N_i})}$ *and* ξ_j^2. The prolong $\mathfrak{sle}(n; \underline{N}) := (\mathrm{id}, \mathfrak{spe}(n))_{*,\underline{N}}$ is singled out by the condition

$$\operatorname{div} \mathrm{Le}_f = 0 \iff \Delta f = 0, \quad \text{where } \Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}.$$

The operator Δ is, therefore, the "Cartan prolong of the supertrace on $\mathfrak{g}_0$" expressed as an operator acting on the space of generating functions.

Modifications in the above description for $p = 2$. If $p = 2$, the analogs of symplectic (resp. periplectic) Lie (super)algebras accrue additional elements: if the matrix of the bilinear form $\mathcal{B}$ is Π_{2n} (resp. $\Pi_{n|n}$), then $\mathfrak{aut}(\mathcal{B})$ consists of the (super)matrices of the form

$$X = \begin{pmatrix} A & B \\ C & A^{\mathrm{t}} \end{pmatrix}, \quad \text{where } B \text{ and } C \text{ are symmetric}, A \in \mathfrak{gl}(n). \tag{2.27}$$

Denote the *general* Lie (super)algebra preserving the form $\mathcal{B}$ as follows:

$$\mathfrak{aut}(\mathcal{B}) = \begin{cases} \mathfrak{o}_{\mathrm{gen}}(2n) & \text{for } \mathcal{B} = \Pi_{2n}, \\ \mathfrak{pe}_{\mathrm{gen}}(n) & \text{for } \mathcal{B} = \Pi_{n|n}. \end{cases}$$

Let

   ZD denote the space of symmetric matrices with zeros on their main diagonals.

The derived Lie (super)algebra $\mathfrak{aut}^{(1)}(\mathcal{B})$ consists of the (super)matrices of the form (2.27), where $B, C \in ZD$. In other words, these Lie (super)algebras resemble the orthogonal Lie algebras.

On these Lie (super)algebras $\mathfrak{aut}^{(1)}(\mathcal{B})$ the following (super)trace (*half-trace*) is defined:

$$\mathrm{htr}\colon \begin{pmatrix} A & B \\ C & A^{\mathrm{t}} \end{pmatrix} \longrightarrow \operatorname{tr} A.$$

The half-traceless Lie sub(super)algebra of $\mathfrak{aut}^{(1)}(\mathcal{B})$ is isomorphic to $\mathfrak{aut}^{(2)}(\mathcal{B})$.

There is, however, an algebra $\widetilde{\mathfrak{aut}}(\mathcal{B})$, such that $\mathfrak{aut}^{(1)}(\mathcal{B}) \subset \widetilde{\mathfrak{aut}}(\mathcal{B}) \subset \mathfrak{aut}(\mathcal{B})$, consisting of (super)matrices of the form (2.27), where $B \in ZD$, and any symmetric C (or isomorphic to it version of the Lie superalgebra with $C \in ZD$, and any symmetric B). We suggest that it be denoted as follows:

$$\widetilde{\mathfrak{aut}}(\mathcal{B}) = \begin{cases} \mathbf{F}(\mathfrak{pe})(2n) & \text{for } \mathcal{B} \text{ even,} \\ \mathfrak{pe}(n) & \text{for } \mathcal{B} \text{ odd,} \end{cases}$$

$$\mathfrak{aut}^{(1)}(\mathcal{B}) = \{X \in \widetilde{\mathfrak{aut}}(\mathcal{B}) \mid \operatorname{htr} X = 0\} = \begin{cases} \mathbf{F}(\mathfrak{spe})(2n) & \text{for } \mathcal{B} \text{ even,} \\ \mathfrak{spe}(n) & \text{for } \mathcal{B} \text{ odd.} \end{cases}$$

2.7 Central extensions

There is only one non-trivial central extension of $\mathfrak{spe}(n)$ for $p \neq 2, 3$ existing only for $n = 4$. We denote it $\mathfrak{as}$ because it was discovered by A. Sergeev (1970s, unpublished). For numerous non-trivial central extensions of versions of $\mathfrak{spe}(n)$ and its simple subquotients for $p = 2, 3$, see [6].

Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$, and z is the central element. The bracket in $\mathfrak{as}$ is

$$\left[\begin{pmatrix} a & b \\ c & -a^{\mathrm{t}} \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -(a')^{\mathrm{t}} \end{pmatrix} + d' \cdot z \right] = \left[\begin{pmatrix} a & b \\ c & -a^{\mathrm{t}} \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -(a')^{\mathrm{t}} \end{pmatrix} \right] + \operatorname{tr}\ c\widetilde{c'} \cdot z, \qquad (2.28)$$

where $\widetilde{}$ is extended via linearity from matrices $c_{ij} = E_{ij} - E_{ji}$ on which $\widetilde{c}_{ij} = c_{kl}$ for any even permutation $(1234) \longmapsto (ijkl)$. Recall that $b = b^{\mathrm{t}}$ and $b' = (b')^{\mathrm{t}}$ in (2.28), whereas $c = -c^{\mathrm{t}}$ and $c' = -(c')^{\mathrm{t}}$.

The Lie superalgebra $\mathfrak{as}$ can also be described in terms of the spinor representation. For this, we need several vectorial superalgebras. Consider $\mathfrak{po}(0|6)$, the Lie superalgebra whose superspace is the Grassmann superalgebra $\Lambda(\xi, \eta)$ generated by $\xi = (\xi_1, \xi_2, \xi_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ with the Poisson bracket.

Recall that $\mathfrak{h}(0|6) = \operatorname{Span}(H_f \mid f \in \Lambda(\xi, \eta))$. Now, observe that $\mathfrak{spe}(4)$ can be embedded into $\mathfrak{h}(0|6)$. Indeed, setting $\deg \xi_i = \deg \eta_i = 1$ for all i we introduce a $\mathbb{Z}$-grading on $\Lambda(\xi, \eta)$ which, in turn, induces a $\mathbb{Z}$-grading on $\mathfrak{h}(0|6)$ of the form $\mathfrak{h}(0|6) = \underset{i \geq -1}{\oplus} \mathfrak{h}(0|6)_i$. Since $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$, we can identify $\mathfrak{spe}(4)_0$ with $\mathfrak{h}(0|6)_0$.

It is not difficult to see that the elements of degree -1 in the standard gradings of $\mathfrak{spe}(4)$ and $\mathfrak{h}(0|6)$ constitute isomorphic $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$-modules. It is subject to a direct verification that it is possible to embed $\mathfrak{spe}(4)_1$ into $\mathfrak{h}(0|6)_1$.

Sergeev's extension $\mathfrak{as}$ is the result of the restriction to $\mathfrak{spe}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$; the through maps $T_\lambda \colon \mathfrak{as} \longrightarrow \mathfrak{po}(0|6) \longrightarrow \mathfrak{gl}(\Lambda(\xi))$ are representations of $\mathfrak{as}$ in the $4|4$-dimensional modules $\operatorname{spin}_\lambda$ isomorphic to each other for all $\lambda \neq 0$. The explicit form of T_λ is as follows:

$$T_\lambda \colon \begin{pmatrix} a & b \\ c & -a^{\mathrm{t}} \end{pmatrix} + d \cdot z \longmapsto \begin{pmatrix} a & b - \lambda\widetilde{c} \\ c & -a^{\mathrm{t}} \end{pmatrix} + \lambda d \cdot 1_{4|4},$$

where $1_{4|4}$ is the unit matrix and $\widetilde{c}$ is defined in the line under equation (2.28). Clearly, T_λ is an irreducible representation for any λ.

2.8 Prolongs

The Lie superalgebra $\mathfrak{pe}_{\mathrm{gen}}(n)$ is larger than $\mathfrak{pe}(n)$: both B and C in $\mathfrak{pe}_{\mathrm{gen}}(n)$ are symmetric, see (2.27). Observe that $\mathfrak{pe}_{\mathrm{gen}}(n) \subset \mathfrak{sl}(n|n)$. Denote

$$\mathfrak{le}_{\mathrm{gen}}(n; \underline{N}|n) := (\mathrm{id}, \mathfrak{pe}_{\mathrm{gen}}(n))_{*,\underline{N}}.$$

Clearly, if $\underline{N} = \underline{N}_\infty$, see (1.8), then $\mathfrak{le}_{\text{gen}}(n; \underline{N}|n)$ consists of the following two parts, cf. equation (2.26):

$$\text{Reg}_{\text{gen}} = \text{Span}(\text{Le}_f \mid f \in \mathcal{O}(q; \underline{N}|\xi)), \quad \text{Irreg}_{\text{gen}} = \text{Span}(B_i := \xi_i \partial_{q_i})_{i=1}^n. \quad (2.29)$$

The part $\text{Irreg}_{\text{gen}}$ corresponds to the *nonexisting* generating functions ξ_i^2. Clearly, $\mathfrak{le}_{\text{gen}}(n; \underline{N}|n)$ is contained in $\mathfrak{svect}(n; \underline{N}|n)$, and therefore

$$\mathfrak{le}_{\text{gen}}(n; \underline{N}|n) = \mathfrak{sle}_{\text{gen}}(n; \underline{N}|n).$$

The difference between $\mathfrak{le}_{\text{gen}}(n; \underline{N}|n)$ and $\mathfrak{le}(n; \underline{N}|n)$ is constituted by the space $\text{Irreg}_{\text{gen}}$. The nonexisting generating functions $\xi_i^{(2)}$ generate linear vector fields corresponding to the diagonal elements of the matrices B in (2.27), like the $q_i^{(2)}$ generate linear vector fields corresponding to the diagonal elements of the matrices C in (2.27), but these two sets of elements are different in their nature: there are no elements of degree > 0 in $(\text{id}, \mathfrak{pe}_{\text{gen}})_{*,\underline{N}}$ whose brackets with $\mathfrak{g}_{-1}$ give the B_i in (2.29).

The *correct $p = 2$ analogs* of the complex Lie superalgebras $\mathfrak{sle}(n)$ and $\mathfrak{spe}(n)$ are, respectively, $\left(\text{id}, (\mathfrak{pe}(n))^{(1)}\right)_{*,\underline{N}}$ and $\mathfrak{pe}(n)^{(1)}$.

In [48], Lebedev considered $\mathfrak{g} = \mathfrak{pe}(n)$, the derived algebras $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$, and the Cartan prolongs of these derived algebras playing the role of $\mathfrak{g}_0$, whereas for $\mathfrak{g}_{-1}$ he considered the tautological $\mathfrak{g}_0$-module id. Clearly, $\mathfrak{g}^{(1)}$ consists of supermatrices of the form (2.27) with zero-diagonal matrices B and C, whereas $\mathfrak{g}^{(2)}$ is singled out of $\mathfrak{g}^{(1)}$ by the condition $\text{htr} = 0$. The corresponding Cartan prolongs only have the regular parts:

$$\left(\text{id}, \mathfrak{g}^{(1)}\right)_{*,\underline{N}} = \text{Span}\left(\text{Le}_f \mid f \in \mathcal{O}(q; \mathbb{1}|\xi)\right);$$
$$\left(\text{id}, \mathfrak{g}^{(2)}\right)_{*,\underline{N}} = \text{Span}\left(\text{Le}_f \mid f \in \mathcal{O}(q; \mathbb{1}|\xi) \text{ and } \Delta f = 0\right).$$

Let a non-degenerate (anti)symmetric bilinear form $\mathcal{B}$ be defined on a superspace V; let $\mathbf{F}(\mathcal{B})$ be the same form considered on $\mathbf{F}(V)$, the same space with superstructure forgotten. Let $\mathfrak{h}_{\mathcal{B}}(a; \underline{N}|b)$ denote the Hamiltonian Lie superalgebra – the Cartan prolong of the ortho-orthogonal Lie superalgebra $\mathfrak{oo}_{\mathcal{B}}(a|b)$ preserving the non-degenerate form B; its desuperization is $\mathfrak{h}_{\mathbf{F}(\mathcal{B})}(a+b; \underline{\widetilde{N}})$, where $\underline{\widetilde{N}}$ has no critical coordinates.

Remark 2.7. For $\underline{N}$ with $N_i < \infty$ for all i and $p = 2$, the Lie superalgebra $\mathfrak{le}^{(1)}(n; \underline{N}|n)$ is spanned by the elements $f \in \mathcal{O}(q; \underline{N}|\xi)$, whereas each of the "virtual" generating functions $q_i^{(2^{N_i})} \notin \mathcal{O}(q; \underline{N}|\xi)$ determines an outer derivation of $\mathfrak{le}^{(1)}(n; \underline{N}|n)$.

2.8.1   Divergence-free subalgebras $\mathfrak{g}$ of series $\mathfrak{h}$ and $\mathfrak{le}$ in the standard W-grading

These subalgebras are prolongations of subalgebras of 0th components of $\mathfrak{h}$ and $\mathfrak{le}$ consisting of traceless subalgebras; that is how these (super)algebras were described in [48].

It is possible, however, to describe various subalgebras of $\mathfrak{h}_0$ or $\mathfrak{le}_0$, generated by (linear combinations of) quadratic monomials, by eliminating squares of indeterminates from the set of functions generating $\mathfrak{g}_0$. In other words,

constraints imposed on the shearing vector $\underline{N}$ corresponding to the space of generating functions determine various divergence-free subalgebras of $\mathfrak{h}(n; \underline{N})$ and $\mathfrak{le}(n; \underline{N})$.

2.8.2 $\mathfrak{svect}_{a,b}(0|n)$

For $p > 0$, let $\mathfrak{svect}_{a,b}(0|n)$ denote $\mathfrak{svect}(0|n) \ltimes \mathbb{K}(az + bd)$, where the element $d := \sum \xi_i \partial_{\xi_i}$ determines the standard $\mathbb{Z}$-grading of $\mathfrak{svect}(0|n)$, while z is an element generating the trivial center commuting with $\mathfrak{svect}(0|n) \ltimes \mathbb{K} \cdot d$.

2.8.3 $\mathfrak{spe}_{a,b}(n)$

For $p = 0$, the meaning of $\mathfrak{spe}_{a,b}(n)$ is similar to that of $\mathfrak{svect}_{a,b}(0|n)$, but with $d := \mathrm{diag}(1_n, -1_n)$. To define the analog of $\mathfrak{spe}_{a,b}(n)$ for $p = 2$, see line $N = 7$ in Table 25.2, observe that the codimension of $\mathfrak{spe}(n)$ in $\mathfrak{m}_0$, where $\mathfrak{m} := \mathfrak{m}(n)$ is considered in its standard $\mathbb{Z}$-grading, is equal to 2.

So, to pass from $\mathfrak{spe}(n)$ to $\mathfrak{m}_0$, we have to add *two linearly independent* elements, whereas to pass to $\mathfrak{spe}_{a,b}$ we have to add a linear combination of these elements with coefficients a and b. The question is: "can we single out these elements in a canonical way?"

For $p = 0$. The identity operator (in matrix realization) is one of these elements. How to select the other element? There is no distinguished element in $\mathfrak{pe}(n) \setminus \mathfrak{spe}(n)$. But, if $p = 0$, there is an element $\mathrm{diag}(1_n, -1_n)$ corresponding to a "most symmetric" generating function $\sum q_i \xi_i$.

For $p > 2$ this "most symmetric" element lies in $\mathfrak{spe}(n)$ if p divides n and the choice of the linearly independent second element from $\mathfrak{pe}(n) \setminus \mathfrak{spe}(n)$ becomes a matter of taste.

For $p = 2$, the situation becomes completely miserable. Now, the restriction of $M_{\sum q_i \xi_i}$ to $\mathfrak{m}_{-1}$ not only lies in $\mathfrak{spe}(n)$ for n even, it coincides with the identity operator. So, in this case, there is no distinguished operator not lying in $\mathfrak{spe}(n)$. What to do?

We suggest considering the elements of $\mathfrak{m}_0$ as operators acting not just on $\mathfrak{m}_{-1}$, but on the whole $\mathfrak{m}_-$. If $\mathfrak{m}_0$ is thus understood, there are two well-defined linear forms ℓ and μ that single out $\mathfrak{spe}(n)$ in $\mathfrak{m}_0$:

for any operator $A \in \mathfrak{m}_0$, let $A_i = \mathrm{ad}_A |_{\mathfrak{m}_i}$; then $A_{-2} = \ell(A) \cdot \mathrm{id}$ and A_{-1} is as in (2.27),

$$\mu(A) = \begin{cases} \mathrm{htr}(A_{-1}) & \text{for } p = 2, \\ \mathrm{str}(A_{-1}) & \text{for } p > 2. \end{cases} \tag{2.30}$$

Now, $\mathfrak{spe}(n)$ is singled out by conditions $\ell(A) = \mu(A) = 0$, while

$$\mathfrak{spe}_{a,b}(n) := \{X \in \mathfrak{pe}(n) = \mathfrak{m}_0 \,|\, (a\mu + b\ell)(X) = 0\}.$$

2.9   On $\mathfrak{m}$ and $\mathfrak{b}$

To pass from $\mathfrak{b}(n; \underline{N}|n)$ to $\mathfrak{m}(n; \underline{N}|n+1)$, we have to add to $\mathfrak{b}(n)_0 = \mathfrak{pe}(n)$ the central element; it will serve as a grading operator of the prolong. We see that $\mathfrak{m}$ is the generalized Cartan prolong of $(\mathfrak{b}(n)_-, \mathfrak{cb}(n)_0)$.

The commutant of $\mathfrak{m}(n; \underline{N}|n + 1)_0$ is the like that of $\mathfrak{b}(n)_0 = \mathfrak{pe}(n)$, so is of codimension 2. Hence there are two traces on $\mathfrak{m}(n; \underline{N}|n + 1)_0$, namely htr and ℓ, see (2.30), and therefore there are two divergences on $\mathfrak{m}$. One of them is given by the operator

∂_τ,   more precisely $D_\tau := \partial_\tau \circ \mathrm{sign}$,

i.e., $D_\tau(f) = (-1)^{p(f)} \partial_\tau(f)$ for any $f \in \mathcal{O}(q; \underline{N}|\tau, \xi)$

since this should be the mapping *commuting* (not *super*commuting) with $\mathfrak{m}_-$, see [66]. The condition $D_\tau(f) = 0$, i.e., just $\partial_\tau(f) = 0$ singles out precisely $\mathfrak{b}(n)$.

The other divergence is given by the operator (2.32).

2.9.1 $\mathfrak{sb}$

The definition of $\mathfrak{sb}(n; \underline{N})$ is the same for any characteristic p (in terms of generating "functions" from an appropriate space $\mathcal{F}$, see (1.9)):

$$\mathfrak{sb}(n; \underline{N}) = \mathrm{Span}(f \in \mathcal{F} \,|\, \Delta(f) = 0).$$

2.9.2 $\mathfrak{b}_{a,b}$ for $p = 2$

The direct analog of trace on $\mathfrak{m}_0$ is htr. On $\mathfrak{le}$, the prolong of htr is the operator Δ. But Δ does not commute with the whole of $\mathfrak{m}_-$. To obtain the $\mathfrak{m}_-$-invariant prolong of this trace on $\mathfrak{m}_0$, we have to express htr in terms of the operators commuting with $\mathfrak{m}_-$ (Y-*type vectors* in terms of [66]). Taking $\mathfrak{m}_-$ spanned by the elements

$$\mathfrak{m}_{-2} = \mathbb{K} \cdot \partial_\tau, \quad \mathfrak{m}_{-1} = \mathrm{Span}(\partial_{q_i} + \xi_i \partial_\tau, \; \partial_{\xi_i})_{i=1}^n,$$

we see that the operators commuting with $\mathfrak{m}_-$ are spanned by

$$\partial_\tau, \quad \partial_{q_i}, \quad \partial_{\xi_i} + q_i \partial_\tau.$$

In terms of these operators, the vector field M_f takes the form:

$$M_f = f\partial_\tau + \sum_i \left(\partial_{q_i}(f)(\partial_{\xi_i} + q_i\partial_\tau) + (\partial_{\xi_i} + q_i\partial_\tau)(f)\partial_{q_i} \right) \tag{2.31}$$

and the invariant prolong of htr – the direct analog of *divergence* – takes the form:

$$\Delta^{\mathfrak{m}}(f) = \sum_i ((\partial_{\xi_i} + q_i\partial_\tau)\partial_{q_i}(f) = \Delta(f) + E_q\partial_\tau(f), \quad \text{where } E_q := \sum_i q_i\partial_{q_i}. \tag{2.32}$$

The condition $\Delta^{\mathfrak{m}}(f) = 0$ singles out the $p = 2$ analog of $\mathfrak{sm}$, whereas the condition

$$b\partial_\tau(f) + a\Delta^{\mathfrak{m}}(f) = (b + aE_q)\partial_\tau(f) + a\Delta(f) = 0 \tag{2.33}$$

singles out the $p = 2$ analog of $\mathfrak{b}_{a,b}$, cf. (2.24).

Setting

$$\mathfrak{po}_{a,b}\big(2n; \widetilde{\underline{N}}\big) := \mathbf{F}(\mathfrak{b}_{a,b}(n; \underline{N}))$$

we single out a subalgebra in the Lie algebra of contact vector fields *which has no analogs for* $p \neq 2$.

Let us figure out how the parameter λ of the regrading $\mathfrak{po}_\lambda(2n; \underline{N}) := \mathbf{F}(\mathfrak{b}_\lambda(n; \underline{N}))$ depends on parameters a, b above; for a summary, see $N = 6, 7$ in Table 25.2. The space of $\mathfrak{b}_{a,b}(n; \underline{N})$ consists of vector fields (2.31) whose generating functions satisfy equation (2.33); the regrading

$$\deg \tau = \deg q_i = 1, \quad \deg \xi_i = 0 \text{ for all } i$$

turns $\mathfrak{b}_{a,b}(n; \underline{N})$ into the Lie superalgebra $\mathfrak{b}_\lambda(n; \underline{N}; n)$ whose 0th component is isomorphic to $\mathfrak{vect}(0|n)$ and the (-1)st component is isomorphic to the $\mathfrak{vect}(0|n)$-module Vol^λ of weighted λ-densities. Set

$$\mathfrak{po}_\lambda\big(2n + 1; \widetilde{\underline{N}}\big) := \mathbf{F}(\mathfrak{b}_\lambda(n; \underline{N}; n)).$$

To express λ in terms of the parameters a, b, we take an element in the 0th component of $\mathfrak{b}_{a,b}(n; \underline{N})$ not lying in $\mathfrak{b}(n; \underline{N})$ and see how it acts on M_1.

Let $f = \alpha\tau + \beta q_1\xi_1$. Equation (2.33) implies that $\alpha b + \beta a = 0$, so we can take $f = a\tau + bq_1\xi_1$. Observe that $M_{bq_1\xi_1}$ acts on the (-1)st component as the vector field $D = b\xi_1\partial_{\xi_1}$ and

$$[M_{a\tau}, M_1] = [a\tau\partial_\tau, \partial_\tau] = a\partial_\tau = \frac{a}{b}(\mathrm{div}\, D)M_1, \quad \text{hence } \lambda = \frac{a}{b}.$$

The $p = 2$ version of equation (2.25) are the following exact sequences that single out the simple Lie superalgebras (recall that $\mathfrak{b}(n; \underline{N}) = \mathfrak{b}_\lambda(n; \underline{N})$ for $\lambda = 0$):

$$0 \longrightarrow \mathbb{K}M_1 \longrightarrow \mathfrak{b}(n; \underline{N}) \longrightarrow \mathfrak{le}(n; \underline{N}) \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{b}_1^{(1)}(n; \underline{N}) \longrightarrow \mathfrak{b}_1(n; \underline{N}) \longrightarrow \mathbb{K} \cdot M_{\xi_1 \dots \xi_n} \longrightarrow 0,$$
$$0 \longrightarrow \mathfrak{b}_\infty^{(1)}(n; \underline{N}) \longrightarrow \mathfrak{b}_\infty(n; \underline{N}) \longrightarrow \mathbb{K} \cdot M_{\tau \xi_1 \dots \xi_n} \longrightarrow 0. \qquad (2.34)$$

The following statement follows directly.

Proposition 2.8 (two exceptional deforms of the Poisson algebra)**.** *Desuperization of the simple Lie superalgebras introduced in* (2.34) *yields* 2 *exceptional serial Lie algebras that have no analogs for $p \neq 2$:*

$$\mathfrak{po}_1^{(1)}(n; \underline{N}) := \boldsymbol{F}\big(\mathfrak{b}_1^{(1)}(n; \underline{N})\big) \quad and \quad \mathfrak{po}_\infty^{(1)}(n; \underline{N}) := \boldsymbol{F}\big(\mathfrak{b}_\infty^{(1)}(n; \underline{N})\big).$$

Problem 2.9. *In* [58], *we described the deformations of the Buttin algebra $\mathfrak{b}(n)$ over $\mathbb{C}$, having corrected a result due to Kochetkov* [42] *who described exceptional deformations of $\mathfrak{b}_\lambda(n)$ at certain values of λ; some of these deformations having an odd parameter. It is an open problem to obtain a version of* [58] *in the modular case.*

2.10 On $\mathfrak{k}$ and $\mathfrak{po}$ for $p = 2$

Observe that $\mathfrak{po}_I(n; \underline{N})$, a central extension of the Lie algebra $\mathfrak{h}_I(n; \underline{N})$, *is not a Lie algebra.* Indeed, the bracket should be anti-symmetric, i.e., alternate, while $\{x_i, x_i\}_I = 1$, not 0, so $\mathfrak{po}_I(n; \underline{N})$ is a *Leibniz* algebra, not Lie algebra. Only $\mathfrak{h}_\Pi(2n; \underline{N})$ has an analog of the familiar central extension; *this* nontrivial central extension is a correct direct analog of the complex Poisson Lie (super)algebra $\mathfrak{po}(2n|0)$.

To pass from $\mathfrak{po}(0|n)$ to $\mathfrak{k}(1; \underline{N}|2n)$, we have to add, as a direct summand, a central element to $\mathfrak{po}(0|n)_0 = \mathfrak{o}_\Pi^{(1)}(n)$; it will act on the prolong of $\big(\mathfrak{po}(0|n)_-, \mathfrak{co}_\Pi^{(1)}(n)\big)$ as a grading operator. We see that the generalized Cartan prolong of $\big(\mathfrak{po}(0|n)_-, \mathfrak{co}_\Pi^{(1)}(n)\big)$ is $\mathfrak{k}(1; \underline{N}|2n)$.

• The commutant of $\mathfrak{k}_0(1; \underline{N}|2n)$ is isomorphic to that of $\mathfrak{po}_0(0|n) = \mathfrak{o}_\Pi^{(1)}(n)$, so it is of codimension 2 in $\mathfrak{k}_0(1; \underline{N}|2n)$. Thus, there are *two* traces on $\mathfrak{k}_0(1; \underline{N}|2n)$, and hence there are *two* divergences on $\mathfrak{k}(1; \underline{N}|2n)$, like on $\mathfrak{m}(n; \underline{L}|n)$. These divergences are given by almost the same formulas as for $\mathfrak{m}(n; \underline{L}|n)$, where $\underline{L} = (N, \mathbb{1})$ and 2^N is the hight of t, "almost" because ∂_t should replace ∂_τ.

For $p \neq 2$, we have two divergences on $\mathfrak{k}(1; \underline{N}|2n)$ only if $n = 1$, see Section 2.5.1.

2.11 Exceptional simple vectorial Lie superalgebras for $p = 2$ analogous to their namesakes over $\mathbb{C}$

We give detailed description of all exceptional simple vectorial Lie superalgebras over $\mathbb{C}$ and fields of characteristic 2 in the main text; for a summary, see Section 25. These Lie superalgebras constitute two non-intersecting sets as follows.

The complete Cartan prolong of its *negative* part: such is every Lie superalgebra of series $\mathfrak{vect}$, $\mathfrak{k}$ and $\mathfrak{m}$ in the standard grading, see (2.18) and each simple exceptional Lie superalgebra $\mathfrak{g}$ of depth > 1, whose negative part in its W-grading is different from the negative part of the Lie superalgebras of series $\mathfrak{k}$ or $\mathfrak{m}$ in their respective standard gradings.

The complete Cartan prolong of its *nonpositive* part: such are the exceptional vectorial Lie superalgebras, and their desuperizations, see Tables 25.3 and 25.5, other than in the above paragraph; the corresponding gradings are explicitly given in Table (25.4).

The desuperizations of two nonisomorphic Lie superalgebras realized by vector fields on supervarieties of different superdimension might turn out to be vectorial Lie algebras realized on varieties of the same dimension. We distinguish these cases by indicating their depths as an index at the name ($\mathfrak{mb}_2(11; \underline{N})$ and $\mathfrak{mb}_3(11; \underline{N})$, and also $\mathfrak{kle}_2(20; \underline{N})$ and $\mathfrak{kle}_3(20; \underline{N})$); for the case of equal depths, we distinguish non-isomorphic algebras by a tilde: $\mathfrak{vle}(9; \underline{N})$ and $\widetilde{\mathfrak{vle}}(9; \underline{L})$, as well as $\mathfrak{kas}(7; \underline{N})$ and $\widetilde{\mathfrak{kas}}(7; \underline{L})$; for details, see respective sections.

2.12   A technical remark: natural generators of vectorial Lie superalgebras

This subsection is needed for calculations only. Let $\mathfrak{g} = \oplus \mathfrak{g}_i$ be a Weisfeiler grading of a given simple vectorial Lie superalgebra. We see that $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$-module with highest-weight vector H, and $\mathfrak{g}_1$ is the direct sum of indecomposable (sometimes, irreducible) $\mathfrak{g}_0$-modules with lowest-weight vectors v_i.

Over $\mathbb{C}$, and over $\mathbb{K}$ for $\underline{N} = \mathbb{1}$, the simple Lie superalgebra $\mathfrak{g}$ is generated (bar a few exceptions) by the generators of $\mathfrak{g}_0$, the vector H, and the v_i. (For other values of $\underline{N}$, we have to add the $\mathfrak{g}_0$-lowest-weight vectors $v_j^k \in \mathfrak{g}_j$ for some $j > 1$ to the above generators; these cases are not considered.) So we have to describe the generators of $\mathfrak{g}_0$, or rather of its quotient modulo its center.

If $\mathfrak{g}_0$ is of the form $\mathfrak{g}(A)$ or its simple subquotient, we select its Chevalley generators, see [10].

If $\mathfrak{g}_0$ is an almost simple "lopsided", see Section 3.2.2 (in particular, of type $\mathfrak{pe}$, $\mathfrak{spe}$), but $\mathbb{Z}$-graded Lie superalgebra, we apply the above-described procedure to $\mathfrak{g}_0$: first, take *its* 0th components and its generators, then the highest and lowest-weight vectors in *its* components of degree ± 1, etc.

If $\mathfrak{g}_0$ is semi-simple of the form $\mathfrak{s} \otimes \Lambda(r) \ltimes \mathfrak{vect}(0|k)$, where $\mathfrak{s}$ is almost simple, then we take the already described generators of $\mathfrak{vect}(0|k)$ and apply the above procedures to $\mathfrak{s}$.

For a list of defining relations for many simple Lie superalgebras over $\mathbb{C}$, and their relatives, see [27, 29]. For defining relations for Lie algebras with Cartan matrix over $\mathbb{K}$, see [5].

3   Introduction: overview of the scenery

In the Introduction (divided into two parts to ease digesting it) we give a brief sketch of the main constructions and ideas; for basic background, see Section 1. For further details, see [48, 55, 56]. All voluminous computations are performed with the help of the *SuperLie* package, see [26].

3.1   Goal: classification of simple finite-dimensional Lie algebras over $\mathbb{K}$ a.k.a. *modular*

In 1960s, Kostrikin and Shafarevich suggested a method for producing simple finite-dimensional Lie algebras over $\mathbb{K}$ for any $p > 0$, together with the final list for $p > 5$. This list is explicit for simple $\mathbb{Z}$-graded algebras; for the rest, it is somewhat implicit ("and deforms of $\mathbb{Z}$-graded algebras"), see [40]. The above-mentioned deforms are often deforms of non-simple algebras the stock of which was not clearly described; this made this part of the KSh-method rather vague.

3.1.1   The original KSh-method

The initial ingredients are simple Lie algebras over $\mathbb{C}$ of two types:

$$\text{finite-dimensional, i.e., of the form } \mathfrak{g}(A), \text{ where } A \text{ is a Cartan matrix,} \tag{3.1}$$

infinite-dimensional vectorial types ($\mathfrak{vect}$, $\mathfrak{svect}$, $\mathfrak{h}$, and $\mathfrak{k}$)

$$\text{with polynomial coefficients.} \tag{3.2}$$

Next, one, respectively,

$$\text{takes a } \mathbb{Z}\text{-form } \mathfrak{g}(A)_{\mathbb{Z}} \text{ of } \mathfrak{g}(A) \text{ corresponding to the Chevalley basis,}$$

$$\text{and tensors with } \mathbb{K} \text{ to get } \mathfrak{g}(A)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}, \tag{3.3}$$

considers modular, over $\mathbb{K}$, analogs of simple vectorial Lie algebras over $\mathbb{C}$ with divided

$$\text{powers as coefficients of } \textit{distinguished} \text{ partial derivatives, see equation (1.10).} \tag{3.4}$$

The ingredient (3.1) yields via (3.3) one finite-dimensional Lie algebra; the ingredient (3.2) yields via (3.4) an infinite family of finite-dimensional Lie algebras over $\mathbb{K}$ depending on the shearing vector $\underline{N}$. Each of the finite-dimensional Lie algebras thus obtained is either simple, or a "relative" of the simple Lie algebra over $\mathbb{K}$ (a central extension or a subalgebra in the algebra of derivations). Some of these simple Lie algebras can be deformed.

$$\text{To describe the } \textit{deforms} \text{ is a } \textit{rather complicated} \text{ part of the KSh-method.} \tag{3.5}$$

Let us clarify claim (3.5). Tables (3.8) and (3.11) show that some of simple Lie algebras are filtered deforms not of the simple $\mathbb{Z}$-graded algebras, but of certain non-simple subalgebras of their Cartan prolongs (since their dimensions differ from those of simple algebras). The list of deforms was obtained in a roundabout way, avoiding computing the cohomology that describes a filtered deformation:

1) In [76], Wilson sharpened a result due to Kac [32, Proposition 7.2] and classified equivalence classes of volume forms for $p > 5$; later, it turned out that the description works for $p > 2$. (Earlier Tyurin published a solution of the same problem [74], but got more classes than Wilson: Tyurin missed some equivalences.)

2) Skryabin [69, 70] classified (for $p > 2$) all equivalence classes of symplectic forms (Skryabin called them Hamiltonian forms); some of Skryabin's difficult-to-obtain results hold for $p = 2$ as well.

Types of Lie algebras $\mathfrak{svect}$ described by Tyurin and Wilson [74, 76]. In the mid-1970s, Kac observed in [32] that the Lie algebra that preserves the volume element of the form $h\,\mathrm{vol}$, where $h \in \widehat{\mathcal{O}}(\widehat{\underline{N}}_i)$ is invertible, can be a subalgebra of $\mathfrak{vect}(m; \underline{N})$ with finite coordinates of $\underline{N}$. Let $p > 2$ and suppose that

$$N_1 = \cdots = N_{t_1} < N_{t_1+1} = \cdots = N_{t_2} < \cdots < N_{t_{s-1}+1} = \cdots = N_{t_s} = N_m. \tag{3.6}$$

The results of Tyurin and Wilson, correct for $p > 3$, state that there are only the following three types of non-equivalent classes of volume forms, and hence filtered deforms with parameter $\varepsilon \in \mathbb{K}^\times$ of divergence-free algebras preserving them:

$$\mathfrak{svect}_h(m; \underline{N}) := \{D \in \mathfrak{vect}(m; \underline{N}) \mid L_D(h\,\mathrm{vol}) = 0\}, \quad \text{where } h \text{ is one of the following:}$$

$$h = \begin{cases} 1, \\ 1 + \varepsilon\bar{u}, \quad \text{where } \bar{u} := \prod \bar{u}_i \text{ and } \bar{u}_i := u_i^{(p^{N_i}-1)}, \\ \exp\left(\varepsilon u_{t_i}^{\left(p^{N_{t_i}}\right)}\right) := \sum_{j \geq 0} \left(\varepsilon u_{t_i}^{\left(p^{N_{t_i}}\right)}\right)^{(j)} \in \widehat{\mathcal{O}}(\widehat{\underline{N}}_{t_i}). \end{cases} \tag{3.7}$$

For brevity, set $\mathfrak{svect}_{\exp_i}(m; \underline{N}) := \mathfrak{svect}_h(m; \underline{N})$, where $h = \exp\left(\varepsilon u_{t_i}^{\left(p^{N_{t_i}}\right)}\right)$.

Remarks 3.1.

1. For $p = 3$ and 2, these deformations of $\mathfrak{svect}$ are also possible. For $p = 3$, nobody knows if there are other deforms, whereas for $p = 2$, there definitely is at least one more deform: its existence is the most spectacular result of this paper, see Section 14.

2. S. Tyurin described the Lie algebras of divergence-free type and got an extra type of volume form, as compared with Wilson's list (3.7), cf. [74].

3. S. Kirillov [38] verified Skryabin's remark in passing [71] for which i the ith derived algebra from Wilson's list (3.7) is simple, and found the dimensions of these simple Lie algebras:

$$\begin{array}{|l|c|}
\hline
\dim \mathfrak{svect}_{\exp_j}(m; \underline{N}) = (m-1)p^{|\underline{N}|} & i = 0 \\
\dim \mathfrak{svect}_{1+\bar{u}}^{(1)}(m; \underline{N}) = \dim \mathfrak{svect}^{(1)}(m; \underline{N}) = (m-1)p^{|\underline{N}|} - m + 1 & i = 1 \\
\hline
\end{array} \tag{3.8}$$

Hamiltonian Lie algebras $\mathfrak{h}$ described and classified by S. Skryabin [69, 70]. Let $\mathfrak{h}(2k; \underline{N})$ or $\mathfrak{h}_{\omega_0}(2k; \underline{N})$ be the $\mathbb{Z}$-graded Lie algebra preserving the symplectic form

$$\omega_0 = \sum_{1 \leq i \leq k} du_i \wedge du_{k+i}.$$

The only non-isomorphic filtered deforms of $\mathfrak{h}_{\omega_0}(2k; \underline{N})$ with parameter $\varepsilon \in \mathbb{K}^\times$ are $\mathfrak{h}_{\omega_i}(2k; \underline{N})$, where $i = 1, 2$, which preserve the following respective forms (of *type* 1 *and* 2 in Skryabin's terminology):

$$\omega_{2,j} = d \left(\exp(\varepsilon u_j) \sum_{1 \leq i \leq k} u_i du_{k+i} \right), \text{ where } j = t_1, \ldots, t_s, \text{ see (3.6)}, \tag{3.9}$$

$$\omega_{1,A} = \omega_0 + \varepsilon \sum_{1 \leq i,j \leq 2k} A_{i,j} d(\bar{u}_i) \wedge d(\bar{u}_j), \quad \text{where } \bar{u}_i := u_i^{(p^{N_i})} \text{ for the shearing vector } \underline{N},$$

and where the non-equivalent normal shapes of the indecomposable matrices $A = (A_{i,j})$ can only be equal for $p > 2$ to one of the following:

type of A	form of A	detailed notation of ω_1
$J_k(0)$	antidiag $\left(J_k(0), -J_k(0)^\mathrm{T} \right)$	$\omega_{1,0}$ for $k > 1$
$J_{k,r}(\lambda)$, where $\lambda \neq 0$	antidiag $\left(J_{k,r}(\lambda), -(J_{k,r}(\lambda))^\mathrm{T} \right)$	$\omega_{1,r,\lambda}$ for $k = rn$ for $r, n \geq 1$
C_k	antidiag $\left(C_k, -C_k^\mathrm{T} \right)$	$\omega_{1,C}$ for $k > 1$

where $J_k(\lambda)$ is a Jordan $k \times k$ block with eigenvalue λ, and $J_{k,r}(\lambda)$ is a $k \times k$ block matrix with blocks of size $r \times r$, so $k = r \times n$ for some $r, n \geq 1$:

$$J_{k,r}(\lambda) = \begin{pmatrix} 0_r & 1_r & & 0 \\ & & \ddots & \\ 0 & 0 & & 1_r \\ J_r(\lambda) & 0 & & 0_r \end{pmatrix},$$

and

$$C_k = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ 1 & 0 & & 0 \end{pmatrix} \text{ is of size } k \times k \text{ for } k > 1.$$

The two conditions on $J_{k,r}(\lambda)$ and C_k. 1) The case with $J_{k,r}(\lambda)$ occurs only when

$$N_1 + \cdots + N_{nr} = N_{nr+1} + \cdots + N_{2nr} \quad \text{(recall that } k = rn) \tag{3.10}$$

and, furthermore, $N_{ir-j} = N_{ir}$ for all $i = 1, \ldots, 2n$ and all $j = 1, \ldots, r-1$; i.e., r indeterminates in each of the $2n$ successive groups have equal heights.

The case with C_k occurs only when condition (3.10) is violated.

2) Let G be the group generated by the cyclic permutations of the row vectors of length k. Then, the identity element is the only permutation in G that simultaneously fixes the two vectors

$$a = (N_1, \ldots, N_k) \quad \text{and} \quad b = (N_{k+1}, \ldots, N_{2k}).$$

It suffices to consider representatives of equivalence classes of pairs (a, b) under the G-action.

Remarks 3.2.

1. Over $\mathbb{C}$ the supervarieties of parameters of deformations of Poisson and Hamiltonian Lie superalgebras can differ, see [58]. For $p = 2$, there is at least one new type of deform: a 1-parametric family of non-isomorphic deforms different from the above – desuperisations of $\mathfrak{b}_{a,b}(n; \underline{N})$.

2. S. Kirillov [38] determined the i for which the ith derived algebra of the Hamiltonian Lie algebra from Skryabin's list [69] is simple and what its dimension is equal to:

$$
\dim \mathfrak{h}_\omega^{(i)}(2k; \underline{N}) = \begin{cases}
p^{|\underline{N}|} - 2 & \text{if } \omega = \omega_0 & i = 2, \\
p^{|\underline{N}|} & \text{if } \omega = \omega_2, \text{where } k + 1 \not\equiv 0 \pmod{p} & i = 0, \\
p^{|\underline{N}|} - 1 & \text{if } \omega = \omega_2, \text{where } k + 1 \equiv 0 \pmod{p} & i = 1, \\
p^{|\underline{N}|} - 1 & \text{if } \omega = \omega_{1,A}, \text{where } \det A \neq 0 & i = 1, \\
p^{|\underline{N}|} - 2 & \text{if } \omega = \omega_{1,A}, \text{where } \det A = 0 & \\
& \text{(type } J_s(0)) & i = 2.
\end{cases}
\tag{3.11}
$$

3.1.2 True and semi-trivial deforms

In particular, the amount of infinitesimal deformations is overwhelming and even frightening as p becomes small ($p = 3$ or – a horrible case – $p = 2$). We recall reasons not to be too frightened; besides, the KSh-method had been improved over the past years.

The abundance of deforms of simple Lie (super)algebras for $p > 0$, especially overwhelming for $p = 2$, is somewhat misleading. It is occasioned by *semi-trivial deforms* each of which is given by a cocycle representing a nontrivial cohomology class but, though integrable, yields a deform isomorphic to the initial algebra. For a description of many semi-trivial deforms, see [13]. We say that a nontrivial and nonsemi-trivial deform is a *true deform*.

The Lie (super)algebra $\mathfrak{g}$ is said to be *rigid* if it has no true deforms; until recently we thought that semi-trivial deforms existed only if $p > 0$, but a more careful study of the literature shows they are a universal phenomenon [61].

If $p > 3$, the classification has been completed, mainly due to Premet and Strade [3, 73], based on explicit description of deforms [4, 70, 76].

If $p = 3$, we conjecture the classification: the examples obtained by Cartan prolongation (see Section 1.6) of appropriate parts of Lie algebras with Cartan matrix [7, 28], exhaust the list of "standard" examples some of which were discovered by Frank, Ermolaev and, mainly, Skryabin. For an (incomplete at the moment) list of true deforms of several "standard" algebras, see [8, 45, 46, 47, 70], and [14] in which an earlier claim concerning deforms is corrected.

If $p = 2$, we are still completing the stock of "standard" examples.

3.1.3 Amendments to the formulation of the goal

On several occasions P. Deligne told us what we understood as follows (for Deligne's own words, and several open problems, see [49]):

> "In positive characteristic, the problem "classify ALL simple Lie (super)algebras, and their representations" is, perhaps, not very reasonable, and definitely very tough; investigate first the *restricted* case related to geometry, and hence meaningful."

Following Deligne's advice, we investigated several plausible notions of restrictedness for $p = 2$ in [12] and gave explicit expressions of the restriction maps for several types of simple Lie algebras and superalgebras in [11]. Nevertheless, even to describe restricted Lie (super)algebras one often needs nonrestricted ones; for more serious examples of their usage, see [40].

In this paper, we concentrate on *simple* Lie (super)algebras, keeping in mind that algebras of the following types are no less important than simple ones:

- Lie (super)algebras of the form $\mathfrak{g}(A)$ where A is indecomposable, see [7, 10].

- Central extensions and algebras of derivations of the known simple Lie (super)algebras (see Section 14.1 and [6]). The algebras of these two types will be called *relatives* of the corresponding simple Lie (super)algebras and each other.

- The generalized Cartan prolongs $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}}$ with $\mathfrak{g}_0$ close to simple, see Sections 1.6 and 14.1.

- True deforms (for definition, see Section 3.1.2) of Lie (super)algebras, see [41, 8].

- Restricted closures of nonrestricted simple Lie algebras.

3.2 Improvements of the KSh-method

3.2.1 "Standard" modular Lie algebras

Dzhumadildaev and Kostrikin [41] suggested simplifying the KSh-method by skipping the step over $\mathbb{C}$ and considering certain "standard" modular Lie algebras from the very beginning, further deforming them and their "relatives". On the other hand, the stock of "standard" examples should include, if $p < 7$, certain non-simple Lie algebras, see [25, 41, 72]. The snag is: we have no idea how to select them.

Until the year 2000 or so, it was believed that the initial KSh-method produces all simple Lie algebras only if $p > 5$. This belief was based on insufficient study of deformations and too narrow a choice of "standard" examples: as shown in [41], the Melikyan algebra, indigenous for $p = 5$, are *deforms* of Poisson Lie algebra which should be considered "standard" and processed via the KSh-scheme (3.1)–(3.5).

What examples should qualify as "standard"? In [51], the improvement of the KSh-method suggested in [41] was developed further by eliminating the vectorial simple Lie algebras from the input of the KSh-method thus diminishing the stock of "standard" simple Lie algebras. In the new procedure, the role of *generalized Cartan prolongation* (complete or partial), see Section 1.6 and especially Section 1.6.1, becomes even more important than in the KSh-procedure. This approach definitely works for $p > 3$, and conjecturally works for $p = 3$.

The stock of "standard" (not necessarily simple) Lie algebras must be enlarged with examples found after [41] was published; for $p = 3$, see [28]; for $p = 2$, see [7, 13, 22, 25, 48, 72], and this paper.

3.2.2 Splitting the problem into smaller chunks

All simple Lie algebras are of the following two types: the root system of a "*symmetric*" algebra contains the root $-\sigma$ of the same multiplicity as that of σ for any root σ; the algebras with root systems without this property are said to be "*lopsided*".

This paper is devoted to the study of lopsided algebras, but "symmetric" Lie (super)algebra will be needed in the process

Symmetric algebras. A significant quantity of symmetric simple Lie algebras consists of algebras $\mathfrak{g}(A)$ *with indecomposable Cartan matrix A or simple "relatives" of such algebras* of the form $\mathfrak{g}^{(i)}(A)/\mathfrak{c}$, where[4] $\mathfrak{g}^{(i)}(A)$ is the ith derived algebra of $\mathfrak{g}(A)$ and $\mathfrak{c}$ is the center of $\mathfrak{g}^{(i)}(A)$.

For any p, finite-dimensional Lie algebras $\mathfrak{g}(A)$ with indecomposable Cartan matrix A, and their simple relatives, were classified in [75] with an omission; for corrections, see [34, 71], where

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[4]This is shorter and more graphic than the correct notation $(\mathfrak{g}(A))^{(i)}$; usually, we will similarly place subscripts designating the degree (closer to the "family name" $\mathfrak{g}$).

no claim was made that these were the only corrections needed; for this claim with a proof, a classification of Lie superalgebras of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix A, and their simple relatives, and precise definitions of related notions, see [10].

   Lopsided algebras: the set they constitute is a virtually virgin territory a part of which – vectorial Lie (super)algebras – we investigate through the whole of this text.

3.2.3   Cartan prolongations of Lie (super)algebras with Cartan matrix

It turns out that every known $\mathbb{Z}$-graded simple Lie algebra for $p > 2$ is obtained as a (generalized, perhaps partial) Cartan prolong of the non-positive part of a Lie algebra $\mathfrak{g}(A)$ with an indecomposable Cartan matrix A. For $p > 3$ this follows from the classification.

Conjecture 3.3. *The above $\mathbb{Z}$-graded simple Lie algebras and simple relatives of their deforms constitute the list of simple finite-dimensional Lie algebras for $p = 3$ as well.*

3.3   Super goal

Although Lie *super*algebras appeared in topology in the 1940s (over finite fields, often over $\mathbb{Z}/2$), the understanding of their importance dawned only in the 1970s, thanks to their applications in physics. This understanding put the problem "classify simple Lie superalgebras" on the agenda of researchers. Over $\mathbb{C}$, the finite-dimensional simple Lie superalgebras were classified by several teams of researchers, see reviews [33, 37]. The classification of certain types of simple vectorial Lie superalgebras was explicitly announced in [33], together with a conjecture listing all *primitive* vectorial Lie superalgebras; for the first counterexamples, see [1, 50].

   A classification of the simple vectorial Lie superalgebras over $\mathbb{C}$ was implicitly announced when the first *exceptional* examples were given [63, 64, 65] and explicitly at a conference in honor of Buchsbaum [57]. The claim of [33] was corrected in [55] (the correction contained both the complete list of simple vectorial algebras, bar one exception later described in [64], and the method of classification of simple $\mathbb{Z}$-graded Lie superalgebras of depth 1) and in a series of papers [17, 19, 20, 21, 35, 36], where the proof in the case of $\mathbb{Z}$-grading compatible with parity was given; for further corrections and proofs, see Section 13. The classification is not completed till today: there is no classification of deformations with odd parameters.

   Although to complete the classification of the simple finite-dimensional Lie superalgebras over $\mathbb{K}$ for p "sufficiently big" (say > 7) will be a more cumbersome and excruciating task than that for Lie algebras, the answer (conjectural, but doubtless) is obvious: to get *restricted* superalgebras, take the obvious modular analogs of the complex simple Lie superalgebras (of both finite-dimensional and of infinite-dimensional vectorial considered for the shearing vector $\underline{N} = \mathbb{1}$, see definition (1.8)) passing to the derived algebra and quotients modulo center if needed; to get *nonrestricted* superalgebras, consider *true deforms*, see Section 3.1.2, of the above-mentioned analogs (for $\underline{N}$ unconstrained, speaking about vectorial algebras). For p "small", the classification problem becomes more and more involved, see, e.g., [9, 30]. Nevertheless, in the two cases the classification is obtained:

- *For any p, the super goal is reached for Lie superalgebras of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrices A or its "relative"*, see [10]. Either $\mathfrak{g} = \mathfrak{g}(A)$ or its "relative" of the form $\mathfrak{g}^{(i)}/\mathfrak{c}$, where $\mathfrak{g}^{(i)}$ is the ith derived algebra of $\mathfrak{g}$ and $\mathfrak{c}$ its center, is simple. For deforms, see [8].

- *Amazingly, the super goal is reached if $p = 2$*, see [12], with a catch: *modulo* the classification of simple Lie algebras, i.e., without an explicit list of all examples.[5] Here, we contribute to a conjectural list of "standard" simple Lie algebras (conjecturally a tame

---

[5]Thus, [12] resembles the classification of restricted Lie algebras for $p > 5$ in the paper [4]: there are no explicit

problem); in particular, we explicitly describe simple vectorial Lie algebras analogous to those over $\mathbb{C}$.

- $p > 5$. The conjecture [51] is easy to formulate ("take direct characteristic-p versions of the simple complex Lie superalgebras of the form $\mathfrak{g}(A)$, queer, and vectorial with polynomial coefficients, and their deformations"), but to describe deformations of even the symmetric ones (of the form $\mathfrak{g}(A)$ and queer) is not easy (for partial results, see [8]).

- $p = 5$. Most plausible conjecture is like for $p > 5$. Observe that there are indigenous $p = 5$ examples of the form $\mathfrak{g}(A)$,

- $p = 3$. We have discovered several new vectorial Lie superalgebras, see [8, 15], and of the form $\mathfrak{g}(A)$, see [10].

3.4   Getting simple Lie algebras from simple Lie superalgebras if $p = 2$

If $p = 2$, there are two methods for constructing a simple Lie superalgebra from a simple Lie algebra, and every simple Lie superalgebra is obtained by one of these two methods; for an amazingly short proof, see [12]. Reversing the process we recover a simple Lie algebra given any simple Lie superalgebra.

Even before these two methods were known, it was clear that one can get a Lie algebra from any Lie superalgebra as follows. Observe that for any odd element $x \in \mathfrak{g}$ in any Lie superalgebra $\mathfrak{g}$ over any field $\mathbb{K}$, we have $[x, x] := 2x^2 \in U(\mathfrak{g})$. That is why if $p = 2$, then one needs a squaring $x \longmapsto x^2$ for any odd $x \in \mathfrak{g}$; together with the brackets of even elements with all other elements, it is the squaring that defines the multiplication in any Lie superalgebra (for details, see Section 1.2.3), while the bracket of odd elements is the polarization of the squaring. Hence,

> For $p = 2$, every Lie superalgebra with the bracket as multiplication –
> we forget the squaring – is a $\mathbb{Z}/2$-graded Lie algebra.   (3.12)

To classify simple Lie superalgebras is a much more difficult task than to classify simple Lie algebras of the same type: the former is based on the latter as well as on careful study of the representation theory of Lie algebras. In [39], it was, nevertheless, suggested – for $p = 2$ – to reverse the process:

> Let $\boldsymbol{F}$ be the desuperization functor forgetting squaring, see (3.12).
> To obtain simple Lie algebras for $p = 2$,
> (A) apply the functor $\boldsymbol{F}$ to every simple Lie superalgebra $\mathfrak{g}$;   (3.13)
> (B) single out the simple Lie subalgebra $\mathfrak{s}(\boldsymbol{F}(\mathfrak{g}))$ of $\boldsymbol{F}(\mathfrak{g})$.

Clearly, $\mathfrak{s}(\boldsymbol{F}(\mathfrak{g}))$ is uniquely recoverable by inverting one of the two superization processes (either queerification or "method 2") that had lead to $\mathfrak{g}$, see [12].

Observe immediately that the idea of [39] just to apply $\boldsymbol{F}$ to the simple Lie superalgebra $\mathfrak{g}$ to get a simple Lie algebra, see (3.12), was naive and partly wrong: the example of $\mathfrak{psq}(n)$ should have hinted at importance of item (B) in the process (3.13). Understanding of this subtlety came together with the description of the two methods of superization of any simple Lie algebra as the only means to obtain any simple Lie superalgebra, see [12]. For the simple vectorial Lie

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formulas for p-structures of simple Lie algebras of Hamiltonian series to this day, see Strade's lamentations in [73, Vol. 1, p. 357]: "The problem of restrictedness is approached. ... [But] the family of Hamiltonian algebras ... is not yet handable". This is no wonder: although Skryabin classified symplectic forms in 1985, the answer was published only in 1991, see [70], three years after [4] appeared (and the details of [70], obtained in 1985, became available only recently, see [69]). The explicit formula for the bracket in any of the deformed Lie algebra of Hamiltonian vector fields is not published to this day and can be found only in Kirillov's Ph.D. Thesis only (in Russian), not in its published summary [38].

superalgebras, in particular, exceptional ones, just by forgetting squaring we get a simple Lie algebra. Let

$\mathbf{F}^{-1} = \mathfrak{s}(\mathbf{F}(-))$ denote the *complete desuperization* —

the composition of $\mathbf{F}$ and application of item (B) of (3.13).

Two reasons to take the direction of study opposite to a seemingly reasonable one:

(a) Although the classification of the simple vectorial Lie superalgebras over $\mathbb{C}$ was only conjectured at the time [39] was written, the list of known examples was already wider than that of known simple vectorial Lie algebras for $p = 2$, and was (and *is*, as we demonstrate in this paper) able to provide new simple examples.

(b) The results of [25, 72] show that a "frontal attack" on the classification for $p = 2$ is likely to be much more excruciating than that performed for $p > 3$ by Premet and Strade. Even to classify *restricted* Lie algebras for $p = 2$ will be much more difficult problem than that Block and Wilson solved for $p > 5$, see [4]. (Even if we confine ourselves to the classical definition of restrictedness, while certain examples, which should be considered as "classical", have another version of restrictedness, see [12].) So, a plausibly complete inventory of simple examples will be helpful. Our interpretations of the Lie (super)algebras are of independent interest.

Here, after a long break, we continue exploring method (3.13). It provides us with new examples of simple vectorial Lie algebras of the form $\mathbf{F}(\mathfrak{g})$, where $\mathfrak{g}$ is a modular, indigenous for $p = 2$, version of a simple vectorial Lie superalgebra over $\mathbb{C}$. The two methods of superization (see [12]) applied to $\mathbf{F}(\mathfrak{g})$ bring many more simple Lie superalgebras than $\mathfrak{g}$, most of them new.

3.5   Forgetting the superstructure if $p = 2$

Applying $\mathbf{F}$ to the serial vectorial Lie superalgebra $\mathfrak{g}(m; \underline{N}|n)$ we get the Lie algebra $\mathbf{F}(\mathfrak{g})\big(m + n; \underline{\widetilde{N}}\big)$, see Table 25.2; these Lie algebras are not necessarily simple, but their simple derived algebras are; here $\underline{\widetilde{N}} = (\underline{N}, 1, \ldots, 1)$ with the last n coordinates equal to 1.

3.5.1   Parameters of the Lie superalgebra that change under desuperization

Here are several examples:

- The unconstrained shearing vector $\underline{\widetilde{N}}^u$, see Section 1.9, of the vectorial Lie algebra $\mathbf{F}(\mathfrak{g})$ may depend on more parameters than the shearing vector $\underline{N}^u$ of $\mathfrak{g}$.

  For example, $\dim \underline{N} = \operatorname{Par} \underline{N}^u = a$ for $\mathfrak{vect}(a; \underline{N}|b)$, whereas

  $$\dim \underline{\widetilde{N}}^u := \operatorname{Par} \underline{\widetilde{N}}^u = a + b, \quad \text{where } \underline{\widetilde{N}} := (\underline{N}, 1 \ldots, 1)$$

  for $\mathbf{F}(\mathfrak{vect}(a; \underline{N}|b)) = \mathfrak{vect}\big(a + b; \underline{\widetilde{N}}\big)$. In all cases, except for $\mathfrak{vle}\big(4; \underline{\widetilde{M}}|3\big)$, the tilde over any shearing vector $\underline{L}$ is understood in the above sense: it enlarges the set of coordinates of $\underline{L}$ acquiring the coordinates of the desuperized odd indeterminates.

  The same applies to the desuperizations of the Lie superalgebras of the series $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{m}$, $\mathfrak{le}$ and their divergence-free subalgebras.

  The abstract Lie superalgebra $\mathfrak{g}$ realized as vectorial Lie superalgebra, $\mathfrak{g}(a; \underline{N}|b)$, depending on a even and b odd indeterminates, can be realized in several ways as $\mathfrak{g}(a; \underline{N}|b; r)$ by means of Weisfeiler filtrations or associated regradings r, see Section 2.4. This $\mathfrak{g}(a; \underline{N}|b; r)$ can be interpreted as the (generalized) Cartan prolong of the nonpositive part of $\mathfrak{g}$ in the corresponding grading, see Section 1.6.1.

- The Lie algebra obtained by desuperization might acquire new properties which its name-sakes for $p \neq 2$ do not have. For example, the Lie algebra $\mathfrak{po}(2n; \underline{N}) = \mathbf{F}(\mathfrak{b}_\lambda(n; n; \underline{N}))$ has a deformation depending on a parameter $\lambda \in \mathbb{K}P^1$; the corresponding non-isomorphic (for different values of λ) deforms are additional to the well-known one, which for $p = 0$ is called the result of the *quantization.*

- Desuperization $\mathbf{F}$ might turn distinct (types of) Lie superalgebras into one (type of) Lie algebras:

  – the Lie superalgebras of types $\mathfrak{k}$ and $\mathfrak{m}$ in the standard grading (2.18) turn into $\mathfrak{k}$;
  – the Lie superalgebras of types $\mathfrak{h}_\Pi$ and $\mathfrak{le}$ in the standard grading (2.18) turn into $\mathfrak{h}_\Pi$.

3.6   Comment on Volichenko algebras in characteristic 2

The notion of *Volichenko algebra*[6], which is an inhomogeneous (relative to parity) subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie superalgebra $\mathfrak{g}$ closed with respect to the superbracket in $\mathfrak{g}$, was introduced in [39]. For a classification of simple (without any nontrivial ideals, both one-sided and two-sided) finite-dimensional Volichenko algebras over $\mathbb{C}$ under a certain (hopefully, inessential) technical assumption, and examples of certain infinite-dimensional algebras, see [53, 54].

   The results of [31] suggested that we look at the definition of the Lie superalgebra for $p = 2$, see Section 1.2.3, more carefully. If one does this, it is not difficult to deduce that

$$\textit{if } p = 2, \textit{ the Volichenko algebras are, actually, Lie algebras.} \tag{3.14}$$

   In [31, 39], the fact (3.14) had not been understood, and therefore there is no need to consider these papers or Volichenko algebras in characteristic 2 while searching for simple Lie algebras. Unlike desuperizations of Lie superalgebras, which are worth investigating.

4   Introduction, continued. Our strategy, main results and open problems

4.1   Generalized Cartan prolongation

This is a principal procedure for getting vectorial Lie (super)algebras over $\mathbb{C}$, the following fact is well-known [77]:

   given a simple Lie algebra of the form $\mathfrak{g}(A)$, *and its* $\mathbb{Z}$-*grading, the generalized prolong of the nonpositive (with respect to that grading) part of* $\mathfrak{g}(A)$ *is isomorphic to* $\mathfrak{g}(A)$, *bar two series of exceptions corresponding to certain simplest gradings of* (4.1) *the embedded algebras* – $\mathfrak{sl}(n + 1) \subset \mathfrak{vect}(n)$ *and* $\mathfrak{sp}(2n + 2) \subset \mathfrak{k}(2n + 1)$ – *and the ambients are the exceptional prolongs.*

   In [51], it is shown how to obtain simple Lie algebras over $\mathbb{C}$ of the two types of prime importance for the classification procedure over $\mathbb{K}$: finite-dimensional and vectorial. Namely, by induction and using (generalized, in particular, partial) Cartan prolongation, see Section 1.6. First, one thus obtains all finite-dimensional simple Lie algebras (each of them has a Cartan matrix); during the next step one obtains all four series of simple vectorial Lie algebras, by considering not only complete Cartan prolongs as in equation (4.1), but also partial ones.

   The same method works to obtain $\mathbb{Z}$-graded simple Lie algebras for p sufficiently big and with new standard examples added. After that, there still remain considerable technical problems: namely, to describe the deforms and to classify non-isomorphic deforms.

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[6]In memory of I. Volichenko who was the first to study inhomogeneous subalgebras in Lie superalgebras.

For any characteristic, the super version of classification of simple Lie algebras is much more complicated than its non-super counterpart: we have to supply the input with several more types of algebras, but the main procedures are still the same: generalized, especially partial, prolongations and deformations.

In several papers (e.g., [8, 7, 15]), we have considered simple Lie (super)algebras, and have investigated the prolongs of the nonpositive parts relative to their $\mathbb{Z}$-gradings with one (or two if $p = 2$) pair(s) of Chevalley generators being of degree ± 1, and the other generators being of degree 0.

Here we consider serial and exceptional simple vectorial Lie superalgebras over $\mathbb{C}$ and desuperizations of their analogs for $p = 2$. Realization of a given Lie (super)algebra $\mathfrak{g}$ in terms of vector fields implies that $\mathfrak{g}$ is endowed with a filtration; one of these filtrations, called the *Weisfeiler filtration*, is the most important, see equation (2.17). Associated with the filtration is the grading; for brevity, the *Weisfeiler* filtrations and gradings are referred to, respectively, as *W-filtration* and *W-grading*. For several vectorial Lie algebras over fields of characteristic 2, we investigate the following problem answered by the fact (4.1) over $\mathbb{C}$:

when $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}^u} \simeq \mathfrak{g}$ and when the prolong strictly contains $\mathfrak{g}$?

We consider only the $\mathbb{Z}$-gradings of the finite-dimensional vectorial algebras corresponding to the W-gradings of their infinite-dimensional versions corresponding to $\underline{N}^u$. For examples of Lie (super)algebras $\mathfrak{g}$ that differ from the prolong of the nonpositive part of a regrading of $\mathfrak{g}$, see [7].

4.2 "Hidden supersymmetries" of Lie algebras

It is sometimes possible to endow the space of a given simple Lie algebra $\mathfrak{g}$ with (several) Lie superalgebra structures. For example, for $\mathfrak{g} = \mathfrak{sl}(n)$ over any ground field, consider any distribution of parities (of the pairs corresponding to positive and negative simple roots) of the Chevalley generators; we thus get Lie superalgebras $\mathfrak{sl}(a|b)$ for $a + b = n$ in various supermatrix formats. The sets of defining relations between the Chevalley generators corresponding to different formats are different.

It is, clearly, possible to perform such changes of parities of (pairs of) Chevalley generators for any simple Lie algebra but, except for $\mathfrak{sl}(n)$, the simple Lie superalgebras obtained by factorization modulo the ideal of relations [10] are infinite-dimensional unless $p = 2$.

If $p = 2$, the following fact is obvious (here $x \longmapsto x^{[2]}$ is the restriction mapping and $x \longmapsto x^2$ is the squaring):[7]

> *any classically restricted and $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ can be turned*
>
> *into a Lie superalgebra by setting $x^2 := x^{[2]}$ for any $x \in \mathfrak{g}_-$.*   (4.2)

In [12], a (rather unexpected) generalization of the possibility (4.2) is described: *every* simple Lie algebra $\mathfrak{g}$ can be turned into a simple Lie superalgebra by slightly enlarging its space if $\mathfrak{g}$ is not restricted. This generalization, and a "queerification", are the two methods which, from every simple Lie algebra, produce simple Lie superalgebras, and every simple Lie superalgebra can be obtained in one of these two ways, as proved in [12]. These two methods applied to the simple Lie algebras we describe in this paper yield a huge quantity of new simple Lie superalgebras, both serial and exceptional. We do not list them; this is routine, modulo the far from routine, as shown in [43], job of describing all $\mathbb{Z}/2$-gradings of our newly found simple Lie algebras.

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[7]In [12], in addition to the classical restrictedness, we introduced other, indigenous to $p = 2$, versions of restrictedness; their meaning is yet unclear, but since they pertain to classical and often used algebras, e.g., $\mathfrak{o}(2n + 1)$ and $\mathfrak{h}(2n + 1; \underline{1})$, we believe that these new "restrictednesses" are important.

4.3   From $\mathbb{C}$ to $\mathbb{K}$

We consider the W-grading (of the desuperized $p = 2$ version of the simple vectorial Lie superalgebra over $\mathbb{C}$) for which the nonpositive part, especially the 0th component, is most clear. Then, we consider the regradings described in Table (25.4), and perform generalized Cartan prolongation of the nonpositive (or negative) part of the regraded algebra in the hope of getting a new simple Lie (super)algebra, as in (4.1).

When this approach is inapplicable since there is no visible analog, suitable for $p = 2$, of the Lie (super)algebras we worked with over $\mathbb{C}$, we consider the description of the Lie superalgebras as the sum of its even and odd parts, desuperize this description and only after this consider the W-grading of the desuperization.

4.4   Our main results

Observe that every modular analog of one vectorial Lie (super)algebra over $\mathbb{C}$ is usually a *family* of algebras depending on $\underline{N}$; by abuse of speech we often skip the word "family" (of algebras) and talk about *one* algebra having in mind the extra parameter $\underline{N}$.

Of the simple Lie superalgebras that can be obtained by the two methods described in [12] from the simple Lie algebras we describe here all but one (the initial one, the one we desuperized) are new. In more details the Lie algebras obtained by desuperization are described in the following Theorems 4.1–4.5 summarizing respective sections with proofs.

Theorem 4.1 (an exceptional deform of $\mathfrak{svect}(5; \underline{N})$). *All W-regradings of $\mathfrak{mb}_3(11; \underline{N})$ are W-regradings of a previously unknown true deform of $\mathfrak{svect}(5; \underline{N})$.*

Proof. See Section 14.   ∎

Theorem 4.2 (desuperizations of $\mathfrak{b}_{a,b}(n)$ and $\widetilde{\mathfrak{sb}}_\nu\big(2^{n-1} - 1|2^{n-1}\big)$ for n even, as well as $\widetilde{\mathfrak{sb}}_\nu\big(2^{n-1}|2^{n-1} - 1\big)$ for n odd). *As (generalized) Cartan prolongs, the desuperizations of the characteristic-2 analogs of complex Lie superalgebras $\mathfrak{b}_{a,b}(n)$ and $\widetilde{\mathfrak{sb}}_\nu\big(2^{n-1} - 1|2^{n-1}\big)$ for n even, as well as $\widetilde{\mathfrak{sb}}_\nu\big(2^{n-1}|2^{n-1} - 1\big)$ for n odd, yield $\widetilde{\mathfrak{sb}}_\nu\big(2^n - 1; \underline{N}\big)$ and*

$$\mathfrak{po}_\lambda(2n + 1; \underline{N}), \text{ the serial simple (for } \frac{a}{b} \text{ generic) Lie algebras } \mathfrak{po}_{a,b}(2n; \underline{N}),$$

$$\text{their simple relatives for } \frac{a}{b} = 0, \ 1, \text{ and } \infty \text{ (for } a \neq 0, b = 0\text{), see (2.34).}$$

Proof. See Sections 2.9.2 and 21.   ∎

Theorem 4.3 (desuperizations of the 15 exceptional simple vectorial Lie superalgebras). *The generalized Cartan prolongs of the nonpositive (or negative) parts of all 15 W-gradings, see Table (25.3), of the 5 exceptional simple vectorial Lie algebras – desuperizations of the characteristic 2 analogs of complex exceptional simple vectorial Lie superalgebras, see Section 2.11 – yield three simple Lie algebras, see Table (25.5) while the other two, $\mathfrak{vle}(9; \underline{N})$ and $\mathfrak{mb}_3(11; \underline{N})$, are described from another point of view in [7].*

Theorem 4.4 (isomorphisms between desuperizations of the 15 exceptional simple vectorial Lie superalgebras).

(i) *For $p \neq 2$, the characteristic-p analogs of the 15 W-graded analogs of the complex exceptional simple vectorial Lie superalgebras constitute, as abstract Lie superalgebras, 5 Lie superalgebras.*

(ii) *For $p = 2$, all their finite-dimensional analogs, and their desuperizations, remain regradings of each other, with one exception: the 3 indeterminates of $\widetilde{\mathfrak{kas}}(7; \underline{\widetilde{K}})$ are defined to be constrained,* Par $\underline{\widetilde{K}}^u = 3$.

One – for $p \neq 2$ – exceptional family ($\mathfrak{kas}$) yields – for $p = 2$ – two families:

$\mathfrak{vle}\left(7; \underline{\widetilde{L}}\right) \simeq \mathfrak{vle}\left(9; \underline{\widetilde{M}}\right) \simeq \widetilde{\mathfrak{vle}}\left(9; \underline{\widetilde{N}}\right)$	Par $\underline{\widetilde{L}}^u = $ Par $\underline{\widetilde{M}}^u = $ Par $\underline{\widetilde{N}}^u = 3$
$\mathfrak{mb}_3\left(9; \underline{\widetilde{L}}\right) \simeq \mathfrak{mb}_3\left(11; \underline{\widetilde{M}}\right) \simeq \mathfrak{mb}_2\left(11; \underline{\widetilde{N}}\right)$	Par $\underline{\widetilde{L}}^u = $ Par $\underline{\widetilde{M}}^u = $ Par $\underline{\widetilde{N}}^u = 5$
$\mathfrak{kle}\left(15; \underline{\widetilde{K}}\right) \simeq \mathfrak{kle}\left(15; \underline{\widetilde{L}}\right) \simeq \mathfrak{kle}_3\left(20; \underline{\widetilde{M}}\right) \simeq \mathfrak{kle}_2\left(20; \underline{\widetilde{N}}\right)$	Par $\underline{\widetilde{K}}^u = $ Par $\underline{\widetilde{L}}^u = $ Par $\underline{\widetilde{M}}^u = $ Par $\underline{\widetilde{N}}^u = 5$
$\mathfrak{kas}^{(1)}\left(7; \underline{\widetilde{L}}\right) \simeq \mathfrak{kas}^{(1)}\left(8; \underline{\widetilde{M}}\right) \simeq \mathfrak{kas}^{(1)}\left(10; \underline{\widetilde{N}}\right)$	Par $\underline{\widetilde{L}}^u = $ Par $\underline{\widetilde{M}}^u = $ Par $\underline{\widetilde{N}}^u = 7$
$\widetilde{\mathfrak{kas}}^{(1)}\left(7; \underline{\widetilde{K}}\right)$	Par $\underline{\widetilde{K}}^u = 3$
$\mathfrak{vas}^{(1)}(8; \underline{N})$	Par $\underline{N}^u = 4$

Proof of Theorems 4.3 and 4.4. For $\mathfrak{vle}\left(9; \underline{\widetilde{N}}\right)$, see Section 7. For $\mathfrak{vle}(5; \underline{N}|4)$ and $\widetilde{\mathfrak{vle}}\left(9; \underline{\widetilde{N}}\right)$, see Section 8. For $\widetilde{\mathfrak{vle}}\left(15; \underline{\widetilde{N}}\right)$, see Section 9. For $\mathfrak{kle}_3\left(20; \underline{\widetilde{N}}\right)$, see Section 11. For $\mathfrak{mb}\left(9; \underline{\widetilde{N}}\right)$, see Section 15. For $\widetilde{\mathfrak{kas}}(5; \underline{N}|5)$, see Section 20.

For $\mathfrak{vle}\left(7; \underline{\widetilde{L}}\right)$, see Section 6. For $\mathfrak{mb}_2\left(11; \underline{\widetilde{N}}\right)$, see Section 16. For $\mathfrak{kle}_2\left(20; \underline{\widetilde{N}}\right)$, see Section 12. For $\mathfrak{kas}\left(8; \underline{\widetilde{M}}\right)$, see Section 19. For $\widetilde{\mathfrak{kas}}\left(7; \underline{\widetilde{M}}\right)$, see Section 18. For $\mathfrak{vas}(4; \underline{N}|4)$ and $\mathfrak{vas}(8; \underline{N})$, see Section 22. ∎

Theorem 4.5 (desuperizations of $\mathfrak{kas}$). *For $p = 2$, the analogs of the W-graded simple vectorial Lie superalgebra $\mathfrak{kas}$ over $\mathbb{C}$ are not simple; they contain a simple ideal of codimension 1, the derived algebra.*

Proof. See Section 17.4. ∎

Other results. We explicitly describe characteristic-2 Shen's version (see [68] and Section 23 here) of both $\mathfrak{g}(2)$ and the Melikyan algebra as vectorial Lie algebras.

We single out the divergence-free subalgebra $\mathfrak{sh}_\Pi\left(2n; \underline{\widetilde{N}}\right)$ of the Lie algebra of Hamiltonian vector fields $\mathfrak{h}_\Pi(2n; \underline{M}) = \mathbf{F}(\mathfrak{le}(n; \underline{N}|n))$, discovered in [48], by imposing constraints on $\underline{M}$.

4.4.1 Open problems

1. In this paper, for serial simple vectorial Lie superalgebras, we considered 2 W-gradings of $\mathbf{F}\left(\widetilde{\mathfrak{sb}}(2^n - 1)\right)$ (the standard and of depth 1). We did not consider 32 W-gradings of the remaining simple vectorial Lie superalgebras, see Section 1.16 in [56]. The fact (4.1) suggests that we should investigate if these W-gradings yield new simple Lie algebra.

2. Describe all the $\mathbb{Z}/2$-grading of the newly found simple Lie algebras in characteristic 2 (see [7, 13, 22, 25, 48, 72], and this paper) to obtain new simple Lie superalgebras. For first results, see [43].

3. Are the partial prolongs described in Section 19.1 and in equation (21.1) isomorphic to known simple Lie algebras?

4. See Problems 2.6, 2.9, 22.1.

5 The Lie superalgebra $\mathfrak{vle}(4|3)$ over $\mathbb{C}$ and its $p > 2$ versions

5.1 Recapitulations, see [67]

In the realization of $\mathfrak{le}(3)$ by means of generating functions, we identify the space of $\mathfrak{le}(3)$ with $\Pi(\mathbb{C}[\theta, q]/\mathbb{C} \cdot 1)$, where before the functor Π is applied $\theta = (\theta_1, \theta_2, \theta_3)$ are odd and $q = (q_1, q_2, q_3)$

are even, see (2.7). In the *standard grading* $\deg_{\text{Lie}}$ of $\mathfrak{le}(3)$, we assume that $\deg q_i = \deg \theta_i = 1$ for all i, and that the grading is given by the formula

$$\deg_{\text{Lie}}(f) := \deg \text{Le}_f = \deg f - 2 \quad \text{for any monomial } f \in \mathbb{C}[\theta, q].$$

The *nonstandard* grading $\deg_{\text{Lie};3}$ of $\mathfrak{g} = \mathfrak{le}(3;3)$ is determined by the formulas

$$\deg_3 \theta_i = 0 \quad \text{and} \quad \deg_3 q_i = 1 \quad \text{for } i = 1, 2, 3,$$
$$\deg_{\text{Lie};3}(f) = \deg_3 f - 1 \quad \text{for any monomial } f \in \mathbb{C}[\theta, q].$$

This grading of $\mathfrak{g} = \mathfrak{le}(3;3)$ is of depth 1, and its homogeneous components are of the form:

$$\mathfrak{g}_{-1} = \Pi(\mathbb{C}[\theta_1, \theta_2, \theta_3]/\mathbb{C} \cdot 1), \quad \mathfrak{g}_k = \Pi(\mathbb{C}[\theta_1, \theta_2, \theta_3]) \otimes S^{k+1}(q_1, q_2, q_3) \quad \text{for } k \geq 0.$$

In particular, $\mathfrak{g}_0 \simeq \mathfrak{vect}(0|3)$, and $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module with the lowest-weight vector q_1^2. The whole Lie superalgebra $\mathfrak{le}(3;3)$ is the Cartan prolong of its nonpositive part and the component $\mathfrak{g}_1$ generates the whole positive part.

To obtain $\mathfrak{vle}(4|3)$, we add the central element d to the 0th component of $\mathfrak{le}(3;3)$; so that ad_d is the grading operator on the Cartan prolong of its nonpositive part; this prolong is strictly bigger than $\mathfrak{le}(3;3) \ltimes \mathbb{C} \cdot d$.

This Cartan prolong is the exceptional simple Lie superalgebra $\mathfrak{vle}(4|3)$.

Its component $\mathfrak{vle}_1$ is reducible, but indecomposable $\mathfrak{vle}_0$-module such that

$$\mathfrak{vle}_1/\mathfrak{le}(3;3)_1 \simeq (\mathfrak{le}(3;3)_{-1})^*.$$

The other, not lying in $\mathfrak{le}(3;3)_1$, lowest-weight vector in $\mathfrak{vle}_1$ is the element dual to the highest-weight vector in $\mathfrak{g}_{-1}$, i.e., to $\Pi(\theta_1\theta_2\theta_3)$.

5.1.1   Introducing indeterminate y (as well as x_i and ξ_i)

Under the identification

$$\Pi(\theta_1\theta_2\theta_3) \longmapsto -\partial_y, \quad \Pi(\theta_i) \longmapsto -\partial_{x_i}, \quad \Pi\left(\frac{\partial(\theta_1\theta_2\theta_3)}{\partial \theta_i}\right) \longmapsto -\partial_{\xi_i}$$

each vector field $D \in \mathfrak{vle}(4|3)$ is of the form

$$D_{f,\,g} = \text{Le}_f + yB_f - (-1)^{p(f)}\left(y\Delta(f) + y^2 \frac{\partial^3 f}{\partial \xi_1 \partial \xi_2 \partial \xi_3}\right)\partial_y$$

$$+ B_g - (-1)^{p(g)}\left(\Delta(g) + 2y\frac{\partial^3 g}{\partial \xi_1 \partial \xi_2 \partial \xi_3}\right)\partial_y, \tag{5.1}$$

where $f, g \in \mathbb{C}[x, \xi]$; the operators B_g and Δ are given by the formulas

$$B_g = \frac{\partial^2 g}{\partial \xi_2 \partial \xi_3}\frac{\partial}{\partial \xi_1} + \frac{\partial^2 g}{\partial \xi_3 \partial \xi_1}\frac{\partial}{\partial \xi_2} + \frac{\partial^2 g}{\partial \xi_1 \partial \xi_2}\frac{\partial}{\partial \xi_3}, \quad \Delta = \sum_{1 \leq i \leq 3} \frac{\partial^2}{\partial x_i \partial \xi_i}. \tag{5.2}$$

There are two embeddings of $\mathfrak{le}(3)$ into $\mathfrak{vle}(4|3)$. The embedding $i_1 \colon \mathfrak{le}(3) \longrightarrow \mathfrak{vle}(4|3)$ corresponds to the grading $\mathfrak{le}(3;3)$. Let us reproduce the explicit formulas from [67]:

Let us clarify notation of indeterminates. At the beginning, the indeterminates describing $\mathfrak{le}$ in any grading are denoted by q and θ, while the indeterminates describing $\mathfrak{vle}$ are denoted by x and ξ. Hence, introduce the passage i from one set of indeterminates to the other, and $\hat{f}$:

$$i(q_1, q_2, q_3, \theta_1, \theta_2, \theta_2) := (x_1, x_2, x_3, \xi_1, \xi_2, \xi_3),$$
$$\hat{f}(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) := f(i(q_1, q_2, q_3, \theta_1, \theta_2, \theta_2)).$$

a) If $f = f(q)$, then

$$i_1(\mathrm{Le}_f) = \mathrm{Le}_{\sum \left(\frac{\partial \hat{f}}{\partial x_i}\right)\xi_j\xi_k - y\hat{f}},$$

where y is treated as a parameter and $(i, j, k) \in A_3$ (even permutations of $\{1, 2, 3\}$).

b) If $f = \sum f_i(q)\theta_i$, then

$$i_1(\mathrm{Le}_f) = \mathrm{Le}_{\hat{f}} - \varphi(x) \sum \xi_i \partial_{\xi_i} + \left(-\varphi(x)y + \Delta(\varphi(x)\xi_1\xi_2\xi_3)\right)\partial_y,$$

where $\varphi(x) = \Delta(\hat{f})$.

c) If $f = \psi_1(q)\theta_2\theta_3 + \psi_2(q)\theta_3\theta_1 + \psi_3(q)\theta_1\theta_2$, then

$$i_1(\mathrm{Le}_f) = -\Delta(\hat{f})\partial_y - \sum_{1 \leq i \leq 3} \psi_i(x)\frac{\partial}{\partial \xi_i}.$$

d) If $f = \psi(q)\theta_1\theta_2\theta_3$, then

$$i_1(\mathrm{Le}_f) = -\psi(x)\partial_y.$$

The embedding $i_2 \colon \mathfrak{le}(3) \longrightarrow \mathfrak{vle}(4|3)$ corresponds to the standard grading of $\mathfrak{le}(3)$. In terms of generating functions this embedding is of the form

$$i_2(f(q, \theta)) \longrightarrow D_{f(i(q,\theta)),\, 0}. \tag{5.3}$$

As vector spaces, we have

$$\mathfrak{vle}(4|3) = i_1(\mathfrak{le}(3;3)) + i_2(\mathfrak{le}(3)) \quad \text{while} \quad i_1(\mathfrak{le}(3;3)) \cap i_2(\mathfrak{le}(3)) \simeq \mathfrak{sle}^{(1)}(3). \tag{5.4}$$

By abuse of notation, denote the operator $\sum\limits_{1 \leq i \leq 3} \frac{\partial^2}{\partial q_i \partial \theta_i}$ acting on the space of functions (divided powers if $p > 0$) in q_i, θ_i also by Δ. In this notation, we have

$$i_1(f(q)) = i_2(\Delta(f(q)\theta_1\theta_2\theta_3)), \quad i_1(\Delta(f(q)\theta_1\theta_2\theta_3)) = -i_2(f(q)), \tag{5.5}$$

and

$$i_1(f) = i_2(f) \quad \text{if} \quad f = \sum_{1 \leq i \leq 3} f_i(q)\theta_i \quad \text{and} \quad \Delta f = 0. \tag{5.6}$$

The formulas (5.3), (5.5), and (5.6) are valid for any p, in particular, for $p = 2$.

The lowest-weight vectors in $\mathfrak{vle}_1$ are $i_1(q_1^2)$ and $i_2(\theta_1\theta_2\theta_3)$. We have

$$\mathrm{sdim}\,\mathfrak{vle}_1 = 28|27. \tag{5.7}$$

Proposition 5.1 (passage from $\mathbb{C}$ to $\mathbb{K}$ for $p > 2$)**.** *The situation described in the previous subsection does not change under passage from the ground field $\mathbb{C}$ to any field of characteristic 0 and also to $\mathbb{K}$ if char $\mathbb{K} = p > 2$ provided the coordinates of the shearing vector $\underline{M}$ of the algebra of coefficients of the vector fields are such that $\underline{M}_i = \infty$ for each even indeterminate x_i.*

In all these cases, the Lie superalgebra $\mathfrak{g}$ – the Cartan prolong of the nonpositive part of $\mathfrak{vle}$ – remains simple and of infinite dimension. The component $\mathfrak{g}_1$ also retains its structure as a $\mathfrak{g}_0$-module, but it does not generate the whole of $\mathfrak{g}$ (since the x_i do not generate $\mathcal{O}(x; \underline{M})$ if $\underline{M} \neq \mathbb{1}$, see (1.8)).

Proof. Proof follows directly from the formulas of Section 5.1.   ∎

In what follows we will investigate the case of shearing vectors with finite coordinates.

Theorem 5.2 ($\mathfrak{vle}$ exists for any $p > 0$)**.** *A characteristic-p version* $\mathfrak{vle}\big(4; \widetilde{M}|3\big)$ *of the Lie superalgebra* $\mathfrak{vle}(4|3)$ *exists for any* $p > 0$ *and any shearing vector* $\widetilde{M} = (\underline{M}, M_y)$ *provided* $M_y = 1$.

For the case $p = 2$, see Sections 6, 7, 8. For the case $p > 2$, see the next proposition.

Proposition 5.3 (description of $\mathfrak{vle}\big(4; \widetilde{M}|3\big)$)**.** *If* $p > 2$ *and* $\underline{M}_i < \infty$ *for* $i = 1, 2, 3$, *then* $\mathfrak{vle}\big(4; \widetilde{M}|3\big) = \mathrm{Span}(D_{f, g})$, *where* $\widetilde{M} = (\underline{M}, M_y)$ *and* $M_y = 1$, *and*

$$f \in \mathcal{O}(x; \underline{M}|\xi) \oplus \mathrm{Span}\big(x_i^{(s_i)}\big)_{i=1}^3 \quad and \quad g \in \mathcal{O}(x; \underline{M}|\xi) \oplus \mathbb{K} \cdot x_1^{(s_1)} x_2^{(s_2-1)} x_3^{(s_3-1)} \xi_1.$$

Proof. For $p > 2$, the component $\mathfrak{g}_1$ also retains its structure as $\mathfrak{g}_0$-module even if any (or all) of the coordinates of the shearing vector $\underline{M} = (\underline{M}_1, \underline{M}_2, \underline{M}_3)$ become finite. If $\underline{M}_i < \infty$ for all i, the Cartan prolong is finite-dimensional. It can be described by means of equation (5.1), but we should investigate when $D_{f, g} \in \mathfrak{vect}\big(4; \widetilde{M}|3\big)$, where

$$\widetilde{M} = (\underline{M}, M_y) \quad \text{and} \quad M_y = 1.$$

Direct observation gives the answer:

$$f \in \mathcal{O}(x; \underline{M}|\xi) \oplus \mathrm{Span}\big(x_i^{(s_i)}\big)_{i=1}^3, \quad \text{where } s_i = p^{M_i};$$

i.e., we should add "virtual" (non-existing for the given $\underline{M}$) elements $f = x_i^{(s_i)}$ for $i = 1, 2, 3$. Since due to (5.5)

$$D_{x_i^{(s_i)},0} = i_2\big(q_i^{(s_i)}\big) = i_1\big(\Delta\big(q_i^{(s_i)}\theta_1\theta_2\theta_3\big)\big),$$

we see that $i_2\big(q_i^{(s_i)}\big) \in \mathfrak{vle}^{(1)}\big(4; \widetilde{M}|3\big)$ though $q_i^{(s_i)} \notin \mathfrak{le}^{(1)}(3; \underline{M})$.

Now, let us investigate the generating functions g. We have

$$\mathrm{Ker}(D_{0,-} : g \longmapsto D_{0,g}) = \mathrm{Span}\left(D_{0,g} = 0 \,\Big|\, g = g(x) \text{ or } g = \sum_i g_i(x)\xi_i \text{ with } \Delta g = 0\right). \quad (5.8)$$

If $g = \sum_i g_i(x)\xi_i$, but $\Delta g = h(x) \neq 0$, then $D_{0,g} = h(x)\partial_y$ depends on h only. It is clear that any function $h \in \mathcal{O}(x; \underline{M})$ can be expressed as $h = \Delta g$ for some $g \in \mathcal{O}(x; \underline{M}|\xi)$, except for

$$h_s = x_1^{(s_1-1)} x_2^{(s_2-1)} x_3^{(s_3-1)}, \quad \text{where } s = (s_1, s_2, s_3).$$

To obtain $D = h_s\partial_y$, we should add to the space of generating functions any of the "virtual" functions

$$g_{s,i} = \xi_i \partial_j \partial_k\big(x_1^{(s_1)} x_2^{(s_2)} x_3^{(s_3)}\big) \quad \text{for } i = 1, 2, 3, \text{ and } j, k \neq i, \ j \neq k.$$

Modulo the kernel (5.8) of the map $D_{0,-}$ only one "extra" generator suffices; for definiteness, we select $g_s := x_1^{(s_1)} x_2^{(s_2-1)} x_3^{(s_3-1)} \xi_1$. Formula (5.1) shows that D_{0,g_s} lies in the homogeneous component of degree

$$s_1 + (s_2 - 1) + (s_3 - 1) + 1 - 3 \equiv -4 \mod p.$$

Since the Lie superalgebra $\mathfrak{vle}(4; \widetilde{M}|3)$ has a grading operator, it follows that

$$D_{0,g_s} \in \mathfrak{vle}^{(1)}\big(4; \widetilde{M}|3\big)$$

for $p > 2$. Moreover, as was shown in [67] for $p = 0$ (but the formulas remain true for any $p > 0$),

$$D_{0,g_s} = i_1(-h_s\xi_1\xi_2\xi_3), \quad \text{and hence } D_{0,g_s} \in i_1\big(\mathfrak{le}^{(1)}(3; \underline{M})\big) \text{ for } p > 2. \quad ∎$$

If $p > 2$, the fact (5.4) does not hold. We have

$$D_{0,\,h_s\xi_1\xi_2\xi_3} \in \mathfrak{vle}\big(4; \widetilde{\underline{M}}|3\big), \quad \text{but} \quad D_{0,\,h_s\xi_1\xi_2\xi_3} = i_1(f) - i_2(f),$$

where

$$f = q_1^{(s_1)}q_2^{(s_2-1)}q_3^{(s_3-1)}\theta_1 + q_1^{(s_1-1)}q_2^{(s_2)}q_3^{(s_3-1)}\theta_2 + q_1^{(s_1-1)}q_2^{(s_2-1)}q_3^{(s_3)}\theta_3 \notin \mathcal{O}(q; \underline{M}|\theta).$$

If $\underline{M}_i > 1$ for $i = 1, 2, 3$, then $\operatorname{sdim}\mathfrak{g}_1 = 28|27$ remains the same as over $\mathbb{C}$ and any other $p > 2$.

6   A description of $\mathfrak{vle}\big(7; \widetilde{\underline{M}}\big) := \mathrm{F}(\mathfrak{vle}(4; \underline{M}|3))$ for $p = 2$

In this and two next sections we prove Theorem 5.2 for the case $p = 2$.

There are three W-gradings of $\mathfrak{vle}$ with unconstrained shearing vector. In this and two next sections, we consider each of these gradings for $p = 2$, describe the corresponding 0th and 1st components of the regraded Lie superalgebra $\mathfrak{vle}$, and calculate partial prolongs. Next, we consider desuperizations of each of them. As a result, we get a new simple Lie superalgebra and a new simple Lie algebra in $p = 2$. Unfortunately (we'd like to get new simple algebras!), there are no partial prolongs.

The Lie superalgebra $\mathfrak{vle}\big(4; \widetilde{\underline{M}}|3\big)$ for $p = 2$ is a direct reduction modulo 2 of the integer form, with divided powers as coefficients, of the complex vectorial Lie superalgebra $\mathfrak{vle}(4|3)$.

First of all, let us define squares of odd elements for the Lie superalgebra $\mathfrak{le}(n; \underline{M})$, cf. equation (2.14):

$$f^2 := \sum_{1 \le i \le n} \frac{\partial f}{\partial q_i}\frac{\partial f}{\partial \theta_i}$$

and have in mind that if $p = 2$ and $\underline{M}_i < \infty$ for all i, the Lie superalgebra $\mathfrak{g} = \mathfrak{le}(n; \underline{M})$ is not simple: the generating function of the maximal degree $q_1^{(s_1-1)}q_2^{(s_2-1)}\cdots q_n^{(s_n-1)}\theta_1\theta_2\cdots\theta_n$ does not belong to $\mathfrak{g}^{(1)}$, the latter being simple.

For $p > 2$, we just reduce the expression (5.1) modulo p.

For $p = 2$, we cannot just reduce the expression (5.1) modulo 2; we should modify it. Indeed, the system of equations on the coefficients of the field $D \in \mathfrak{vle}(4|3)$ whose solution is given by the formula (5.1) contain coefficients $\frac{1}{2}$, see [67]. The vector field $D \in \mathfrak{vle}\big(4; \widetilde{\underline{M}}|3\big)$ is of the form (recall formula (5.2) for B_g):

$$D_{f,\,g} = \mathrm{Le}_f + yB_f + y\Delta(f)\partial_y + B_g + \Delta(g)\partial_y, \quad \text{where } \widetilde{\underline{M}} = (\underline{M}, M_y) \text{ and } M_y = 1. \quad (6.1)$$

Let us explain how we got this formula: we just rewrote equations from [67] without $\frac{1}{2}$ (multiplied the equations by 2) and solved them in the same way as it was done in [67].

For $p = 2$, unlike the case $p \ne 2$, this Lie superalgebra $\mathfrak{g} = \mathfrak{vle}\big(4; \widetilde{\underline{M}}|3\big)$ is not simple, but $\mathfrak{g}^{(1)}$ is simple, its codimension in $\mathfrak{g}$ is equal to 2: for $f = q_1^{(s_1-1)}q_2^{(s_2-1)}q_3^{(s_3-1)}\theta_1\theta_2\theta_3$, we have

$$D_{f,0} = i_2(f) \notin \mathfrak{g}^{(1)}, \quad D_{0,\,x_1^{(s_1)}x_2^{(s_2-1)}x_3^{(s_3-1)}\xi_1} = i_1(f) \notin \mathfrak{g}^{(1)}.$$

For $p = 2$, the structure of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ differs drastically from that for $p \ne 2$. Instead of two lowest-weight vectors, we have three of them. Besides, these three lowest-weight vectors do not describe the whole complexity of the module.

The submodule $i_1(\mathfrak{le}(3;3)_1)$ has a complicated structure. To describe it, observe that for any vectorial Lie (super)algebra expressed in terms of generating functions, the shearing vector can be considered either

(1) on the level of generating functions (let us denote it $\underline{M}$ in this case) or

(2) on the level of coefficients of vector fields they generate (let us denote it $\widetilde{\underline{M}}$ in this case).

In case(1), we obtain the "underdeveloped" Lie superalgebra

$$\mathfrak{le}(n; \underline{M}) := \mathrm{Span}(\mathrm{Le}_f \mid f \in \mathcal{O}(n; \underline{M}|n));$$

in case (2), we get the correct $\mathfrak{le}(n; \widetilde{\underline{M}}) = \mathrm{Span}\left(\mathrm{Le}_f \in \mathfrak{vect}(n; \widetilde{\underline{M}}|n)\right)$. We have

$$\mathfrak{le}(n; \widetilde{\underline{M}}) = \mathfrak{le}(n; \underline{M}) \ltimes \mathrm{Span}\left(q_i^{(s_i)} \mid i = 1, \ldots, n\right).$$

Accordingly, for $\widetilde{\underline{M}}$ unconstrained, the component $\mathfrak{le}(3; \widetilde{\underline{M}}; 3)_1$ is of the form:

$$\mathfrak{le}(3; \widetilde{\underline{M}}; 3)_1 = \Pi\left(\mathbb{K}[\theta_1\theta_2\theta_3]\right) \otimes S^2(q_1, q_2, q_3).$$

This component contains submodules corresponding to the minimal values $\underline{M}_i = 1$ for some i. To describe $\mathfrak{g}_1$ as $\mathfrak{g}_0$-module, consider the submodule

$$W_0 := \mathfrak{le}^{(1)}(3; 3)_1 = \mathrm{Span}(q_i q_j \varphi(\theta) \text{ for any } i, j \text{ and any function } \varphi).$$

It is irreducible. It is glued to the submodules

$$W_i := W_0 \ltimes \mathbb{K} \cdot q_i^{(2)}$$

in each of which W_0 is a submodule, but not a direct summand. Each W_i can be further enlarged to the module

$$W_{i,\theta} := W_i \ltimes \mathrm{Span}\left(q_i^{(2)}\varphi(\theta)\right) \quad \text{corresponding to } \underline{M}_i > 1, \text{ where } \underline{M}_j = 1 \text{ for } j \neq i$$

with shearing performed on the level of generating functions.

Let us describe the subalgebras of $\mathfrak{vle}(4; \widetilde{1}|3)$, the partial prolongs. In what follows we will often use the following

6.1 Notational convention: on partial prolongs

Let v_i be a lowest-weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$
and V_i the submodule generated by v_i.
Let $\mathfrak{g}_{k,(i)}$ be the kth prolong "in the direction of $V_i \subset \mathfrak{g}_1$", i.e.,
kth prolong of $(\mathfrak{g}_-, \mathfrak{g}_0, V_i)$. $\hspace{3cm}$ (6.2)

Consider the $\mathfrak{g}_0$-submodules $W \subset \mathfrak{g}_1$ not contained in $i_1(\mathfrak{le}(3; 3))$. There is only one such submodule $V_3 \subset W$ generated by v_3, see (6.3). The $\mathfrak{g}_0$-module $\mathfrak{g}_{1,M}$ has the following three lowest-weight vectors expressed in the form $D_{f,g}$, and also as $i_1(-)$ or $i_2(-)$:

v_1	$x_1 y \partial_{\xi_2} + x_1 \xi_3 \partial_{x_1} + x_1 \xi_1 \partial_{x_3} + x_2 y \partial_{\xi_1}$ $+ x_2\xi_3\partial_{x_2} + x_2\xi_2\partial_{x_3} + \xi_3\xi_2\partial_{\xi_2} + \xi_3\xi_1\partial_{\xi_1}$	$i_1(q_1 q_2)$	$D_{x_1\xi_1\xi_3 + x_2\xi_2\xi_3, \, 0}$
v_2	$x_1 y \partial_{\xi_1} + x_1 \xi_3 \partial_{x_2} + x_1\xi_2\partial_{x_3} + \xi_3\xi_2\partial_{\xi_1}$	$i_1\left(q_1^{(2)}\right)$	$D_{x_1\xi_2\xi_3, \, 0}$
v_3	$y\xi_3\partial_{\xi_3} + y\xi_2\partial_{\xi_2} + y\xi_1\partial_{\xi_1} + \xi_3\xi_2\partial_{x_1}$ $+ \xi_3\xi_1\partial_{x_2} + \xi_2\xi_1\partial_{x_3}$	$i_2(\theta_1\theta_2\theta_3)$	$D_{\xi_1\xi_2\xi_3, \, 0}$

(6.3)

By increasing the value of some of the coordinates $\underline{M}_i$ we enlarge $\mathfrak{g}_{1,(3)} = \mathfrak{vle}\big(4; \widetilde{\mathbb{1}}|3\big)_1$. As $\mathfrak{g}_0$-module, $\mathfrak{g}_{1,(3)}$ is of the following form:

$$W_0 \subset (W_1 + W_2 + W_3) \subset \mathfrak{g}_{1,(3)},$$

where $\mathfrak{g}_{1,(3)}/(W_1 + W_2 + W_3) \simeq (\mathfrak{g}_{-1})^*$, and $(W_1 + W_2 + W_3)/W_0$ is the trivial $0|3$-dimensional module, and where sdim $W_0 = 12|12$.

The superdimensions of the positive components of $\mathfrak{vle}(4; \mathbb{1}|3)$ (and its derived algebra in parentheses) are given in the following table:

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$
sdim	16|18	10(9)|12	4|3	1(0)|0

6.2 Partial prolongs as subalgebras of $\mathfrak{vle}\big(4; \underline{\widetilde{M}}|3\big)$

We have $[\mathfrak{g}_{-1}, \mathfrak{g}_{1,(i)}] \simeq \mathfrak{vect}(0|3)$ for $i = 1, 2$. (Actually, $\mathfrak{g}_{1,(2)} = \mathfrak{g}_{1,(1)} \ltimes \mathbb{K} \cdot i_1\big(q_1^{(2)}\big)$.)

6.2.1 Convention: on partial prolongs "of no interest"

In what follows, we do not investigate partial prolongs with $[\mathfrak{g}_{-1}, \mathfrak{g}_{1,(i)}]$, see (6.2), smaller than $\mathfrak{g}_0$ if the $[\mathfrak{g}_{-1}, \mathfrak{g}_{1,(1)}]$-module $\mathfrak{g}_{-1}$ is not irreducible: no such prolong can be a simple Lie (super)algebra with the given nonpositive part.

6.3 Desuperization

We have $\mathfrak{g}_0 \simeq \mathfrak{c}(\mathfrak{vect}(3; \mathbb{1}))$, and $\mathfrak{g}_{-1} \simeq \mathcal{F}/\mathbb{K}$.

For $\underline{N}$ unconstrained, we have $\dim \mathfrak{g}_1 = 55$. (Compare with (5.7) and (6.4): for $\underline{N} = \mathbb{1}$, the dimension drops.)

Critical coordinates of the unconstrained shearing vector: $\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3, \widetilde{M}_7$.

The dimensions of the positive components of $\mathfrak{vle}(7; \widetilde{\mathbb{1}})$ and its simple derived algebra (in parentheses) are given in the following table; so $\dim \mathfrak{vle}^{(1)}\big(7; \widetilde{\mathbb{1}}\big) = 94$:

	$\mathfrak{g}_1$	$\mathfrak{g}_2$ (or $\mathfrak{g}_1^{\cdot 2}$)	$\mathfrak{g}_3 = \mathfrak{g}_1^{\cdot 3}$	$\mathfrak{g}_4$ or $-$
dim	34	22 (21)	7	1 $(-)$

(6.4)

6.4 Partial prolongs as subalgebras of $\mathfrak{vle}\big(7; \underline{\widetilde{M}}\big)$

(i) We have $\dim(\mathfrak{g}_1') = 34$. (If $\underline{N} = \mathbb{1}$, then $\dim(\mathfrak{g}_{1,(1)}) = \dim \mathfrak{g}_1$; otherwise, $\mathfrak{g}_{1,(1)} \subsetneqq \mathfrak{g}_1$.) The partial Cartan prolong

$$\mathfrak{vle}'\big(7; \underline{\widetilde{M}}\big) := \big(\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_{1,(1)}\big)_{*;\underline{\widetilde{M}}}$$

is such that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \simeq \mathfrak{cvect}(3; \mathbb{1})$; this prolong is $\mathfrak{vle}\big(7; \widetilde{\mathbb{1}}\big)$.

(ii) The partial Cartan prolong $(\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_{1,(i)})_{*;\underline{M}}$ is such that $[\mathfrak{g}_{-1}, \mathfrak{g}_{1,(i)}] \simeq \mathfrak{vect}(3; \mathbb{1})$ for $i = 2, 3$. *By Convention 6.2.1, we do not investigate this partial prolong.*

Conclusion. We have found a new simple Lie superalgebra $\mathfrak{vle}^{(1)}\big(4; \underline{\widetilde{M}}|3\big)$ and a new simple Lie algebra $\mathfrak{vle}^{(1)}\big(7; \underline{\widetilde{M}}\big)$. Partial prolongs do not yield new simple Lie (super)algebras.

7 $\mathfrak{vle}\big(9; \widetilde{\underline{N}}\big) := \mathbf{F}(\mathfrak{vle}(3; \underline{N}|6))$, where $\mathfrak{vle}(3; \underline{N}|6) := \mathfrak{vle}(4; \underline{M}|3; K)$ for $p = 2$

The Lie superalgebra $\mathfrak{vle}(3; \underline{N}|6) := \mathfrak{vle}(4; \underline{M}|3; K)$ is the complete prolong of its negative part, see Section 2.11. A realization of the weight basis of the nonpositive components by vector fields is as follows, where the w_i is a shorthand notation for convenience:

$\mathfrak{g}_i$	the basis elements
$\mathfrak{g}_{-2}$	$\partial_1,\ \partial_2,\ \partial_3$
$\mathfrak{g}_{-1}$	$\partial_4,\ \partial_5,\ \partial_6,\ w_7 = x_5\partial_3 + x_6\partial_2 + \partial_7,$
	$w_8 = x_4\partial_3 + x_6\partial_1 + \partial_8,\ w_9 = x_4\partial_2 + x_5\partial_1 + \partial_9$
$\mathfrak{g}_0 \cong$	$X_1^+ = x_2\partial_1 + x_4\partial_5 + x_7\partial_8,\ X_1^- = x_1\partial_2 + x_5\partial_4 + x_8\partial_7,\ X_3^\pm = \big[X_1^\pm, X_2^\pm\big],$
$\mathfrak{sl}(3) \oplus \mathfrak{gl}(2)$	$X_2^+ = x_3\partial_2 + x_5\partial_6 + x_8\partial_9,\ X_2^- = x_2\partial_3 + x_6\partial_5 + x_9\partial_8,\ H_i = \big[X_i^+, X_i^-\big]$ for $i = 1, 2;$
	$\widetilde{X}_1^+ = x_7\,x_8\partial_3 + x_7\,x_9\partial_2 + x_8\,x_9\partial_1 + x_7\partial_4 + x_8\partial_5 + x_9\partial_6,$
	$\widetilde{X}_1^- = x_4x_5\partial_3 + x_4x_6\partial_2 + x_5x_6\partial_1 + x_4\partial_7 + x_5\partial_8 + x_6\partial_9,$
	$d = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + x_5\partial_5 + x_6\partial_6,\ \widetilde{H}_1 = \big[\widetilde{X}_1^+, \widetilde{X}_1^-\big]$

The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ has the following lowest- weight vectors:

$$v_1 = x_1x_4\partial_3 + x_1x_6\partial_1 + x_2x_5\partial_3 + x_2x_6\partial_2 + x_1\partial_8 + x_2\partial_7 + x_4x_6\partial_4 + x_4x_9\partial_7$$
$$+ x_5x_6\partial_5 + x_5x_9\partial_8 + x_6x_7\partial_7 + x_6x_8\partial_8,$$
$$v_2 = x_1x_5\partial_3 + x_1x_6\partial_2 + x_1\partial_7 + x_5x_6\partial_4 + x_5x_9\partial_7 + x_6x_8\partial_7.$$

7.1   No simple partial prolongs

For $\underline{N}$ unconstrained, $\dim \mathfrak{g}_1 = 18$. The module V_1 generated by v_1 is 6-dimensional, and the module V_2 generated by v_2 is 8-dimensional; $V_1 \subset V_2$.

    Critical coordinates of the unconstrained shearing vector: $\underline{\widetilde{N}}_4, \ldots, \underline{\widetilde{N}}_9$.

8   A description of $\widetilde{\mathfrak{vle}}\big(9; \widetilde{\underline{N}}\big) := \mathbf{F}(\mathfrak{vle}(5; \underline{N}|4))$ for $p = 2$

Whenever possible in this section, we do not indicate the shearing vectors. This Lie superalgebra is the complete prolong of its negative part, see Section 2.11.

    For $p = 0$, the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ is that on the tensor product of a 2-dimensional space on the space of semi–densities in 2 odd indeterminates. So it is not possible to just reduce modulo 2 the formulas derived for the characteristic 0. We have to understand how $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ when $p = 2$. For this, we use the explicit form of elements of $\mathfrak{vle}(4|3)$ for $p = 2$, see equation (6.1).

    Note that the mapping $\mathrm{Le}_f \longmapsto D_{(f,0)}$ determines a Lie superalgebra isomorphism between $\mathfrak{le}(3)$ and its image in $\mathfrak{vle}$. However, first, the mapping $D_{0,-} : g \longmapsto D_{(0,\,g)}$ has the kernel:

$$\mathrm{Ker}(D_{0,-}) = \{g \in \mathcal{O}(x; \underline{M}|\xi) \,|\, \deg_\xi g < 2 \text{ and } \Delta g = 0\}, \tag{8.1}$$

and, second, certain coincidences $D_{(f,0)} = D_{(0,g)}$ might occur. Formula (6.1) makes it clear that such a coincidence takes place if and only if (recall formula (5.2) for B_g)

$$B_f = 0, \quad \Delta f = \Delta g = 0, \quad \text{and} \quad \mathrm{Le}_f = B_g.$$

Taking equation (5.2) into account, these conditions are equivalent to following conditions:

$$f = f(x), \quad g = \sum_{(i,j,k)\in A_3} \frac{\partial f}{\partial x_i}\xi_j\xi_k. \tag{8.2}$$

The grading of the Lie superalgebra we are interested in is induced by the following grading of the space of generating functions:

$$\deg \xi_1 = 0, \quad \deg x_1 = 2, \quad \deg \xi_2 = \deg \xi_3 = \deg x_2 = \deg x_3 = 1, \quad \deg y = 0. \tag{8.3}$$

Clearly, $\deg D_{(f,g)} = \deg f - 2 = \deg g - 2$. Therefore (here we introduce the 9 indeterminates z_1, z_2, z_3, z_8, z_9 (even) and z_4, z_5, z_6, z_7 (odd) of the ambient Lie superalgebra $\mathfrak{vect}(5; \underline{N}|4)$ containing our $\mathfrak{g}$) and $\partial_i := \partial_{z_i}$

$\mathfrak{g}_i$	the basis elements	in terms of $\mathfrak{vect}(5; \underline{N}|4)$
$\mathfrak{g}_{-2}$	$D_{(f,0)}$, where $f = \xi_1$	∂_1
$\mathfrak{g}_{-1}$	$D_{(f,0)}$, where $f = x_i, \ \xi_1\xi_i \,|\, \xi_i, \ \xi_1 x_i$ for $i = 2,3$	$x_i \longleftrightarrow \partial_{2+i}, \ \xi_1\xi_i \longleftrightarrow \partial_{4+i} + z_{2+i}\partial_1$
		$\xi_i \longleftrightarrow \partial_i, \ \xi_1 x_i \longleftrightarrow \partial_{6+i} + z_i\partial_1$

because, for the nonzero vector fields of the form $D_{(0,g)}$ lying in $\mathfrak{g}_-$, we have, thanks to conditions (8.2), the following identifications:

$$D_{(0,\xi_1\xi_2)} = D_{(x_3,0)}, \quad D_{(0,\xi_1\xi_3)} = D_{(x_2,0)}.$$

Because the tautological representation of $\mathfrak{sl}(2)$ is isomorphic to its dual, we identify

$$\mathfrak{g}_{-1} \simeq W = V \otimes \Lambda(\xi, \eta), \quad \text{where} \quad V = \mathrm{Span}(v_1, v_2)$$

using the rules listed in Table (8.4). The table also contains the explicit form of the vector fields $D_{(f,0)} \in \mathfrak{g}_{-1}$ needed to calculate the action of the fields of the form $D_{(0,g)} \in \mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ (the action of the fields of the form $D_{(f,0)} \in \mathfrak{g}_0$ can be computed in terms of generating functions and the bracket in $\mathfrak{k}$).

f	$D_{(f,0)}$	the image in W	f	$D_{(f,0)}$	the image in W
ξ_2	∂_{x_2}	v_1	$\xi_1\xi_2$	$\xi_2\partial_{x_1} + \xi_1\partial_{x_2} + y\partial_{\xi_3}$	$v_1 \otimes \xi$
ξ_3	∂_{x_3}	v_2	$\xi_1\xi_3$	$\xi_3\partial_{x_1} + \xi_1\partial_{x_3} + y\partial_{\xi_2}$	$v_2 \otimes \xi$
x_2	∂_{ξ_2}	$v_2 \otimes \eta$	$\xi_1 x_2$	$\xi_1\partial_{\xi_2} + x_2\partial_{x_1}$	$v_2 \otimes \xi\eta$
x_3	∂_{ξ_3}	$v_1 \otimes \eta$	$\xi_1 x_3$	$\xi_1\partial_{\xi_3} + x_3\partial_{x_1}$	$v_1 \otimes \xi\eta$

$$(8.4)$$

Proposition 8.1 (the component $\mathfrak{g}_0$ of $\mathfrak{g} = \mathfrak{vle}(5; \underline{N}|4) := \mathfrak{vle}(4; \underline{N}|3; 1)$ and its action on $\mathfrak{g}_{-1}$)**.** *The component $\mathfrak{g}_0$ of $\mathfrak{g} = \mathfrak{vle}(5; \underline{N}|4) := \mathfrak{vle}(4; \underline{N}|3; 1)$ consists of vector fields $D_{(f,\,g)}$, where $\deg f = \deg g = 2$ in the grading (8.3). Table (8.5) shows the correspondence between pair of generating functions (f, g) and operators in $\mathrm{End}(W)$*

(f, g)	its image in $\mathrm{End}(W)$	(f, g)	its image in $\mathrm{End}(W)$
$f = \sum\limits_{i,j=2,3} a_{ij}x_i\xi_j,$ $\Delta(f) = 0, \ g = 0$	$\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \otimes \mathbb{1}$ $a_{22} + a_{33} = 0$	$f = \xi_1 \sum\limits_{i,j=2,3} a_{ij}x_i\xi_j,$ $\Delta(f) = 0, \ g = 0$	$\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \otimes \xi$ $a_{22} + a_{33} = 0$
$(x_2^{(2)}, 0)$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \eta$	$(\xi_1 x_2^{(2)}, 0)$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \xi\eta$
$(x_3^{(2)}, 0)$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \eta$	$(\xi_1 x_3^{(2)}, 0)$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \xi\eta$
$(x_2 x_3, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \eta$	$(\xi_1 x_2 x_3, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \xi\eta$
$(x_1, 0)$ $(\xi_2\xi_3, 0)$	∂_ξ ∂_η	$(\xi_1 x_1, 0)$ $(\xi_1\xi_2\xi_3, 0)$	$\xi\partial_\xi$ $\xi\partial_\eta$
$(x_2\xi_2, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1} + \eta\partial_\eta$	$(\xi_1 x_2\xi_2, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \xi + \xi\eta\partial_\eta$
$(0, \xi_1 x_1)$	$\eta\partial_\xi$	$(0, \xi_1\xi_2\xi_3)$	$\xi\partial_\xi + \eta\partial_\eta$
$(0, \xi_1\xi_2 x_2)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \eta + \xi\eta\partial_\xi$		

$$(8.5)$$

Thus, $\mathfrak{g}_0 \simeq \mathfrak{d}(\mathfrak{sl}(2) \otimes \Lambda(2) \ltimes \mathfrak{vect}(0|2))$, where the operator of outer derivation added to the ideal $\mathfrak{sl}(2) \otimes \Lambda(2) \ltimes \mathfrak{vect}(0|2)$ is $\mathcal{D} = D \otimes \mathbb{1}$, where $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and the subalgebra $\mathfrak{vect}(0|2)$ acts as $X \mapsto \mathbb{1} \otimes X + D \otimes \mathrm{div}(X)$ for any $X \in \mathfrak{vect}(0|2)$. Hence, $\mathfrak{g}_{-1} \simeq \mathrm{Vol}(0|2) \oplus \Lambda(2)$, as $\mathfrak{vect}(0|2)$-module.

Proof. Let us begin with fields of the form $D_{(f,0)}$.

As we have already noted above, $[D_{(f_1,0)}, D_{(f_2,0)}] = D_{(\{f_1,f_2\},0)}$, and hence the action of such fields can be described in terms of the generating functions and the Buttin bracket.

If $f = \sum\limits_{i,j=2,3} a_{ij} x_i \xi_j$ and $\Delta(f) = 0$, then f acts on $\mathfrak{g}_{-1}$ as $\sum\limits_{i,j=2,3} a_{ij}(\xi_j \partial_{\xi_i} + x_i \partial_{x_j})$ which, thanks to our identification, corresponds to the action of the operator $\left(\begin{smallmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{smallmatrix}\right) \otimes \mathbb{1}$; i.e., the elements of this form span the subspace $\mathfrak{sl}(V) \otimes \mathbb{1} \in \mathrm{End}(W)$.

Analogously, the functions of the form $f = \xi_1 \sum\limits_{i,j=2,3} a_{ij} x_i \xi_j$ such that $\Delta(f) = 0$ act on $\mathfrak{g}_{-1}$ as $\xi_1 \sum\limits_{i,j=2,3} a_{ij}(\xi_j \partial_{\xi_i} + x_i \partial_{x_j})$ which corresponds to the action of the operator $\left(\begin{smallmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{smallmatrix}\right) \otimes \xi$; i.e., the elements of this form span the subspace $\mathfrak{sl}(V) \otimes \xi \in \mathrm{End}(W)$.

The correspondence between the other operators $D_{(f,g)}$ and operators in $\mathrm{End}(W)$ is not so evident. To find it, in tables (8.6)–(8.8) and (8.9)–(8.10) we indicate nonzero actions of an operator $D_{(f,g)}$ in the first row of these tables and the same actions in the basis of the space W in the second row. This allows us to reestablish the operator in $\mathrm{End}(W)$.

For example, Table (8.6) describes the action of the generating function $f = x_2^{(2)}$. In terms of generating functions, f acts as Le_f, i.e., as the vector field (operator) $X = x_2 \partial_{\xi_2}$. Looking at Table (8.4) we deduce that this X acts as non-zero only on ξ_2 and $\xi_1\xi_2$. Explicitly, $X(\xi_2) = x_2$, and $X(\xi_1\xi_2) = \xi_1 x_2$. This is precisely what is written in the first row of Table (8.6).

In the 2nd row there is the same action in terms of the basis of W. Indeed, looking again at Table (8.4) we see that in W the incarnation of ξ_2 is denoted by v_1, whereas x_2 by $v_2 \otimes \eta$; i.e., the equality $X(\xi_2) = x_2$ in the new notation takes the form $X(v_1) = v_2 \otimes \eta$:

$$
\begin{array}{|c|c|}
\hline
\xi_2 \longmapsto x_2 & \xi_1\xi_2 \longmapsto \xi_1 x_2 \\
\hline
v_1 \longmapsto v_2 \otimes \eta & v_1 \otimes \xi \longmapsto v_2 \otimes \xi\eta \\
\hline
\end{array}
\tag{8.6}
$$

Thus, $f = x_2^{(2)}$ acts as $x_2 \partial_{\xi_2}$, i.e., as $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \eta \in \mathrm{End}(W)$.

Similarly, $f = x_3^{(2)}$ acts as $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \otimes \eta \in \mathrm{End}\,W$, whereas $f = x_2 x_3$ acts as the operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \eta \in \mathrm{End}(W)$.

Thus, the functions of the form $f = f(x_2, x_3)$ span $\mathfrak{sl}(V) \otimes \eta \in \mathrm{End}(W)$.

Analogously, the functions of the form $f = \xi_1 f(x_2, x_3)$ span $\mathfrak{sl}(V) \otimes \xi\eta \in \mathrm{End}(W)$.

Clearly, $f = x_1$ acts on $\mathfrak{g}_{-1}$ as ∂_{ξ_1} which corresponds to the operator $\partial_\xi \in \mathrm{End}(W)$, and $f = \xi_1 x_1$ acts on $\mathfrak{g}_{-1}$ as $\xi_1 \partial_{\xi_1}$ which corresponds to the operator $\xi\partial_\xi \in \mathrm{End}(W)$. The element $f = \xi_2\xi_3$ acts on $\mathfrak{g}_{-1}$ as $\xi_2 \partial_{x_3} + \xi_3 \partial_{x_2}$, i.e., as $\partial_\eta \in \mathrm{End}(W)$:

$$
\begin{array}{|c|c|c|c|}
\hline
x_3 \longmapsto \xi_2 & x_2 \longmapsto \xi_3 & \xi_1 x_3 \longmapsto \xi_1\xi_2 & \xi_1 x_2 \longmapsto \xi_1\xi_3 \\
\hline
v_1 \otimes \eta \longmapsto v_1 & v_2 \otimes \eta \longmapsto v_2 & v_1 \otimes \xi\eta \longmapsto v_1 \otimes \xi & v_2 \otimes \xi\eta \longmapsto v_2 \otimes \xi \\
\hline
\end{array}
\tag{8.7}
$$

Analogously, $f = \xi_1\xi_2\xi_3$ acts as the action of $\xi\partial_\eta \in \mathrm{End}(W)$.

Finally, $f = x_2\xi_2$ acts as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes \mathbb{1} + \eta\partial_\eta$:

$$
\begin{array}{|c|c|c|c|}
\hline
\xi_2 \longmapsto \xi_2 & \xi_1\xi_2 \longmapsto \xi_1\xi_2 & x_2 \longmapsto x_2 & \xi_1 x_2 \longmapsto \xi_1 x_2 \\
\hline
v_1 \longmapsto v_1 & v_1 \otimes \xi \longmapsto v_1 \otimes \xi & v_2 \otimes \eta \longmapsto v_2 \otimes \eta, & v_2 \otimes \xi\eta \longmapsto v_2 \otimes \xi\eta \\
\hline
\end{array}
\tag{8.8}
$$

Analogously, $f = \xi_1 x_2 \xi_2$ acts as the operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes \xi + \xi\eta\partial_\eta$.

Now, let us describe the action of operators $D_{(0,g)}$. First, taking equation (8.2) into account, we have

$$D_{(0,\xi_2\xi_3)} = D_{(x_1,0)}, \quad D_{(0,\xi_1\xi_2x_3)} = D_{(x_3^{(2)},0)},$$
$$D_{(0,\xi_1\xi_3x_2)} = D_{(x_2^{(2)},0)}, \quad D_{(0,\xi_1\xi_2x_2+\xi_1\xi_3x_3)} = D_{(x_2x_3,0)}.$$

Now, taking the kernel (8.1) into account, we only have to establish the three operators corresponding to the functions $g = \xi_1 x_1$, $\xi_1\xi_2\xi_3$, and $\xi_1\xi_2 x_2$.

For $g = \xi_1 x_1$, the operator $D_{(0,g)} = \partial_y$ corresponds to $\eta\partial_\xi \in \mathrm{End}(W)$:

$\xi_1\xi_2 \longmapsto x_3$	$\xi_1\xi_3 \longmapsto x_2$
$v_1 \otimes \xi \longmapsto v_1 \otimes \eta$	$v_2 \otimes \xi \longmapsto v_2 \otimes \eta$

$$(8.9)$$

If $g = \xi_1\xi_2\xi_3$, then $D_{(0,g)} = \sum\limits_{1\leq i\leq 3} \xi_i\partial_i$, which corresponds to $\xi\partial_\xi + \eta\partial_\eta \in \mathrm{End}(W)$.

Finally, let $D_{(0,\xi_1\xi_2x_2)} = x_2\partial_{\xi_3} + \xi_1\partial_y$. This operator acts as follows:

$\partial_{x_2} = D_{(\xi_2,0)} \longmapsto \partial_{\xi_3} = D_{(x_3,0)}$	$\xi_3\partial_{x_1} + \xi_1\partial_{x_3} + y\partial_{\xi_2} = D_{(\xi_1\xi_3,0)}$
	$\longmapsto \xi_1\partial_{\xi_2} + x_2\partial_{x_1} = D_{(\xi_1x_2,0)}$
$v_1 \longmapsto v_1 \otimes \eta$	$v_2 \otimes \xi \longmapsto v_2 \otimes \xi\eta$

$$(8.10)$$

which corresponds to the action of the operator $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \otimes \eta + \xi\eta\partial_\xi \in \mathrm{End}(W)$.

$D_{(x_1\xi_1+x_2\xi_2,\xi_1\xi_2\xi_3)}$ corresponds to the operator $\mathcal{D} := \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \otimes \mathbb{1} \in \mathrm{End}(W)$, and $D_{(x_2\xi_2,\xi_1\xi_2\xi_3)}$ corresponds to the operator $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \otimes \mathbb{1} + \xi\partial_\xi \in \mathrm{End}(W)$. ∎

Critical coordinates of $\underline{\widetilde{N}}^u$: $\underline{\widetilde{N}}_1$, $\underline{\widetilde{N}}_2$, $\underline{\widetilde{N}}_3$, and $\mathrm{sdim}\,\mathfrak{g}_1 = 20|20$. There are five lowest-weight vectors in $\mathfrak{g}_1$:

$$v_1 = z_1z_3\partial_1 + z_1\partial_9 + z_3z_6\partial_6 + z_3z_7\partial_7 + z_3z_8\partial_8 + z_3z_9\partial_9,$$
$$v_2 = z_3^{(2)}\partial_3 + z_3z_4\partial_4 + z_3z_6\partial_6 + z_3z_9\partial_9 + z_4z_6\partial_9 + z_1z_3\partial_1 + z_1\partial_9 + z_2z_3\partial_2$$
$$\qquad + z_2z_4\partial_5 + z_2z_7\partial_6 + z_2z_8\partial_9 + z_3z_6\partial_6 + z_3z_7\partial_7 + z_4z_7\partial_8,$$
$$v_3 = z_3^{(2)}\partial_2 + z_3z_4\partial_5 + z_3z_7\partial_6 + z_3z_8\partial_9 + z_4z_7\partial_9,$$
$$v_4 = z_1z_2\partial_1 + z_1\partial_8 + z_2z_6\partial_6 + z_2z_7\partial_7 + z_2z_8\partial_8 + z_2z_9\partial_9 + z_2^{(2)}\partial_2$$
$$\qquad + z_2z_3\partial_3 + z_2z_7\partial_7 + z_3z_5\partial_4 + z_3z_6\partial_7 + z_3z_9\partial_8 + z_5z_6\partial_9 + z_1z_2\partial_1 + z_1\partial_8$$
$$\qquad + z_2z_4\partial_4 + z_2z_7\partial_7 + z_2z_9\partial_9 + z_4z_6\partial_8 + z_2z_5\partial_5 + z_2z_6\partial_6 + z_5z_7\partial_8,$$
$$v_5 = z_3^{(2)}\partial_3 + z_3z_4\partial_4 + z_3z_6\partial_6 + z_3z_9\partial_9 + z_4z_6\partial_9 + z_3z_5\partial_5 + z_3z_7\partial_7 + z_3z_9\partial_9 + z_5z_7\partial_9.$$

No simple partial prolongs. The module generated by either one of v_1, v_2 is $\mathfrak{g}_1$. The modules V_i generated by either of v_3, v_4 are of dimension $4|4$ and $\mathrm{sdim}([\mathfrak{g}_{-1}, V_i]) = 4|4$; $\mathrm{sdim}\,V_5 = 8|8$ and $\mathrm{sdim}([\mathfrak{g}_{-1}, V_i]) = 8|10$, so there are no new simple partial prolongs, see 6.2.1.

8.1 Desuperization

For $\underline{N}$ unconstrained, the critical coordinates are those that correspond to the formerly odd indeterminates.

9 $\mathfrak{kle}\big(15; \widetilde{\underline{N}}\big) := \mathbf{F}(\mathfrak{kle}(5; \underline{N}|10))$

Whenever possible in this section, we do not indicate the shearing vectors. The Lie superalgebra $\mathfrak{kle}(5; \underline{N}|10)$ is the complete prolong of its negative part, see Section 2.11.

Recall that for $\mathfrak{g} = \mathfrak{kle}(5|10)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{svect}(5|0) \simeq d\Omega^3(5|0)$ and $\mathfrak{g}_{\bar{1}} = \Pi\big(d\Omega^1(5|0)\big)$ with the natural $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{g}_{\bar{1}}$ and the bracket of any two odd elements being their product, where we identify

$$dx_i \wedge dx_j \wedge dx_k \wedge dx_l \otimes \mathrm{vol}^{-1} = \mathrm{sign}(ijklm)\partial_{x_m} \text{ for any permutation } (ijklm) \text{ of } (12345).$$

Let x_i, where $1 \leq i \leq 5$, be the even indeterminates, $\partial_i := \partial_{x_i}$. Let θ_{ab}, where $1 \leq a, b \leq 5$, be an odd indeterminate such that $\theta_{ab} = -\theta_{ba}$; in particular, $\theta_{aa} = 0$ and we may assume that $a < b$. Let $\delta_{ab} := \partial_{\theta_{ab}}$. Let $\mathfrak{g}_0 = \mathfrak{sl}(5) = \mathfrak{sl}(V)$ act on $\mathfrak{g}_{-2}$ as on its tautological 5-dimensional module V. Let $E^2(V)$ be the 2nd exterior power of V. For a basis of the nonpositive components of $\mathbf{F}(\mathfrak{kle}(5; \underline{N}|10))$ we take the following elements (only Chevalley generators are given for $\mathfrak{g}_0$):

$\mathfrak{g}_i$	the basis elements
$\mathfrak{g}_{-2} = V$	$\partial_1, \partial_2, \partial_3, \partial_4, \partial_5$
$\mathfrak{g}_{-1} = E^2(V)$	$w_{12} = \delta_{12} + \theta_{34}\partial_5 + \theta_{45}\partial_3 - \theta_{35}\partial_4, \ \ w_{13} = \delta_{13} + \theta_{25}\partial_4 - \theta_{24}\partial_5 - \theta_{45}\partial_2,$
	$w_{14} = \delta_{14} + \theta_{23}\partial_5 + \theta_{35}\partial_2 - \theta_{25}\partial_3, \ \ w_{15} = \delta_{15} + \theta_{24}\partial_3 - \theta_{23}\partial_4 - \theta_{34}\partial_2,$
	$w_{23} = \delta_{23} + \theta_{45}\partial_1, \ \ w_{24} = \delta_{24} + \theta_{35}\partial_1, \ \ w_{25} = \delta_{25} + \theta_{34}\partial_1,$
	$\delta_{34}, \ \ \delta_{35}, \ \ \delta_{45}$
$\mathfrak{g}_0 = \mathfrak{sl}(V)$	$Z_1 = x_1\partial_2 + \theta_{23}\theta_{24}\partial_5 + \theta_{23}\theta_{25}\partial_4 + \theta_{24}\theta_{25}\partial_3 + \theta_{23}\delta_{13} + \theta_{24}\delta_{14} + \theta_{25}\delta_{15},$
	$Z_2 = x_2\partial_3 + \theta_{34}\theta_{35}\partial_1 + \theta_{13}\delta_{12} + \theta_{34}\delta_{24} + \theta_{35}\delta_{25},$
	$Z_3 = x_3\partial_4 + \theta_{14}\delta_{13} + \theta_{24}\delta_{23} + \theta_{45}\delta_{35}, \ \ Z_4 = x_4\partial_5 + \theta_{15}\delta_{14} + \theta_{25}\delta_{24} + \theta_{35}\delta_{34}$
	$H_1 = [Z_1, Y_1], \ \ H_2 = [Z_2, Y_2], \ \ H_3 = [Z_3, Y_3], \ \ H_4 = [Z_4, Y_4]$
	$Y_1 = x_2\partial_1 + \theta_{13}\theta_{14}\partial_5 + \theta_{13}\theta_{15}\partial_4 + \theta_{14}\theta_{15}\partial_3 + \theta_{13}\delta_{23} + \theta_{14}\delta_{24} + \theta_{15}\delta_{25},$
	$Y_2 = x_3\partial_2 + \theta_{24}\theta_{25}\partial_1 + \theta_{12}\delta_{13} + \theta_{24}\delta_{34} + \theta_{25}\delta_{35}$
	$Y_3 = x_4\partial_3 + \theta_{13}\delta_{14} + \theta_{23}\delta_{24} + \theta_{35}\delta_{45}, \ \ Y_4 = x_5\partial_4 + \theta_{14}\delta_{15} + \theta_{24}\delta_{25} + \theta_{34}\delta_{35},$

The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible of dimension 40. The lowest-weight vector is

$$v_1 = \theta_{12}\theta_{13}\delta_{15} + \theta_{12}\theta_{23}\delta_{25} + \theta_{13}\theta_{23}\delta_{35} + \theta_{14}\theta_{23}\delta_{45} + \theta_{12}\theta_{13}\theta_{23}\partial_4 + \theta_{12}\theta_{14}\theta_{23}\partial_3$$
$$+ \theta_{13}\theta_{14}\theta_{23}\partial_2 + \theta_{13}\theta_{23}\theta_{24}\partial_1 + x_5\delta_{45}.$$

No simple partial prolongs. Critical coordinates of the shearing vector for $\mathfrak{kle}\big(15; \widetilde{\underline{N}}\big)$ *are those corresponding to the formerly odd indeterminates.*

10 $\widetilde{\mathfrak{kle}}\big(15; \widetilde{\underline{N}}\big) := \mathbf{F}(\mathfrak{kle}(9; \underline{N}|6))$

The construction of $\mathfrak{kle}(9; \underline{N}|6)$, and its desuperization $\widetilde{\mathfrak{kle}}\big(15; \widetilde{\underline{N}}\big)$, resemble that of $\mathfrak{kas}$, see Section 17.

However, $\mathfrak{kas}$ is a partial prolong of $\mathfrak{k}(1|6)_{\leq 0} \oplus \mathfrak{kas}_1$, where $\mathfrak{kas}_1$ is a "half" of $\mathfrak{k}(1|6)_1$. Indeed, $\mathfrak{kas}_1$ is one of the two irreducible modules whose direct sum is $\mathfrak{k}(1|6)_1$, and the Lie superalgebra $\mathfrak{kle}(9; \underline{N}|6)$ is the prolong of $\mathfrak{k}(9; \underline{N}|6)_- \oplus \mathfrak{kle}(9; \underline{N}|6)_0$, the latter summand constituting a half of $\mathfrak{k}(9; \underline{N}|6)_0$; this half corresponds to either of the two possible embeddings $\mathfrak{svect}(0|4) \longrightarrow \mathfrak{osp}(6|8)$ corresponding to the representations of $\mathfrak{svect}(0|4)$ in the space $T_0^0(0|4) := \mathrm{Vol}_0(0|4)/\mathbb{K} \cdot \mathrm{vol}$, see (1.13), and in its dual.

This is why $\mathfrak{kle}(9; \underline{N}|6)$ is NOT the complete prolong of its negative part, see Section 2.11.

To determine the component $\mathfrak{g}_0$ of $\mathfrak{g}$, we have to consider a linear combination of two elements: the central element Z commuting with the image of $\mathfrak{svect}(0|4)$ in $\mathfrak{osp}(6|8)$ and an outer derivation, say $D = \xi_1\partial_{\xi_1} \in \mathfrak{vect}(0|4)$.

Let $\mathfrak{G}$ be the prolong of the nonpositive part where $\mathfrak{G}_0 := (\mathfrak{svect}(0|4) \ltimes \mathbb{K}D) \bigoplus \mathbb{K}Z$ and $\mathfrak{G}_- := \mathfrak{g}_-$. Having computed $[\mathfrak{G}_1, \mathfrak{g}_{-1}]$ we determine the coefficients in the linear combination $aZ + bD$ that should belong to $\mathfrak{g}_0 := \mathfrak{svect}(0|4) \ltimes \mathbb{K}(aD + bZ)$ from the condition $[\mathfrak{G}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$.

To realize the Lie superalgebra $\mathfrak{g}$ by vector fields, we use the representation of the even part of $\mathfrak{g}$ as $\mathfrak{svect}(5; \underline{M})$ and its odd part as $\Pi\big(d\Omega^1(5; \underline{M})\big)$: whatever the $\mathbb{Z}$-grading of $\mathfrak{g}$, the components $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ have the needed nonpositive part. For convenience, we use $\mathfrak{gl}(5)$-weights of the elements of $\mathfrak{g}$, having added the outer derivation – the grading operator – to $\mathfrak{svect}(5; \underline{M})$.

Let $u_1, \dots, u_5$ be a basis of the space U we used to define $\mathfrak{svect}(U)$ and $d\Omega^1(U)$. In our grading, $\deg(u_5) = 2$ and $\deg(u_i) = 1$ for $i < 5$. Let $x_1, \dots, x_{15}$ be the desuperized indeterminates. Then,

$$
\begin{aligned}
&\partial_{x_1} + \cdots \longleftrightarrow \partial_{u_1}, \quad \dots, \quad \partial_{x_4} + \cdots \longleftrightarrow \partial_{u_4}, \\
&\partial_{x_5} + \cdots \longleftrightarrow \Pi(du_1 du_2), \quad \dots, \quad \partial_{x_{10}} \longleftrightarrow \Pi(du_3 du_4), \\
&\partial_{x_{11}} \longleftrightarrow u_1 \partial_{u_5}, \quad \dots, \quad \partial_{x_{14}} \longleftrightarrow u_4 \partial_{u_5}, \quad \partial_{x_{15}} \longleftrightarrow \partial_{u_5}.
\end{aligned}
\tag{10.1}
$$

The functor Π is interpreted as multiplication (tensoring) by the 1-dimensional module whose generator Π has the following weight w to make the weight and degree compatible:

$$
w(\Pi) = \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right),
$$
$$
\deg(\Pi) = -\frac{5}{2}.
$$

To get rid of fractions, we multiply all weights by 2; assuming that $\deg du_i = \deg u_i$ we have

$$
\begin{aligned}
w(x_5) &= w(\Pi) + w(du_1) + w(du_2) \\
&= (-1, -1, -1, -1, -1) + (2, 0, 0, 0, 0) + (0, 2, 0, 0, 0) \\
&= (1, 1, -1, -1, -1).
\end{aligned}
$$

Now, the weights are symmetric in the sense that if there is an element of weight $(2, 0, 0, 0, 0)$, there should be elements whose weight have all coordinates but one equal to 0, one coordinate being equal to 2. This symmetry helps to find correct expressions of the vector fields in each component. Thus, the weights of the indeterminates in the new grading are as follows:

$$
\begin{array}{lll}
x_1 \to \{-2, 0, 0, 0, 0\}, & x_6 \to \{1, -1, 1, -1, -1\}, & x_{11} \to \{2, 0, 0, 0, -2\}, \\
x_2 \to \{0, -2, 0, 0, 0\}, & x_7 \to \{1, -1, -1, 1, -1\}, & x_{12} \to \{0, 2, 0, 0, -2\}, \\
x_3 \to \{0, 0, -2, 0, 0\}, & x_8 \to \{-1, 1, 1, -1, -1\}, & x_{13} \to \{0, 0, 2, 0, -2\}, \\
x_4 \to \{0, 0, 0, -2, 0\}, & x_9 \to \{-1, 1, -1, 1, -1\}, & x_{14} \to \{0, 0, 0, 2, -2\}, \\
x_5 \to \{1, 1, -1, -1, -1\}, & x_{10} \to \{-1, -1, 1, 1, -1\}, & x_{15} \to \{0, 0, 0, 0, -2\}.
\end{array}
$$

The degree is equal to one half of (the sum of the first 4 coordinates plus the doubled fifth one).

In equation (10.2) we give the basis of the negative part and generators of the 0th component. It is possible to generate the semi-simple part of $\mathfrak{g}_0$ by just 1 positive and 4 negative generators, or 4 positive and 1 negative ones, but for symmetry we give 4 and 4 of them. These 8 generators

do not generate the element $D + Z \in [\mathfrak{g}_1, \mathfrak{g}_{-1}]$ of weight $(0, 0, 0, 0, 0)$, so we give it separately.

$\mathfrak{g}_i$	the generators
$\mathfrak{g}_{-2}$	∂_{15}
$\mathfrak{g}_{-1} \simeq \mathbf{F}\big(T^0_0(0\|4)\big)$,	$\partial_1 + x_{11}\partial_{15}, \ \partial_2 + x_{12}\partial_{15}, \ \partial_3 + x_{13}\partial_{15}, \ \partial_4 + x_{14}\partial_{15}, \ \partial_5 + x_{10}\partial_{15},$
see (1.13)	$\partial_6 + x_9\partial_{15}, \ \partial_7 + x_8\partial_{15}, \ \partial_8, \ldots, \partial_{14}$
$\mathfrak{g}_0 \simeq \mathbb{K}(D + Z)$	$\{-1, -1, -1, 1, 1\} \to x_2\partial_6 + x_9\partial_{12} + x_3\partial_5 + x_{10}\partial_{13} + x_1\partial_8$
$\rtimes \mathfrak{svect}(4; \mathbb{1})$	$\qquad\qquad + x_7\partial_{11} + x_1 x_7\partial_{15}$
	$\{-1, -1, 1, -1, 1\} \to x_2\partial_7 + x_8\partial_{12} + x_4\partial_5 + x_{10}\partial_{14} + x_1\partial_9$
	$\qquad\qquad + x_6\partial_{11} + x_1 x_6\partial_{15}$
	$\{-1, 1, -1, -1, 1\} \to x_3\partial_7 + x_8\partial_{13} + x_4\partial_6 + x_9\partial_{14} + x_1\partial_{10}$
	$\qquad\qquad + x_5\partial_{11} + x_1 x_5\partial_{15}$
	$\{1, -1, -1, -1, 1\} \to x_2\partial_{10} + x_5\partial_{12} + x_2 x_5\partial_{15} + x_3\partial_9 + x_6\partial_{13}$
	$\qquad\qquad + x_3 x_6\partial_{15} + x_4\partial_8 + x_7\partial_{14} + x_4 x_7\partial_{15}$
	$\{0, 0, 0, 0, 0\} \to x_1\partial_1 + x_8\partial_8 + x_9\partial_9 + x_{10}\partial_{10} + x_{12}\partial_{12} + x_{13}\partial_{13}$
	$\qquad\qquad + x_{14}\partial_{14} + x_{15}\partial_{15}$
	$\{0, 0, 0, 4, -2\} \to x_{14}\partial_4 + x_{14}^{(2)}\partial_{15}, \quad \{0, 0, 4, 0, -2\} \to x_{13}\partial_3 + x_{13}^{(2)}\partial_{15}$
	$\{0, 4, 0, 0, -2\} \to x_{12}\partial_2 + x_{12}^{(2)}\partial_{15}, \quad \{4, 0, 0, 0, -2\} \to x_{11}\partial_1 + x_{11}^{(2)}\partial_{15}$

$$(10.2)$$

No simple partial prolongs. Critical coordinates for $\widetilde{\mathfrak{kle}}(15; \underline{\widetilde{N}})$ are those corresponding to formerly odd indeterminates.

11 $\mathfrak{kle}_3\big(20; \underline{\widetilde{N}}\big) := \mathbf{F}(\mathfrak{kle}(9; \underline{N}|11))$

Whenever possible in this section, we do not indicate the shearing vectors. The Lie superalgebra $\mathfrak{kle}(9; \underline{N}|11)$ is the complete prolong of its negative part, see Section 2.11.

11.1   Description of $\mathfrak{kle}(9; \underline{N}|11)_-$

We consider the realization of $\mathfrak{g} = \mathfrak{kle}$ as the direct sum of $\mathfrak{g}_{\bar{0}} = \mathfrak{svect}(U)$ and $\mathfrak{g}_{\bar{1}} = \Pi\big(d\Omega^1(U)\big)$, where $U = \mathrm{Span}(u_1, \ldots, u_5)$. Let $i, j = 1, 2$, while $a, b, c = 3, 4, 5$. Let $\{ijabc\} = \{12345\}$ as sets, $\partial_\alpha := \partial_{u_\alpha}$ for any index α. Set, cf. (25.4):

$$\deg u = (3, 3, 2, 2, 2), \quad \deg du = (0, 0, -1, -1, -1), \quad \text{where } u = (u_1, \ldots, u_5). \qquad (11.1)$$

Then,

$$\mathfrak{g}_{-3} = \mathrm{Span}(\partial_1, \partial_2),$$
$$\mathfrak{g}_{-2} = \mathrm{Span}(\partial_a, du_a \wedge du_b \text{ for any } a, b = 3, 4, 5),$$
$$\mathfrak{g}_{-1} = \mathrm{Span}(u_a\partial_i, du_i \wedge du_a \text{ for any } i = 1, 2, a = 3, 4, 5).$$

The brackets are as in grading K, see (25.4):

$$[\mathfrak{g}_{-1}, \ \mathfrak{g}_{-1}]: \ [du_1 \wedge dux_a, du_2 \wedge du_b] = \partial_c \text{ for } \{a, b, c\} = \{3, 4, 5\},$$
$$[du_i \wedge du_a, u_b\partial_i] = du_a \wedge du_b,$$
$$[\mathfrak{g}_{-1}, \ \mathfrak{g}_{-2}]: \ [du_i \wedge du_a, \ du_b \wedge du_c] = \partial_j, \ [\partial_a, \ u_a\partial_i] = \partial_i.$$

11.1.1 Description of $\mathfrak{kle}(9; \underline{N}|11)_-$ in terms of vector fields

We use the realization of Section 10 with the same weights and degrees (11.1).

Recall that $\mathfrak{g} := \mathfrak{kle}(9; \underline{N}|11)$ is the prolong of its *negative* part, and $\mathfrak{g}_{\bar{0}} = \mathfrak{svect}(5; \underline{M})$. For a basis of the negative part we take the following elements, where we denote the 20 indeterminates by x, set $\delta_i := \partial_{x_i}$:

$\mathfrak{g}_i$	the generators
$\mathfrak{g}_{-3}$	$\{2,0,0,0,0\} \to \delta_{19}, \quad \{0,2,0,0,0\} \to \delta_{20}$
$\mathfrak{g}_{-2}$	$\{0,0,2,0,0\} \to \delta_{13} + x_7\delta_{19} + x_8\delta_{20}, \quad \{0,0,0,2,0\} \to \delta_{14} + x_9\delta_{19} + x_{10}\delta_{20},$
	$\{0,0,0,0,2\} \to \delta_{15} + x_{11}\delta_{19} + x_{12}\delta_{20},$
	$\{1,1,-1,-1,1\} \to \delta_{16}, \quad \{1,1,-1,1,-1\} \to \delta_{17}, \quad \{1,1,1,-1,-1\} \to \delta_{18}$
$\mathfrak{g}_{-1} \simeq$	$\{-1,1,-1,1,1\} \to \delta_1 + x_5\delta_{15} + x_6\delta_{14} + x_9\delta_{16} + x_{11}\delta_{17} + x_{18}\delta_{20}$
	$\{-1,1,1,-1,1\} \to \delta_2 + x_4\delta_{15} + x_6\delta_{13} + x_7\delta_{16} + x_{11}\delta_{18} + x_{17}\delta_{20}$
	$\{-1,1,1,1,-1\} \to \delta_3 + x_4\delta_{14} + x_5\delta_{13} + x_7\delta_{17} + x_9\delta_{18} + x_{16}\delta_{20}$
	$\{1,-1,-1,1,1\} \to \delta_4 + x_{10}\delta_{16} + x_{12}\delta_{17} + x_{18}\delta_{19}$
	$\{1,-1,1,-1,1\} \to \delta_5 + x_8\delta_{16} + x_{12}\delta_{18} + x_{17}\delta_{19}$
	$\{1,-1,1,1,-1\} \to \delta_6 + x_8\delta_{17} + x_{10}\delta_{18} + x_{16}\delta_{19}$
	$\{2,0,-2,0,0\} \to \delta_7, \quad \{0,2,-2,0,0\} \to \delta_8, \quad \{2,0,0,-2,0\} \to \delta_9, \quad \{0,2,0,-2,0\} \to \delta_{10}$
	$\{2,0,0,0,-2\} \to \delta_{11}, \quad \{0,2,0,0,-2\} \to \delta_{12}$

No simple partial prolongs. Critical coordinates of $\underline{\widetilde{N}}^u$ for $\mathfrak{kle}_3(20; \widetilde{\underline{N}})$ are those corresponding to formerly odd indeterminates.

12 $\mathfrak{kle}_2(20; \widetilde{\underline{N}}) := \mathbf{F}(\mathfrak{kle}(11; \underline{N}|9))$

Whenever possible in this section, we do not indicate the shearing vectors. The Lie superalgebra $\mathfrak{kle}(11; \underline{N}|9)$ is the complete prolong of its negative part, see Section 2.11.

12.1 Description of $\mathfrak{kle}(11; \underline{N}|9)_-$

In [67], the Lie superalgebra $\mathfrak{kle}$ was constructed from a central extension of $\mathfrak{sle}^{(1)}(4)$ with central element further denoted by c. The algebra $\mathfrak{sle}^{(1)}(4)$ was considered in the grading where the degrees of the odd indeterminates are all 0. The regradings of this realization are listed in equation (25.4). Let us give details.

Let the degrees of the generating functions of $\mathfrak{sle}^{(1)}(4)$ be determined as follows:

$$\deg \xi_3 = \deg \xi_4 = 0, \quad \deg q_3 = \deg q_4 = 2, \quad \deg q_i = \deg \xi_i = 1 \quad \text{for } i = 1, 2.$$

Then, (recall that the parities of the function are opposite to the "natural" ones, and c is even)

$$\mathfrak{g}_{-2} = \mathrm{Span}(c, \xi_3, \xi_4, \xi_3\xi_4), \qquad \mathfrak{g}_{-1} = \mathrm{Span}(\xi_1, \xi_2, q_1, q_2) \otimes \Lambda(\xi_3, \xi_4),$$

with the nonzero brackets of the generating functions f and g in ξ_3 and ξ_4 being as follows:

$$[f\xi_1, \ g\xi_2] = c \int_\xi (fg\xi_1\xi_2), \quad \text{where } \int_\xi F = \text{ coeff. of } \xi_1\xi_2\xi_3\xi_4 \text{ in the expansion of } F,$$

$$[f\xi_i, \ gq_i] = \begin{cases} 0 & \text{if } f, g \in \mathbb{K}, \\ fg & \text{otherwise} \end{cases} \quad \text{for } i = 1, 2.$$

12.1.1   Description of $\mathfrak{kle}(11; \underline{N}|9)_-$ in terms of vector fields

The above was a description easy to understand for humans. To compute with the aid of *SuperLie*, we use the realization of Section 11 with the same weights and the degrees given by (compare with (11.1))

$$\deg u = (2,2,2,1,1), \quad \deg du = (0,0,0,-1,-1), \quad \text{where } u = (u_1, \ldots, u_5).$$

Let us express the basis of $\mathfrak{g}_{-1}$ in terms of the u_i introduced in (10.1):

$$\partial_{x_1} + \cdots \longleftrightarrow \partial_{u_4}, \quad \partial_{x_2} + \cdots \longleftrightarrow \partial_{u_5},$$
$$\partial_{x_3} + \cdots \longleftrightarrow \Pi(du_1 du_4), \quad \ldots, \quad \partial_{x_8} + \cdots \longleftrightarrow \Pi(du_3 du_5),$$
$$\partial_{x_9} \longleftrightarrow u_4 \partial_{u_1}, \quad \ldots, \quad \partial_{x_{14}} \longleftrightarrow u_5 \partial_{u_3},$$
$$\partial_{x_{15}} \longleftrightarrow \Pi(u_4 du_4 du_5), \quad \partial_{x_{16}} \longleftrightarrow \Pi(u_5 du_4 du_5),$$
$$\partial_{x_{17}} \longleftrightarrow \partial_{u_1}, \quad \ldots, \quad \partial_{x_{19}} \longleftrightarrow \partial_{u_3}, \quad \partial_{x_{20}} \longleftrightarrow \Pi(du_4 du_5). \tag{12.1}$$

The Lie superalgebra $\mathfrak{kle}(11; \underline{N}|9)$ is the prolong of the *negative* part. For a basis of the negative part we take the following elements, see (12.2). For their weights we take

$$w(u_i) = w(du_i) = (0, \ldots, 2, \ldots, 0), \quad w(\Pi) = (-1, \ldots, -1).$$

We select the degree of Π so as to ensure the correct degrees of the ∂_{x_i}, see (12.2), where by abuse of notation $\partial_i := \partial_{x_i}$. Looking at the expression of $\partial_{x_{20}}$, see (12.1), we set $\deg(\Pi) = -4$. Likewise, the weights of ∂_{15} and ∂_{16}, see (12.2), are deduced from their expressions in terms of the u_i, see (12.1):

$\mathfrak{g}_i$	the generators
$\mathfrak{g}_{-2}$	$\{-2,0,0,0,0\} \to \partial_{17}, \ \{0,-2,0,0,0\} \to \partial_{18}, \ \{0,0,-2,0,0\} \to \partial_{19},$
	$\{-1,-1,-1,1,1\} \to \partial_{20}$
$\mathfrak{g}_{-1} \simeq$	$\{0,0,0,-2,0\} \to \partial_1 + x_9\partial_{17} + x_{10}\partial_{18} + x_{11}\partial_{19} + x_{15}\partial_{20},$
	$\{0,0,0,0,-2\} \to \partial_2 + x_{12}\partial_{17} + x_{13}\partial_{18} + x_{14}\partial_{19} + x_{16}\partial_{20},$
	$\{1,-1,-1,1,-1\} \to \partial_3 + x_8\partial_{18} + x_6\partial_{19} + x_{12}\partial_{20},$
	$\{1,-1,-1,-1,1\} \to \partial_4 + x_7\partial_{18} + x_5\partial_{19} + x_9\partial_{20},$
	$\{-1,1,-1,1,-1\} \to \partial_5 + x_8\partial_{17} + x_{13}\partial_{20},$
	$\{-1,1,-1,-1,1\} \to \partial_6 + x_7\partial_{17} + x_{10}\partial_{20},$
	$\{-1,-1,1,1,-1\} \to \partial_7 + x_{14}\partial_{20}, \ \{-1,-1,1,-1,1\} \to \partial_8 + x_{11}\partial_{20},$
	$\{-2,0,0,2,0\} \to \partial_9, \ \{0,-2,0,2,0\} \to \partial_{10}, \ \{0,0,-2,2,0\} \to \partial_{11}$
	$\{-2,0,0,0,2\} \to \partial_{12}, \ \{0,-2,0,0,2\} \to \partial_{13}, \ \{0,0,-2,0,2\} \to \partial_{14},$
	$\{-1,-1,-1,3,1\} \to \partial_{15}, \ \{-1,-1,-1,1,3\} \to \partial_{16}$

$$\tag{12.2}$$

No simple partial prolongs. The critical coordinates of the shearing vector for $\mathfrak{kle}_2(20; \widetilde{\underline{N}})$ are those corresponding to the formerly odd indeterminates. Explicitly: noncritical coordinates of the shearing vector correspond to $x_1, x_2, x_{17}, x_{18}, x_{19}$.

13   The Lie superalgebra $\mathfrak{mb}(4|5)$ over $\mathbb{C}$

In this section, we illustrate the algorithm presented in detail in [66], verify and rectify one formula from [17]. This algorithm allows one to describe vectorial Lie superalgebras by means of differential equations. In [64, 65] the algorithm was used to describe the exceptional simple vectorial Lie superalgebras over $\mathbb{C}$.

The Lie superalgebra $\mathfrak{mb}(4|5)$ has three realizations as a transitive and primitive (i.e., not preserving invariant foliations on the space where it is realized by means of vector fields) vectorial Lie superalgebra. Speaking algebraically, the requirement that it should be transitive and primitive vectorial Lie superalgebra is the same as to have a W-filtration, so $\mathfrak{mb}(4|5)$ has three W-filtrations.

Two of these W-filtrations are of depth 2, and one is of depth 3. In each realization this Lie superalgebra is the complete prolong of its negative part, see Section 2.11. In this section we consider the case of depth 3 (the grading K); i.e., we explicitly solve the differential equations singling out our Lie superalgebra. We thus explicitly obtain the expressions for the elements of $\mathfrak{mb}(4|5; K)$.

In this realization, the Lie superalgebra $\mathfrak{g} = \mathfrak{mb}(3|8) = \mathfrak{mb}(4|5; K)$ is the complete prolong of its negative part $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, where

$$\operatorname{sdim} \mathfrak{g}_{-3} = 0|2, \quad \operatorname{sdim} \mathfrak{g}_{-2} = 3|0, \quad \operatorname{sdim} \mathfrak{g}_{-1} = 0|6.$$

We would like to embed $\mathfrak{g}_-$ into the Lie superalgebra

$$\mathfrak{v} := \mathfrak{vect}(3|8) = \mathfrak{der}\mathbb{C}[u_1, u_2, u_3; \eta_1, \eta_2, \eta_3, \zeta_1, \zeta_2, \zeta_3, \chi_1, \chi_2]$$

considered with the grading

$$\deg \eta_i = \deg \xi_i = 1, \quad \deg u_i = 2, \quad \deg \chi_j = 3 \quad \text{for any } i, j.$$

According to the algorithm described in [66], we find in $\mathfrak{v}_-$ two mutually commuting families of elements: X-vectors (the basis of $\mathfrak{g}_-$) and Y-vectors. The table of correspondences, where $i = 1, 2, 3$ and $j = 1, 2$:

k	basis in $\mathfrak{mb}_{-k}$	X	Y
-1	q_1, q_2, q_3	X_{η_i}	Y_{η_i}
	$\xi_2\xi_3, \xi_3\xi_1, \xi_1\xi_2$	X_{ζ_i}	Y_{ζ_i}
-2	ξ_1, ξ_2, ξ_3	X_{u_i}	Y_{u_i}
-3	$1, \hat{1}$	X_{χ_j}	Y_{χ_j}

The nonzero commutation relations for the X-vectors are of the form $((i, j, k) \in A_3)$:

$$[X_{\eta_i}, X_{u_i}] = -X_{\chi_1}, \quad [X_{\zeta_i}, X_{u_i}] = -X_{\chi_2}, \quad [X_{\eta_i}, X_{\zeta_k}] = -X_{u_j}, \quad [X_{\eta_i}, \ X_{\zeta_j}] = X_{u_k}.$$

The nonzero commutation relations for the Y-vectors correspond to the negative of the above structure constants:

$$[Y_{\eta_i}, Y_{u_i}] = Y_{\chi_1}, \quad [Y_{\zeta_i}, Y_{u_i}] = Y_{\chi_2}, \quad [Y_{\eta_i}, Y_{\zeta_k}] = Y_{u_j}, \quad [Y_{\eta_i}, Y_{\zeta_j}] = -Y_{u_k}.$$

Let us represent an arbitrary vector field $D \in \mathfrak{vect}(3|8)$ in the form

$$X = F_1 Y_{\chi_1} + F_2 Y_{\chi_2} + \sum_{1 \le i \le 3} (f_{\zeta_i} Y_{\zeta_i} + f_{\eta_i} Y_{\eta_i} + f_{u_i} Y_{u_i}). \tag{13.1}$$

As it was shown in [66], any $X \in \mathfrak{mb}(3|8)$ is completely determined by a pair of functions F_1, F_2 by means of equations, where $i = 1, 2, 3$ and $(i, j, k) \in A_3$:

$$Y_{\zeta_i}(F_1) = 0, \quad Y_{\eta_i}(F_1) = -(-1)^{p(f_{u_i})} f_{u_i} = Y_{\zeta_i}(F_2), \quad Y_{\eta_i}(F_2) = 0, \tag{13.2}$$

$$Y_{\zeta_i}(f_{u_i}) = Y_{\eta_i}(f_{u_i}) = 0, \tag{13.3}$$

$$Y_{\zeta_i}(f_{u_j}) = (-1)^{p(f_{\eta_k})} f_{\eta_k}, \quad Y_{\eta_i}(f_{u_j}) = -(-1)^{p(f_{\zeta_k})} f_{\zeta_k}, \tag{13.4}$$

$$Y_{\zeta_i}(f_{u_k}) = -(-1)^{p(f_{\eta_j})} f_{\eta_j}, \quad Y_{\eta_i}(f_{u_k}) = (-1)^{p(f_{\zeta_j})} f_{\zeta_j}. \tag{13.5}$$

(Comment: since $(i, j, k) \in A_3$, i.e., is an even permutation, the formulas (13.4) and (13.5) are different; of course one can express the system by one formula inserting the sign of permutation.) Therefore the functions F_1, F_2 must satisfy the following three groups of equations:

$$Y_{\zeta_i}(F_1) = 0, \quad Y_{\eta_i}(F_1) = Y_{\zeta_i}(F_2), \quad Y_{\eta_i}(F_2) = 0 \quad \text{for } i = 1, 2, 3. \tag{13.6}$$

The relations (13.2), (13.4) determine the remaining coordinates while the relations (13.3), (13.5) follow from (13.2), (13.4) and the commutation relations that the Y-vectors obey. Indeed, since $p(f_{u_j}) = p(f_{u_i}) = p(X)$, we have

$$Y_{\zeta_i}(f_{u_j}) = \begin{cases} -(-1)^{p(f_{u_i})} Y_{\zeta_i} Y_{\zeta_i}(F_2) = 0 & \text{for } i = j, \\ -(-1)^{p(f_{u_j})} Y_{\zeta_i} Y_{\zeta_j}(F_2) = (-1)^{p(f_{u_i})} Y_{\zeta_j} Y_{\zeta_i}(F_2) = -Y_{\zeta_j}(f_{u_i}) & \text{for } i \neq j. \end{cases}$$

Besides,

$$f_{\zeta_k} = -(-1)^{p(f_{\zeta_k})} Y_{\eta_i}(f_{u_j}) = -Y_{\eta_i} Y_{\zeta_j}(F_2) = Y_{\zeta_j} Y_{\eta_i}(F_2) + Y_{u_k}(F_2) = Y_{u_k}(F_2).$$

We similarly get the expressions for the remaining coordinates:

$$f_{\eta_k} = Y_{u_k}(F_1).$$

Therefore, an arbitrary element $X \in \mathfrak{mb}(3|8)$ is of the form

$$X = X^F = F_1 Y_{\chi_1} + F_2 Y_{\chi_2} + \sum_{1 \leq i \leq 3} \left(Y_{u_i}(F_2) Y_{\zeta_i} + Y_{u_i}(F_1) Y_{\eta_i} - (-1)^{p(X)} Y_{\zeta_i}(F_2) Y_{u_i} \right), \tag{13.7}$$

where the pair of functions $F = \{F_1, F_2\}$ satisfies the system of equations (13.6).

We select the Y-vectors so that the equations (13.6) the functions F_1, F_2 should satisfy were as simple as possible. For example, take the following Y-vectors, where $i = 1, 2, 3$, $s = 1, 2$, $(i, j, k) \in A_3$:

$$Y_{\eta_i} = \partial_{\eta_i} + \zeta_k \partial_{u_j} - \zeta_j \partial_{u_k} + (\zeta_k \eta_j - \zeta_j \eta_k) \partial_{\chi_1} - \zeta_j \zeta_k \partial_{\chi_2}, \quad Y_{\zeta_i} = \partial_{\zeta_i},$$
$$Y_{u_i} = \partial_{u_i} + \eta_i \partial_{\chi_1} + \zeta_i \partial_{\chi_2}, \qquad\qquad\qquad\qquad\qquad\qquad Y_{\chi_s} = \partial_{\chi_s}.$$

Then, the corresponding X-vectors are of the form

$$X_{\eta_i} = \partial_{\eta_i} + u_i \partial_{\chi_1}, \quad X_{\zeta_i} = \partial_{\zeta_i} - \eta_j \partial_{u_k} + \eta_k \partial_{u_j} - \eta_j \eta_k \partial_{\chi_1} + u_i \partial_{\chi_2},$$
$$X_{u_i} = \partial_{u_i}, \qquad\qquad X_{\chi_s} = \partial_{\chi_s}.$$

The Lie superalgebra $\mathfrak{mb}(3|8)$ consists of the vector fields preserving the distribution determined by the following equations for the vector field D of the form (13.1):

$$f_{u_1} = f_{u_2} = f_{u_3} = F_1 = F_2 = 0. \tag{13.8}$$

Let us express the coordinates f of the field D in the Y-basis in terms of the standard coordinates in the basis of partial derivatives:

$$D = g_{\chi_1} \partial_{\chi_1} + g_{\chi_2} \partial_{\chi_1} + \sum_{1 \leq i \leq 3} (g_{\zeta_i} \partial_{\zeta_i} + g_{\eta_i} \partial_{\eta_i} + g_{u_i} \partial_{u_i}).$$

We get

$$f_{u_i} = g_{u_i} + g_{\eta_j} \zeta_k - g_{\eta_k} \zeta_j \quad \text{for } 1 \leq i \leq 3 \text{ and } (i, j, k) \in A_3,$$

$$F_1 = g_{\chi_1} - \sum g_{u_i}\eta_i, \quad F_2 = g_{\chi_2} - \sum g_{u_i}\zeta_i - \sum_{1 \le i \le 3, \ (i,j,k) \in A_3} g_{\eta_i}\zeta_j\zeta_k.$$

Therefore, in the standard coordinates, the distribution singled out by conditions (13.8) is given by the equations:

$$g_{u_i} + g_{\eta_j}\zeta_k - g_{\eta_k}\zeta_j = 0 \quad \text{for } i = 1, 2, 3,$$
$$g_{\chi_1} - \sum g_{u_i}\eta_i = 0, \quad g_{\chi_2} - \sum g_{u_i}\zeta_i - \sum_{1 \le i \le 3, \ (i,j,k) \in A_3} g_{\eta_i}\zeta_j\zeta_k = 0. \tag{13.9}$$

The three equations determined by the first line of (13.9) allow one to express g_{u_i} and substitute into the third line to get

$$g_{\chi_2} + \sum_{1 \le i \le 3, \ (i,j,k) \in A_3} g_{\eta_i}\zeta_j\zeta_k = 0.$$

Assuming that the pairing of the space of vector fields with that of 1-forms is given by the formula

$$\langle f\partial_\xi, \ gd\xi \rangle = (-1)^{p(g)} fg \quad \text{for any } f, g \in \mathcal{F},$$

we see that the distribution is singled out by Pfaff equations given by the following 1-forms[8]:

$$du_i + \zeta_j d\eta_k - \zeta_k d\eta_j, \quad \text{where } (i, j, k) \in A_3,$$
$$d\chi_1 - \sum \eta_i du_i, \quad d\chi_2 + \sum_{i \text{ such that } (i,j,k) \in A_3} \zeta_j\zeta_k d\eta_i.$$

Let us now solve the system (13.6).

Since $Y_{\zeta_i} = \partial_{\zeta_i}$, the condition $Y_{\zeta_i}(F_1) = 0$ implies that $F_1 = F_1(u, \eta, \chi)$. The condition $Y_{\zeta_i}(F_2) = Y_{\eta_i}(F_1)$ takes the form:

$$\frac{\partial F_2}{\partial \zeta_i} = \frac{\partial F_1}{\partial \eta_i} + \left(\zeta_k \frac{\partial F_1}{\partial u_j} - \zeta_j \frac{\partial F_1}{\partial u_k} \right) + (\zeta_k \eta_j - \zeta_j \eta_k) \frac{\partial F_1}{\partial \chi_1} - \zeta_j \zeta_k \frac{\partial F_1}{\partial \chi_2},$$

wherefrom (since F_1 does not depend on ξ) we see that

$$F_2 = \sum_{1 \le i \le 3} \zeta_i \frac{\partial F_1}{\partial \eta_i} - \sum_{i=1,2,3, \ (i,j,k) \in A_3} \zeta_i \zeta_j \left(\frac{\partial F_1}{\partial u_k} + \eta_k \frac{\partial F_1}{\partial \chi_1} \right) - \zeta_1 \zeta_2 \zeta_3 \frac{\partial F_1}{\partial \chi_2} + \alpha_2, \tag{13.10}$$

where $\alpha_2 = \alpha_2(u, \eta, \chi)$, i.e., does not depend on ζ.

Let us consider the last group of equations (13.6):

$$Y_{\eta_i}(F_2) = 0 \quad \text{for } i = 1, 2, 3. \tag{13.11}$$

To solve this system, take the expression (13.10) for F_2 and apply the operator Y_{η_i}. As a result, we get a function depending on various indeterminates, in particular, on ζ_j. By virtue of (13.11), the coefficients of all monomials in ζ should vanish. Observe that the coefficient of $\zeta_1\zeta_2\zeta_3$ vanishes automatically. The terms of degree 0 in ζ are of the form:

$$\frac{\partial \alpha_2}{\partial \eta_i} = 0 \implies \alpha_2 = \alpha_2(u, \chi).$$

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[8]We do not use the formulas thus obtained in THIS text. However, they describe the algebra in meaningful terms, "as preserving a distribution" and explicitly define this distribution. So we provide these formulas, and keep them for future use.

Now, let us look at the degree 1 terms in ζ. To get them we should either take the term independent of ζ in expression (13.10) for F_2 (and this is α_2), and apply to it the degree 1 terms in ζ of Y_{η_i}, i.e.,

$$\zeta_k \partial_{u_j} - \zeta_j \partial_{u_k} + (\zeta_k \eta_j - \zeta_j \eta_k) \partial_{\chi_1},$$

or, the other way round, take the degree 1 terms in ζ in (13.10), i.e., $\sum_s \zeta_s \frac{\partial F_1}{\partial \eta_s}$, and apply to it the degree 0 in ζ term of the operator Y_{η_i}, i.e., ∂_{η_i}.

Therefore, the terms of degree 1 in ζ are of the form:

$$\zeta_k \frac{\partial \alpha_2}{\partial u_j} - \zeta_j \frac{\partial \alpha_2}{\partial u_k} + (\zeta_k \eta_j - \zeta_j \eta_k) \frac{\partial \alpha_2}{\partial \chi_1} = \zeta_j \frac{\partial^2 F_1}{\partial \eta_i \partial \eta_j} + \zeta_k \frac{\partial^2 F_1}{\partial \eta_i \partial \eta_k},$$

implying that

$$F_1 = \sum_{i=1,2,3,(i,j,k)\in A_3} \eta_i \eta_j \frac{\partial \alpha_2}{\partial u_k} + \eta_1 \eta_2 \eta_3 \frac{\partial \alpha_2}{\partial \chi_1} + \alpha_1(u,\chi) + \sum_{1\leq i\leq 3} f_i(u,\chi) \eta_i.$$

So, the functions F_1, F_2 are completely determined by the 5 functions α_1, α_2, f_1, f_2, f_3 that depend only on u and χ.

The terms of degree 2 in ζ follow from the same expression (13.10) and the same explanation as in the above paragraph leads to the equation (the coefficient of $\zeta_j \zeta_k$):

$$\sum_{1\leq s\leq 3} \left(\frac{\partial^2 F_1}{\partial u_s \partial \eta_s} - \eta_s \frac{\partial}{\partial \eta_s} \frac{\partial F_1}{\partial \chi_1} \right) + \frac{\partial F_1}{\partial \chi_1} + \frac{\partial \alpha_2}{\partial \chi_2} = 0. \tag{13.12}$$

Let us expand this equation in parts corresponding to degrees in η. In degree 0 we have:

$$\sum_{1\leq i\leq 3} (-1)^{p(f_i)} \frac{\partial f_i}{\partial u_i} + \frac{\partial \alpha_1}{\partial \chi_1} + \frac{\partial \alpha_2}{\partial \chi_2} = 0. \tag{13.13}$$

In degrees 1, 2, 3 in η the equation (13.12) is automatically satisfied.

Let us express equation (13.13) in the following more lucid way. We designate

$$f_i := f_i^0 + f_i^1 \chi_1 + f_i^2 \chi_2 + f_i^{12} \chi_1 \chi_2, \quad \alpha_s := \alpha_s^0 + \alpha_s^1 \chi_1 + \alpha_s^2 \chi_2 + \alpha_s^{12} \chi_1 \chi_2.$$

The equation (13.13) is equivalent to the following system of four equations:

$$\sum_{1\leq i\leq 3} \frac{\partial f_i^{12}}{\partial u_i} = 0, \quad \alpha_1^{12} - \sum_{1\leq i\leq 3} \frac{\partial f_i^2}{\partial u_i} = 0, \quad \alpha_2^{12} + \sum_{1\leq i\leq 3} \frac{\partial f_i^1}{\partial u_i} = 0, \quad \alpha_1^1 + \alpha_2^2 + \sum_{1\leq i\leq 3} \frac{\partial f_i^0}{\partial u_i} = 0.$$

Let us describe the commutation relations in $\mathfrak{mb}(3|8)$ more explicitly. Let us represent the vector field (13.7) as

$$X^F = x^F + \sum_{1\leq i\leq 3} \left(f_{\zeta_i} Y_{\zeta_i} + f_{\eta_i} Y_{\eta_i} - (-1)^{p(X)} f_{u_i} Y_{u_i} \right), \quad \text{where } x^F = F_1 \partial_{\chi_1} + F_2 \partial_{\chi_2}, \tag{13.14}$$

and observe that, taking relation (13.2) and (13.4) into account, we have

$$\left[X^F, X^G \right] = X^H, \quad \text{where}$$
$$H_1 = \left[x^F, x^G \right]_1 + \sum_{1\leq i\leq 3} \left(f_{u_i} g_{\eta_i} - (-1)^{p(X^G)} f_{\eta_i} g_{u_i} \right),$$

$$H_2 = \left[x^F, x^G\right]_2 + \sum_{1 \leq i \leq 3} \left(f_{u_i} g_{\zeta_i} - (-1)^{p(X^G)} f_{\zeta_i} g_{u_i}\right).$$

Observe that it suffices to compute only the *defining* components of F, G, and H:

the pair	determined by the set
F	$\{\alpha_s, f_i \mid s = 1, 2, \ i = 1, 2, 3\}$
G	$\{\beta_s, g_i \mid s = 1, 2, \ i = 1, 2, 3\}$
H	$\{\gamma_s, h_i \mid s = 1, 2, \ i = 1, 2, 3\}$

Then, we get

$$\gamma_1 = \sum_{1 \leq i \leq 3} \left(-f_i \frac{\partial \beta_1}{\partial u_i} + (-1)^{p(X^G)} \frac{\partial \alpha_1}{\partial u_i} g_i\right)$$
$$+ \left(\alpha_1 \frac{\partial \beta_1}{\partial \chi_1} + \alpha_2 \frac{\partial \beta_1}{\partial \chi_2}\right) - (-1)^{p(X^F)p(X^G)} \left(\beta_1 \frac{\partial \alpha_1}{\partial \chi_1} + \beta_2 \frac{\partial \alpha_1}{\partial \chi_2}\right),$$

$$\gamma_2 = \sum_{1 \leq i \leq 3} \left(-f_i \frac{\partial \beta_2}{\partial u_i} + (-1)^{p(X^G)} \frac{\partial \alpha_2}{\partial u_i} g_i\right)$$
$$+ \left(\alpha_1 \frac{\partial \beta_2}{\partial \chi_1} + \alpha_2 \frac{\partial \beta_2}{\partial \chi_2}\right) - (-1)^{p(X^F)p(X^G)} \left(\beta_1 \frac{\partial \alpha_2}{\partial \chi_1} + \beta_2 \frac{\partial \alpha_2}{\partial \chi_2}\right),$$

$$h_i = - \sum_{1 \leq r \leq 3} f_r \frac{\partial g_i}{\partial u_r} + \sum_{1 \leq r \leq 3} \frac{\partial f_i}{\partial u_r} g_r$$
$$- (-1)^{p(X^G)} \left(\frac{\partial \alpha_2}{\partial u_j} \frac{\partial \beta_1}{\partial u_k} - \frac{\partial \alpha_2}{\partial u_k} \frac{\partial \beta_1}{\partial u_j} - \frac{\partial \alpha_1}{\partial u_j} \frac{\partial \beta_2}{\partial u_k} + \frac{\partial \alpha_1}{\partial u_k} \frac{\partial \beta_2}{\partial u_j}\right)$$
$$+ \sum_{s=1,2} \alpha_s \frac{\partial g_i}{\partial \chi_s} - (-1)^{p(X^F)p(X^G)} \sum_{s=1,2} \beta_s \frac{\partial f_i}{\partial \chi_s} \quad \text{for } i = 1, 2, 3, \ (i, j, k) \in A_3. \tag{13.15}$$

In what follows we identify the vector field X^F with the collection

$$\{\alpha_s, \ f_i \mid s = 1, 2, \ i = 1, 2, 3\}. \tag{13.16}$$

The bracket of vector fields corresponds to the bracket of such collections given by equations (13.15).

Consider now the even part $\mathfrak{mb}(3|8)_{\bar{0}}$ of our algebra. Since $p(F_1) = p(F_2) = \bar{1}$, it follows that $p(\alpha_s) = \bar{1}$ and $p(f_i) = \bar{0}$ for all s and i. The component $\mathfrak{mb}(3|8)_{\bar{0}}$ has the three subspaces:

$$\mathfrak{mb}(3|8)_{\bar{0}} = V_1 \oplus V_2 \oplus V_3.$$

The subspace V_1 is determined by the collection (13.16) such that

$$\{\alpha_1 = \alpha_2 = 0, \ f_i = f_i(u)\chi_1\chi_2 \mid \sum_{i=1,2,3} \frac{\partial f_i}{\partial u_i} = 0\}.$$

Equations (13.15) imply that the vector fields generated by such functions form a commutative ideal in $\mathfrak{mb}(3|8)_{\bar{0}}$; we will identify this ideal with $d\Omega^1(3)$:

$$\{0, 0, f_i \mid i = 1, 2, 3\} \longmapsto - \sum_{i \text{ such that } (ijk) \in A_3} f_i du_j \wedge du_k.$$

The subspace V_2 is determined by the collection (13.16) such that $f_i = 0$ for $i = 1, 2, 3$. We will identify this space with $\Omega^0(3) \otimes \mathfrak{sl}(2)$, by setting

$$\{\alpha(u)(a\chi_1 + b\chi_2), \ \alpha(u)(c\chi_1 - a\chi_2), \ 0, 0, 0\} \longmapsto \alpha(u) \otimes \begin{pmatrix} a & c \\ b & -a \end{pmatrix},$$

where $\alpha \in \Omega^0(3)$, $a, b, c \in \mathbb{C}$. Equations (13.15) imply that the subspaces V_1 and V_2 commute with each other whereas the brackets of two collections from V_2 is in our notation of the form

$$[f \otimes A, g \otimes B] = fg \otimes [A, B] + df \wedge dg \cdot \operatorname{tr} AB.$$

Concerning V_3, we have the following three natural ways to describe it: in all three cases we take $f_i = f_i(u)$ for all i, whereas for the α_s, we select one of the following:

(a) $\alpha_1 = -\sum \dfrac{\partial f_i}{\partial u_i}\chi_1,$ $\quad \alpha_2 = 0,$

(b) $\alpha_1 = 0,$ $\qquad\qquad \alpha_2 = -\sum \dfrac{\partial f_i}{\partial u_i}\chi_2,$

(c) $\alpha_1 = -\dfrac{1}{2}\sum \dfrac{\partial f_i}{\partial u_i}\chi_1,$ $\quad \alpha_2 = -\dfrac{1}{2}\sum \dfrac{\partial f_i}{\partial u_i}\chi_2.$ $\qquad\qquad$ (13.17)

For $p \neq 2$, the case (c) is more convenient to simplify the brackets. Thus, we identify V_3 with $\mathfrak{vect}(3)$, by means of the mapping

$$\left\{ -\frac{1}{2}\sum \frac{\partial f_i}{\partial u_i}\chi_1, \ -\frac{1}{2}\sum \frac{\partial f_i}{\partial u_i}\chi_2, \ f_1(u), \ f_2(u), \ f_3(u) \right\} \longmapsto D_f = -\sum f_i(u)\partial_{u_i}.$$

The actions of D_f on the subspace V_1 (as on the space of 2-forms) and V_2 (as on the space $\mathcal{F} \otimes \mathfrak{sl}(2)$ of $\mathfrak{sl}(2)$-valued functions) are natural. The bracket of two elements if the form D_f is, however, quite different from the usual bracket thanks to an extra term:

$$[D_f, D_g] = D_f D_g - D_g D_f - \frac{1}{2} d(\operatorname{div} D_f) \wedge d(\operatorname{div} D_g).$$

Consider now the odd part: $\mathfrak{mb}(3|8)_{\bar{1}}$. We have $p(F_1) = p(F_2) = \bar{0}$, and hence

$$p(\alpha_s) = \bar{0}, \quad p(f_i) = \bar{1}.$$

Let V_4 consist of collections (13.16) with $f_i = 0$. We identify V_4 with $\Omega^0(3)\operatorname{vol}^{-1/2}\otimes\mathbb{C}^2$, by setting

$$\{(\alpha(u)w_1, \ \alpha(u)w_2, \ 0, 0, 0)\} \longmapsto \alpha(u)\operatorname{vol}^{-1/2}\otimes \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix}.$$

Let V_5 consist of the collections (13.16), where

$$f_i = f_i(u)(v_1\chi_1 + v_2\chi_2), \quad \alpha_1 = v_2\sum \frac{\partial f_i}{\partial u_i}\chi_1\chi_2, \quad \alpha_2 = -v_1\sum \frac{\partial f_i}{\partial u_i}\chi_1\chi_2. \qquad (13.18)$$

We identify V_5 with $\Omega^2(3)\operatorname{vol}^{-1/2}\otimes\mathbb{C}^2$, by assigning to the collection (13.18) the element

$$\omega\operatorname{vol}^{-1/2}\otimes\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{where } \omega = -\!\!\!\sum_{i \text{ such that } (ijk)\in A_3}\!\!\! f_i du_j \wedge du_k.$$

Let us sum up a description of the spaces V_i and their elements, see Table (13.19).

Having explicitly computed the brackets using expressions (13.15) and presenting the result by means of correspondences (13.19), we obtain the formulas almost identical to those offered in [17]. The difference, however, is vital: the Jacobi identity either holds or not.

We have already given the brackets of the even elements. The brackets of elements of $\mathfrak{mb}_{\bar{0}}$ and $\mathfrak{mb}_{\bar{1}}$ are of the form:

$$[V_1, V_4]\colon \ \left[\omega, \alpha\operatorname{vol}^{-1/2}\otimes v\right] = \alpha \cdot \omega\operatorname{vol}^{-1/2}\otimes v \in V_5,$$

The space	α_1	α_2	f_i	the element of V_i
$V_1 \cong d\Omega^1(3)$	0	0	$f_i(u)\chi_1\chi_2,$ $\sum \frac{\partial f_i}{\partial u_i} = 0$	$\omega = \sum f_i du_j \wedge du_k,$ $d\omega = 0$
$V_2 \cong$ $\Omega^0(3) \otimes \mathfrak{sl}(2)$	$\alpha(u)(a\chi_1 + b\chi_2)$	$\alpha(u)(c\chi_1 - a\chi_2)$	0	$\alpha(u) \otimes \begin{pmatrix} a & c \\ b & -a \end{pmatrix}$
$V_3 \cong$ $\mathfrak{vect}(3)$	$-\frac{1}{2}f(u)\chi_1$	$-\frac{1}{2}f(u)\chi_2$	$f_i(u)$ $f(u) = \sum \frac{\partial f_i}{\partial u_i}$	$D = -\sum f_i(u)\partial_{u_i}$ $\operatorname{div} D = -f(u)$
$V_4 \cong$ $\Omega^0 \operatorname{vol}^{-1/2} \otimes \mathbb{C}^2$	$\alpha(u)w_1$	$\alpha(u)w_2$	0	$\frac{\alpha(u)}{\operatorname{vol}^{1/2}} \otimes \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix}$
$V_5 \cong$ $\Omega^2 \operatorname{vol}^{-1/2} \otimes \mathbb{C}^2$	$v_2 f(u)\chi_1\chi_2$	$-v_1 f(u)\chi_1\chi_2$	$f_i(u)(v_1\chi_1 + v_2\chi_2)$ $f(u) = \sum \frac{\partial f_i}{\partial u_i}$	$\frac{\omega}{\operatorname{vol}^{1/2}} \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\omega = \sum f_i du_j \wedge du_k$

$$(13.19)$$

$[V_2, V_4]: \quad \left[f \otimes A, \alpha \operatorname{vol}^{-1/2} \otimes v \right] = f\alpha \operatorname{vol}^{-1/2} \otimes Av - df \wedge d\alpha \operatorname{vol}^{-1/2} \otimes Av \in V_4 \oplus V_5,$

$[V_3, V_4]: \quad \left[D, \alpha \operatorname{vol}^{-1/2} \otimes v \right] = (D(\alpha) - \frac{1}{2} \operatorname{div} D \cdot \alpha) \operatorname{vol}^{-1/2} \otimes v$

$\qquad\qquad + \frac{1}{2} d(\operatorname{div} D) \wedge d\alpha \cdot \operatorname{vol}^{-1/2} \otimes v \in V_4 \oplus V_5,$

$[V_1, V_5] = 0,$

$[V_2, V_5]: \quad \left[f \otimes A, \omega \operatorname{vol}^{-1/2} \otimes v \right] = f\omega \operatorname{vol}^{-1/2} \otimes Av \in V_5,$

$[V_3, V_5]: \quad \left[D, \omega \operatorname{vol}^{-1/2} \otimes v \right] = (L_D\omega - \frac{1}{2} \operatorname{div} D \cdot \omega) \operatorname{vol}^{-1/2} \otimes v \in V_5.$

To describe in these terms the bracket of two odd elements, perform the following natural identifications:

$$\frac{\Omega^2(3)}{\operatorname{vol}} \cong \mathfrak{vect}(3): \quad \frac{\omega}{\operatorname{vol}} \longleftrightarrow D_\omega,$$

$$i_{D_\omega}(\operatorname{vol}) = \omega, \text{ i.e.,} \quad \sum_{\{i,j,k\}=\{1,2,3\} \text{ such that } (ijk) \in A_3} f_i dx_j \wedge dx_k \longleftrightarrow \sum f_i \partial_i,$$

$$\Lambda^2 \mathbb{C}^2 \cong \mathbb{C}: \quad v \wedge w \longleftrightarrow \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix},$$

$$S^2(\mathbb{C}^2) \cong \mathfrak{sl}(2): \quad v \cdot w \longleftrightarrow \begin{pmatrix} -v_1 w_2 - v_2 w_1 & 2v_1 w_1 \\ -2v_2 w_2 & v_1 w_2 + v_2 w_1 \end{pmatrix}.$$

The bracket of two odd elements is of the form:

$[V_4, V_4]: \quad \left[\frac{f}{\operatorname{vol}^{1/2}} \otimes v, \frac{g}{\operatorname{vol}^{1/2}} \otimes w \right] = \frac{df \wedge dg \otimes v \wedge w}{\operatorname{vol}} \in \mathfrak{vect}(3) = V_3,$

$[V_5, V_5]: \quad \left[\frac{\omega_1}{\operatorname{vol}^{1/2}} \otimes v, \frac{\omega_2}{\operatorname{vol}^{1/2}} \otimes w \right] = (D_{\omega_1}(\omega_2) - (\operatorname{div} D_{\omega_2}) \cdot \omega_1) v \wedge w \in V_1,$

$[V_4, V_5]: \quad \left[\frac{f}{\operatorname{vol}^{1/2}} \otimes v, \frac{\omega}{\operatorname{vol}^{1/2}} \otimes w \right]$

$\qquad = \frac{f\omega}{\operatorname{vol}} \otimes v \wedge w - \frac{1}{2}(fd\omega - \omega \wedge df) \otimes v \cdot w + df \wedge d(\operatorname{div} D_\omega) \otimes v \wedge w.$ $\qquad(13.20)$

In the last line above, the first summand lies in V_3, the second one in V_2, and the third one in V_1. *The difference as compared with* [17]: the coefficient of the third summand in the last line on (13.20) should be 1 whereas in [17] it is equal to $\frac{1}{2}$.

To verify, compute the Jacobi identity (it holds for 1 and not for $\frac{1}{2}$) for the triple

$$u_3 du_2 \wedge du_3 \in V_1, \quad \frac{u_1}{\mathrm{vol}^{1/2}} \otimes e_1, \quad \text{and} \quad \frac{u_2}{\mathrm{vol}^{1/2}} \otimes e_2 \in V_4, \quad \text{where } e_1, e_2 \text{ span } \mathbb{C}^2.$$

For $p = 2$, when case (c) in (13.17) is not defined, we can select any one of the cases (a) or (b), we take case (a) for definiteness. In these cases (a) and (b), we get two embeddings $\mathfrak{vect}(3) \subset \mathfrak{mb}(3|8)_{\bar{0}}$.

14 The Lie algebra $\mathbf{F}(\mathfrak{mb}(3; \underline{N}|8))$ is a true deform of $\mathfrak{svect}(5; \underline{\widetilde{N}})$

In this section, we describe the analog of the complex Lie superalgebra $\mathfrak{mb}(3|8)$ for $p = 2$ and consider its desuperization. For brevity, whenever possible we do not indicate the shearing vectors.

In Section 13, we showed that an arbitrary vector field $X^F \in \mathfrak{g}$, where $\mathfrak{g} = \mathfrak{mb}(3|8)$, is of the form (13.14) and is determined by 5 functions $(\alpha_1, \alpha_2, f_1, f_2, f_3)$ in indeterminates χ_1, χ_2, u_1, u_2, u_3. Now, in discussing $\mathbf{F}(\mathfrak{mb}(3|8))$, we assume that all these indeterminates are even.

For consistency we replace χ_i with u_{3+i}, and α_i with f_{3+i}. Accordingly we denote X^F by X^f, where $f = (f_1, f_2, f_3, f_4, f_5)$. The equation (13.13) takes the form

$$\sum_{1 \le i \le 5} \frac{\partial f_i}{\partial u_i} = 0. \tag{14.1}$$

The equation (14.1) is the only condition imposed on the functions f_i, and hence there are no restrictions on the values of coordinates of the shearing vector corresponding to the indeterminates u_i, including u_4 and u_5.

Consider the mapping

$$\varphi \colon \mathfrak{g} \longrightarrow \mathfrak{vect}(5), \quad X^f \longmapsto D^f := \sum f_i \partial_{u_i}. \tag{14.2}$$

Clearly, this is a linear injective mapping. Formula (14.1) implies that $\varphi(\mathfrak{g}) = \mathfrak{svect}(5)$. The mapping φ is not, however, an isomorphism of Lie algebras $\mathfrak{g}$ and $\mathfrak{svect}(5)$. Indeed, equations (13.15) rewritten in new notation imply the following equality (since $p = 2$, we skip the signs):

$$\varphi([X^f, X^g]) = [D^f, D^g] + \sum_{(i,j,k) \in S_3} \left(\frac{\partial f_4}{\partial u_i} \frac{\partial g_5}{\partial u_j} + \frac{\partial f_5}{\partial u_i} \frac{\partial g_4}{\partial u_j} \right) \frac{\partial}{\partial u_k}. \tag{14.3}$$

Realization of $\mathfrak{kle}$ convenient in what follows: for $\mathfrak{g} = \mathfrak{kle}(5|10)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{svect}(5|0) \simeq d\Omega^3$ and $\mathfrak{g}_{\bar{1}} = \Pi(d\Omega^1)$ with the natural $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{g}_{\bar{1}}$, while the bracket of any two odd elements is their product naturally identified with a divergence-free vector field.

For any $D = \sum_{1 \le i \le 5} f_i \partial_{u_i} \in \mathfrak{svect}(5)$, we define

$$Z_i(D) := du_i \wedge df_i \in d\Omega^1(5)$$

and construct the embedding (as a vector space)

$$\psi \colon \mathfrak{svect}(5|0) \longrightarrow \mathrm{F}(\mathfrak{kle}), \quad D \longmapsto D + Z_4(D) + Z_5(D). \tag{14.4}$$

Let us compute the bracket of two fields of the form (14.4):

$$\left[D^f + Z_4(D^f) + Z_5(D^f), D^g + Z_4(D^g) + Z_5(D^g) \right].$$

In order not to write too lengthy expressions, let us compute, separately, the brackets of individual summands. First, let $i = 1, 2, 3$, and $k = 4, 5$:

$$
\begin{aligned}
[f_i \partial_{u_i}, g_k \partial_{u_k} + Z_k(g_k \partial_{u_k})] &= [f_i \partial_{u_i}, g_k \partial_{u_k} + du_k \wedge dg_k] \\
&= \left(f_i \frac{\partial g_k}{\partial u_i} \right) \frac{\partial}{\partial u_k} + du_k \wedge d \left(f_i \frac{\partial g_k}{\partial u_i} \right) + g_k \frac{\partial f_i}{\partial u_k} \frac{\partial}{\partial u_i} \\
&= [f_i \partial_i, g_k \partial_k] + Z_k([f_i \partial_i, g_k \partial_k]).
\end{aligned}
\tag{14.5}
$$

Here we applied the Leibniz formula for the action of a vector field on a 2-form, and the expressions for the Lie derivative along the vector field X:

$$L_{f_i \partial_i}(du_k) = 0 \quad \text{and} \quad L_X \circ d = d \circ L_X.$$

Now, let $k = 4$ or 5:

$$
\begin{aligned}
[f_k \partial_{u_k} + Z_k(f_k \partial_{u_k}), g_k \partial_{u_k} + Z_k(g_k \partial_{u_k})] &= [f_k \partial_{u_k} + du_k \wedge df_k, g_k \partial_{u_k} + du_k \wedge dg_k] \\
&= [f_k \partial_{u_k}, g_k \partial_{u_k}] + L_{f_k \partial_{u_k}}(du_k \wedge dg_k) + L_{g_k \partial_{u_k}}(du_k \wedge df_k) + [du_k \wedge df_k, du_k \wedge dg_k] \\
&= \left(f_k \frac{\partial g_k}{\partial u_k} + g_k \frac{\partial f_k}{\partial u_k} \right) \partial_{u_k} + df_k \wedge dg_k + du_k \wedge d \left(f_k \frac{\partial g_k}{\partial u_k} \right) + dg_k \wedge df_k + du_k \wedge d \left(g_k \frac{\partial f_k}{\partial u_k} \right) \\
&= [f_k \partial_{u_k}, g_k \partial_{u_k}] + Z_k([f_k \partial_{u_k}, g_k \partial_{u_k}]).
\end{aligned}
\tag{14.6}
$$

Finally, let $i = 4$ and $k = 5$:

$$
\begin{aligned}
[f_4 \partial_{u_4} + Z_4(f_4 \partial_{u_4}), g_5 \partial_{u_5} + Z_5(g_5 \partial_{u_5})] &= [f_4 \partial_{u_4} + du_4 \wedge df_4, g_5 \partial_{u_5} + du_5 \wedge dg_5] \\
&= \left(f_4 \frac{\partial g_5}{\partial u_4} \right) \frac{\partial}{\partial u_5} + du_5 \wedge d \left(f_4 \frac{\partial g_5}{\partial u_4} \right) + g_5 \frac{\partial f_4}{\partial u_5} \frac{\partial}{\partial u_4} + du_4 \wedge d \left(g_5 \frac{\partial f_4}{\partial u_5} \right) \\
&\quad + \frac{du_4 \wedge df_4 \wedge du_5 \wedge dg_5}{\text{vol}} \\
&= [f_4 \partial_4, g_5 \partial_5] + Z_4([f_4 \partial_4, g_5 \partial_5]) + Z_5([f_4 \partial_4, g_5 \partial_5]) + \sum_{(i,j,k) \in S_3} \frac{\partial f_4}{\partial u_i} \frac{\partial g_5}{\partial u_j} \frac{\partial}{\partial u_k}.
\end{aligned}
\tag{14.7}
$$

The expressions (14.5), (14.6), and (14.7) show that the through mapping $\psi \circ \varphi$ determines an embedding $\mathfrak{g} \longrightarrow \mathfrak{klc}$, and hence the Lie algebra $\mathfrak{g}$ is isomorphic to the thus-constructed Lie subalgebra of $\mathfrak{klc}$.

Remark 14.1. Note that, thanks to formulas (14.5) and (14.6), the image of Lie algebra $\mathfrak{svect}(5)$ under the embedding

$$\mathfrak{svect}(5) \longrightarrow \mathfrak{klc}, \quad D \longmapsto D + Z_k(D) \quad \text{for any } k$$

is isomorphic to $\mathfrak{svect}(5)$. The image of the embedding with three additional terms

$$\mathfrak{svect}(5) \longrightarrow \mathfrak{klc}, \quad D \longmapsto D + Z_3(D) + Z_4(D) + Z_5(D) \tag{14.8}$$

is not a proper subalgebra of $\mathfrak{klc}$: it generates the whole $\mathfrak{klc}$. Indeed: take the bracket of the images of two fields of the form $f \partial_4, g \partial_5 \in \mathfrak{svect}(5)$; we see, thanks to equation (14.7), that the image of $\mathfrak{svect}(5)$ under the mapping (14.8) must contain 2-forms such as $du_3 \wedge dh$ for certain h, and hence this image is not a subalgebra. Since the $\mathfrak{svect}(5)$-module $d\Omega^1(5)$ is irreducible, the image of (14.8) generates the whole $\mathfrak{klc}$.

14.1   The Lie algebra $\mathbf{F}(\mathfrak{mb}(3; \underline{N}|8))$ is a true deform of $\mathfrak{svect}(5; \widetilde{\underline{N}})$

Indeed, for the shearing vectors of the form $\underline{N}_\infty$, all W-gradings of $\mathfrak{mb}$ are the same as over $\mathbb{C}$. None of them has a maximal subalgebra of codimension 5, whereas $\mathfrak{svect}(5)$ has such a subalgebra; cf. deforms described in [74, 76] as well.

We consider $\mathfrak{g}$ as a deform of $\mathfrak{svect}(5)$ with the grading

$$\deg u_a = 2, \quad \deg u_i = 3, \quad \text{where} \quad a = 1, 2, 3, \ i = 4, 5$$

and the new bracket (14.3) designated $[[-, -]]$:

$$[[D^f, D^g]] = [D^f, D^g] + c(D^f, D^g), \tag{14.9}$$

where $D^F = \sum f_i \partial_i \in \mathfrak{svect}(5)$, $[-, -]$ is the usual bracket of vector fields, and the cocycle that determines the deform is

$$c(D^f, D^g) = \sum_{(i,j,k) \in S_3} \left(\frac{\partial f_4}{\partial u_i} \frac{\partial g_5}{\partial u_j} + \frac{\partial f_5}{\partial u_i} \frac{\partial g_4}{\partial u_j} \right) \frac{\partial}{\partial u_k}.$$

All calculations in this realization are rather simple. We have (observe that thanks to formulas (14.2) and (14.9) brackets between the elements of $\mathfrak{g}_{-1}$ are nontrivial, and $\mathfrak{g}_{-1}$ generates the negative part)

$\mathfrak{g}_i$	its basis
$\mathfrak{g}_{-3}$	$\partial_4, \ \partial_5$
$\mathfrak{g}_{-2}$	$\partial_1, \ \partial_2, \ \partial_3$
$\mathfrak{g}_{-1}$	$u_a \partial_i$, where $a = 1, 2, 3, \ i = 4, 5$

We also have

$$\mathfrak{g}_0 = \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathbb{K}(u_1 \partial_1 + u_5 \partial_5), \quad \text{where}$$

$$\mathfrak{sl}(3) = \left\{ \sum_{a,b=1,2,3} \alpha_{ab} u_a \partial_b \ \Big| \ \sum_{1 \le a \le 3} \alpha_{aa} = 0 \right\}, \quad \mathfrak{sl}(2) = \mathrm{Span}(u_4 \partial_5, u_5 \partial_4, u_4 \partial_4 + u_5 \partial_5).$$

14.1.1   The deforms of $\mathfrak{svect}(n; \underline{N})$ for $p > 3$

These deforms are described in [76].

14.2   Partial prolongs

The Lie algebra $\mathfrak{g} = \mathbf{F}(\mathfrak{mb}(3|8))$ constructed above is the *complete* prolong of its negative part, see Section 2.11; let us investigate if there is a *partial* prolong inside $\mathfrak{g}$. The component $\mathfrak{g}_1 = V_1 \oplus V_2$ is the direct sum of the following $\mathfrak{g}_0$-invariant subspaces:

$$V_1 = \mathrm{Span}(u_i \partial_a \,|\, i = 4, 5, \ a = 1, 2, 3), \quad V_2 = \mathrm{Span}(u_a u_b \partial_i \,|\, a, b = 1, 2, 3, \ i = 4, 5).$$

The $\mathfrak{g}_0$-module V_1 is irreducible.

The $\mathfrak{g}_0$-module V_2 contains an irreducible $\mathfrak{g}_0$-submodule $V_2^0 = \mathrm{Span}(x_a x_b \partial_i \,|\, a \neq b) \subset V_2$ and $\mathfrak{g}_0$ acts in the quotient space as follows: $\mathfrak{sl}(3)$ acts in V_2/V_2^0 by zero and $\mathfrak{sl}(2)$ acts as $\mathrm{id}_{\mathfrak{sl}(2)}$ with multiplicity 3, so $\dim V_2/V_2^0 = 8$.

Using (14.9) it is easy to see that

$$[[V_1, \mathfrak{g}_{-1}]] = \mathfrak{g}_0, \quad [[V_2, \mathfrak{g}_{-1}]] \subset \mathfrak{sl}(3).$$

This means that only partial prolongs with $\widetilde{\mathfrak{g}}_1 \subset \mathfrak{g}_1$ containing V_1 can be simple.

For $\widetilde{\mathfrak{g}}_1 = V_1 \oplus V_2^0$, the partial prolong with the unconstrained shearing vector which is of the form $\underline{N}^u = (1, 1, 1, \infty, \infty)$ is a deform of $\mathfrak{svect}(5; \underline{N}^u)$.

For $\widetilde{\mathfrak{g}}_1 = V_1 \oplus V_2^0 \oplus \mathrm{Span}\left(u_1^{(2)}\partial_i \,|\, i = 4, 5\right)$, the partial prolong is a deform of $\mathfrak{svect}(5; \underline{N}^u)$ with $\underline{N}^u = (\infty, 1, 1, \infty, \infty)$.

For $\widetilde{\mathfrak{g}}_1 = V_1 \oplus V_2^0 \oplus \mathrm{Span}\left(u_a^{(2)}\partial_i \,|\, a = 1, 2, \ i = 4, 5\right)$, the partial prolong is a deform of $\mathfrak{svect}(5; \underline{N}^u)$ with $\underline{N}^u = (\infty, \infty, 1, \infty, \infty)$.

The subspace V_1 is commutative and the partial prolong $\mathfrak{h}$ with V_1 as the first component is trivial, i.e., $\mathfrak{h} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \left(\widetilde{\mathfrak{g}}_1 = V_1\right)$. Since $[[V_1, \mathfrak{g}_{-2}]] = 0$, it follows that $\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2}$ is an ideal in $\mathfrak{h}$. The simple 24-dimensional quotient obtained is isomorphic to $\mathfrak{sl}(5)$ with the degrees of Chevalley generators being $(0, \pm 1, 0, 0)$.

Conclusion. There are no new algebras as partial prolongs.

15 $\mathfrak{mb}\left(9; \widetilde{\underline{M}}\right)$ and analogs of semi-densities for $p = 2$

The Lie algebra $\mathfrak{mb}\left(9; \widetilde{\underline{M}}\right)$ is the desuperization of $\mathfrak{mb}(4; \underline{M}|5)$, the $p = 2$ analog of $\mathfrak{mb}(4|5)$ over $\mathbb{C}$. It can be obtained from the Lie algebra $\mathbf{F}(\mathfrak{mb}(3; \underline{N}|8))$, a deform of $\mathfrak{svect}(5; \widetilde{\underline{N}})$, by regrading of the latter:

$$\deg u_5 = 2, \quad \deg u_i = 1 \quad \text{for } i = 1, 2, 3, 4.$$

Let us recall a description of $\mathfrak{mb}(4|5)$ as the Lie superalgebra that preserves something.

Over $\mathbb{C}$, the Lie superalgebra $\mathfrak{mb}(4|5)$ was initially constructed as follows. Consider the Lie superalgebra $\mathfrak{m}(3; 3)$: this is the regrading of $\mathfrak{m}(3)$ which preserves the distribution given by the Pfaff equation with the form $d\tau + \sum q_i d\xi_i$; this regrading is a $\mathbb{Z}$-grading of depth 1, see (2.18). We have (assuming $p\left(\mathrm{vol}^{1/2}\right) = \bar{1}$)

$$\mathfrak{m} = (\mathfrak{m}_{-1}, \mathfrak{m}_0)_*,$$

where

$$\mathfrak{m}_0 = \mathfrak{vect}(\xi) \ltimes \Lambda(\xi)\tau \quad \text{and} \quad \mathfrak{m}_{-1} = \Lambda(\xi) \otimes \mathrm{vol}^{1/2} \overset{\text{as spaces}}{\simeq} \Pi(\Lambda(\xi)).$$

Here $\mathfrak{vect}(\xi) = \mathrm{Span}\left(\sum f_i(\xi)q_i\right)$. Denote $\mathfrak{n} := \Lambda(\xi)\tau$.

Considering $\mathfrak{m}_{-1}$ as a $\mathfrak{vect}(\xi)$-module, we preserve the multiplication of the Grassmann algebra $\Lambda(\xi)$; i.e., the $\mathfrak{vect}(\xi)$-action satisfies the Leibniz rule, whereas the ideal $\mathfrak{n}$ of $\mathfrak{m}_0$ does not preserve this multiplication. However, there is an isomorphism of $\mathfrak{vect}(\xi)$-modules $\sigma \colon \mathfrak{n} \longrightarrow \Pi(\mathfrak{m}_{-1})$ and the action of $\mathfrak{n}$ on $\mathfrak{m}_{-1}$ is accomplished with the help of this isomorphism[9]:

$$[f, g] = \sigma(f) \cdot g \quad \text{for any } f \in \mathfrak{n}, \ g \in \mathfrak{m}_{-1}.$$

---

[9]Speaking informally, although $\mathfrak{n}$ does not preserve the multiplication in $\mathfrak{m}_{-1}$ considered as the Grassmann algebra, $\mathfrak{n}$ "remembers" this multiplication. And since $\mathfrak{m}$ is the Cartan prolong, it also somehow "remembers" this structure.

The bilinear form ω with which we construct the central extension $\mathfrak{m}_- = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ is the Berezin integral (the coefficient of the highest term) of the product of the two functions:

$$\omega(g_1, g_2) = \int g_1 g_2 \, \mathrm{vol} \quad \text{for any } g_1, g_2 \in \mathfrak{m}_{-1};$$

i.e., it also "remembers" the multiplication in $\mathfrak{m}_{-1}$.

The "right" question therefore is not "which elements of $\mathfrak{m}_0$ preserve ω?", but rather

"which elements of $\mathfrak{m}_0$ preserve ω *conformally*, up to multiplication by a scalar?"   (15.1)

It is precisely these elements which are derivations of the Lie superalgebra $\mathfrak{m}_-$, and since $\mathfrak{m}$ is the maximal algebra that "remembers" the multiplication, it follows that the whole of $\mathfrak{der}(\mathfrak{m}_-)$ lies inside $\mathfrak{m}_0$.

Let us give an interpretation of the analog of the space of semi-densities for $p = 2$.

For $p \neq 2$, we know the answer to the question (15.1): these are elements of the two types:

(a) the elements of $(\mathfrak{b}_{1/2}(3))_0$, the space of linear vector fields preserving the form ω, i.e., elements of the form

$$D + \frac{1}{2}\,\mathrm{div}\,D, \quad \text{where } D \in \mathfrak{vect}(\xi);$$

(b) the elements of the form $c \cdot \tau \in \mathfrak{n}$ which multiply ω by $2c \in \mathbb{K}$.

For $p = 2$, the elements of the form $c \cdot 1 \in \mathfrak{n}$, where $c \in \mathbb{K}$, multiply ω by $2c = 0 \in \mathbb{K}$. Moreover, any function $f \in \mathfrak{n}$ preserves ω as well:

$$f\omega(g_1, g_2) = \int ((fg_1)g_2 + g_1(fg_2))\,\mathrm{vol} = 0 \quad \text{for any } f \in \mathfrak{n},\ g_1, g_2 \in \mathfrak{m}_{-1}.$$

Thus, the form ω is preserved by $\mathfrak{svect}(\xi) \rtimes \Lambda(\xi)$ which is isomorphic to the subalgebra of linear (degree 0) vector fields in $\mathfrak{b}_\infty$. This should have been expected: since $2 = 0$, then $\frac{1}{2} = \infty$.

The elements conformally preserving ω are precisely $\xi_i \partial_i \longleftrightarrow q_i \xi_i$, so we have to add their sum to the 0th part and calculate the Cartan prolong.

Now we are able to obtain the basis of the nonpositive components of $\mathfrak{mb}(9; \underline{M})$. A realization of the weight basis of the negative components and generators of the 0th component by vector fields is as follows, see Section 2.12 ($X_i^\pm$ are the Chevalley generators of $\mathfrak{sl}(3) = \mathfrak{svect}(0|3)_0$):

$\mathfrak{g}_i$	the generators
$\mathfrak{g}_{-2}$	∂_1
$\mathfrak{g}_{-1} \simeq$	$\xi_1 \longleftrightarrow \partial_2,\ \xi_2 \longleftrightarrow \partial_3,\ \xi_3 \longleftrightarrow \partial_4,\ \xi_1\xi_2\xi_3 \longleftrightarrow \partial_9 + x_8\partial_1,$
$\mathcal{O}(3; \mathbb{1})$	$1 \longleftrightarrow \partial_8,\ \xi_1\xi_2 \longleftrightarrow \partial_5 + x_4\partial_1,\ \xi_1\xi_3 \longleftrightarrow \partial_6 + x_3\partial_1,\ \xi_2\xi_3 \longleftrightarrow \partial_7 + x_2\partial_1$
$\mathfrak{g}_0 \simeq \mathbb{K}D \rtimes$	$\partial_{\xi_3} \longleftrightarrow x_6x_7\partial_1 + x_4\partial_8 + x_6\partial_2 + x_7\partial_3 + x_9\partial_5,$
$(\mathfrak{svect}(3; \mathbb{1}) \rtimes \mathcal{O}(3; \mathbb{1})$	$X_1^- \longleftrightarrow x_3\partial_2 + x_7\partial_6,\ X_1^+ \longleftrightarrow x_2\partial_3 + x_6\partial_7,$
	$X_2^- \longleftrightarrow x_4\partial_3 + x_6\partial_5,\ X_2^+ \longleftrightarrow x_3\partial_4 + x_5\partial_6,$
	$\xi_1\xi_2\partial_3 \longleftrightarrow x_4^{(2)}\partial_1 + x_4\partial_5,\ \xi_1\xi_3\partial_2 \longleftrightarrow x_3^{(2)}\partial_1 + x_3\partial_6,$
	$\xi_2\xi_3\partial_1 \longleftrightarrow x_2^{(2)}\partial_1 + x_2\partial_7,\ \xi_1\xi_2\xi_3 \longleftrightarrow x_8^{(2)}\partial_1 + x_8\partial_9,$
	$D = x_1\partial_1 + x_2\partial_2\ + x_5\partial_5 + x_6\partial_6 + x_9\partial_9$

For $\underline{M}$ unconstrained, $\dim \mathfrak{g}_1 = 64$. The lowest-weight vectors in $\mathfrak{g}_1$ are

$$v_1 = x_2x_5x_6\partial_1 + x_2^{(2)}\partial_8 + x_2x_5\partial_3 + x_2x_6\partial_4 + x_2x_9\partial_7 + x_5x_6\partial_7,$$

$$v_2 = x_2x_3x_4\partial_1 + x_2x_3\partial_5 + x_2x_4\partial_6 + x_3x_4\partial_7 + x_2x_7\partial_9 + x_2x_8\partial_2 + x_4x_5\partial_9 + x_4x_8\partial_4$$
$$+ x_5x_8\partial_5 + x_7x_8\partial_7 + x_1x_8\partial_1 + x_1\partial_9 + x_2x_7\partial_9 + x_2x_8\partial_2 + x_6x_8\partial_6$$
$$+ x_8x_9\partial_9 + x_3x_6\partial_9 + x_3x_8\partial_3 + x_8^{(2)}\partial_8,$$

$$v_3 = x_2x_5x_7\partial_1 + x_3x_5x_6\partial_1 + x_2x_3\partial_8 + x_2x_5\partial_2 + x_2x_7\partial_4 + x_2x_9\partial_6 + x_3x_5\partial_3 + x_3x_6\partial_4$$
$$+ x_3x_9\partial_7 + x_5x_6\partial_6 + x_5x_7\partial_7,$$

$$v_4 = x_1\partial_8 + x_5x_6\partial_2 + x_5x_7\partial_3 + x_5x_9\partial_5 + x_6x_7\partial_4 + x_6x_9\partial_6 + x_7x_9\partial_7.$$

Critical coordinates: $\underline{M}_5 = \underline{M}_6 = \underline{M}_7 = \underline{M}_9 = 1$. This Lie algebra is a regrading of $\mathfrak{mb}_3(11; \underline{N})$.

15.1 No simple partial prolongs with the whole $\mathfrak{g}_0$

There are remarkable elements in $\mathfrak{g}_1$:

$$v_1 = x_2^{(2)}\partial_7 + x_2^{(3)}\partial_1, \quad v_2 = x_3^{(2)}\partial_6 + x_3^{(3)}\partial_1, \quad v_3 = x_4^{(2)}\partial_5 + x_4^{(3)}\partial_1, \quad v_4 = x_8^{(2)}\partial_9 + x_8^{(3)}\partial_1.$$

Each of the first three vectors generates a submodule of dim $= 32$; any two of the first three generate a submodule of dim $= 40$; all three together generate a submodule of dim $= 48$. The last one generates a submodule of dim $= 8$. All 4 together generate a submodule of $\widetilde{\mathfrak{g}}_1$ of dim $= 56$. The quotient $\mathfrak{g}_1/\widetilde{\mathfrak{g}}_1$ is an irreducible $\mathfrak{g}_0$-module. We have $\dim([\mathfrak{g}_{-1}, \widetilde{\mathfrak{g}}_1]) = 25$ while $\dim \mathfrak{g}_0 = 26$; absent is the vector of weight 0:

$$x_1\partial_1 + x_2\partial_2 + x_5\partial_5 + x_6\partial_6 + x_9\partial_9.$$

Note that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

For the 24-dimensional intersection $\mathfrak{g}_{1i}$ of the 32-dimensional submodules, we see that $\mathfrak{g}_{-1}$ is irreducible over $[\mathfrak{g}_{-1}, \mathfrak{g}_{1i}]$, and hence over $[\mathfrak{g}_{-1}, \widetilde{\mathfrak{g}}_1]$; we have $\dim([\mathfrak{g}_{-1}, \mathfrak{g}_{1i}]) = 21$.

The elements absent in $[\mathfrak{g}_{-1}, \mathfrak{g}_{1i}]$ as compared with $\mathfrak{g}_0$:

$$x_8^2\partial_1 + x_8\partial_9, \quad x_2^2\partial_1 + x_2\partial_7, \quad x_3^2\partial_1 + x_3\partial_6, \quad x_4^2\partial_1 + x_4\partial_5.$$

16   A description of $\mathfrak{mb}_2\big(11; \underline{\widetilde{N}}\big) := \mathbf{F}(\mathfrak{mb}(5; \underline{N}|6))$

Whenever possible in this section, we do not indicate the shearing vectors. The Lie superalgebra $\mathfrak{mb}(5; \underline{N}|6)$ is the complete prolong of its negative part, see Section 2.11.

Let us consider $\mathfrak{g} := \mathbf{F}(\mathfrak{mb}(5; \underline{N}|6))$ as a deform of $\mathfrak{svect}(5)$ with the grading

$$\deg u_1 = \deg u_2 = 1, \quad \deg u_3 = \deg u_4 = \deg u_5 = 2.$$

Let us describe the complete prolong of this negative part of this Lie superalgebra, see Section 2.11. We deduce the form of the vector fields forming a basis of the negative part of $\mathfrak{mb}(5; \underline{N}|6)$ from nonzero commutation relations between ∂_k and $x_i\partial_a$, where $k = 1, \ldots, 5$, $a = 3, 4, 5$, and $i = 1, 2$, cf. (14.3), (14.9), considered as elements of $\mathbf{F}(\mathfrak{mb}(5; \underline{N}|6))$:

$$[[\partial_i, u_i\partial_a]] = \partial_a, \quad [[u_1\partial_4, u_2\partial_5]] = [[u_1\partial_5, u_2\partial_4]] = \partial_3. \tag{16.1}$$

For a basis we take realization in vector fields in 5 indeterminates z_k, where $k = 1, \ldots, 5$, and 6 indeterminates z_{ia}, where $a = 3, 4, 5$ and $i = 1, 2$, of which z_1, z_2, z_3, z_{13}, z_{23} are even while z_4, z_5, z_{14}, z_{15}, z_{24}, z_{25} are odd and $\delta_i := \partial_{z_i}$:

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-2}$	$\delta_3 \mid \delta_4,\ \delta_5$
$\mathfrak{g}_{-1}$	$\delta_1,\ \delta_2,\ \delta_{13} + z_1\delta_3,\ \delta_{23} + z_2\delta_3 \mid \delta_{14} + z_1\delta_4 + z_{25}\delta_3,\ \delta_{24} + z_2\delta_4,\ \delta_{15} + z_1\delta_5 + z_{24}\delta_3,\ \delta_{25} + z_2\delta_5$

Because the bracket (16.1) is a deformation that does not preserve the grading given by the torus in $\mathfrak{gl}(5)$, we consider the part of the weights that is salvaged, namely, we just exclude the 3rd coordinate of the weight; whereas the weight of x_3 is defined to be equal to $(-1, -1, 1, 1)$.

The dimension of $\mathfrak{g}_0$ is the same for all p; it is the expressions of the elements that differ. The raising operators in $\mathfrak{g}_0$ are those of weight $(1, -1, 0, 0)$ or $(0, 0, 1, -1)$, and those with a positive sum of coordinates of the weight, $\dim(\mathfrak{g}_0^+) = 13$; we skip their explicit description (it is commented with % marks in the TeX file available in arXiv).

The lowering operators in $\mathfrak{g}_0$ are those of weight $(-1, 1, 0, 0)$ or $(0, 0, -1, 1)$, and those with a negative sum of coordinates of the weight; $\dim(\mathfrak{g}_0^-) = 4$:

$$\boxed{\{\text{-1,-1,0,1}\}} \to \{z_{13}\delta_{14} + z_{23}\delta_{24} + z_{15}\delta_2 + z_{25}\delta_1 + z_3\delta_4 + z_{15}z_{23}\delta_3 + z_{15}z_{24}\delta_4 + z_{15}z_{25}\delta_5\},$$

$$\boxed{\{\text{-1,-1,1,0}\}} \to \{z_{13}\delta_{15} + z_{23}\delta_{25} + z_{14}\delta_2 + z_{24}\delta_1 + z_3\delta_5 + z_{14}z_{23}\delta_3 + z_{14}z_{24}\delta_4 + z_{14}z_{25}\delta_5\},$$

$$\boxed{\{\text{-1,1,0,0}\}} \to \{z_2\delta_1 + z_{13}\delta_{23} + z_{14}\delta_{24} + z_{15}\delta_{25} + z_{14}z_{15}\delta_3\},$$

$$\boxed{\{\text{0,0,1,-1}\}} \to \{z_{14}\delta_{15} + z_{24}\delta_{25} + z_4\delta_5\}. \tag{16.2}$$

Noncritical coordinates: N_1, N_2, N_3.

For the unconstrained shearing vector, $\operatorname{sdim} \mathfrak{g}_1 = 20|20$ with the lowest-weight vector

$$v_1 = x_3\delta_1 + x_6x_7\delta_7 + x_6x_8\delta_8 + x_6x_{10}\delta_{10} + x_7x_8\delta_9 + x_7x_{10}\delta_{11}$$
$$+ x_8x_{10}\delta_2 + x_8x_9x_{10}\delta_4 + x_8x_{10}x_{11}\delta_5$$

generating the whole $\mathfrak{g}_0$-module $\mathfrak{g}_1$. All other highest- and lowest-weight vectors together generate a submodule V of $\mathfrak{g}_1$ of superdimension $16|16$ such that $[\mathfrak{g}_{-1}, V]$ is a 20-dimensional subalgebra of $\mathfrak{g}_0$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1/V$ is irreducible.

16.1   Desuperization

Its 0th component is the same as in (16.2) with parities forgotten.

Noncritical coordinates: $N_1, \ldots, N_5$.

17   On analogs of $\mathfrak{kas}$ for $p = 2$

In this section, whenever possible, we do not indicate the shearing vectors. All computations in this section are performed for $p = 2$; however, for comparison, we also recall expressions obtained earlier over $\mathbb{C}$ in [64, 65, 67]. These expressions do not differ, usually, from those for $p > 2$.

The Lie superalgebra $\mathfrak{kas}$ over $\mathbb{C}$ was the last example needed to complete the list of simple W-graded vectorial Lie superalgebras, see [64, 65]. Its nonpositive part is the same as that of $\mathfrak{g} := \mathfrak{k}(1|6)$ (generated by the functions in the even t and 6 odd indeterminates) in its standard $\mathbb{Z}$-grading while the component $\mathfrak{g}_1$ is exceptional, as a $\mathfrak{g}_0$-module, among various $\mathfrak{k}(1|n)$: only for $n = 6$ does $\mathfrak{g}_1$ split into 3 irreducible components: one depends on t, the other two are dual to each other. For any $p \neq 2$, we define two copies of $\mathfrak{kas}$; each of them is the partial prolong generated by the nonpositive part, and the two submodules of $\mathfrak{g}_1$: the one that depends on t, and any one of the other two submodules.

To distinguish between these two isomorphic copies of $\mathfrak{kas}$, we denote by $\mathfrak{kas}^\xi$ the one whose space of generating functions contains the product $\xi_1\xi_2\xi_3$; let $\mathfrak{kas}^\eta$ be the one whose space of generating functions contains the product $\eta_1\eta_2\eta_3$. We always consider only $\mathfrak{kas}^\xi$, see (17.4), so we skip the superscript.

For $p = 2$, the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module is rather complicated, and it is not clear what should one take for an analog of $\mathfrak{kas}$. Let us investigate.

17.1 The component $\mathfrak{g}_1$ as $\mathfrak{g}_0$-module for $\mathfrak{g} := \mathfrak{k}(1|6)$ if $p = 2$

Let $\mathfrak{g} = \mathfrak{k}(1|6)$ be described in terms of generating functions of ξ, η, t, where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$, with the bracket

$$\{f, g\}_{\text{k.b.}} = \frac{\partial f}{\partial t}(1 + E')(g) + (1 + E')(f)\frac{\partial g}{\partial t} + \sum_{1 \leq i \leq 3}\left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \xi_i}\right), \qquad (17.1)$$

where $E' = \sum \xi_i \partial_{\xi_i}$, and the standard grading $\deg t = 2$, $\deg \xi_i = \deg \eta_i = 1$.

We have $\mathfrak{g}_0 \simeq \mathfrak{d}\big(\mathfrak{o}_\Pi^{(1)}(4)\big)$. However, since $\mathfrak{G}_0 \simeq \mathfrak{d}(\mathfrak{o}_\Pi(4))$ for $\mathfrak{G} := \mathfrak{k}(7; \underline{N})$, we have to investigate how can we enlarge $\mathfrak{d}\big(\mathfrak{o}_\Pi^{(1)}(4)\big)$ to get a "correct" version of $\mathbf{F}(\mathfrak{kas})_0$.

Consider the subalgebra $\mathfrak{h} := \mathfrak{o}_\Pi^{(1)}(6) \simeq \Lambda^2(\xi, \eta) = \mathfrak{h}_{-2} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_2$ of $\mathfrak{g}_0 = \mathfrak{d}(\mathfrak{o}_\Pi^{(1)}(6))$, where

$$\mathfrak{h}_{-2} = \Lambda^2(\xi), \quad \mathfrak{h}_0 = \text{Span}(\xi_i \eta_j \mid i, j = 1, 2, 3) \simeq \mathfrak{gl}(3), \quad \mathfrak{h}_2 = \Lambda^2(\eta). \qquad (17.2)$$

Set $\Phi := \sum \xi_i \eta_i$. For $p \neq 2$ and the contact bracket (2.6), we have $\text{ad}_\Phi\big|_{\mathfrak{h}_i} = i\,\text{id}$, hence the grading in (17.2). For $p = 2$, the elements Φ and t interchange their roles: Φ commutes with $\mathfrak{g}_0 = \mathfrak{co}_\Pi^{(1)}(6)$, while t is a grading operator on $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{h}_1$, where $\mathfrak{h}_1 = \mathfrak{h}_{-2} \oplus \mathfrak{h}_2$.

17.2 Desuperization

Under desuperization $\mathfrak{k}(1; \underline{N}|6)$ turns into $\mathfrak{G} := \mathfrak{k}\big(7; \widetilde{N}\big)$, whereas the Lie algebra $\mathfrak{h}$, see (17.2), turns into $\mathfrak{H} := \mathfrak{o}_\Pi(6) = S^2(\xi, \eta) \subset \mathfrak{G}_0 \simeq \mathfrak{d}(\mathfrak{o}_\Pi(4))$, where

$$\mathfrak{H}_{-2} = S^2(\xi), \quad \mathfrak{H}_0 = \text{Span}(\xi_i \eta_j \mid i, j = 1, 2, 3) \simeq \mathfrak{gl}(3), \quad \mathfrak{H}_2 = S^2(\eta).$$

The highest-weight vectors of the $\mathfrak{G}_0$-module $S^3(\xi, \eta)$ are as follows (in parentheses are the dimensions of the respective $\mathfrak{G}_0$-modules these vectors generate)

$$\xi_1^{(3)}(26), \quad \xi_1 \xi_2^{(2)}(26), \quad \xi_1 \xi_2 \xi_3(14), \quad \xi_1(\xi_2 \eta_2 + \xi_3 \eta_3)(6).$$

The lowest-weight vectors and the dimensions of the $\mathfrak{G}_0$-modules these vectors generate are the same with the replacement $\xi \longleftrightarrow \eta$. However, since the modules generated by lowest or highest-weight vectors do not span the whole of $\mathfrak{G}_1$ if $p = 2$, it is more natural to describe this component differently, as follows.

Bracketing $\xi_i^{(3)}$ (resp. $\eta_j^{(3)}$) with $\mathfrak{g}_{-1}$ yields $\xi_i^{(2)}$ (resp. $\eta_j^{(2)}$), and *since each of the 26-dimensional modules generated by any cube contains only one cube, to have all squares in $\mathbf{F}(\mathfrak{kas})_0$, we have to take for $\mathbf{F}(\mathfrak{kas})_1$ the module generated by all cubes. But we cannot do this: the prolong of the module containing all cubes is equal to $\mathfrak{k}(7)$.* Let us establish which cubes should be absent in the correct version of $\mathbf{F}(\mathfrak{kas})_1$ and how many versions are there.

At this stage we do not yet know what shall we eliminate in $\mathfrak{G}_0$ to get a correct version of $\mathbf{F}(\mathfrak{kas})_0$, so we consider modules over $\mathfrak{g}_0$.

The $\mathfrak{g}_0$-submodules of $E^3(\xi, \eta)$ and $\mathfrak{g}_1$. The submodule $V = \text{Span}(\xi_i \Phi, \eta_i \Phi)_{i=1}^3 \subset E^3(\xi, \eta)$ is the smallest; observe that in $E^3(\xi, \eta)$ all squares vanish. By adding any of the following 8 one-dimensional modules

$$\text{spanned by expressions } x_1 x_2 x_3, \text{ where } x_i \text{ is any of } \xi_i \text{ or } \eta_i, \qquad (17.3)$$

we can enlarge V and still have a $\mathfrak{g}_0$-submodule. Together these modules span a 14-dimensional submodule W. The quotients $E^3(\xi, \eta)/W \simeq V^*$ and W/V are irreducible $\mathfrak{g}_0$-modules.

Now, let us involve t. Set $P := \text{Span}(t\xi_j, \eta_i(t + \xi_j \eta_j) \mid i \neq j)$. As is easy to see, $\dim V \cap P = 3$. The $\mathfrak{g}_0$-module generated by $t\xi_i$ is of dimension 16, as space it is the direct sum

$$(V + P) \oplus \text{ a 4-dimensional subspace of the 8-dimensional space (17.3)}.$$

The $\mathfrak{g}_0$-module generated by $t\xi_i$ and $t\eta_j$ is of dimension 26; as a vector space it is the direct sum

$(V + P) \oplus$ the 8-dimensional space (17.3).

The $\mathfrak{g}_0$-submodules of $S^3(\xi, \eta)$ and $\mathfrak{G}_1$. There are 6 modules of dimension 26 each, each of them is generated by one cube $\left(\xi_i^{(3)} \text{ or } \eta_j^{(3)}\right)$. The intersection of ≥ 2 of these modules is a 20-dimensional module $E^3(\xi, \eta)$. Unions of several of these modules form 32-, 38-, 44-, and 50-dimensional submodules, or the whole $S^3(\xi, \eta)$.

The module generated by $\xi_1^{(3)}$ is of dimension 16, and contains 5 elements with $\xi_1^{(2)}$, V and 4 elements of the form $\xi_1 x_2 x_3$, see (17.3). The intersection of all 6 such modules generated by cubes is equal to $W = V \oplus$ the 8-dimensional space (17.3).

The dimension of the union of the modules generated by $\xi_i^{(3)}$ and $\xi_j^{(3)}$ for $i \neq j$ is equal to 24.

The dimension of the union of the modules generated by $\xi_i^{(3)}$ and $\eta_j^{(3)}$ for any i and j is equal to 26.

The dimension of the union of the modules generated by $\xi_i^{(3)}$ and $\eta_j^{(3)}$ for all i and j is equal to 50, it is all $\mathfrak{G}_1$ except for V^*. *Verdict:*

> $\mathbf{F}(\mathfrak{kas}(1; \underline{N}|6))$ is the prolong of $\mathfrak{k}(7)_-$ and $Y \oplus (V + P)$,
>
> where Y is the 8-dimensional module (17.3),
>
> $V := \text{Span}(\eta_i \Phi, \xi_j \Phi)_{i,j=1}^3, \quad P := \text{Span}(t\xi_j, \eta_i(t + \xi_j \eta_j) \,|\, i \neq j).$ \hfill (17.4)

17.3 Remark: useful formulas for manual computations

The lowest-weight vectors of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ are as follows:

$$\eta_1 \eta_2 \xi_3, \quad \eta_1 \eta_2 \eta_3, \quad t\eta_1 \quad \text{for } p = 0,$$
$$\eta_1 \eta_2 \xi_3, \quad \eta_1 \eta_2 \eta_3, \quad \eta_1 \eta_2 \xi_2 + \eta_1 \eta_3 \xi_3 = \eta_1 \Phi \quad \text{for } p = 2.$$

Clearly, $\Lambda^3(\xi, \eta)$ is a $\mathfrak{g}_0$-submodule. Let us describe it.

Let $X_0 := \eta_1 \eta_2 \eta_3$. The subalgebra $\mathfrak{h}_2 \subset \mathfrak{g}_0$ commutes with X_0, and $\mathfrak{h}_0$ acts on X_0 by scalar operators, so $U(\mathfrak{g}_0)X_0 = U(\mathfrak{h}_{-2})X_0$. Denote $V_1 := \text{Span}(\eta_1 \Phi, \eta_2 \Phi, \eta_3 \Phi)$.

We have

$$\{\xi_1 \xi_2, X_0\} = \xi_2 \eta_2 \eta_3 + \xi_1 \eta_1 \eta_3 = \eta_3(\xi_1 \eta_1 + \xi_2 \eta_2) = \eta_3 \Phi.$$

Similar computations show that $[\mathfrak{h}_{-2}, X_0] = V_1$.

Let us now describe $[\mathfrak{h}_{-2}, V_1]$. Clearly, see (17.1),

$$\{\xi_i \xi_j, \eta_\alpha \Phi\} = \{\xi_i \xi_j, \eta_\alpha\} \cdot \Phi + \eta_\alpha \{\xi_i \xi_j, \Phi\}.$$

We have

$$\{\xi_i \xi_j, \Phi\} = \begin{cases} 2\xi_i \xi_j & \text{for } p \neq 2, \\ 0 & \text{for } p = 2, \end{cases}$$

and respectively we have

$$\{\xi_1 \xi_2, \eta_1 \Phi\} = \begin{cases} \xi_2(-\xi_1 \eta_1 + \xi_3 \eta_3), \\ \xi_2 \Phi, \end{cases} \qquad \{\xi_1 \xi_3, \eta_1 \Phi\} = \begin{cases} \xi_3(-\xi_1 \eta_1 + \xi_2 \eta_2), \\ \xi_3 \Phi, \end{cases}$$

$$\{\xi_2 \xi_3, \eta_1 \Phi\} = \begin{cases} 2\eta_1 \xi_2 \xi_3, \\ 0. \end{cases}$$

We set $V_2 := [\mathfrak{h}_{-2}, V_1] = \mathrm{Span}(\xi_1\Phi, \xi_2\Phi, \xi_3\Phi)$ and see that $[\mathfrak{h}_{-2}, V_2] = 0$.

Therefore, $\mathbb{K}X_0 \oplus V_1 \oplus V_2$ is a $\mathfrak{g}_0$-submodule.

We have $\{\eta_i\eta_j, \eta_\alpha\Phi\} = 0$; i.e., $V = V_1 \oplus V_2$ is a submodule to which $\mathbb{K}X_0$ us glued "from above".

Absolutely analogously, if $Y_0 := \xi_1\xi_2\xi_3$, then Y_0 generates the submodule $\mathbb{K}Y_0 \oplus V_1 \oplus V_2$, and $\mathbb{K}Y_0$ is now a submodule glued to V "from below".

Now, let $W_1 := \mathrm{Span}(\xi_2\xi_3\eta_1, \xi_1\xi_3\eta_2, \xi_1\xi_2\eta_3)$ and $W_2 := \mathrm{Span}(\xi_1\eta_2\eta_3, \xi_2\eta_1\eta_3, \xi_3\eta_1\eta_2)$. Then,

$$\xi_i\xi_j\colon W_1 \longrightarrow 0, \quad \xi_i\eta_j\colon W_1 \longrightarrow \begin{cases} V_2 & \text{for } i \neq j, \\ W_1 & \text{for } i = j, \end{cases} \quad \eta_i\eta_j\colon W_1 \longrightarrow V_1,$$

$$\xi_i\xi_j\colon W_2 \longrightarrow V_2, \quad \xi_i\eta_j\colon W_2 \longrightarrow \begin{cases} V_1 & \text{for } i \neq j, \\ W_2 & \text{for } i = j, \end{cases} \quad \eta_i\eta_j\colon W_2 \longrightarrow 0.$$

We see that $U := V \oplus W_1 \oplus W_2$ is a submodule and $\mathfrak{h}$ annihilates the quotient U/V, and hence it is possible to glue any element of $W_1 \oplus W_2$ to V.

Finally, the tautological representation of $\mathfrak{o}_\Pi^{(1)}(6)$ is realized in the 6-dimensional quotient of $\Lambda^3(\xi, \eta)$; it cannot be, however, singled out as a SUBmodule, moreover, it is glued to the whole submodule U, including the elements X_0 and Y_0.

Now, look at the elements of $\mathfrak{g}_1$ whose expressions contain t.

The Lie algebra $\mathfrak{h}_0 = \mathfrak{gl}(3)$ acts in the same way as for $p = 0$ (as on the direct sum of the tautological $\mathfrak{gl}(3)$-module and its dual). Whereas

$$\{\eta_i\eta_j, t\eta_k\} = \begin{cases} \eta_i\eta_j\eta_k = X_0 & \text{if the indices } i, j, k \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Further,

$$\{\xi_1\xi_2, t\eta_1\} = \xi_2(t + \xi_1\eta_1), \quad \{\xi_1\xi_3, t\eta_1\} = \xi_3(t + \xi_1\eta_1), \quad \{\xi_2\xi_3, t\eta_1\} = \xi_2\xi_3\eta_1,$$

and $\{\xi_i\xi_j, \xi_2(t + \xi_1\eta_1)\} = 0$ for all i, j.

Thus, under the action of $\mathfrak{h}$ the space $Q_1 = \mathrm{Span}(t\eta_1, t\eta_2, t\eta_3)$ generates the space

$$Q_2 = \mathrm{Span}(\xi_1(t + \xi_2\eta_2), \xi_2(t + \xi_3\eta_3), \xi_3(t + \xi_1\eta_1)),$$

as well as V_2, W_1, and $\mathbb{K}X_0$.

We can try to twist the elements $t\eta_i$ by adding something to them to enable the subalgebra $\mathfrak{h}_2$ annihilate them. Such twisted elements span the space $P_1 := \mathrm{Span}(\eta_i(t + \xi_j\eta_j) \mid i \neq j)$. Under the action of $\mathfrak{g}_{-2}$ we obtain from P_1 the spaces $P_2 := \mathrm{Span}(t\xi_1, t\xi_2, t\xi_3)$, W_2, and $\mathbb{K}Y_0$.

17.4　The simple ideal $\mathfrak{kas}^{(1)}(1; \underline{N}|6)$ in $\mathfrak{kas}$

Let $\mathfrak{g} := \mathfrak{kas}(1; \underline{N}|6)$ considered with the standard $\mathbb{Z}$-grading. In $\mathfrak{g}_0$, the subalgebra $\mathfrak{h}$, see (17.2), is not simple: it contains the ideal $\mathfrak{o}_\Pi^{(2)}(6)$ of codimension 1, consisting of matrices $\left(\begin{smallmatrix} A & B \\ C & A^\mathrm{t} \end{smallmatrix}\right)$ with zero-diagonal symmetric matrices B and C and $A \in \mathfrak{sl}(6)$ whereas $\mathfrak{h}$ consists of the same type matrices with $A \in \mathfrak{gl}(6)$. Therefore, $\mathfrak{g} = \mathfrak{kas}(1; \underline{N}|6)$ contains a simple ideal $\mathfrak{kas}^{(1)}(1; \underline{N}|6)$ of codimension 1, its outer derivation being the outer derivation of $\mathfrak{g}_0$. This derivation is present in the versions of $\mathfrak{kas}$ considered in the next three sections.

17.5   Desuperizations of $\mathfrak{kas}(1; \underline{N}|6)$

For one of the W-gradings of $\boldsymbol{F}(\mathfrak{kas})$, we do not require presence of all squares in $\boldsymbol{F}(\mathfrak{kas})_0$, but rather require their absence; this affects the number of parameters the shearing vector depends on.

Critical coordinates. The shearing vector $\widetilde{\underline{N}}$ of the desuperization $\mathfrak{k} := \mathfrak{k}(7; \widetilde{\underline{N}})$, the ambient of the desuperized $\mathfrak{kas}(1; \underline{N}|6)$, has no critical coordinates.

18 $\widetilde{\mathfrak{kas}}(7; \underline{M}) := \mathbf{F}\big(\mathfrak{kas}\big(4; \widetilde{\underline{N}}|3\big)\big)$, where $\mathfrak{kas}(4|3) := \mathfrak{kas}(1|6; 3\eta)$

element of $\mathfrak{g}_0$	its action on $\mathfrak{g}_{-1} \simeq \mathrm{Vol}_0$	the corresponding vector field $\in \mathfrak{vect}(4; \underline{N}|3)$
ξ_1	∂_{η_1}	$z_1 \partial_0 + z_{12} \partial_2 + z_{13} \partial_3$
ξ_2	∂_{η_2}	$z_2 \partial_0 + z_{12} \partial_1 + z_{23} \partial_3$
ξ_3	$\boxed{\partial_{\eta_3}}$	$z_3 \partial_0 + z_{13} \partial_1 + z_{23} \partial_2$
$t \longleftrightarrow 1, \ \xi_i \eta_i \longleftrightarrow \eta_i \partial_{\eta_i} \Longrightarrow t + \xi_i \eta_i \longleftrightarrow \eta_i \partial_{\eta_i} + \mathrm{div}(\eta_i \partial_{\eta_i})$		
t	1	$\sum z_s \partial_s$ for any index s
$\xi_1 \eta_1$	$\eta_1 \partial_{\eta_1}$	$z_1 \partial_1 + z_{12} \partial_{12} + z_{13} \partial_{13}$
$\xi_2 \eta_2$	$\eta_2 \partial_{\eta_2}$	$z_2 \partial_2 + z_{12} \partial_{12} + z_{23} \partial_{23}$
$\xi_3 \eta_3$	$\eta_3 \partial_{\eta_3}$	$z_3 \partial_3 + z_{13} \partial_{13} + z_{23} \partial_{23}$
$\xi_1 \eta_2$	$\boxed{\eta_2 \partial_{\eta_1}}$	$z_1 \partial_2 + z_{13} \partial_{23}$
$\xi_1 \eta_3$	$\eta_3 \partial_{\eta_1}$	$z_1 \partial_3 + z_{12} \partial_{23}$
$\xi_2 \eta_1$	$\eta_1 \partial_{\eta_2}$	$z_2 \partial_1 + z_{23} \partial_{13}$
$\xi_2 \eta_3$	$\boxed{\eta_3 \partial_{\eta_2}}$	$z_2 \partial_3 + z_{12} \partial_{13}$
$\xi_3 \eta_1$	$\eta_1 \partial_{\eta_3}$	$z_3 \partial_1 + z_{23} \partial_{12}$
$\xi_3 \eta_2$	$\eta_2 \partial_{\eta_3}$	$z_3 \partial_2 + z_{13} \partial_{12}$
$\xi_i \eta_j \eta_k \longleftrightarrow \eta_j \eta_k \partial_{\eta_i} \in \mathfrak{svect}(\eta), \quad i \neq j \neq k$		
$\xi_1 \eta_2 \eta_3$	$\eta_2 \eta_3 \partial_{\eta_1}$	$z_1 \partial_{23}$
$\xi_2 \eta_1 \eta_3$	$\eta_1 \eta_3 \partial_{\eta_2}$	$z_2 \partial_{13}$
$\xi_3 \eta_1 \eta_2$	$\eta_1 \eta_2 \partial_{\eta_3}$	$z_3 \partial_{12}$
$\eta_i \Phi \longleftrightarrow \eta_i (\eta_j \partial_{\eta_j} + \eta_k \partial_{\eta_k}) \in \mathfrak{svect}(\eta), \quad i \neq j \neq k$		
$\eta_1 \Phi$	$\eta_1 (\eta_2 \partial_{\eta_2} + \eta_3 \partial_{\eta_3})$	$z_2 \partial_{12} + z_3 \partial_{13}$
$\eta_2 \Phi$	$\eta_2 (\eta_1 \partial_{\eta_1} + \eta_3 \partial_{\eta_3})$	$z_1 \partial_{12} + z_3 \partial_{23}$
$\eta_3 \Phi$	$\eta_3 (\eta_2 \partial_{\eta_2} + \eta_1 \partial_{\eta_1})$	$z_1 \partial_{13} + z_2 \partial_{23}$
$\eta_i (t + \xi_j \eta_j) \longleftrightarrow \eta_i \eta_j \partial_{\eta_j} + \eta_i = \eta_i \eta_j \partial_{\eta_j} + \mathrm{div}(\eta_i \eta_j \partial_{\eta_j})$		
$\eta_1 (t + \xi_2 \eta_2)$	$\eta_1 \eta_2 \partial_{\eta_2} + \eta_1$	$z_0 \partial_1 + z_3 \partial_{13}$
$\eta_2 (t + \xi_3 \eta_3)$	$\eta_2 \eta_3 \partial_{\eta_3} + \eta_2$	$z_0 \partial_2 + z_1 \partial_{12}$
$\eta_3 (t + \xi_1 \eta_1)$	$\eta_1 \eta_3 \partial_{\eta_1} + \eta_3$	$z_0 \partial_3 + z_2 \partial_{23}$
$\eta_i \eta_i (t + \xi_k \eta_k) \longleftrightarrow \eta_1 \eta_2 \eta_3 \partial_{\eta_k} + \eta_i \eta_j = \eta_1 \eta_2 \eta_3 \partial_{\eta_k} + \mathrm{div}(\eta_1 \eta_2 \eta_3 \partial_{\eta_k})$		
$\eta_1 \eta_2 (t + \xi_3 \eta_3)$	$\eta_1 \eta_2 \eta_3 \partial_{\eta_3} + \eta_1 \eta_2$	$z_0 \partial_{12}$
$\eta_1 \eta_3 (t + \xi_2 \eta_2)$	$\eta_1 \eta_2 \eta_3 \partial_{\eta_2} + \eta_1 \eta_3$	$z_0 \partial_{13}$
$\eta_2 \eta_3 (t + \xi_1 \eta_1)$	$\eta_1 \eta_2 \eta_3 \partial_{\eta_1} + \eta_2 \eta_3$	$z_0 \partial_{23}$

(18.1)

For bases in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ we take the following elements:

$$\mathfrak{g}_{-1}: \quad \begin{aligned} &\partial_0 \longleftrightarrow 1, \quad \partial_{12} \longleftrightarrow \eta_1\eta_2, \quad \partial_{13} \longleftrightarrow \eta_1\eta_3, \quad \partial_{23} \longleftrightarrow \eta_2\eta_3, \\ &\partial_1 \longleftrightarrow \eta_1, \quad \partial_2 \longleftrightarrow \eta_2, \quad \partial_3 \longleftrightarrow \eta_3. \end{aligned} \tag{18.2}$$

Let $\mathfrak{k}(1; \underline{N}|6)$ be considered as preserving the distribution given by the form $dt + \sum \xi_i d\eta_i$ with the contact bracket (17.1) and the grading of the generating functions given by, see Table (25.4):

$$\deg t = \deg \xi_i = 1, \quad \deg \eta_i = 0 \quad \text{for } i = 1, 2, 3, \quad \text{hence } \deg_{\mathrm{Lie}}(f) = \deg(f) - 1.$$

For $\mathfrak{g} := \mathfrak{kas}\big(4; \widetilde{\underline{N}}|3\big)$, we have $\mathfrak{g}_{-1} \simeq \mathrm{Vol}_0 = \mathrm{Span}(f(\eta) \,|\, \int f = 0)$, i.e, all polynomials of η without the product of the three of them. In (18.2), (18.1) we express these fields in terms of the 7 indeterminates z; we set $\partial_i := \partial_{z_i}$, $\partial_{ij} := \partial_{z_{ij}}$. We have (recall the definition of Vol_0, see (1.13))

$$\mathfrak{g}_0 \simeq \mathfrak{c}(\mathfrak{vect}(0|3)),$$

with $\mathfrak{vect}(0|3)$ acting on $\mathfrak{g}_{-1}$ as on the space of volume forms, i.e., $D \longmapsto D + \mathrm{div}(D)$, and the element t generating the center of $\mathfrak{g}_0$ acts on $\mathfrak{g}$ as the grading operator. To simplify notation, we redenote the indeterminates as follows:

$$z_0 \longleftrightarrow 1 \longleftrightarrow x_1, \quad z_{12} \longleftrightarrow \eta_1\eta_2 \longleftrightarrow x_2, \quad z_{13} \longleftrightarrow \eta_1\eta_3 \longleftrightarrow x_3, \quad z_{23} \longleftrightarrow \eta_2\eta_3 \longleftrightarrow x_4,$$
$$z_1 \longleftrightarrow \eta_1 \longleftrightarrow x_5, \quad z_2 \longleftrightarrow \eta_2 \longleftrightarrow x_6, \quad z_3 \longleftrightarrow \eta_3 \longleftrightarrow x_7.$$

18.1 Partial prolongs

The unconstrained shearing vector only depends on N_1, we have $\mathrm{sdim}\,\mathfrak{g}_1 = 16|18$.

Let V_i be the $\mathfrak{g}_0$-submodule in $\mathfrak{g}_1$ generated by v_i. For the unconstrained shearing vector, we have $\mathrm{sdim}\,V_1 = 8|6$, $\mathrm{sdim}\,V_2 = 12|9$ with $[\mathfrak{g}_{-1}, V_i] = \mathfrak{g}_0$, and $V_1 \subset V_2$.

The prolong in the direction of V_1, see (6.2), is trivial, namely $\mathfrak{g}_2^{(V_1)} = 0$.

The prolong in the direction of V_2, see (6.2), gives $\mathrm{sdim}\,\mathfrak{g}_2^{(V_2)} = 4|4$, and $\mathrm{sdim}\,\mathfrak{g}_2^{(V_2)} = 0|1$. $\mathrm{sdim}([V_2, V_2]) = 3|4$, while $[V_2, \mathfrak{g}_2^{(V_2)}] = 0$.

There are also 3 highest-weight vectors that generate nested modules $W_1 \subset W_2 \subset W_3$; we have

$$W_1 = V_1, \quad \mathrm{sdim}(W_2) = 8|7 \quad \text{and} \quad \mathrm{sdim}(W_3) = 12|12.$$

The prolong in the direction of W_2 is trivial, as is the prolong in the direction of V_1.

The prolong in the direction of the $12|10$-dimensional module $V_2 + W_2$ is equal to the prolong in the direction of V_2.

The prolong in the direction of W_3: $\mathrm{sdim}\,\mathfrak{g}_i^{(W_3)} = 12|12$ for every $i > 1$; and $\underline{N}$ depends on one parameter: N_1.

The prolong in the direction of the $16|15$-dimensional module $V_2 + W_3$ is the same as for the whole of $\mathfrak{g}_1$, and $\mathrm{sdim}\,\mathfrak{g}_i^{(V_2+W_3)} = 16|16$ for every $i > 1$; and hence $\underline{N}$ depends on one parameter: N_1.

The lowest-weight vectors in $\mathfrak{g}_1$ are

$$\begin{aligned} v_1 ={}& x_1x_2\partial_6 + x_1x_3\partial_7 + x_1x_5\partial_1 + x_2x_5\partial_2 + x_3x_5\partial_3 + x_4x_5\partial_4 \\ &+ x_2x_7\partial_4 + x_3x_6\partial_4 + x_5x_6\partial_6 + x_5x_7\partial_7, \\ v_2 ={}& x_2x_3\partial_3 + x_2x_4\partial_4 + x_2x_5\partial_5 + x_2x_6\partial_6 + x_3x_6\partial_7 + x_4x_5\partial_7 + x_5x_6\partial_1. \end{aligned}$$

The highest-weight vectors in $\mathfrak{g}_1$ are

$$w_1 = x_1^{(2)}\partial_5 + x_1x_6\partial_2 + x_1^{(2)}\partial_5 + x_1x_7\partial_3, \quad w_2 = x_1x_7\partial_2, \quad w_3 = x_1^{(2)}\partial_2.$$

18.2   Desuperization

For the unconstrained shearing vector, we have $\dim \mathfrak{g}_1 = 55$ with three lowest-weight vectors. The first two are as above, and the third one is

$$v_3 = x_2 x_3 \partial_4 + x_2 x_5 \partial_6 + x_3 x_5 \partial_7 + x_5^{(2)} \partial_1.$$

The unconstrained shearing vector is of the form $\underline{M} = (m, 1, 1, 1, n, s, t)$.

18.2.1   Partial prolongs

For the unconstrained shearing vector, we have

$$\dim V_1 = 14, \quad \dim V_2 = 21, \quad \dim V_3 = 31 \quad \text{and} \quad \dim(V_3 + W_3) = 40$$

with $[\mathfrak{g}_{-1}, V_i] = \mathfrak{g}_0$ for every i, and $V_1 \subset V_2 \subset V_3$, $W_2 \subset V_3$.

The unconstrained shearing vector for the prolong in the direction of V_3 depends on 1 parameter N_5, and $\dim \mathfrak{g}_i^{(V_3)} = 32$ for every $i > 1$.

The unconstrained shearing vector for the prolong in the direction of $V_3 + W_3$ depends on 2 parameters N_1, N_5, and

$$\dim \mathfrak{g}_2^{(V_3+W_3)} = 56, \quad \dim \mathfrak{g}_3^{(V_3+W_3)} = 72, \quad \dim \mathfrak{g}_4^{(V_3+W_3)} = 88.$$

19 $\mathfrak{kas}(8; \underline{M}) := \mathrm{F}(\mathfrak{kas}(4; \underline{N}|4))$

Let $\mathfrak{k}(1; \underline{N}|6)$ be considered as preserving the distribution given by the form $dt + \sum \xi_i d\eta_i$ with the contact bracket (17.1) and the grading of the generating functions given by, see Table (25.4):

$$\deg t = \deg \eta_i = 1, \quad \deg \xi_i = 0 \quad \text{for } i = 1, 2, 3, \quad \text{hence } \deg_{\mathrm{Lie}}(f) = \deg(f) - 1.$$

For the subalgebra $\mathfrak{g} = \mathfrak{kas}\big(4; \widetilde{\underline{N}}|4\big)$ of $\mathfrak{k}(1; \underline{N}|6; 3\xi)$, see (17.4), we have

$$\mathfrak{g}_0 \simeq \begin{cases} \mathfrak{sl}(1|3) \rtimes \mathcal{O}(0|3), & \text{where } \mathfrak{sl}(1|3) \subset \mathfrak{vect}(0|3), & \text{for } p \neq 2, \\ \mathfrak{d}(\mathfrak{svect}^{(1)}(0|3)) \rtimes \mathcal{O}(0|3), & \text{see (19.2)}, & \text{for } p = 2, \end{cases}$$

with the natural action of $\mathfrak{sl}(1|3)$ if $p \neq 2$, or $\mathfrak{d}(\mathfrak{svect}^{(1)}(0|3))$ if $p = 2$, on $\mathcal{O}(0|3)$.

Indeed, the element $(t + \Phi)f(\xi) \in \mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ as the operator of multiplication by $f(\xi)$. Additionally $\mathfrak{g}_0$ contains the following operators:

$$\eta_i \longleftrightarrow \partial_{\xi_i}, \quad \xi_i \eta_j \longleftrightarrow \xi_i \partial_{\xi_j}, \quad \xi_i \Phi \longleftrightarrow \xi_i(\xi_j \partial_{\xi_j} + \xi_k \partial_{\xi_k}). \tag{19.1}$$

For $p \neq 2$, the last 3 elements in (19.1) do not belong to $\mathfrak{svect}(0|3)$ and the elements (19.1) generate $\mathfrak{sl}(1|3)$.

For $p = 2$, the last 3 elements in (19.1) do belong to $\mathfrak{svect}(0|3)$, whereas the elements $\xi_i \eta_i$ do not; it is the sum of any two of them that belongs to $\mathfrak{svect}(0|3)$. So the elements (19.1) generate a subalgebra $\mathfrak{d}(\mathfrak{svect}^{(1)}(0|3))$ in the Lie algebra $\mathfrak{der}(\mathfrak{svect}^{(1)}(0|3))$ of all derivations of $\mathfrak{svect}^{(1)}(0|3)$.

In (19.2) we introduce 8 indeterminates x needed to express $\mathfrak{kas}(4; \underline{N}|4)$ and its desuperization in terms of vector fields as the prolong; $\partial_i := \partial_{x_i}$. For a basis of the nonpositive part (the $X_i^{\pm}$

are the Chevalley generators of what is $\mathfrak{sl}(1|3)$ for $p \neq 2$ and turns into $\mathfrak{svect}^{(1)}(0|3)$ for $p = 2$) we take:

$\mathfrak{g}_{-1}$	even: $1 \longleftrightarrow \partial_1,\ \xi_1\xi_2 \longleftrightarrow \partial_2,\ \xi_1\xi_3 \longleftrightarrow \partial_3,\ \xi_2\xi_3 \longleftrightarrow \partial_4$
	odd: $\xi_1 \longleftrightarrow \partial_5,\ \xi_2 \longleftrightarrow \partial_6,\ \xi_3 \longleftrightarrow \partial_7,\ \xi_1\xi_2\xi_3 \longleftrightarrow \partial_8$
	$t + \Phi \longleftrightarrow \sum x_i\partial_i,\ (t+\Phi)\xi_1 \longleftrightarrow x_1\partial_5 + x_4\partial_8 + x_6\partial_2 + x_7\partial_3,$
	$(t+\Phi)\xi_2 \longleftrightarrow x_1\partial_6 + x_3\partial_8 + x_5\partial_2 + x_7\partial_4,$
	$(t+\Phi)\xi_3 \longleftrightarrow x_1\partial_7 + x_2\partial_8 + x_5\partial_3 + x_6\partial_4,$
$\mathfrak{g}_0 \simeq \mathcal{O}(0\|3)$	$(t+\Phi)\xi_1\xi_2 \longleftrightarrow x_1\partial_2 + x_7\partial_8,\ (t+\Phi)\xi_1\xi_3 \longleftrightarrow x_1\partial_3 + x_6\partial_8,$
	$(t+\Phi)\xi_2\xi_3 \longleftrightarrow x_1\partial_4 + x_5\partial_8,\quad t\xi_1\xi_2\xi_3 \longleftrightarrow x_1\partial_8$
$\ltimes \mathfrak{d}(\mathfrak{svect}^{(1)}(0\|3))$	$\eta_1 \longleftrightarrow x_2\partial_6 + x_3\partial_7 + x_5\partial_1 + x_8\partial_4,\ \eta_2 \longleftrightarrow x_2\partial_5 + x_4\partial_7 + x_6\partial_1 + x_8\partial_3,$
	$X_1^- = \eta_3 \longleftrightarrow x_3\partial_5 + x_4\partial_6 + x_7\partial_1 + x_8\partial_2,$
	$X_1^+ = \xi_1\Phi \longleftrightarrow x_6\partial_2 + x_7\partial_3,\ \xi_2\Phi \longleftrightarrow x_5\partial_2 + x_7\partial_4,\ \xi_3\Phi \longleftrightarrow x_5\partial_3 + x_6\partial_4,$
	$X_2^- = \eta_1\xi_2 \longleftrightarrow x_3\partial_4 + x_5\partial_6,\ \eta_1\xi_3 \longleftrightarrow x_2\partial_4 + x_5\partial_7,$
	$X_2^+ = \eta_2\xi_1 \longleftrightarrow x_4\partial_3 + x_6\partial_5$
	$X_3^- = \eta_2\xi_3 \longleftrightarrow x_2\partial_3 + x_6\partial_7,\ \eta_3\xi_1 \longleftrightarrow x_4\partial_2 + x_7\partial_5,$
	$X_3^+ = \eta_3\xi_2 \longleftrightarrow x_3\partial_2 + x_7\partial_6$
	$\eta_1\xi_1 \longleftrightarrow x_2\partial_2 + x_3\partial_3 + x_5\partial_5 + x_8\partial_8,\ \eta_2\xi_2 \longleftrightarrow x_2\partial_2 + x_4\partial_4 + x_6\partial_6 + x_8\partial_8$
	$\eta_3\xi_3 \longleftrightarrow x_3\partial_3 + x_4\partial_4 + x_7\partial_7 + x_8\partial_8$

$$(19.2)$$

For the unconstrained shearing vector, $\text{sdim}\,\mathfrak{g}_1 = 16|16$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ splits into the 2 irreducible submodules: $\mathfrak{g}_1 = V_1 \oplus V_2$, where $\text{sdim}(V_1) = 12|12$, and $\text{sdim}(V_2) = 4|4$. There are the 2 highest-weight vectors in $\mathfrak{g}_1$:

$$h_1 = x_1 x_6 \partial_2 + x_1 x_7 \partial_3 + x_6 x_7 \partial_8, \quad h_2 = x_1^{(2)}\partial_8;$$

and the 2 lowest-weight vectors:

$$v_1 = x_2 x_3 \partial_3 + x_2 x_4 \partial_4 + x_2 x_5 \partial_5 + x_2 x_6 \partial_6 + x_3 x_6 \partial_7 + x_4 x_5 \partial_7 + x_5 x_6 \partial_1$$
$$+ x_5 x_8 \partial_3 + x_6 x_8 \partial_4,$$
$$v_2 = x_1^{(2)}\partial_1 + x_1 x_2 \partial_2 + x_1 x_7 \partial_7 + x_1 x_8 \partial_8 + x_2 x_7 \partial_8 + x_1^{(2)}\partial_1 + x_1 x_3 \partial_3 + x_1 x_6 \partial_6$$
$$+ x_1 x_8 \partial_8 + x_3 x_6 \partial_8 + x_1^{(2)}\partial_1 + x_1 x_4 \partial_4 + x_1 x_5 \partial_5 + x_1 x_8 \partial_8 + x_4 x_5 \partial_8 + x_5 x_6 \partial_2$$
$$+ x_5 x_7 \partial_3 + x_6 x_7 \partial_4.$$

19.1 Partial prolongs

We have $\text{sdim}([\mathfrak{g}_{-1}, \mathfrak{g}_1]) = 12|10$, as it should be (having in mind the outer derivation of $\mathfrak{kas}$); for its representative, we can take $x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4$.

In the direction of V_1, we have $\text{sdim}([\mathfrak{g}_{-1}, V_1]) = 12|9$ (apart from the outer derivative, $x_1\partial_8$ is absent); the $[\mathfrak{g}_{-1}, V_1]$-module $\mathfrak{g}_{-1}$ is irreducible, $\text{sdim}(\mathfrak{g}_2^{(V_1)}) = 4|7$, and $\mathfrak{g}_3^{(V_1)} = 0$. More precisely, $\text{sdim}([V_1, V_1]) = 3|3$ and the $[\mathfrak{g}_{-1}, V_1]$-modules V_1 and $[V_1, V_1]$ are irreducible. Thus, the superdimension of this simple prolong is $31|28$.

In the direction of V_2 we have $\text{sdim}([\mathfrak{g}_{-1}, V_2]) = 4|4$, the $[\mathfrak{g}_{-1}, V_1]$-module $\mathfrak{g}_{-1}$ is not irreducible, so no new simple partial prolongs exist in this direction.

Critical coordinate: only N_1.

19.2 Desuperization

The same as above, with dimension $a + b$ instead of superdimension $a|b$.

20   The Lie superalgebra $\mathfrak{kas}\big(5;\widetilde{\underline{N}}|5\big) \subset \mathfrak{k}(1;\underline{N}|6;1\xi)$

Whenever possible we do not indicate the shearing vector.

Let $\mathfrak{k}(1;\underline{N}|6)$ be the Lie superalgebra which preserves the distribution given by the form $dt+\sum \xi_i d\eta_i$. Then, $\mathfrak{k}(1;\underline{N}|6)$ is endowed with the contact bracket (17.1); set $\deg K_f = \deg(f)-2$, where the grading of the generating functions is given by

$$\deg t = \deg \eta_1 = 2, \quad \deg \xi_1 = 0, \quad \deg \eta_i = \deg \xi_i = 1 \quad \text{for } i = 2,3.$$

We identify $\mathfrak{g}_{-1}$ with $V(\Lambda)\otimes W$, where $\Lambda := \Lambda[\xi_1]$ and $V(\Lambda) := V \otimes \Lambda$; let $V = \mathrm{Span}(v_1, v_2)$, $W = \mathrm{Span}(w_1, w_2)$. For a basis of the nonpositive part of $\mathfrak{g}$, we take the elements listed in (20.2). The component

$$\mathfrak{g}_0 \cong \begin{cases} \mathfrak{d}\big(\big(\widetilde{\mathfrak{sl}}(W) \oplus (\mathfrak{gl}(V;\Lambda) \ltimes \mathfrak{vect}(0|1))/\mathbb{K}Z\big)\big), \\ \quad \text{where } \mathfrak{d} = \mathbb{K}D, \text{ see } (20.2) & \text{if } p = 2, \\ \widetilde{\mathfrak{sl}}(W) \oplus (\mathfrak{gl}(V;\Lambda) \ltimes \mathfrak{vect}(0|1)) & \text{if } p \neq 2 \end{cases} \tag{20.1}$$

of the subalgebra $\mathfrak{g} := \mathfrak{kas}\big(5;\widetilde{\underline{N}}|5\big) \subset \mathfrak{k}(1;\underline{N}|6;1\xi)$, see (17.4), is rather complicated for $p = 2$. To describe this component, we compare it with the complete prolong of the negative part, see Section 2.11. The 0th component of this prolong is equal to the 0th component of $\mathfrak{k}(1;\underline{N}|6;1\xi)$. Its 3 elements that do not belong to $\mathfrak{g}_0$ are easy to find from the description of $\mathfrak{kas}$ given in Section 17 (they are boxed):

$\mathfrak{g}_i$	the basis elements
$\mathfrak{g}_{-2} \simeq \Lambda$	$1 \longleftrightarrow \partial_2 \,\vert\, \xi_1 \longleftrightarrow \partial_1$
$\mathfrak{g}_{-1} \simeq \mathrm{id}_{\mathfrak{sl}(W)}$ $\otimes \, \mathrm{id}_{\mathfrak{gl}(V;\Lambda)}$	$\xi_1\xi_2 \longleftrightarrow \xi_1 v_1 \otimes w_1 \longleftrightarrow \partial_3, \ \xi_1\xi_3 \longleftrightarrow \xi_1 v_1 \otimes w_2 \longleftrightarrow \partial_4,$ $\xi_1\eta_2 \longleftrightarrow \xi_1 v_2 \otimes w_2 \longleftrightarrow \partial_5, \ \xi_1\eta_3 \longleftrightarrow \xi_1 v_2 \otimes w_1 \longleftrightarrow \partial_6 \,\vert$ $\xi_2 \longleftrightarrow v_1 \otimes w_1 \longleftrightarrow x_5\partial_1 + \partial_7, \ \xi_3 \longleftrightarrow v_1 \otimes w_2 \longleftrightarrow x_6\partial_1 + \partial_8,$ $\eta_2 \longleftrightarrow v_2 \otimes w_2 \longleftrightarrow x_3\partial_1 + x_7\partial_2 + \partial_9, \ \eta_3 \longleftrightarrow v_2 \otimes w_1 \longleftrightarrow x_4\partial_1 + x_8\partial_2 + \partial_{10}$
$\mathfrak{g}_0$	$\xi_1 E \otimes \mathbb{1} \longleftrightarrow \xi_1 \Phi \longleftrightarrow x_7\partial_3 + x_9\partial_5 + x_8\partial_4 + x_{10}\partial_6 + x_7x_9\partial_1 + x_8x_{10}\partial_1$ $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \xi_1 \end{smallmatrix}\right) \otimes \mathbb{1} \longleftrightarrow t\xi_1 \longleftrightarrow x_7\partial_3 + x_8\partial_4 + x_2\partial_1$ $\xi_1 X^- \otimes \mathbb{1} \longleftrightarrow \xi_1\eta_2\eta_3 \longleftrightarrow x_7\partial_6 + x_8\partial_5 + x_7x_8\partial_1$ $\xi_1 X^+ \otimes \mathbb{1} \longleftrightarrow \xi_1\xi_2\xi_3 \longleftrightarrow x_9\partial_4 + x_{10}\partial_3 + x_9x_{10}\partial_1$ $X^+ \longleftrightarrow \xi_2\xi_3 \longleftrightarrow x_5\partial_4 + x_6\partial_3 + x_9\partial_8 + x_{10}\partial_7 + x_9x_{10}\partial_2$ $\widetilde{X}^+ \longleftrightarrow \xi_3\eta_2 \longleftrightarrow x_3\partial_4 + x_6\partial_5 + x_7\partial_8 + x_{10}\partial_9$ $\boxed{X^-} \longleftrightarrow \eta_2\eta_3 \longleftrightarrow x_3\partial_6 + x_4\partial_5 + x_7\partial_{10} + x_8\partial_9 + x_7x_8\partial_2$ $\boxed{\widetilde{X}^-} \longleftrightarrow \xi_2\eta_3 \longleftrightarrow x_4\partial_3 + x_5\partial_6 + x_8\partial_7 + x_9\partial_{10}$ $D := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes \mathbb{1} + \mathbb{1} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \longleftrightarrow \xi_2\eta_2 \longleftrightarrow x_3\partial_3 + x_5\partial_5 + x_7\partial_7 + x_9\partial_9$ $E \longleftrightarrow \xi_2\eta_2 + \xi_3\eta_3 \longleftrightarrow x_3\partial_3 + x_4\partial_4 + x_5\partial_5 + x_6\partial_6 + x_7\partial_7$ $\quad + x_8\partial_8 + x_9\partial_9 + x_{10}\partial_{10}$ $\xi_1\partial_{\xi_1} \longleftrightarrow \xi_1\eta_1 \longleftrightarrow x_4\partial_4 + x_6\partial_6 + x_8\partial_8 + x_{10}\partial_{10}$ $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \mathbb{1} \longleftrightarrow t + \xi_1\eta_1 \longleftrightarrow x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + x_9\partial_9 + x_{10}\partial_{10}$ $\boxed{\partial_{\xi_1}} \longleftrightarrow \eta_1 \longleftrightarrow x_3x_5\partial_1 + x_4x_6\partial_1 + x_5x_7\partial_2 + x_6x_8\partial_2$ $\quad + x_1\partial_2 + x_3\partial_7 + x_4\partial_8 + x_5\partial_9 + x_6\partial_{10}$

(20.2)

The component $\mathfrak{g}_0$ contains two copies of $\mathfrak{sl}(2)$; to distinguish them, we endow one of them with a tilde: $\widetilde{\mathfrak{sl}}(2) = \mathfrak{sl}(W)$ generated by $\widetilde{X}^+$ and $\widetilde{X}^-$, the other copy being $\mathfrak{sl}(V)$ generated

by X^+ and X^-. These two copies of $\mathfrak{sl}(2)$ are "glued"; their glued sum has a common center spanned by E; i.e., their direct sum is factorized by a 1-dimensional subalgebra $\mathbb{K}Z$ in their 2-dimensional center, the explicit form of Z is inessential for us at the moment. Observe that $D, \xi_1 \partial_{\xi_1} \notin [\mathfrak{g}_{-1}, \mathfrak{g}_1]$; only their sum $D + \xi_1 \partial_{\xi_1} \longleftrightarrow \xi_1 \eta_1 + \xi_2 \eta_2$ belongs to the commutant.

In (20.2), we expressed the nonpositive part of $\mathfrak{g}$ by means of vector fields in 10 indeterminates x setting $\partial_i := \partial_{x_i}$.

The reader wishing to verify our computations will, of course, use the contact bracket and generating functions to compute inside $\mathfrak{g}_0$. The realization by vector fields is only needed to compute $\mathfrak{g}_i$ for $i > 0$ (with computer's aid to speed up the process).

The only noncritical coordinate of the shearing vector $\underline{N}$ is N_2; it corresponds to what used to be t.

For the unconstrained shearing vector, we have sdim $\mathfrak{g}_1 = 8|8$. The only lowest-weight vector (w.r.t. the boxed operators) of $\mathfrak{g}_1$ that generates $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module is

$$u_1 = x_1\partial_9 + x_3x_4\partial_4 + x_3x_6\partial_6 + x_3x_7\partial_7 + x_3x_9\partial_9 + x_4x_6\partial_5 + x_4x_7\partial_8$$
$$+ x_4x_{10}\partial_9 + x_6x_7\partial_{10} + x_6x_8\partial_9 + x_1x_3\partial_1 + x_1x_7\partial_2 + x_6x_7x_8\partial_2.$$

The other lowest-weight vector and the only highest-weight vector (together and separately) generate a submodule V which, together with $\mathfrak{g}_{-1}$, generate an 8-dimensional part of $\mathfrak{g}_0$. The quotient $\mathfrak{g}_1/V$ is an irreducible $\mathfrak{g}_0$-module.

20.1 Desuperization of $\mathfrak{las}(5; \underline{\widetilde{N}}|5)$

The only critical coordinates are N_1 and N_2. (For the unconstrained shearing vector, dim $\mathfrak{g}_1 = 16$, $\dim(\mathfrak{g}_2) = 20$, $\dim(\mathfrak{g}_3) = 24$, $\dim(\mathfrak{g}_4) = 28$.)

21 A description of $\widetilde{\mathfrak{sb}}(2^n - 1; \underline{\widetilde{N}})$ for $p = 2$

21.1 Recapitulation: $p = 0$, n even

Let $q = (q_1, \ldots, q_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. We consider the subsuperspace of functions $\mathbb{C}[q, \xi]$ of the form

$$\{(1 + \Xi)f(q, \xi) \,|\, \Delta(f) = 0 \quad \text{and} \quad \int_\xi f \, \mathrm{vol}_\xi = 0\}, \quad \text{where } \Xi := \xi_1 \cdots \xi_n,$$

with the Buttin bracket. In this section we use only this bracket and omit the index "B.b".

Let us compute the bracket in $(1 + \Xi)\mathfrak{sb}^{(1)}(n; n)$ realized by elements of $\mathfrak{sb}^{(1)}(n)$. We have

$$\{(1 + \Xi)f, (1 + \Xi)g\}$$
$$= \begin{cases} \{f, g\} = (1 + \Xi)\{f, g\} & \text{if } \deg_\xi(f), \deg_\xi(g) > 0; \\ \{(1 + \Xi)f, g\} = (1 + \Xi)\{f, g\} & \text{if } \deg_\xi(f) = 0, \deg_\xi(g) > 1; \\ \{(1 + \Xi)f, g\} = \\ (1 + \Xi)\{f, g\} + \sum \partial_{\xi_i}\Xi f \partial_{q_i}(g_i)\xi_i & \text{if } \deg_\xi(f) = 0, \deg_\xi(g) = 1, \\ \text{\scriptsize (since } \sum \partial_{\xi_i}\Xi f \partial_{q_i}(g_i)\underline{\underline{\xi_i}} = \Xi f \sum \partial_{q_i}(g_i) = 0) \\ (1 + \Xi)\{f, g\} & \text{if } g = \sum g_i(q)\xi_i \text{ and } \Delta(g) = 0; \\ \sum \partial_{\xi_i}\Xi(\partial_{q_i}(f)g - \partial_{q_i}(g)f) & \text{if } \deg_\xi(f) = \deg_\xi(g) = 0. \end{cases}$$

In the $\mathbb{Z}$-grading of $\mathfrak{g} = \widetilde{\mathfrak{sb}}(n; n)$ by degrees of the q shifted by -1, we have:

- $\mathfrak{g}_{-1}$ is spanned by monomials in ξ of degrees 1 through $n-1$, and by $1 + \Xi$;

- $\mathfrak{g}_0$ is spanned by functions of the form $g = (1 + \Xi) \sum g_i(\xi) q_i$, where $\sum \partial_{\xi_i} g_i = 0$.

The $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ is as follows. If $\deg_\xi(g_i) > 0$, then we can ignore Ξ in the factor $1 + \Xi$ since Ξ annihilates g_i, and hence ad_g acts on $\mathfrak{g}_{-1}$ as the vector field $\sum g_i \partial_{\xi_i}$ acts on the space of functions in ξ.

If $g = (1 + \Xi) q_i$, then the ad_g-acts on $\mathfrak{g}_{-1}$ precisely as an element of $\widetilde{\mathfrak{svect}}(0|n)$ acts on the space Vol_ξ:

$$\{g, \xi_j\} = (1 + \Xi)\delta_{ij}, \quad \{g, 1 + \Xi\} = \partial_{\xi_i}(\Xi), \quad \mathrm{ad}_g(f) = \partial_{\xi_i}(f) \quad \text{for monomials } f = f(\xi).$$

Since $(1 - \Xi)$ vol is the invariant subspace in Vol_ξ, it follows that, in the quotient space, we can take for a basis elements of the form $f(\xi)\, \mathrm{vol}_\xi$, where monomials f differ from 1 and Ξ, and either 1 or Ξ. For reasons unknown, *SuperLie* selected Ξ, not 1.

21.1.1   Recapitulation: $p = 0$, n odd

Everything is as above but with $\tau\Xi$, where τ an odd parameter, instead of Ξ.

21.2 $\mathbf{F}\big(\widetilde{\mathfrak{sb}}\big(2^{n-1}; \underline{N}|2^{n-1} - 1\big)\big)$ for n odd, $p = 2$

For $p = 2$, it is possible to desuperize deforms with odd parameters and consider them in the category of superspaces, see [12]. We assume that $p(\mathrm{vol}_\xi) \equiv n \mod 2$.

21.2.1   Example: $n = 3$

For a basis in $\mathfrak{g}_{-1}$, where $\partial_i := \partial_{x_i}$, we take:

$$\partial_1 = \xi_1 \, \mathrm{vol}_\xi, \quad \partial_2 = \xi_2 \, \mathrm{vol}_\xi, \quad \partial_3 = \xi_3 \, \mathrm{vol}_\xi, \quad \partial_4 = \xi_1\xi_2 \, \mathrm{vol}_\xi, \quad \partial_5 = \xi_1\xi_3 \, \mathrm{vol}_\xi,$$
$$\partial_6 = \xi_2\xi_3 \, \mathrm{vol}_\xi, \quad \partial_7 = \tau\xi_1\xi_2\xi_3 \, \mathrm{vol}_\xi \,.$$

For a basis of $\mathfrak{g}_0$, where $\delta_i := \partial_{\xi_i}$ we take the following elements, where the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ is given by realizations on the right of the $\longleftrightarrow$:

$$(1 + \tau\xi_1\xi_2\xi_3)\delta_1 \longleftrightarrow x_1\partial_7 + x_4\partial_2 + x_5\partial_3 + x_7\partial_6,$$
$$(1 + \tau\xi_1\xi_2\xi_3)\delta_2 \longleftrightarrow x_2\partial_7 + x_4\partial_1 + x_6\partial_3 + x_7\partial_5,$$
$$(1 + \tau\xi_1\xi_2\xi_3)\delta_3 \longleftrightarrow x_3\partial_7 + x_5\partial_1 + x_6\partial_2 + x_7\partial_4,$$
$$\xi_1\delta_2 \longleftrightarrow x_2\partial_1 + x_6\partial_5,$$
$$\xi_1\delta_3 \longleftrightarrow x_3\partial_1 + x_6\partial_4,$$
$$\xi_2\delta_1 \longleftrightarrow x_1\partial_2 + x_5\partial_6,$$
$$\xi_1\delta_1 + \xi_2\delta_2 \longleftrightarrow x_1\partial_1 + x_2\partial_2 + x_5\partial_5 + x_6\partial_6,$$
$$\xi_1\delta_1 + \xi_3\delta_3 \longleftrightarrow x_1\partial_1 + x_3\partial_3 + x_4\partial_4 + x_6\partial_6,$$

$$\xi_2\delta_3 \longleftrightarrow x_3\partial_2 + x_5\partial_4,$$
$$\xi_3\delta_1 \longleftrightarrow x_1\partial_3 + x_4\partial_6,$$
$$\xi_3\delta_2 \longleftrightarrow x_2\partial_3 + x_4\partial_5,$$
$$\xi_1\xi_2\delta_3 \longleftrightarrow x_3\partial_4,$$
$$\xi_1\xi_3\delta_2 \longleftrightarrow x_2\partial_5,$$
$$\xi_1\xi_2\delta_2 + \xi_1\xi_3\delta_3 \longleftrightarrow x_2\partial_4 + x_3\partial_5,$$
$$\xi_2\xi_3\delta_1 \longleftrightarrow x_1\partial_6,$$
$$\xi_1\xi_3\delta_1 + \xi_2\xi_3\delta_2 \longleftrightarrow x_1\partial_5 + x_2\partial_6,$$
$$\xi_1\xi_2\delta_1 + \xi_2\xi_3\delta_3 \longleftrightarrow x_1\partial_4 + x_3\partial_6.$$

The weights are considered with respect to $\mathfrak{sl}(3) \subset \mathbf{F}\big(\widetilde{\mathfrak{svect}}(0|3)\big)$, i.e.,

$$w(\xi_1) = (1, 0), \quad w(\xi_2) = (-1, 1), \quad w(\xi_3) = (0, -1).$$

The raising elements are those for which either $w_1 + w_2 > 0$ or $w_1 = -w_2 > 0$; the lowering elements are those for which either $w_1 + w_2 < 0$ or $w_1 = -w_2 < 0$. (To find lowering and raising

operators, we could have considered a $\mathbb{Z}$-grading of $\widetilde{\mathfrak{svect}}(0|n)$ by setting $\deg \xi_n = -n+1$ and $\deg \xi_1 = \cdots = \deg \xi_{n-1} = 1$ with ensuing natural division into "positive" and "negative" parts.)

The highest-weight vectors of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ are

$$
\begin{aligned}
w_1 &= x_2 x_3 \partial_7 + x_2 x_5 \partial_1 + x_2 x_6 \partial_2 + x_2 x_7 \partial_4 + x_3 x_4 \partial_1 + x_3 x_6 \partial_3 \\
&\quad + x_3 x_7 \partial_5 + x_4 x_6 \partial_4 + x_5 x_6 \partial_5, \\
w_2 &= x_2 x_3 \partial_1 + x_2 x_6 \partial_4 + x_3 x_6 \partial_5, \\
w_3 &= x_3^{(2)} \partial_1 + x_3 x_6 \partial_4.
\end{aligned}
$$

The lowest-weight vectors of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ are

$$
\begin{aligned}
v_1 &= x_1^{(2)} \partial_1 + x_1 x_2 \partial_2 + x_1 x_5 \partial_5 + x_1 x_6 \partial_6 + x_2 x_5 \partial_6 + x_1^{(2)} \partial_1 + x_1 x_3 \partial_3 \\
&\quad + x_1 x_4 \partial_4 + x_1 x_6 \partial_6 + x_3 x_4 \partial_6, \\
v_2 &= x_1^{(2)} \partial_1 + x_1 x_2 \partial_2 + x_1 x_5 \partial_5 + x_1 x_6 \partial_6 + x_2 x_5 \partial_6 + x_1^{(2)} \partial_1 + x_1 x_3 \partial_3 \\
&\quad + x_1 x_4 \partial_4 + x_1 x_6 \partial_6 + x_3 x_4 \partial_6, \\
v_3 &= x_1^{(2)} \partial_6,
\end{aligned}
$$

Partial prolongs: The elements of $\mathfrak{g}_0$ absent in $\widetilde{\mathfrak{g}}_0 := [\mathfrak{g}_1, \mathfrak{g}_{-1}]$ are $\xi_1 \xi_2 \delta_3$, $\xi_1 \xi_3 \delta_2$, $\xi_2 \xi_3 \delta_1$. The $\widetilde{\mathfrak{g}}_0$-module $\mathfrak{g}_{-1}$ is irreducible.

Let V_i and W_i denote the $\widetilde{\mathfrak{g}}_0$-modules generated by v_i and w_i, respectively. We have

$$
\begin{aligned}
&\dim \mathfrak{g}_1 = 31, \quad \dim \mathfrak{g}_2 = 49, \quad \dim \mathfrak{g}_3 = 71, \\
&\dim V_1 = \dim W_1 = 7, \quad \dim V_2 = \dim W_2 = 8, \quad \dim V_3 = \dim W_3 = 16, \\
&V_1 = W_1, \quad V_1 \subset V_2 \subset V_3, \quad W_1 \subset W_2 \subset W_3, \\
&\dim(V_2 + W_2) = 9, \quad \dim(V_2 + W_3) = \dim(V_3 + W_2) = 17, \quad \dim(V_3 + W_3) = 24.
\end{aligned}
$$

The brackets with $\mathfrak{g}_{-1}$:

$$
\begin{aligned}
&\dim([\mathfrak{g}_{-1}, V_1]) = \dim[\mathfrak{g}_{-1}, V_2 + W_2] = 14, \\
&\dim([\mathfrak{g}_{-1}, V_3]) = 15 \quad \text{(absent are } \xi_1 \xi_2 \delta_3, \ \xi_1 \xi_3 \delta_2), \\
&\dim([\mathfrak{g}_{-1}, W_3]) = 15 \quad \text{(absent are } \xi_1 \xi_3 \delta_2, \ \xi_2 \xi_3 \delta_1), \\
&\dim([\mathfrak{g}_{-1}, V_3 + W_3]) = 16 \quad \text{(absent is } \xi_1 \xi_3 \delta_2).
\end{aligned}
$$

Therefore (recall the convention (6.2))

Partial prolongs in the direction of	dimensions
V_1 or V_2 or $V_2 + W_2$	$\dim \mathfrak{g}_2 = 1, \ \mathfrak{g}_3 = 0$
V_3 or $V_3 + W_2$	$\dim \mathfrak{g}_2 = \dim \mathfrak{g}_3 = 16$
$V_3 + W_3$	$\dim \mathfrak{g}_2 = 32, \quad \dim \mathfrak{g}_3 = 40$
$[\mathfrak{g}_{-1}, V_1]$ or $[\mathfrak{g}_{-1}, V_2 + W_2]$	$\dim \mathfrak{g}_1 = 10$ absent are v_3 and w_3, $\dim \mathfrak{g}_2 = 1, \ \mathfrak{g}_3 = 0$.

(21.1)

Critical coordinates of $\widetilde{\mathfrak{sb}}(7; \underline{\widetilde{N}})$ are N_4, N_5, N_6, and N_7, as follows from (21.1).

21.3 $\mathrm{F}(\widetilde{\mathfrak{sb}}(2^{n-1} - 1; \underline{N}|2^{n-1}))$ for n even, $p = 2$

For the unconstrained shearing vector $\underline{N}^u$, the dimensions of homogeneous components of $\mathfrak{g} = \widetilde{\mathfrak{sb}}(2^n - 1; \underline{N}^u)$ are the same as those of $\mathfrak{sb}^{(1)}(n)$ in the nonstandard grading $\mathfrak{sb}^{(1)}(n; n)$ for $p = 0$.

The main idea: $\mathfrak{sb}^{(1)}(n) = \operatorname{Im}\Delta|_{\mathfrak{b}(n)}$, where $\Delta = \sum \frac{\partial^2}{\partial q_i \partial \xi_i}$. The dimensions of homogeneous components for n even are:

i	sdim $\mathfrak{g}_i$	sdim $\mathfrak{sb}^{(1)}(n; n)_i$
-1	$2^{n-1}\vert 2^{n-1}$	$2^{n-1}\vert 2^{n-1} - 1$
0	$n\left(2^{n-1}\vert 2^{n-1}\right)$	$(n-1)\left(2^{n-1}\vert 2^{n-1}\right) + 1\vert 0$
1	$\frac{1}{2}n(n+1)\left(2^{n-1}\vert 2^{n-1}\right)$	$\frac{1}{2}\left(n^2 - n + 2\right)\left(2^{n-1}\vert 2^{n-1}\right) - 0\vert 1$

Let the weights of ξ_i be $w(\xi_i) = (0,\dots,0,1,0,\dots,0)$ with a 1 on the ith place for $i < n$ and $w(\xi_n) = (-1,\dots,-1)$.

21.3.1   Example: $n = 4$

For a basis (even | odd) of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1} \simeq \frac{\Pi(\text{Vol}(0|n))}{\mathbb{K}(1+\xi_1\cdots\xi_4)\,\text{vol}_\xi}$, where $\mathfrak{g}_0 \simeq \widetilde{\mathfrak{svect}}(0|4))$, we take:

$$\partial_1 := \xi_1\,\text{vol}_\xi, \qquad \partial_2 := \xi_2\,\text{vol}_\xi, \qquad \Big| \qquad \partial_9 := \xi_1\xi_2\,\text{vol}_\xi, \qquad \partial_{10} := \xi_1\xi_3\,\text{vol}_\xi,$$
$$\partial_3 = \xi_3\,\text{vol}_\xi, \qquad \partial_4 = \xi_4\,\text{vol}_\xi, \qquad \Big| \qquad \partial_{11} := \xi_1\xi_4\,\text{vol}_\xi, \qquad \partial_{12} := \xi_2\xi_3\,\text{vol}_\xi,$$
$$\partial_5 = \xi_1\xi_2\xi_3\,\text{vol}_\xi, \quad \partial_6 = \xi_1\xi_2\xi_4\,\text{vol}_\xi, \Big| \quad \partial_{13} := \xi_2\xi_4\,\text{vol}_\xi, \qquad \partial_{14} := \xi_3\xi_4\,\text{vol}_\xi,$$
$$\partial_7 = \xi_1\xi_3\xi_4\,\text{vol}_\xi, \quad \partial_8 = \xi_2\xi_3\xi_4\,\text{vol}_\xi, \Big| \quad \partial_{15} := \xi_1\xi_2\xi_3\xi_4\,\text{vol}_\xi\,.$$

Critical coordinates of $\widetilde{\mathfrak{sb}}(15;\widetilde{\underline{N}}) = \boldsymbol{F}\big(\widetilde{\mathfrak{sb}}(8;\underline{N}|7)\big)$ are the same as those of $\widetilde{\mathfrak{sb}}(8;\underline{N}|7)$: $N_5 = N_6 = N_7 = N_8 = 1$, and also all those corresponding to the formerly odd indeterminates.

21.3.2   Partial prolongs

We have sdim $\mathfrak{g}_0 = 25|24$, and $\mathfrak{g}_0$ contains a simple ideal of sdim $= 21|24$, the quotient is commutative; $\mathfrak{g}_{-1}$ is irreducible over this ideal. We have sdim $\mathfrak{g}_1 = 56|55$, there are 3 highest-weight vectors and 2 lowest-weight vectors in $\mathfrak{g}_1$;

$$V_1 = W_1, \quad V_1 \subset V_2, \quad W_1 \subset W_2 \subset W_3,$$
$$\text{sdim}\,V_1 = 24|21, \quad \text{sdim}\,V_2 = \text{sdim}\,W_3 = 32|31, \quad \text{sdim}\,W_2 = 24|22,$$
$$\text{sdim}(V_2 + W_2) = 32|32, \quad \text{sdim}(V_2 + W_3) = 40|40, \quad \text{sdim}\,\mathfrak{g}_2 = 105|104.$$

The highest-weight vector of W_3 is $w_3 = x_4^{(2)}\partial_5$. This answer seems strange: the algebra is symmetric with respect to the permutation of the ξ_i while the list of highest-weight vectors is not. Performing all possible permutations we obtain similar vectors $x_1^{(2)}\partial_8$, $x_2^{(2)}\partial_7$, $x_3^{(2)}\partial_6$ (which are not highest/lowest with respect to the division into positive/negative weights we have selected first), but generate similar submodules Y_1, Y_2, Y_3 (and $Y_4 = W_3$).

We have $Y_1 + Y_2 + Y_3 + Y_4 = \mathfrak{g}_1$ and $\text{sdim}(Y_1 + Y_2 + Y_3) = 48|48$.

Other highest-weight vectors:

$$w_1 = x_3x_4\partial_{10} + x_3x_{13}\partial_5 + x_4^{(2)}\partial_{11} + x_4x_{12}\partial_5 + x_4x_{13}\partial_6$$
$$\qquad + x_2x_4\partial_9 + x_2x_{14}\partial_5 + x_4^{(2)}\partial_{11} + x_4x_{12}\partial_5 + x_4x_{14}\partial_7,$$
$$w_2 = x_3x_4\partial_9 + x_3x_{14}\partial_5 + x_4x_{14}\partial_6. \tag{21.2}$$

The lowest-weight vectors:

$$v_1 = x_1x_2\partial_4 + x_1x_5\partial_7 + x_1x_9\partial_{11} + x_1x_{12}\partial_{14} + x_2x_5\partial_8$$
$$\qquad + x_2x_9\partial_{13} + x_2x_{10}\partial_{14} + x_9x_{10}\partial_7 + x_9x_{12}\partial_8,$$
$$v_2 = x_1^{(2)}\partial_4 + x_1x_5\partial_8 + x_1x_9\partial_{13} + x_1x_{10}\partial_{14} + x_9x_{10}\partial_8.$$

We have

$$\begin{aligned}
&\mathrm{sdim}([\mathfrak{g}_{-1}, V_1]) = 21|24, \quad \mathrm{sdim}([\mathfrak{g}_{-1}, Y_i]) = 22|24, \\
&\mathrm{sdim}([\mathfrak{g}_{-1}, Y_i + Y_j]) = 23|24 \quad \text{for } i \neq j, \\
&\mathrm{sdim}([\mathfrak{g}_{-1}, Y_i + Y_j + Y_k]) = 24|24 \quad \text{for } i \neq j \neq k \neq i.
\end{aligned} \tag{21.3}$$

Partial prolongs of $\mathfrak{g}_0$ and the following parts of $\mathfrak{g}_1$:

- from $V_1 = W_1$ and W_2: $\mathrm{sdim}\,\mathfrak{g}_2 = 11|8$, $\mathrm{sdim}\,\mathfrak{g}_3 = 0|1$, no parameters;
- from V_2: $\mathrm{sdim}\,\mathfrak{g}_2 = 33|32$, 1 parameter: N_1 (same for W_3, parameter N_4);
- from $V_2 + W_3$: $\mathrm{sdim}\,\mathfrak{g}_2 = 56|56$, 2 parameters: N_1 and N_4, similar for $Y_i + Y_j$;
- from $Y_1 + Y_2 + Y_3$: $\mathrm{sdim}\,\mathfrak{g}_2 = 80|80$, 3 parameters: N_1, N_2 and N_3.

Partial prolongs of the following parts of $\mathfrak{g}_0$, see equation (21.3):

- from $(21|24)$: $\mathrm{sdim}\,\mathfrak{g}_1 = 24|27$, $\mathrm{sdim}\,\mathfrak{g}_2 = 11|8$, no parameters;
- from $(22|24)$: $\mathrm{sdim}\,\mathfrak{g}_1 = 32|34$, $\mathrm{sdim}\,\mathfrak{g}_2 = 33|32$, 1 parameter;
- from $(23|24)$: $\mathrm{sdim}\,\mathfrak{g}_1 = 40|41$, $\mathrm{sdim}\,\mathfrak{g}_2 = 56|56$, 2 parameters;
- from $(24|24)$: $\mathrm{sdim}\,\mathfrak{g}_1 = 48|48$, $\mathrm{sdim}\,\mathfrak{g}_2 = 80|80$, 3 parameters.

22 $\mathfrak{vas}(4; \underline{N}|4)$

In this section, we can omit $\underline{N}$ when the arguments do not depend on it.

22.1 For $p \neq 2$

For $\mathfrak{g} = \mathfrak{vas}(4|4)$ described in Table 25.4 as the Cartan prolong of the pair $(\mathrm{id}_{\mathfrak{as}}, \mathfrak{as})$, we have another description: $\mathfrak{g}_{\bar{0}} = \mathfrak{vect}(4|0)$ and $\mathfrak{g}_{\bar{1}} = \Omega^1(4|0) \otimes_{\Omega^0(4|0)} \mathrm{Vol}^{-1/2}(4|0)$ with the natural $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{g}_{\bar{1}}$, and the bracket of odd elements given by

$$\left[\frac{\omega_1}{\sqrt{\mathrm{vol}}}, \frac{\omega_2}{\sqrt{\mathrm{vol}}} \right] = \frac{d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2}{\mathrm{vol}},$$

where we identify

$$\frac{dx_i \wedge dx_j \wedge dx_k}{\mathrm{vol}} = \mathrm{sign}(ijkl)\partial_{x_l} \quad \text{for any permutation } (ijkl) \text{ of } (1234). \tag{22.1}$$

22.2 For $p = 2$

The first impression is that the characteristic-2 version of the Lie superalgebra $\mathfrak{vas}$ does not exist: the cocycle that determines the central extension $\mathfrak{as}$ of $\mathfrak{spe}(4)$ is trivial, see [6]. The following problem is most natural.

Problem 22.1 (on analogs of $\mathfrak{as}$ for $p = 2$). *For $p = 2$, there are 8 analogs of $\mathfrak{pe}(n)$ and 8 analogs of $\mathfrak{spe}(n)$, and lots of their nontrivial central extensions, see [6]. Is there a nontrivial central extension $\mathfrak{e}$ of one of these Lie (super)algebras, and an irreducible $\mathfrak{e}$-module M such that $(M, \mathfrak{e})_1 \neq 0$?*

The above-mentioned "first impression" was, however, too hasty. Define *a character-2 analog of* $\mathfrak{vas}$ in the form $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathfrak{svect}(4|0) \rtimes \mathcal{F}(4|0)$ and $\mathfrak{g}_{\bar{1}} = \Omega^1(4|0)$ with the natural $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{g}_{\bar{1}}$. Define the bracket of odd elements by the formula

$$[\omega_1, \omega_2] = \frac{d(\omega_1 \wedge \omega_2)}{\text{vol}} + \text{div}\left(\frac{d\omega_1 \wedge \omega_2}{\text{vol}}\right)$$

subject to identification (22.1). Define the square of every $\omega = \sum f_i dx_i \in \Omega^1(4|0)$ as follows, where (i, j, k) is a permutation of indices $(1, 2, 3)$:

$$\omega^2 := \frac{d\omega \wedge \omega}{\text{vol}} + \sum_{(i,j,k) \in S_3} \frac{\partial f_i}{\partial x_j}\left(\frac{\partial f_k}{\partial x_4} + \frac{\partial f_4}{\partial x_k}\right).$$

Let $\mathfrak{svect}(4|0) = \mathfrak{svect}(y)$, where $y = (y_1, y_2, y_3, y_4)$. Consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ of depth 1, by setting

$$\deg y_i = 1, \quad \deg(dy_i) = -1.$$

We get an embedding $\mathfrak{g} \longrightarrow \mathfrak{vect}(4|4)$. Let us describe the non-positive components of the embedded algebra. Let the coordinates of the ambient be x and ξ, and let us identify the basis elements of $\mathfrak{g}_{-1}$ with the following vector fields in $\mathfrak{vect}(4|4) = \mathfrak{vect}(x|\xi)$

$$\partial_{y_i} \longleftrightarrow \partial_{x_i}, \quad dy_i \longleftrightarrow \partial_{\xi_i}.$$

Then, $\mathfrak{g}_{-1} = \text{Span}\{\partial_{y_i}, dy_i\}$ for $i = 1, \ldots, 4$, and $(\mathfrak{g}_0)_{\bar{0}}$ consists of the pairs (D, c), where $D = \sum_{i,j} a_{ij} y_i \partial_{y_j}$ is any vector field such that $\text{div} D = 0$, and $c \in \mathbb{K}$, whereas $(\mathfrak{g}_0)_{\bar{1}}$ consists of 1-forms $y_i dy_j$:

Element of $\mathfrak{g}_0$	its non-zero action	the corresponding vector field
$y_i \partial_{y_j}, i \neq j$	$\partial_{y_i} \mapsto \partial_{y_j}, \; dy_j \mapsto dy_i$	$x_i \partial_{x_j} + \xi_j \partial_{\xi_i}, i \neq j$
$y_i \partial_{y_i} + y_j \partial_{y_j},$ $i < j$	$\partial_{y_i} \mapsto \partial_{y_i}, \; \partial_{y_j} \mapsto \partial_{y_j},$ $dy_i \mapsto dy_i, \; dy_j \mapsto dy_j$	$x_i \partial_{x_i} + x_j \partial_{x_j} + \xi_i \partial_{\xi_i} + \xi_j \partial_{\xi_j},$ $i < j$
1	$\begin{cases} \text{id} & \text{on } (\mathfrak{g}_{-1})_{\bar{1}} \\ 0 & \text{on } (\mathfrak{g}_{-1})_{\bar{0}} \end{cases}$	$\sum \xi_i \partial_{\xi_i}$
$y_i dy_i$	$\partial_{y_i} \mapsto dy_i$	$x_i \partial_{\xi_i}$
$y_i dy_j,$ $i \neq j$	$\partial_{y_i} \mapsto dy_j, \; dy_k \mapsto \partial_{y_l}, \; dy_l \mapsto \partial_{y_k},$ $(i, j, k, l) \in S_4$	$x_i \partial_{\xi_j} + \xi_k \partial_{x_l} + \xi_l \partial_{x_k}$

(22.2)

This $\mathfrak{g}_0$ is a characteristic-2 analog of $\mathfrak{as}$. In the basis $\partial_{x_1}, \ldots, \partial_{x_4}, \partial_{\xi_1}, \ldots, \partial_{\xi_4}$ the $\mathfrak{g}_0$-action in $\mathfrak{g}_{-1}$ is given by the following (super)matrix whose correspondence to vector fields we give explicitly only for $(\mathfrak{g}_0)_{\bar{0}}$ since the correspondence with $(\mathfrak{g}_0)_{\bar{1}}$ is too cumbrous to describe; moreover, is not worth the trouble thanks to the explicit table (22.2):

$$\begin{pmatrix} a & c \\ b + \tilde{c} & a^{\text{t}} \end{pmatrix} + d \, \text{diag}(0_4, 1_4), \quad \text{where } b^{\text{t}} = b, \; c \in ZD, \; a \in \mathfrak{sl}(4),$$

$$\tilde{c}_{ij} = E_{kl} \quad \text{for } k < l \text{ and } d \in \mathbb{K}, \quad \text{corresponding to } \sum a_{ij} y_i \partial_{y_j} + d.$$

Claim 22.2 (description of the simple part of $\mathfrak{vas}$).

1) *The Lie superalgebra* $\mathfrak{vas}^{(1)}(4; \underline{N}|4)$ *is simple; its even part is* $\mathfrak{svect}^{(1)}(4; \underline{N}|0) \rtimes \text{Vol}_0(4; \underline{N}|0)$, *see Section* 24.1.2.

2) *The critical coordinates of the shearing vector for the simple Lie algebra* $\mathfrak{vas}^{(1)}(8; \widetilde{\underline{N}})$ – *the desuperization of* $\mathfrak{vas}^{(1)}(4; \underline{N}|4)$ – *are the ones that correspond to formerly odd indeterminates.*

22.2.1 Partial prolongs

In order to investigate possible partial prolongs, we have to consider the $\mathfrak{g}_0$-submodules V_i of $\mathfrak{g}_1$ such that $[\mathfrak{vas}_{-1}, V_i] = \mathfrak{g}_0$. Since it is not clear what is a lowest/highest-weight vector with respect to $\mathfrak{g}_0$, we consider the lowest-weight vectors with respect to $(\mathfrak{g}_0)_{\bar{0}}$, and build the $\mathfrak{g}_0$-submodules from them.

Claim 22.3 (lowest-weight vectors). *There are 7 lowest-weight vectors LWVs:*

LWV	its image in (x, ξ)-model	its image in y-model
v_1	$x_2 x_3 \partial_{\xi_4} + x_2 x_4 \partial_{\xi_3} + x_3 x_4 \partial_{\xi_2}$	$y_2 y_3 dy_4 + y_2 y_4 dy_3 + y_3 y_4 dy_2$
v_2	$x_3 x_4 \partial_{x_1} + x_3 \xi_1 \partial_{\xi_4} + x_4 \xi_1 \partial_{\xi_3}$	$y_3 y_4 \partial_{y_1}$
v_3	$x_3^{(2)} \partial_{\xi_4} + x_3 x_4 \partial_{\xi_3}$	$y_3^{(2)} dy_4 + y_3 y_4 dy_3$
v_4	$x_4^{(2)} \partial_{\xi_4}$	$y_4^{(2)} dy_4$
v_5	$x_4^{(2)} \partial_{x_1} + x_4 \xi_1 \partial_{\xi_4}$	$y_4^{(2)} \partial_{y_1}$
v_6	$x_3 x_4 \partial_{\xi_4} + x_4 \xi_1 \partial_{x_2} + x_4 \xi_2 \partial_{x_1} + \xi_1 \xi_2 \partial_{\xi_4}$	$y_3 y_4 dy_4$
v_7	$x_4 \sum_{i=1}^4 \xi_i \partial_{\xi_i} + \xi_1 \xi_2 \partial_{x_3} + \xi_1 \xi_3 \partial_{x_2} + \xi_2 \xi_3 \partial_{x_1}$	y_4

$$(22.3)$$

Claim 22.4 (no partial prolongs). *Let $\mathfrak{g} := \mathfrak{vas}(4; \underline{N}|4)$.*

1) *Let V_i be the $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$ generated by v_i, see (22.3). Then, $[V_i, \mathfrak{g}_{-1}] = \mathfrak{g}_0$ for all i. For $\underline{N}$ unconstrained, we have $\mathrm{sdim}(\mathfrak{g}_1) = 40|40$, and*

$$V_7 = V_6 = V_2 = V_1 = V_3 \cap V_4, \quad \text{this } \mathfrak{g}_0\text{-module is irreducible}, \quad V_4 = V_5,$$
$$\mathrm{sdim}(V_1) = 24|24, \quad \mathrm{sdim}(V_3) = \mathrm{sdim}(V_4) = 28|28.$$

2) *Let $\mathfrak{g}^{V_i} := (\mathfrak{g}_-, \mathfrak{g}_0, V_i)_*$ be the prolong in the direction of V_i. Then, $\mathfrak{g}^{V_1} = \mathfrak{vas}(4; \mathbb{1}|4)$. We have $\mathrm{sdim}\, \mathfrak{vas}^{(1)}(4; \mathbb{1}|4) = 60|64$.*

3) *In the quotient $\mathfrak{g}_1/V_1$, to each $i \in \{1, 2, 3, 4\}$ there corresponds a $4|4$-dimensional submodule M_i spanned by the images of $y_i^{(2)} \partial_{y_j}$ and $y_i^{(2)} dy_j$ for $j = 1, 2, 3, 4$. Each M_i is irreducible, and the images of M_i and M_j in $\mathfrak{g}_1/V_1$ do not intersect for $i \neq j$. Thus, $\mathfrak{g}_1$ contains a submodule V_1 corresponding to $\underline{N} = \mathbb{1}$, and up to four modules M_i glued to V_1 if $\underline{N} \neq \mathbb{1}$. The partial prolongation in the direction of $(\oplus_{i \in I \subset \{1,2,3,4\}} M_i) \rtimes V_1$ is $\mathfrak{vas}(4; \underline{N}|4)$, where*

$$N_i = \begin{cases} \infty & \text{if } i \in I, \\ 1 & \text{if } i \notin I. \end{cases}$$

Idea of the proof. Since there is no complete reducibility, to prove item 3) we have to consider also highest-weight vectors (HWV) with respect to $(\mathfrak{g}_0)_{\bar{0}}$. Then, we are able to find the two quotients modules M_i invisible in table (22.3) since their LWVs go to V_1 under $(\mathfrak{g}_0)_{\bar{0}}$. We have already encountered similar phenomenon in previous sections considering LWVs and HWVs with respect to the whole $\mathfrak{g}_0$ for respective $\mathfrak{g}$. We skip the table of HWVs analogous to (22.3). $\blacksquare$

23 Cartan prolongs of the Shen algebra; Melikyan algebras for $p = 2$

23.1 Brown's version of the Melikyan algebra in characteristic 2

Brown [16] described characteristic-2 analog of the Melikyan algebra as follows. As spaces, and $\mathbb{Z}/3$-graded Lie algebras, let

$$L(\underline{N}) := \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}} \simeq \mathfrak{vect}(2; \underline{N}) \oplus \mathrm{Vol}(2; \underline{N}) \oplus \mathcal{O}(2; \underline{N}).$$

The $\mathfrak{g}_{\bar{0}}$-action on the $\mathfrak{g}_{\bar{i}}$ is natural (adjoint, on volume forms, and functions, respectively); $\mathcal{O}(2; \underline{N}) = \mathbb{K}[u_1, u_2; \underline{N}]$ is the space of functions; $\mathrm{Vol}(2; \underline{N})$ is the space of volume forms with volume element $\mathrm{vol} := \mathrm{vol}(u)$, where $u = (u_1, u_2)$. Let the multiplication in $L(\underline{N})$ be given, for any $f, g \in \mathcal{O}(2; \underline{N})$, by the following formulas:

$$[f \, \mathrm{vol}, g \, \mathrm{vol}] = 0, \quad [f \, \mathrm{vol}, g] = f H_g, \quad [f, g] := H_f(g) \, \mathrm{vol},$$

where

$$H_f = \frac{\partial f}{\partial u_1} \partial_{u_2} + \frac{\partial f}{\partial u_2} \partial_{u_1}.$$

Define a $\mathbb{Z}$-grading of $L(\underline{N})$ by setting

$$\deg u^{(\underline{r})} \partial_{u_i} = 3|\underline{r}| - 3, \quad \deg u^{(\underline{r})} \, \mathrm{vol} = 3|\underline{r}| - 2, \quad \deg u^{(\underline{r})} = 3|\underline{r}| - 4.$$

Now, set $\mathfrak{me}(5; \underline{N}) := L(\underline{N})/L(\underline{N})_{-4}$, where $L(\underline{N})_{-4}$ is the center (the space of constants). The algebra $\mathfrak{me}(5; \underline{N})$ is not simple, because $\mathrm{Vol}(2; \underline{N})$ has a submodule of codimension 1; but $\mathfrak{me}^{(1)}(5; \underline{N})$ is simple; in [22], Eick denoted what we denote $\mathfrak{me}^{(1)}(5; \mathbb{1})$ by $\mathrm{Bro}_2(1,1)$. This algebra was discovered by Shen Guangyu, see [68], and should be denoted somehow to commemorate his wonderful discovery, we suggest to designate this Shen's analog of $\mathfrak{g}(2)$ by $\mathfrak{gs}(2)$.

There are two $\mathbb{Z}$-gradings of $\mathfrak{g}(2)$ with **one** pair of Chevalley generators of degree ± 1 (the other generators being of degree 0): one $\mathbb{Z}$-grading of depth 2 and the other one of depth 3. As is easy to see, for the grading of depth 3, the nonpositive parts of $\mathfrak{g}(2)$ over fields $\mathbb{K}$ of characteristic $p \neq 3$ and those of $\mathfrak{me}(5; \underline{N})$ are isomorphic. Remarkably, this description holds for any $p \neq 3$, see [66]. For $p = 3$, the positive parts of the prolongation have the same dimensions as those of $\mathfrak{g}(2)$ for $p \neq 2, 5$, but $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathbb{K}1_2$, the center of $\mathfrak{gl}(2)$. (By the way, the realization of the nonpositive components of $\mathfrak{g}(2)$, see equation (23.1), that works for $p \neq 3$, should be modified for $p = 3$, but we skip this since neither the complete prolong nor any partial prolong is simple.)

Let $U[k]$ be the $\mathfrak{gl}(V)$-module which is U as $\mathfrak{sl}(V)$-module, and let the central element $z \in \mathfrak{gl}(V)$ represented by the unit matrix, which acts on $U[k]$ as $k \, \mathrm{id}$, where k should be understood modulo p. Then, the grading of depth 3 is of the form

$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-3}$
$\mathfrak{gl}(2) \simeq \mathfrak{gl}(V)$	$V = V[-1]$	$E^2(V)[-2]$	$V[-3]$

for char $\mathbb{K} \neq 3$.

Set $\partial_i := \partial_{x_i}$ to distinguish it from ∂_{u_i}; we use both representations in terms of x and u, whichever is more convenient. Here is the (borrowed from [66]) description of nonpositive components of $\mathfrak{me}(5; \underline{N})$, which are the same as those of $\mathfrak{gs}(2)$ and $\mathfrak{g}(2)$, by means of vector fields:

$\mathfrak{g}_i$	the basis elements
$\mathfrak{g}_{-3}$	$\partial_{u_1} \longleftrightarrow \partial_1, \ \partial_{u_2} \longleftrightarrow \partial_2$
$\mathfrak{g}_{-2}$	$\mathrm{vol} \longleftrightarrow \partial_3$
$\mathfrak{g}_{-1}$	$X_2^- := u_1 \longleftrightarrow (x_3 + x_4 x_5)\partial_2 + \partial_4, \quad u_2 \longleftrightarrow x_3 \partial_1 + x_4 \partial_3 + \partial_5$
$\mathfrak{g}_0 \simeq$ $\mathfrak{gl}(2)$	$u_1 \partial_{u_1} \longleftrightarrow x_1 \partial_1 + x_3 \partial_3 + x_4 \partial_4,$ $X_1^+ := u_1 \partial_{u_2} \longleftrightarrow x_5^{(3)} \partial_1 + (x_1 + x_4 x_5^{(2)})\partial_2 + x_5^{(2)} \partial_3 + x_5 \partial_4$ $X_1^- := u_2 \partial_{u_1} \longleftrightarrow (x_2 + x_4^{(2)} x_5)\partial_1 + x_4^{(3)} \partial_2 + x_4^{(2)} \partial_3 + x_4 \partial_5$ $u_2 \partial_{u_2} \longleftrightarrow x_2 \partial_2 + x_3 \partial_3 + x_5 \partial_5$

$$(23.1)$$

The highest-weight vector in $\mathfrak{g}_{-1}$ is $X_2^- := u_1$. Consider the positive part of $\mathfrak{g} = \mathfrak{gs}(2)$. The lowest-weight vector in $\mathfrak{g}_1$ is given by the vector field

$$X_2^+ := x_4^{(3)} x_5 \partial_2 + \left(x_2 + x_4^{(2)} x_5 \right) \partial_3 + x_4 x_5 \partial_5 \quad (= u_2 \text{ vol}).$$

So far, the generators and the dimensions of the components look like their namesakes in $\mathfrak{g}(2)$ for $p > 3$; however, the relations are different: To facilitate comparison with presentations in terms of Chevalley generators, set $H_i := [X_i^+, X_i^-]$; i.e.,

$$H_1 = x_1 \partial_1 + x_2 \partial_2 + x_4 \partial_4 + x_5 \partial_5 \quad (= u_1 \partial_{u_1} + u_2 \partial_{u_2}),$$
$$H_2 = x_2 \partial_2 + x_3 \partial_3 + x_5 \partial_5 \quad (= u_2 \partial_{u_2}).$$

Clearly, H_1 is the central element of $\mathfrak{g}_0$; for its grading element we take $u_1 \partial_{u_1}$, see [10].

Lemma 23.1 (the multiplication tables in $\mathfrak{gs}(2)$ and $\mathfrak{g}(2)$). *The multiplication tables in $\mathfrak{gs}(2)$ and $\mathfrak{g}(2)$ are as follows* (for $\mathfrak{g}(2)$, we get $[H_i, X_j^\pm] = \pm A_{ij} X_j^\pm$, not $[H_i, X_j^\pm] = \pm A_{ji} X_j^\pm$; let $X_3^\pm := [X_1^\pm, X_2^\pm]$)

in $\mathfrak{gs}(2)$	*in* $\mathfrak{g}(2)$	*in* $\mathfrak{gs}(2)$	*in* $\mathfrak{g}(2)$	*in* $\mathfrak{gs}(2)$	*in* $\mathfrak{g}(2)$
$[H_1, X_1^+] = 0$	$2X_1^+$	$[H_2, X_1^+] = X_1^+$	$-3X_1^+$	$[H_1, H_2] = 0$	0
$[H_1, X_2^+] = X_2^+$	$-X_2^+$	$[H_2, X_2^+] = 0$	$2X_2^+$	$[X_1^-, X_2^-] = x_3\partial_1 + x_4\partial_3 + \partial_5$	X_3^-
$[H_1, X_1^-] = 0$	$-2X_1^-$	$[H_2, X_1^-] = X_1^-$	$3X_1^-$	$[X_1^+, X_2^+] = u_1 \text{ vol}$	X_3^+
$[H_1, X_2^-] = X_2^-$	X_2^-	$[H_2, X_2^-] = 0$	$-2X_2^-$	$[X_1^\pm, X_2^\mp] = 0$	0

   Critical coordinates of $\mathfrak{me}(5; \underline{N})$: $\underline{N}_3 = 1$.
   The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is generated by the lowest-weight vector X_2^+; we have $\dim \mathfrak{g}_1 = 2$. Since $X_1^\pm$ and X_2^+ contain x_4 and x_5 in degrees 2 and 3, see equation (23.1), the corresponding coordinates of the shearing vector in the generic case are ≥ 2; for the shearing vector with the smallest coordinates still ensuring simplicity; i.e., for $\underline{N} = (1,1,1,2,2)$, the prolong $\mathfrak{g}$ is of dimension 17; it has ideals of dimension 14, 15, 16. The ideal of dimension 14 is simple, see [16, 22, 68].

24   Miscellaneous remarks

24.1   Desuperizations that are nonsimple if $N_i < \infty$ for all i

In Section 17.4, the simple derived algebras of various W-graded versions of $\mathfrak{kas}$ are described; this is new. The results of *this* section are not new (although they were usually considered for $p > 2$); see, e.g., Lemma 2.4 in [32]; we present them for completeness, see also equation (2.34) and Section 23.1 on $\mathfrak{me}^{(1)}$.

24.1.1 $\mathfrak{g} = \mathfrak{svect}(n; \underline{N})$

Let us prove that the elements of the form

$$D_k = \left(\prod_{i \in \{1,\dots,n\}, \ i \neq k} x_i^{(2^{N_i} - 1)} \right) \partial_k$$

do not lie in $\mathfrak{g}^{(1)}$. In what follows we assume that $k = n$, for definiteness. As $\mathfrak{g}$ is a sum of its $\mathbb{Z}^n$-weighted components, it suffices to show that D_n cannot be obtained as the bracket of two elements homogenous with respect to the grading by the weight. As the x_n-weight (i.e., weight with respect to $x_n \partial_n$) of D_n is equal to -1, which is also the minimal possible x_n-weight in $\mathfrak{g}$,

it follows that, in order to obtain D_n as a bracket, one of the factors (we say "factor" speaking about the Lie bracket, just as we do it for an associative multiplication) has to have weight -1 as well. Then, if this factor is homogenous w.r.t. the $\mathbb{Z}^n$-weight, it must be a monomial of the form $a = \left(\prod_{1 \leq i \leq n-1} x_i^{(r_i)} \right) \partial_n$ up to a scalar multiplier, where $0 \leq r_i < 2^{N_i}$. Then, from the weight considerations, the other factor must be of the form

$$b = \sum_{1 \leq i < n,\ r_i > 0} c_i \left(\prod_{1 \leq j < n,\ j \neq i} x_j^{(2^{N_j} - 1 - r_j)} \right) x_i^{(2^{N_i} - r_i)} \partial_i + c_n \left(\prod_{1 \leq j \leq n-1} x_j^{(2^{N_j} - 1 - r_j)} \right) x_n \partial_n.$$

Clearly,

$$[a, b] = \left(\sum_{1 \leq i < n \text{ such that } r_i > 0,\ i = n} c_i \right) D_n,$$

$$\operatorname{div} b = \left(\sum_{1 \leq i < n \text{ such that } r_i > 0,\ i = n} c_i \right) \left(\prod_{1 \leq j \leq n-1} x_j^{(2^{N_j} - 1 - r_j)} \right).$$

So $b \in \mathfrak{g}$ if and only if $[a, b] = 0$, hence $\mathfrak{g}^{(1)}$ contains no elements of the same weight as D_n.

24.1.2 $\mathfrak{vas}$ for $p = 2$

In this case, the even part of the Lie superalgebra $\mathfrak{vas}(4; \underline{N}|4)$, and of its $\mathbb{Z}/2$-graded desuperization, should be diminished to get a simple Lie algebra, namely

$$\mathfrak{vas}^{(1)}(4; \underline{N}|4)_{\bar{0}} = \mathfrak{svect}^{(1)}(4; \underline{N}|0) \rtimes \operatorname{Vol}_0(4; \underline{N}|0).$$

24.1.3   The Lie (super)algebra of contact vector fields

Let $p \neq 2$. As follows from equation (2.19), if $2n + 2 - m \equiv 0 \mod p$, then the Lie superalgebra $\mathfrak{k}(2n + 1; \underline{N}|m)$ is divergence-free, its derived algebra is simple.

If $2n + 2 - m \equiv -2 \mod p$, then $\mathfrak{k}(2n + 1; \underline{N}|m) \simeq \operatorname{Vol}$, and hence not simple; it contains a codimension 1 ideal, $\mathfrak{k}^{(1)}(2n + 1; \underline{N}|m)$.

Let $p = 2$. If $(n, m) \neq (0, 0)$, then the Lie (super)algebra $\mathfrak{k}(2n + 1; \underline{N}|2m)$ is divergence-free if $n + m + 1 \equiv 0 \mod 2$, see equation (2.11).

The *Zassenhaus algebra* $\mathfrak{vect}(1; \underline{N})$ for $p = 2$ is not simple; observe that $\mathfrak{vect}(1; \underline{N}) \simeq \mathfrak{k}(1; \underline{N})$.

24.2   On deforms of $\mathfrak{svect}$ and $\mathfrak{h}$. Quantizations

- In [74], Tyurin described non-isomorphic filtered deforms of the Lie algebras of series $\mathfrak{svect}$ for $p > 3$ considered in the *standard* $\mathbb{Z}$-grading. There are three statements in [74] that should be corrected.

  First, in the introduction to [74], Tyurin wrote that in [32] Kac proved that all deforms of $\mathfrak{svect}$ for $p > 3$ are filtered. Kac did not claim this in [32]. Moreover, Kac did not claim he described *all* filtered deformations, either; Kac writes only about filtered deformations associated with the *standard* $\mathbb{Z}$-gradings.

  Today, when the simple modular Lie algebras are classified for $p > 3$, the list of all their deforms is not needed for *classification*, but is a useful part of *interpretation* of the algebras found, see, e.g., [69, 70]; this is of independent interest, like knowledge about "occasional isomorphisms" $\mathfrak{o}(3) \simeq \mathfrak{sl}(2)$ or $\mathfrak{o}(6) \simeq \mathfrak{sl}(4)$, or $\mathfrak{vect}(1|1) \simeq \mathfrak{m}(1) \simeq \mathfrak{k}(1|2)$, as abstract Lie superalgebras.

Second, for any p, a particular deformation – called *quantization* in physical literature – of the Poisson Lie algebra on 2 indeterminates, induces a deform of $\mathfrak{svect}(2;\underline{N}) \simeq \mathfrak{h}(2;\underline{N})$, at least for $\underline{N}$ of the form (a,a) for any $a \geq 1$, cf. [13]. Therefore, in [74], the claims describing all deformations of $\mathfrak{svect}(m;\underline{N})$ should have been confined to $m > 2$ and, moreover, Tyurin's main theorem should only claim a complete description of non-isomorphic *filtered* deforms related to the *standard* $\mathbb{Z}$-grading; for examples of filtered deforms of $\mathfrak{svect}^{(1)}(3;\mathbb{1}) \simeq \mathfrak{h}^{(1)}(4;\mathbb{1})$ corresponding to distinct $\mathbb{Z}$-gradings, see [18].

Although other deforms of $\mathfrak{h}(2n;\underline{N})$ do not provide us with new Lie algebras, they do provide us with new deforms, non-isomorphic to the filtered deforms.

Third, Wilson [76] corrected the main result of Tyurin who found all normal shapes of volume forms for $p > 2$, but missed an isomorphism. Wilson wrote only about normal shapes of volume forms, thus avoiding any discussion of deforms of $\mathfrak{svect}$.

- The deform of $\mathfrak{svect}(5;\underline{N})$ we describe here is a completely new, exceptional, simple vectorial Lie algebra. It exists only for $p = 2$, the case neither Tyurin nor Wilson considered.

- The characteristic-2 analogs of exceptional deformations of $\mathfrak{h}$ and $\mathfrak{b}$ described in [58] can have both even and odd parameters. The complete description of the deformations is unknown.

25   Tables

25.1   Series of vectorial Lie superalgebras over $\mathbb{C}$; conditions for their simplicity

In Table (25.1), FD indicates finite dimension.

N	the family and conditions for its simplicity
1	$\mathfrak{vect}(m\|n;r)$ for $m \geq 1$ and $0 \leq r \leq n$
2	$\mathfrak{vect}(0\|n;r)$ for $n > 1$ and $0 \leq r \leq n$ (FD)
3	$\mathfrak{svect}(m\|n;r)$ for $m > 1$, $0 \leq r \leq n$
4	$\mathfrak{svect}(0\|n;r)$ for $n > 2$ and $0 \leq r \leq n$ (FD)
5	$\mathfrak{svect}^{(1)}(1\|n;r)$ for $n > 1$, $0 \leq r \leq n$
6	$\widetilde{\mathfrak{svect}}(0\|n)$ for $n > 2$ (FD)
7	$\mathfrak{k}(2m+1\|n;r)$ for $0 \leq r \leq [\frac{n}{2}]$ unless $(m\|n) = (0\|2k)$
	$\mathfrak{k}(1\|2k;r)$ for $0 \leq r \leq k$ except $r = k-1$
8	$\mathfrak{h}(2m\|n;r)$ for $m > 0$ and $0 \leq r \leq [\frac{n}{2}]$
9	$\mathfrak{h}_\lambda(2\|2;r)$ for $\lambda \neq -2, -\frac{3}{2}, -1, \frac{1}{2}, 0, 1, \infty$, and
	$r = 0,\ 1$ and $\mathrm{Reg}_{\mathfrak{h}}$ (see Sect. 1.3.1 in [58])
10	$\mathfrak{h}^{(1)}(0\|n)$ for $n > 3$ (FD)
11	$\mathfrak{m}(n\|n+1;r)$ for $0 \leq r \leq n$ except $r = n-1$
12	$\mathfrak{sm}(n\|n+1;r)$ for $n > 1$, but $n \neq 3$ and $0 \leq r \leq n$ except $r = n-1$
13	$\mathfrak{b}_\lambda(n\|n+1;r)$ for $n > 2$, where $\lambda \neq 0, 1, \infty$ and $0 \leq r \leq n$ except $r = n-1$
14	$\mathfrak{b}_1^{(1)}(n\|n+1;r)$ for $n > 2$ and $0 \leq r \leq n$ except $r = n-1$
15	$\mathfrak{b}_\infty^{(1)}(n\|n+1;r)$ for $n > 2$ and $0 \leq r \leq n$ except $r = n-1$
16	$\mathfrak{le}(n\|n;r)$ for $n > 1$ and $0 \leq r \leq n$ except $r = n-1$
17	$\mathfrak{sle}^{(1)}(n\|n;r)$ for $n > 2$ and $0 \leq r \leq n$ except $r = n-1$
18	$\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1\|2^{n-1})$ for $\mu \neq 0$ and $n > 2$

(25.1)

25.2 Lie algebras $F(\mathfrak{g})$ over $\mathbb{K}$ ($p = 2$) analogous to serial vectorial Lie superalgebras $\mathfrak{g}$ over $\mathbb{C}$ and names of both

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$F(\mathfrak{g}_0)$	$F(\mathfrak{g})$
1	$\mathfrak{vect}(n\lvert m)$ for $mn \neq 0$, $n > 1$ or $m = 0$, $n > 2$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{gl}(n\lvert m) \simeq \mathfrak{gl}(V)$	$\mathfrak{gl}(n+m)$	$\mathfrak{vect}(n+m; \widetilde{N})$
2	$\mathfrak{svect}(n\lvert m)$ for $m, n \neq 1$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{sl}(n\lvert m) \simeq \mathfrak{sl}(V)$	$\mathfrak{sl}(n+m)$	$\mathfrak{svect}(n+m; \widetilde{N})$
3	$\mathfrak{h}_{\mathcal{B}}(2n\lvert m)$, where $mn \neq 0$, $n > 1$,	$-$	id	$\mathfrak{osp}_{\mathcal{B}}(m\lvert 2n)$	$\mathfrak{o}_{F(\mathcal{B})}(m + 2n)$	$\mathfrak{h}_{F(B)}(2n + m; \widetilde{N})$
4	$\mathfrak{k}(2n+1\lvert m)$ for $mn \neq 0$ and m even	$\mathbb{F}$	$\mathrm{id} \simeq V$	$\mathfrak{cosp}_{\mathcal{B}}(m\lvert 2n) \simeq \mathfrak{cosp}(V)$	$\mathfrak{co}_{F(\mathcal{B})}(m + 2n)$	$\mathfrak{k}(2n + m + 1; \widetilde{N})$
5	$\mathfrak{m}(n) := \mathfrak{m}(n\lvert n+1)$ for $n > 1$	$\Pi(\mathbb{F})$	$\mathrm{id} \simeq V$	$\mathfrak{cpe}(n) \simeq \mathfrak{cpe}(V)$	$\mathfrak{cpe}(n)$	$\mathfrak{k}(2n + 1; \widetilde{N})$
6	$\mathfrak{b}_{\lambda}(n; n)$ for $n > 1$	$-$	$\Pi(\mathrm{Vol}^{\lambda}(0\lvert n))$	$\mathfrak{vect}(0\lvert n)$	$\mathfrak{vect}(n; \mathbb{1})$	$\mathfrak{po}_{\lambda}(2n + 1; \widetilde{N})$
6_1	$\mathfrak{b}_1^{(1)}(n; n)$ for $n > 1$	$-$	$\Pi(\mathrm{Vol}_0(0\lvert n))$	$\mathfrak{vect}(0\lvert n)$	$\mathfrak{vect}(n; \mathbb{1})$	$\mathfrak{po}_1^{(1)}(2n + 1; \widetilde{N})$
6_∞	$\mathfrak{b}_\infty^{(1)}(n; n) \simeq \mathfrak{sm}(n; n)$ for $n > 1$	$-$	$\Pi(\mathrm{Vol}_0(0\lvert n))$	$\mathfrak{svect}(0\lvert n) \rtimes \mathrm{Vol}_0(0\lvert n)$	$\mathfrak{svect}(n; \mathbb{1}) \rtimes \mathrm{Vol}_0(n; \mathbb{1})$	$\mathfrak{po}_\infty^{(1)}(2n + 1; \widetilde{N})$
7	$\mathfrak{b}_{a,b}(n)$ for $n > 1$	$\Pi(\mathbb{F})$	id	$\mathfrak{spe}(n)_{a,b}$	$\mathfrak{spe}(n)_{a,b}$	$\mathfrak{po}_{a,b}(2n; \widetilde{N})$
8	$\mathfrak{le}(n) := \mathfrak{le}(n\lvert n)$ for $n > 1$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{pe}(n) \simeq \mathfrak{pe}(V)$	$\mathfrak{pe}(n)$	$\mathfrak{h}_{\Pi}(2n; \widetilde{N})$
9	$\mathfrak{sle}(n) := \mathfrak{sle}(n\lvert n)$ for $n > 1$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{spe}(n) \simeq \mathfrak{spe}(V)$	$\mathfrak{spe}(n)$	$\mathfrak{sh}_{\Pi}(2n; \widetilde{N})$
10	$\widetilde{\mathfrak{sb}}_{\mu}(2^{n-1} - 1\lvert 2^{n-1})$ or $\widetilde{\mathfrak{sb}}_{\mu}(2^{n-1}\lvert 2^{n-1} - 1)$	$-$	$\frac{\Pi(\mathrm{Vol}(0\lvert n))}{\mathbb{F}(1 - \mu\xi_1 \cdots \xi_n)\,\mathrm{vol}_\xi}$	$\widetilde{\mathfrak{svect}}_{\mu}(0\lvert n)$	$\widetilde{\mathfrak{svect}}_{\mu}(n; \mathbb{1})$	$\widetilde{\mathfrak{sb}}_{\mu}(2^n - 1; \widetilde{N})$

$$(25.2)$$

25.2.1 Remarks

In all lines Par $\underline{\widetilde{N}} = \dim \widetilde{N}$, except for the bottom line, see Section 21. To save space, we skip most of the conditions for simplicity in Table (25.2). In columns $\mathfrak{g}_i$ for $i < 0$, obviously, $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{K}$. In lines $N = 6, 7$, we have $\lambda = \frac{2a}{n(a-b)}$ for $p \neq 2$ and $\lambda = \frac{a}{b}$ for $p = 2$. In line 6_∞, we identify Vol_0 with a subspace of the space of functions $\mathcal{F}$. In line 10, the Lie superalgebra

$$\widetilde{\mathfrak{svect}}_{\mu}(0\lvert n) := (1 + \mu\xi_1 \cdots \xi_n)\mathfrak{svect}(0\lvert n) \quad \text{preserves the volume element} \quad (1 - \mu\xi_1 \cdots \xi_n)\,\mathrm{vol}_\xi, \quad \text{where } p(\mu) \equiv n \mod 2.$$

For n even, we can (and do) set $\mu = 1$, whereas μ odd should be considered as an additional indeterminate on which the coefficients depend. The Lie superalgebras $\widetilde{\mathfrak{svect}}_{\mu}(0\lvert n)$ are isomorphic for nonzero μ's; and therefore so are the algebras

$$\widetilde{\mathfrak{sb}}_{\mu}(2^{n-1} - 1\lvert 2^{n-1}) := (1 + \mu\xi_1 \cdots \xi_n)\mathfrak{sb}(n; n) \text{for } n \text{ even}, \quad \widetilde{\mathfrak{sb}}_{\mu}(2^{n-1}\lvert 2^{n-1} - 1) := (1 + \mu\xi_1 \cdots \xi_n)\mathfrak{sb}(n; n) \text{for } n \text{ odd},$$

Recall the definition of $\mathfrak{spe}(n)_{a,b}$ in Section 2.8.2.

To be specified: Some of the Lie superalgebras in Table (25.2) are not simple; it is their quotients modulo their centers or ideals of codimension 1 which are simple (such are $\mathfrak{svect}(1\lvert m)$, $\mathfrak{h}(0\lvert m)$, $\mathfrak{b}_{\lambda}(n)$ for certain values of λ, and $\mathfrak{sle}(n)$); some small values of superdimension should be excluded (like $(1\lvert 1)$ and $(0\lvert m)$, where $m \leq 2$, for the $\mathfrak{svect}$ series; $(0\lvert m)$, where $m \leq 3$, for the $\mathfrak{h}$ series; etc.)

25.3 Exceptional vectorial Lie superalgebras over $\mathbb{C}$

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}(\operatorname{sdim}\mathfrak{g}_-)$
$\mathfrak{vle}(4|3)$	–	$\Pi(\Lambda(3)/\mathbb{C}1)$	$\mathfrak{c}(\mathfrak{vect}(0|3))$	$\mathfrak{vle}(4|3)$
$\mathfrak{vle}(4|3;1)$	$\mathbb{C}[-2]$	$\mathrm{id}_{\mathfrak{sl}(2;\Lambda(2))}\boxtimes\mathrm{vol}^{1/2}$	$\mathfrak{c}(\mathfrak{sl}(2;\Lambda(2))\ltimes T^{1/2}(\mathfrak{vect}(0|2)))$	$\mathfrak{vle}(5|4)$
$\mathfrak{vle}(4|3;K)$	$\mathrm{id}_{\mathfrak{sl}(3)}\boxtimes\mathbb{C}[-2]$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\boxtimes\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{C}[-1]$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$\mathfrak{vle}(3|6)$
$\mathfrak{vas}(4|4)$	–	spin	$\mathfrak{as}$	$\mathfrak{vas}(4|4)$
$\mathfrak{kas}$	$\mathbb{C}[-2]$	$\Pi(\mathrm{id})$	$\mathfrak{co}(6)$	$\mathfrak{kas}(1|6)$
$\mathfrak{kas}(;1\xi)$	$\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathrm{id}_{\mathfrak{gl}(2;\Lambda(1))}$	$\mathfrak{sl}(2)\oplus[\mathfrak{gl}(2;\Lambda(1))\ltimes\mathfrak{vect}(0|1)]$	$\mathfrak{kas}(5|5)$
$\mathfrak{kas}(;3\xi)$	–	$\Lambda(3)$	$\Lambda(3)\oplus\mathfrak{sl}(1|3)$	$\mathfrak{kas}(4|4)$
$\mathfrak{kas}(;3\eta)$	–	$\mathrm{Vol}_0(0|3)$	$\mathfrak{c}(\mathfrak{vect}(0|3))$	$\mathfrak{kas}(4|3)$
$\mathfrak{mb}(4|5)$	$\Pi(\mathbb{C}[-2])$	$\mathrm{Vol}^{1/2}(0|3)$	$\mathfrak{c}(\mathfrak{vect}(0|3))$	$\mathfrak{mb}(4|5)$
$\mathfrak{mb}(4|5;1)$	$\Lambda(2)/\mathbb{C}1$	$\mathrm{id}_{\mathfrak{sl}(2;\Lambda(2))}\boxtimes\mathrm{vol}^{1/2}$	$\mathfrak{c}(\mathfrak{sl}(2;\Lambda(2))\ltimes T^{1/2}(\mathfrak{vect}(0|2)))$	$\mathfrak{mb}(5|6)$
$\mathfrak{mb}(4|5;K)$	$\mathrm{id}_{\mathfrak{sl}(3)}\boxtimes\mathbb{C}[-2]$	$\Pi(\mathrm{id}^*_{\mathfrak{sl}(3)}\boxtimes\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{C}[-1])$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$\mathfrak{mb}(3|8)$
$\mathfrak{kle}(9|6)$	$\mathbb{C}[-2]$	$\Pi(T^0_0(\vec{0}))$	$\mathfrak{svect}_{3,4}(0|4)$	$\mathfrak{kle}(9|6)$
$\mathfrak{kle}(9|6;2)$	$\Pi(\mathrm{id}_{\mathfrak{sl}(1|3)})$	$\mathrm{id}_{\mathfrak{sl}(2;\Lambda(3))}$	$\mathfrak{sl}(2;\Lambda(3))\ltimes\mathfrak{sl}(1|3)$	$\mathfrak{kle}(11|9)$
$\mathfrak{kle}(9|6;K)$	id	$\Pi(\Lambda^2(\mathrm{id}^*))$	$\mathfrak{sl}(5)$	$\mathfrak{kle}(5|10)$
$\mathfrak{kle}(9|6;CK)$	$\mathrm{id}^*_{\mathfrak{sl}(3;\Lambda(1))}$	$\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathrm{id}_{\mathfrak{sl}(3;\Lambda(1))}$	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(3;\Lambda(1))\ltimes\mathfrak{vect}(0|1)$	$\mathfrak{kle}(9|11)$

Depth 3: None of the simple W-graded vectorial Lie superalgebras over $\mathbb{C}$ is of depth > 3 and only two superalgebras are of depth 3:

$$\mathfrak{mb}(3|8)_{-3} = \Pi(\mathbb{C}\boxtimes\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{C}[-3]), \quad \mathfrak{kle}(9|11)_{-3} = \Pi(\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{C}[-3]). \tag{25.3}$$

For the definition of the module Vol_0, see (1.13). Here, $T^{1/2}$ is the representation of $\mathfrak{vect}$ in the module of $\frac{1}{2}$-densities, and $\mathfrak{as}$ is the nontrivial central extension of $\mathfrak{spe}(4)$, cf. [6]. For the definition of $\mathfrak{svect}_{3,4}(0|4)$, see Section 2.8.2. In the 0th term $(\mathfrak{sl}(2)\boxtimes\Lambda(3))\ltimes\mathfrak{sl}(1|3)$ of $\mathfrak{g} = \mathfrak{kle}(11;\underline{N}|9)$, we consider $\mathfrak{sl}(1|3)$ naturally embedded into $\mathfrak{vect}(0|3)$ with its tautological action on the space $\Lambda(3)$ of "functions".

For the notation $\mathbb{C}[i]$, see Section 2.1.1.

25.4 The exceptional simple vectorial Lie superalgebras over $\mathbb{C}$ as Cartan prolongs

For depth 2, we sometimes write $(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ for clarity.

In Table (25.4), there are given indeterminates and their respective degrees in the regrading $R(r)$.

$\mathfrak{vle}(4|3;r) = (\Pi(\Lambda(3))/\mathbb{C}\cdot 1, \mathfrak{cvect}(0|3))_*$	$\subset\mathfrak{vect}(4|3;R(r))$	$r = 0, 1, K$
$\mathfrak{vas}(4|4) = (\mathrm{spin}, \mathfrak{as})_*$	$\subset\mathfrak{vect}(4|4)$	
$\mathfrak{kas}(1|6;r)$	$\subset\mathfrak{k}(1|6;r)$	$r = 0, 1\xi, 3\xi$
$\mathfrak{kas}(1|6;3\eta) = (\mathrm{Vol}_0(0|3), \mathfrak{c}(\mathfrak{vect}(0|3)))_*$	$\subset\mathfrak{svect}(4|3)$	$r = 3\eta$
$\mathfrak{mb}(4|5;r) = (\mathfrak{ba}(4), \mathfrak{cvect}(0|3))_*$	$\subset\mathfrak{m}(4|5;R(r))$	$r = 0, 1, K$
$\mathfrak{kle}(9|6;r) = (\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(0|4))_*$	$\subset\mathfrak{k}(9|6;r)$	$r = 0, 2, \mathrm{CK}$
$\mathfrak{kle}(9|6;K) = (\mathrm{id}_{\mathfrak{sl}(5)}, \Lambda^2(\mathrm{id}^*_{\mathfrak{sl}(5)}), \mathfrak{sl}(5))_*$	$\subset\mathfrak{svect}(5|10;R(K)), r = K$	

$$
\begin{array}{|c|l|}
\hline
\mathfrak{vle}(4|3) & R(0) = \begin{pmatrix} x_1 & x_2 & x_3 & y & | & \xi_1 & \xi_2 & \xi_3 \\ 1 & 1 & 1 & 1 & | & 1 & 1 & 1 \end{pmatrix} \\
\mathfrak{vle}(5|4) & R(1) = \begin{pmatrix} x_1 & x_2 & x_3 & y & | & \xi_1 & \xi_2 & \xi_3 \\ 2 & 1 & 1 & 0 & | & 0 & 1 & 1 \end{pmatrix} \\
\mathfrak{vle}(3|6) & R(K) = \begin{pmatrix} x_1 & x_2 & x_3 & y & | & \xi_1 & \xi_2 & \xi_3 \\ 2 & 2 & 2 & 0 & | & 1 & 1 & 1 \end{pmatrix} \\
\hline
\mathfrak{mb}(4|5) & R(0) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & | & \xi_0 & \xi_1 & \xi_2 & \xi_3 & \tau \\ 1 & 1 & 1 & 1 & | & 1 & 1 & 1 & 1; & 2 \end{pmatrix} \\
\mathfrak{mb}(5|6) & R(1) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & | & \xi_0 & \xi_1 & \xi_2 & \xi_3 & \tau \\ 0 & 2 & 1 & 1 & | & 2 & 0 & 1 & 1; & 2 \end{pmatrix} \\
\mathfrak{mb}(3|8) & R(K) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & | & \xi_0 & \xi_1 & \xi_2 & \xi_3 & \tau \\ 0 & 2 & 2 & 2 & | & 3 & 1 & 1 & 1; & 3 \end{pmatrix} \\
\hline
\mathfrak{kas}(1|6) & R(0) = \begin{pmatrix} t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 2 & | & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\
\mathfrak{kas}(5|5) & R(1\xi) = \begin{pmatrix} t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 2 & | & 0 & 1 & 1 & 2 & 1 & 1 \end{pmatrix} \\
\mathfrak{kas}(4|4) & R(3\xi) = \begin{pmatrix} t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 1 & | & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \\
\mathfrak{kas}(4|3) & R(3\eta) = \begin{pmatrix} t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 1 & | & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \\
\hline
\mathfrak{kle}(9|6) & R(0) = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & p_1 & p_2 & p_3 & p_4 & t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1; & 2 & | & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\
\mathfrak{kle}(11|9) & R(2) = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & p_1 & p_2 & p_3 & p_4 & t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0; & 2 & | & 0 & 1 & 1 & 2 & 1 & 1 \end{pmatrix} \\
\mathfrak{kle}(5|10) & R(K) = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & p_1 & p_2 & p_3 & p_4 & t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1; & 2 & | & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\
\mathfrak{kle}(9|11) & R(CK) = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & p_1 & p_2 & p_3 & p_4 & t & | & \xi_1 & \xi_2 & \xi_3 & \eta_1 & \eta_2 & \eta_3 \\ 3 & 2 & 2 & 2 & 0 & 1 & 1 & 1; & 3 & | & 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix} \\
\hline
\end{array}
$$

$$(25.4)$$

25.5 Exceptional vectorial Lie superalgebras over $\mathbb{K}$ and their desuperizations

$\mathfrak{g}$	$\mathbf{F}(\mathfrak{g}_{-2})$	$\mathbf{F}(\mathfrak{g}_{-1})$	$\mathbf{F}(\mathfrak{g}_0)$	$\mathbf{F}(\mathfrak{g})$	Par $\widetilde{N}$
$\mathfrak{vle}(4;\underline{N}\vert 3)$	$-$	$\mathcal{O}(3;\mathbb{1})/\mathbb{K}1$	$\mathfrak{c}(\mathfrak{vect}(3;\mathbb{1}))$	$\mathfrak{vle}\big(7;\widetilde{N}\big)$	3
$\mathfrak{vle}(3;\underline{N}\vert 6)$	$\mathrm{id}_{\mathfrak{sl}(3)}\boxtimes\mathbb{K}[*]$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\boxtimes\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{K}[*]$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{K}z$	$\mathfrak{vle}\big(9;\widetilde{N}\big)$	3
$\mathfrak{kas}(1;\underline{N}\vert 6)$	$\mathbb{K}[*]$	id	$\mathfrak{co}^{(1)}_\Pi(6)$	$\mathfrak{kas}\big(7;\widetilde{N}\big)$	7
$\mathfrak{kas}(5;\underline{N}\vert 5)$	$\mathcal{O}(1;\mathbb{1})$	$\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathrm{id}_{\mathfrak{gl}(2)}\boxtimes\mathcal{O}(1;\mathbb{1})$	$\mathfrak{d}((\widetilde{\mathfrak{sl}}(W)\oplus(\mathfrak{gl}(V;\mathcal{O})(1;\underline{1}))\ltimes\mathfrak{vect}(1;\mathbb{1}))/\mathbb{K}Z)$, see (20.1)	$\mathfrak{kas}\big(10;\widetilde{N}\big)$	7
$\mathfrak{kas}(4;\underline{N}\vert 4)$	$-$	$\mathcal{O}(3;\mathbb{1})$	$\mathcal{O}(3;\mathbb{1})\ltimes\mathfrak{d}(\mathfrak{svect}^{(1)}(3;\mathbb{1}))$, see (19.2)	$\mathfrak{kas}\big(8;\widetilde{N}\big)$	7
$\mathfrak{kas}(4;\underline{N}\vert 3)$	$-$	$\mathrm{Vol}_0(3;\mathbb{1})$	$\mathfrak{c}(\mathfrak{vect}(3;\mathbb{1}))$	$\widetilde{\mathfrak{kas}}\big(7;\widetilde{N}\big)$	3
$\mathfrak{mb}(4;\underline{N}\vert 5)$	$\mathbb{K}[*]$	$\mathcal{O}(3;\mathbb{1})$	$\mathfrak{svect}(3;\mathbb{1})\rtimes\mathcal{O}(3;\mathbb{1})$	$\mathfrak{mb}\big(9;\widetilde{N}\big)$	5
$\mathfrak{mb}(3;\underline{N}\vert 8)$	$\mathrm{id}_{\mathfrak{sl}(3)}\boxtimes\mathbb{K}[*]$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\boxtimes\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{K}[*]$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{K}z$	$\mathfrak{mb}_3\big(11;\widetilde{N}\big)$	5
$\mathfrak{mb}(5;\underline{N}\vert 6)$	$\mathcal{O}(2;\mathbb{1})/\mathbb{K}1$	$\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathcal{O}(2;\mathbb{1})$	$\mathfrak{c}(\mathfrak{sl}(2)\boxtimes\mathcal{O}(2;\mathbb{1}))\ltimes T^\infty(\mathfrak{svect}(2;\mathbb{1})\rtimes\mathcal{O}(2;\mathbb{1}))$	$\mathfrak{mb}_2\big(11;\widetilde{N}\big)$	5
$\mathfrak{kle}(5;\underline{N}\vert 10)$	id	$\Lambda^2(\mathrm{id}^*)$	$\mathfrak{sl}(5)$	$\mathfrak{kle}\big(15;\widetilde{N}\big)$	5
$\mathfrak{kle}(11;\underline{N}\vert 9)$	$\mathrm{id}_{\mathfrak{sl}(4)}$	$\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathcal{O}(3;\mathbb{1})$	$(\mathfrak{sl}(2)\boxtimes\mathcal{O}(3;\mathbb{1}))\ltimes\mathfrak{pgl}(4)$	$\mathfrak{kle}_2\big(20;\widetilde{N}\big)$	5
$\mathfrak{kle}(9;\underline{N}\vert 11)$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\boxtimes\mathcal{O}(1;\mathbb{1})$	$\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\big(\mathrm{id}_{\mathfrak{sl}(3)}\boxtimes\mathcal{O}(1;\mathbb{1})\big)$	$\mathfrak{sl}(2)\oplus(\mathfrak{sl}(3)\boxtimes\mathcal{O}(1;\mathbb{1}))\ltimes\mathfrak{vect}(1;\mathbb{1}))$	$\mathfrak{kle}_3\big(20;\widetilde{N}\big)$	5
$\mathfrak{vle}(5;\underline{N}\vert 4)$	$\mathbb{K}[*]$	$\mathrm{id}\boxtimes\mathcal{O}(2;\mathbb{1})$	$\mathfrak{c}(\mathfrak{sl}(2)\boxtimes\mathcal{O}(2;\mathbb{1}))\ltimes T^\infty(\mathfrak{vect}(2;\mathbb{1}))$	$\widetilde{\mathfrak{vle}}\big(9;\widetilde{N}\big)$	3
$\mathfrak{kle}(9;\underline{N}\vert 6)$	$\mathbb{K}[*]$	T^0_0	$\mathfrak{svect}(4;\mathbb{1})\ltimes\mathbb{K}(D+Z)$, see (10.2)	$\widetilde{\mathfrak{kle}}\big(15;\widetilde{N}\big)$	5
$\mathfrak{vas}(4;\underline{N}\vert 4)$	$-$	$\mathrm{id}_{\mathbf{F}(\mathfrak{as})}$	$\mathbf{F}(\mathfrak{as})$	$\mathfrak{vas}\big(8;\widetilde{N}\big)$	4

Recall the definition of the module Vol_0, see (1.13) and (2.16); before desuperization we replace (25.3) with

$$\mathfrak{mb}(3\vert 8)_{-3} = \Pi(\mathbb{K}\boxtimes\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{K}[*]), \quad \mathfrak{kle}(9\vert 11)_{-3} = \Pi(\mathrm{id}_{\mathfrak{sl}(2)}\boxtimes\mathbb{K}[*]).$$

To distinguish the two desuperizations of $\mathfrak{kle}$ realized by vector fields on the spaces of the same dimension, we indicate by an index the depths of these algebras, e.g., $\mathfrak{kle}_2\big(20;\underline{\widetilde{N}}\big)$; if both algebras are of the same depth, we cover one of the desuperizations with a tilde. Clearly, under the desuperization we should ignore the change of parity in the negative components of $\mathbf{F}(\mathfrak{g})$.

Acknowledgements

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