

Real Part of Twisted-by-Grading Spectral Triples

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Abstract. After a brief review on the applications of twisted spectral triples to physics, we adapt to the twisted case the notion of *real part* of a spectral triple. In particular, when one twists a usual spectral triple by its grading, we show that – depending on the KO dimension – the real part is either twisted as well, or is the intersection of the initial algebra with its opposite. We illustrate this result with the spectral triple of the standard model.

Key words: noncommutative geometry; twisted spectral triple; standard model

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In honor of Giovanni Landi for its sixtieth birthday

1 Introduction

Twisted spectral triples have been defined by Connes and Moscovici in [13] in order to adapt the theory of spectral triples to type III algebras. Later, it turned out that twists also have applications to models of high energy physics in noncommutative geometry [16], paving the way to models beyond the standard model.

Most of the properties of twisted spectral triples relevant for physics (in particular those regarding the real structure and gauge transformation) have been developed by Gianni Landi together with one of the authors, in a couple of papers [19, 20]. This note first presents a short review of some of these results. Then we show how the notion of *real part* of a spectral triple (defined in [7]) easily adapts to the twisted case (Proposition 1). We then focus on twisted spectral triples obtained by twisting usual spectral triples by their grading. We investigate the behaviour of the real part, stressing the dependence on the KO -dimension. More precisely, as shown in Proposition 2, the real part of the twisted spectral triple is either the twist of the real part of the initial triple, or the intersection of the algebra with its opposite. In the last section, we illustrate this result with the spectral triple of the standard model.

2 Minimal twist by grading

This section is a (partial) review of the work of one of the authors with Gianni Landi, regarding twisted spectral triples [19, 20]. We begin with some considerations on the use of spectral triples in physical models of fundamental interactions, and how the discovery of the Higgs boson in 2012 has motivated some of us to use rather *twisted* spectral triples, as defined earlier by Connes and Moscovici [13]. This led to the *twist-by-grading* procedure, that has been somehow “touched” in [16] and clearly formalised in [19].

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2.1 Motivation

In noncommutative geometry [11], the standard model of fundamental interactions is described by the product (in the sense of spectral triples) of a 4-dimensional closed spin manifold \mathcal{M} with an internal geometry that encodes the gauge degrees of freedom. The latter is given by the finite-dimensional real algebra

$$\mathcal{A}_{\text{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

acting on the finite-dimensional Hilbert space $\mathcal{H}_F = \mathbb{C}^{32n}$ (n is the number of generations of fermions), together with a selfadjoint operator D_F on \mathcal{H}_F that encodes the Yukawa coupling of fermions. We refer the reader to the literature for the details, in particular the *raison d'être* of this algebra (see [7] for the original paper and, e.g., [10] for a recent review). In brief, the algebra is such that its group of unitary elements ($u \in \mathcal{A}_{\text{SM}}$ such that $u^*u = uu^* = \mathbb{I}$) yields back – modulo a unimodularity condition – the gauge group $U(1) \times SU(2) \times SU(3)$ of the standard model.

The product of spectral triples is

$$A = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\text{SM}}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \not{D} \otimes \mathbb{I} + \gamma^5 \otimes D_F, \quad (1)$$

where $C^\infty(\mathcal{M})$ is the algebra of smooth functions on \mathcal{M} , acting by multiplication of the space $L^2(\mathcal{M}, S)$ of square integrable spinors with¹ $\not{D} = -i\gamma^\mu \nabla_\mu$ the Dirac operator associated with the spin structure of \mathcal{M} .

Fermions are elements of \mathcal{H} , bosons are connection 1-forms which, in noncommutative geometry, are of the form

$$A = a^i [D, b_i], \quad a^i, b_i \in \mathcal{A}. \quad (2)$$

For the spectral triple

$$(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D}) \quad (3)$$

of a spin manifold, equation (2) gives back the usual 1-forms. For the product of geometry (1), it gives the gauge bosons of the standard model together with the Higgs field. In noncommutative geometry, the latter is thus obtained on the same footing as the other gauge bosons, that is as a connection 1-form.

Furthermore, the mass of the Higgs boson is not a free parameter and can be computed as a function of the parameters of the model (that is, the entries of the matrix D_F). From the beginning of the model in the 90's, the prediction has always been around $m_H \simeq 170$ GeV. After the discovery of the Higgs boson with $m_H \simeq 125$ GeV, several ways have been explored to accommodate the correct mass (e.g., [1, 2, 3, 8, 9], see [10] for a recent review).

Most of them start from the following observation: in the spectral triple of the standard model, there is a part of the operator D_F – the one that contains the Majorana mass of the neutrinos – that commutes with the algebra and, as such, does not contribute to the bosonic content of the model via (2). However, it turns out that by turning this Majorana mass (which is a constant k_R) into a field, say σ , then one obtains precisely the kind of scalar field proposed in particle physics to stabilize the electroweak vacuum. In addition, by altering the running of the renormalisation group, this extra scalar field makes the computation of the mass of the Higgs boson compatible with its experimental value [6].

The point is then to understand how to turn the constant k_R into a field σ within the framework of noncommutative geometry. Various scenarios have been proposed, one of them

¹As usual we use Einstein convention for the summation on indices repeated in alternate up/down positions.

consists in viewing the total Hilbert space \mathcal{H} of the product (1) as a $(32n \times 4)$ -dimensional space [15], allowing the algebra to act non trivially on the spinorial degrees of freedom. By doing so, still following the classification of all possible internal algebras in [4, 5], one is able to consider an internal algebra bigger than \mathcal{A}_{SM} – called *grand algebra* in [15] – that no longer commutes with the part of D_F that contains k_R , making the latter contribute to the bosonic content of the theory.

However, as a side effect, one gets that the commutator of $\not{D} \otimes \mathbb{I}$ with this grand algebra is no longer bounded. This is in contradiction with one of the basic requirement of spectral triples (namely that $[D, a]$ should be bounded for any $a \in \mathcal{A}$). This kind of problem has already been encountered when one deals with conformal maps on the canonical spectral triple (3) of a manifold. A solution, as explained by Connes and Moscovici in [13], consists in twisting the spectral triple. Quite remarkably (given that twists were originally motivated by purely mathematical reasons), this solution also works for the standard model.

2.2 Twisted spectral triple

A twisted spectral triple [13] is given by an involutive algebra \mathcal{A} acting on a Hilbert space \mathcal{H} , a selfadjoint operator \mathcal{D} on \mathcal{H} with compact resolvent, together with an automorphism ρ of \mathcal{A} such that the *twisted commutator*

$$[\mathcal{D}, a]_\rho := \mathcal{D} a - \rho(a) \mathcal{D}$$

is bounded for any $a \in \mathcal{A}$.

A *grading* is a selfadjoint operator Γ on \mathcal{H} that squares to \mathbb{I} , commutes with the algebra and anti-commutes with D . A *real structure* is an anti-linear isometry J (that is $J^* J = \mathbb{I}$) on \mathcal{H} , that squares to $+\mathbb{I}$ or $-\mathbb{I}$ and commutes or anticommutes with the Dirac operator and the grading:

$$JD = \epsilon' DJ, \quad J\Gamma = \epsilon'' \Gamma J.$$

The choice of these various signs determines the *KO*-dimension of the spectral triple. Notice that since $J^2 = \pm \mathbb{I}$, one has that J is surjective hence unitary: $J^* = J^{-1}$.

As in the non-twisted case, the real structure is asked to implement a representation of the opposite algebra \mathcal{A}° , identifying a° with Ja^*J^{-1} . This action commutes with the one of \mathcal{A} , yielding the *order zero condition*

$$[a, b^\circ] = 0 \quad \forall a, b \in \mathcal{A}. \quad (4)$$

This condition is the same as in the non-twisted case. However, it was shown in [19] that another important condition, the *first order condition*, needs to be modified in the twisted context, yielding the following *twisted first-order condition* (originally introduced in [16]):

$$[[\mathcal{D}, a]_\rho, b^\circ]_{\rho_0} = 0 \quad \forall a, b \in \mathcal{A}, \quad (5)$$

where ρ° is the automorphism of \mathcal{A}° defined by

$$\rho^\circ(a^\circ) := (\rho^{-1}(a))^\circ. \quad (6)$$

One uses ρ^{-1} instead of ρ because the twisting automorphism in [13] is not asked to be a $*$ automorphism but rather to satisfy

$$\rho(a^*) = (\rho^{-1}(a))^*.$$

2.3 Twist by grading

Given a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, requiring that there exists a non-trivial automorphism ρ such that $[D, a]_\rho$ is bounded for any a puts severe constraints on the algebra (actually $a - \rho(a)$ must be bounded for any a [19, Lemma 3.1]). So in order to introduce a twist one needs to modify some of the elements of the triple. Since \mathcal{H} and D encode the fermionic sector of the standard model, and the point is to generate a new scalar field (no new fermions), it makes sense to look for a minimal modification of the spectral triple, letting the Hilbert space and the Dirac operator untouched. Playing only with the representation does not allow much freedom, so one should be allowed to enlarge the algebra, in agreement with the grand algebra idea mentioned in the introduction. We call this procedure a *minimal twist*.

As a matter of fact, to twist the standard model, one considers as a grand algebra twice the algebra (1), that is

$$\mathcal{A} \otimes \mathbb{R}^2, \quad (7)$$

with each of the copies of \mathcal{A} acting independently on the $+1$, -1 eigenspaces of the grading operator Γ . Namely the representation π of (7) on \mathcal{H} is

$$\pi(a, a') = \frac{1}{2}(\mathbb{I} + \Gamma)a + \frac{1}{2}(\mathbb{I} - \Gamma)a' \quad \forall a, a' \in \mathcal{A}, \quad (8)$$

where \mathbb{I} is the identity operator on \mathcal{H} . The twisting automorphism is simply the exchange of the two components of (7):

$$\rho(a, a') = (a', a). \quad (9)$$

This construction is generic: given any real graded spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (with \mathcal{A} a complex algebra to fix the notations), we call its *twist by grading* the twisted spectral triple

$$(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D)_\rho$$

with representation (8) and twist (9). This is a real graded twisted spectral triple, with the same grading Γ and real structure J (hence same KO -dimension) as the initial spectral triple [19, Proposition 3.8].

The flip (9) coincides with the inner automorphism of $\mathcal{B}(\mathcal{H})$ induced by the unitary operator

$$\mathcal{R} = \begin{pmatrix} 0 & \mathbb{I}_+ \\ \mathbb{I}_- & 0 \end{pmatrix},$$

where \mathbb{I}_\pm are the identity operators on the subspace \mathcal{H}_\pm of the grading Γ . To be able to define the fermionic action in a twisted context, we required in [14] a compatibility condition between the automorphism ρ and the real structure, namely one asks that

$$J\mathcal{R} = \epsilon''' \mathcal{R}J \quad \text{for } \epsilon''' = \pm 1. \quad (10)$$

Using the same notation ρ to denote the extension of the flip to the whole of $\mathcal{B}(\mathcal{H})$,

$$\rho(\mathcal{O}) := \mathcal{R}\mathcal{O}\mathcal{R}^\dagger \quad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}),$$

the compatibility condition (10) amounts to

$$\rho(a^\circ) = (\rho(a))^\circ. \quad (11)$$

3 Real part

We now come to the original content of this paper, which is to compute the real part of a twisted-by-grading spectral triple

$$(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D)_\rho. \quad (12)$$

Being graded, such a spectral triple necessarily has an even KO -dimension. We show below that in KO -dimension 0, 4 the real part of (12) is the twist by grading of the real part of the initial spectral triple $(\mathcal{A}, \mathcal{H}, D)$, whereas in KO -dimension 2, 6, the real part is the intersection $\mathcal{A} \cap \mathcal{A}^\circ$ of the algebra with its opposite.

3.1 Real part of a twisted spectral triple

The *real part* of a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$ – as defined in [7] – is the spectral triple $(\mathcal{A}_J, \mathcal{H}, D)$ where \mathcal{A}_J is the subalgebra of \mathcal{A} generated by the elements that commute with the real structure. This definition easily generalizes to the twisted case, thanks to following proposition which is a twisted version of [12, Proposition 1, p. 125].

Proposition 1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})_\rho$ be a real twisted spectral triple with real structure J . Then, the following holds:*

1. *The equality*

$$\mathcal{A}_J = \{a \in \mathcal{A} \mid aJ = Ja\}$$

defines an involutive commutative real subalgebra of the center of \mathcal{A} .

2. *If the twisting automorphism ρ is induced by a unitary operator compatible with the real structure in the sense of (10), then $(\mathcal{A}_J, \mathcal{H}, \mathcal{D})_\rho$ is a real twisted spectral triple.*
3. *Any $a \in \mathcal{A}_J$ twist-commutes with the algebra generated by the sums $a^i[\mathcal{D}, b_i]_\rho$ for a^i, b_i in \mathcal{A} .*

Proof. The proof is a straightforward adaptation of the non-twisted case.

1. By construction \mathcal{A}_J is a real subalgebra of \mathcal{A} (but not a complex one, being J antilinear). Since J is unitary, and remembering that the usual rule of adjoint for a product of operators also holds in the antilinear case as soon as the product involves an even number of antilinear operators, one has

$$(JaJ^{-1})^* = Ja^*J^{-1} \quad \forall a \in \mathcal{A}.$$

Since for $a \in \mathcal{A}_J$ one has $JaJ^{-1} = a$, one gets

$$Ja^*J^{-1} = a^*, \quad (13)$$

meaning $a^* \in \mathcal{A}_J$, that is \mathcal{A}_J is an involutive algebra. To show that it is contained in the center of \mathcal{A} , notice that (13) implies

$$a^\circ = a^* \quad \forall a \in \mathcal{A}_J, \quad (14)$$

so that the order zero condition (4) yields $[b, a^*] = 0$ for any $b \in \mathcal{A}$.

2. Being \mathcal{A}_J a subalgebra of \mathcal{A} , the only point one needs to check is the stability of \mathcal{A}_J under the automorphism ρ , that is

$$\rho(a) \in \mathcal{A}_J \quad \forall a \in \mathcal{A}_J.$$

This is guaranteed by the compatibility (10) of the real structure with ρ : for $a \in \mathcal{A}_J$ one has

$$J\rho(a)J^{-1} = J\mathcal{R}a\mathcal{R}^\dagger J^{-1} = (\epsilon''')^2 \mathcal{R}(JaJ^{-1})\mathcal{R}^\dagger = \mathcal{R}a\mathcal{R}^\dagger = \rho(a).$$

The definition of the twisted commutator yields

$$[a^i[\mathcal{D}, b_i]_\rho, a^\circ]_{\rho^\circ} = a^i [[\mathcal{D}, b_i]_\rho, a^\circ]_{\rho^\circ} + [a^i, \rho^\circ(a^\circ)] [\mathcal{D}, b_i]_\rho.$$

The first term on the right hand side is zero by the twisted first order condition (5), the second vanishes because of the order zero condition (4), remembering that $\rho^\circ(a^\circ) = (\rho(a))^\circ$ (this follows from (6) because the flip automorphism is its own inverse). By (14) one then has that $a_i[\mathcal{D}, b_i]_\rho$ twist-commutes with any a^* , hence the result. \blacksquare

This definitions makes sense also when \mathcal{A} is a real algebra, as for the standard model.

Notice that the compatibility condition (10) between the real structure and the twisting automorphism is important to guarantee the stability of \mathcal{A}_J under ρ . Whether this condition is necessary in order to be able to define the real part of a twisted spectral triple should be further investigated [17].

3.2 Real part of a twisted-by-grading

How the real part behaves under the twist-by-grading heavily depends on the KO -dimension. In KO -dimension 0, 4, the grading commutes with the real structure and we show below that the real part of the twist-by-grading is the twist-by-grading of the real part. In KO -dimension 2, 6, the grading anticommutes with the real structure (as for the standard model which has KO -dimension 2) and the real part is the intersection of the algebra with its opposite.

Proposition 2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a real graded spectral triple with real part \mathcal{A}_J . If its twist-by-grading $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D)_\rho$ is compatible with the real structure, then its real part is either $\mathcal{A}_J \otimes \mathbb{C}^2$ in KO -dimension 0, 4, or it is $\mathcal{A} \cap \mathcal{A}^\circ$ in KO -dimension 2, 6.*

Proof. In KO -dimension 0, 4 the grading and the real structure commute (i.e., $\epsilon'' = 1$) so that

$$[J, \pi(a, a')] = \frac{1}{2}[J, (\mathbb{I} + \Gamma)a] + \frac{1}{2}[J, (\mathbb{I} - \Gamma)a'] = \frac{1}{2}[J, a + a'] + \frac{1}{2}[J, \Gamma(a - a')] \quad (15)$$

reduces to

$$\frac{1}{2}[J, a + a'] + \frac{1}{2}\Gamma[J, a - a'].$$

If a, a' are in \mathcal{A}_J , this is zero, showing that

$$\mathcal{A}_J \otimes \mathbb{R}^2 \subset (\mathcal{A} \otimes \mathbb{C}^2)_J. \quad (16)$$

To show the opposite inclusion, assume that $(a, a') \in (\mathcal{A} \otimes \mathbb{C}^2)_J$. Hence (15), rewritten as

$$\frac{1}{2}(\mathbb{I} + \Gamma)[J, a] + \frac{1}{2}(\mathbb{I} - \Gamma)[J, a']$$

is zero. Multiplying by $\mathbb{I} + \Gamma$ and $\mathbb{I} - \Gamma$ one obtains

$$(\mathbb{I} + \Gamma)[J, a] = 0, \quad (\mathbb{I} - \Gamma)[J, a'] = 0. \quad (17)$$

By compatibility of the real structure with the twist, (a', a) also belongs to $(\mathcal{A} \otimes \mathbb{C}^2)_J$ so that

$$(\mathbb{I} + \Gamma)[J, a'] = 0, \quad (\mathbb{I} - \Gamma)[J, a] = 0. \quad (18)$$

Combining (17), (18), one gets $[J, a] = [J, a'] = 0$, that is a and a' are in \mathcal{A}_J . In other terms, $(\mathcal{A} \otimes \mathbb{C}^2)_J \subset \mathcal{A}_J \otimes \mathbb{R}^2$. Together with (16), this shows the first statement of the proposition.

In KO -dimension 2, 6, grading and real structure anti-commute ($\epsilon'' = -1$) so that

$$\begin{aligned} [J, \pi(a, a')] &= \frac{1}{2}(J(\mathbb{I} + \Gamma)a - (\mathbb{I} + \Gamma)aJ + J(\mathbb{I} - \Gamma)a' - (\mathbb{I} - \Gamma)a'J) \\ &= \frac{1}{2}((\mathbb{I} - \Gamma)Ja - (\mathbb{I} + \Gamma)aJ + (\mathbb{I} + \Gamma)Ja' - (\mathbb{I} - \Gamma)a'J) \\ &= \frac{1}{2}((\mathbb{I} - \Gamma)(Ja - a'J) + (\mathbb{I} + \Gamma)(Ja' - aJ)). \end{aligned} \quad (19)$$

For (a, a') in $(\mathcal{A} \otimes \mathbb{C}^2)_J$, this is zero. The same is true for (a', a) by the invariance of $(\mathcal{A} \otimes \mathbb{C}^2)_J$ by the twist. Multiplying by $\mathbb{I} \pm \Gamma$ one thus obtains

$$\begin{aligned} (\mathbb{I} - \Gamma)(Ja - a'J) &= 0, & (\mathbb{I} + \Gamma)(Ja' - aJ) &= 0, \\ (\mathbb{I} - \Gamma)(Ja' - aJ) &= 0, & (\mathbb{I} + \Gamma)(Ja - a'J) &= 0. \end{aligned}$$

Combining these two sets of equations yields $Ja' = aJ$ and $Ja = a'J$, so that

$$a' = JaJ^{-1} = (a^*)^\circ. \quad (20)$$

Therefore $a = Ja'J^{-1}$ is in \mathcal{A}° , but also in \mathcal{A} by hypothesis. In other terms, any element of $(\mathcal{A} \otimes \mathbb{C}^2)_J$ is of the type $(a, (a^*)^\circ)$ with $a \in \mathcal{A} \cap \mathcal{A}^\circ$.

Conversely, by definition of the opposite algebra any $a \in \mathcal{A} \cap \mathcal{A}^\circ$ is equal to $b^\circ = Jb^*J^{-1}$ for some $b \in \mathcal{A}$. Then $(a^*)^\circ = JaJ^{-1} = b^*$ is in \mathcal{A} , so that $(a, (a^*)^\circ)$ is in $(\mathcal{A} \otimes \mathbb{C}^2)_J$. Inserting in (19) one gets $[J, \pi(a, (a^*)^\circ)] = 0$, meaning that $(a, (a^*)^\circ) \in (\mathcal{A} \otimes \mathbb{C}^2)_J$.

Therefore $(\mathcal{A} \otimes \mathbb{C}^2)_J \simeq \mathcal{A} \cap \mathcal{A}^\circ$, hence the second statement of the proposition. \blacksquare

Obviously \mathcal{A}_J is in $\mathcal{A} \cap \mathcal{A}^\circ$, so the real part of the initial triple is contained in the real part of the twist-by-grading, whatever the KO -dimension. However $(\mathcal{A} \otimes \mathbb{C}^2)_J$ may equal \mathcal{A}_J only in KO -dimension 2, 6, and only if the intersection $\mathcal{A} \cap \mathcal{A}^\circ$ reduces to \mathcal{A}_J . In that case, a' in (20) coincides with a and the flip ρ is the identity automorphism. The spectral triple defined in the second point of Proposition 1 is then a usual (non-twisted) spectral triple. This actually happens with the standard model as discussed in the next section.

3.3 The twisted standard model and its real part

We now compute the real part of the twist-by-grading of the Standard Model, working with one generation of fermions only, that is $n = 1$. Following [7], the 32 degrees of freedom of the finite-dimensional Hilbert space \mathcal{H}_F are labelled by a multi-index $CI\alpha$ where

- $C = 0, 1$ is for particle ($C = 0$) or anti-particle ($C = 1$);
- $I = 0, i$ with $i = 1, 2, 3$ is the lepto-colour index: $I = 0$ means lepton, while $I = 1, 2, 3$ are for the quarks, which exists in three colors;
- $\alpha = \dot{a}, a$ with $a = 1, 2$ is the flavour index:

$$\begin{aligned} \dot{1} &= \nu_R, & \dot{2} &= e_R, & 1 &= \nu_L, & 2 &= e_L & \text{for leptons } (I = 0), \\ \dot{1} &= u_R, & \dot{2} &= d_R, & 1 &= q_L, & 2 &= d_L & \text{for quarks } (I = i). \end{aligned}$$

To deal with the twist, it is convenient to label the degrees of freedom of $L^2(\mathcal{M}, S)$ by two extra-indices $s\dot{s}$ where

- $s = r, l$ is the chirality index;
- $\dot{s} = \dot{0}, \dot{1}$ denotes particle ($\dot{0}$) or anti-particle part ($\dot{1}$).

The grading is

$$\Gamma = \gamma^5 \otimes \gamma_F,$$

where $\gamma^5 = \begin{pmatrix} \delta_s^t & 0 \\ 0 & -\delta_s^t \end{pmatrix}_s^t$ is the product of the four Euclidean Dirac matrices on \mathcal{M} while γ_F takes value $+1$ on right particles and left antiparticles, and -1 on left particles and right antiparticles.

An element a of the double algebra (7) is a pair of elements of \mathcal{A} , namely

$$a = (c, c', q, q', m, m') \quad (21)$$

with

$$c, c' \in C^\infty(\mathcal{M}, \mathbb{C}), \quad q, q' \in C^\infty(\mathcal{M}, \mathbb{H}), \quad m, m' \in C^\infty(\mathcal{M}, M_3(\mathbb{C})).$$

Following the twist by grading procedure of Section 2.3, its action on \mathcal{H} is given by the 128×128 matrix

$$a = \begin{pmatrix} Q & \\ & M \end{pmatrix}_C^D \quad (22)$$

in which the element $(c, q, m) \in \mathcal{A}$ acts on the $+1$ eigenspace of \mathcal{H} (that is $s = r$ with $C = 0$, $\alpha = \dot{a}$ or $C = 1$, $\alpha = a$, and $s = l$ with $C = 0$, $\alpha = a$ or $C = 1$, $\alpha = \dot{a}$) while the element $(c', q', m') \in \mathcal{A}$ acts on the -1 eigenspace of \mathcal{H} (that is $s = r$ with $C = 0$, $\alpha = a$ or $C = 1$, $\alpha = \dot{a}$ and $s = l$ with $C = 0$, $\alpha = \dot{a}$ or $C = 1$, $\alpha = a$). Explicitly,

$$Q = \begin{pmatrix} Q_r & \\ & Q_l \end{pmatrix}_s^t, \quad M = \begin{pmatrix} M_r & \\ & M_l \end{pmatrix}_s^t,$$

are 64×64 matrices whose blocks are the 32×32 matrices

$$Q_s = \delta_s^t \begin{pmatrix} \delta_I^J c_s & \\ & \delta_I^J q_{\bar{s}} \end{pmatrix}_\alpha^\beta, \quad M_s = \delta_s^t \begin{pmatrix} \delta_a^b m_{\bar{s}} & \\ & \delta_a^b m_s \end{pmatrix}_\alpha^\beta,$$

where $s = r, l$, \bar{s} denotes the opposite chirality of s and we define the 2×2 and 4×4 matrices

$$c_s = \begin{pmatrix} c_s & \\ & \bar{c}_s \end{pmatrix}_a^b, \quad m_s = \begin{pmatrix} c_s & \\ & m_s \end{pmatrix}_I^J,$$

whose components are the elements of a :

$$c_r = c, \quad c_l = c', \quad q_r = q, \quad q_l = q', \quad m_r = m, \quad m_l = m', \quad (23)$$

identifying the quaternions q, q' with their usual representation as complex 2×2 matrices.

Note that this representation is not the one used in [18] (where the twisting operator is not the grading), neither the one of [16] (in which only the electroweak sector of the theory had been twisted). A extensive discussion on the various ways of twisting an almost commutative geometry, and the physical consequence for the standard model, is in preparation [17].

Proposition 3. *The real part of the twist-by-grading of the standard model is the (non-twisted) spectral triple $(C^\infty(\mathcal{M}, \mathbb{R}), \mathcal{H}, D)$.*

Proof. For any a as in (22), the conjugation by the real structure amounts to exchanging Q with M and taking the complex conjugate (similar proof as in [18, Proposition 3.1])

$$JaJ^{-1} = \begin{pmatrix} \bar{M} & \\ & \bar{Q} \end{pmatrix}. \quad (24)$$

The twist is the exchange of the primed quantities with the unprimed ones. From (23), this amounts to exchanging Q_r, M_r with Q_l, M_l . This operation extends as an inner automorphism of $\mathcal{B}(\mathcal{H})$, and commutes with the conjugation by the real structure as described above. In other terms,

$$J\rho(a)J^{-1} = \rho(JaJ^{-1}),$$

which shows that the twist of the standard model is compatible with the real structure in the sense of (11). Therefore, from Proposition 1, the real part of the twist-by-grading of the standard model is

$$((\mathcal{A} \otimes \mathbb{C}^2)_J, \mathcal{H}, D)_\rho. \quad (25)$$

By definition, a in (21) being in \mathcal{A}_J is equivalent to $a = JaJ^{-1}$. From (24) this is equivalent to $Q = \bar{M}$ that is, in components,

$$\begin{pmatrix} \delta_I^J \begin{pmatrix} c_s & \\ & \bar{c}_s \end{pmatrix}_a^b & \\ & \delta_I^J (q_{\bar{s}})_a^b \end{pmatrix}_\alpha^\beta = \begin{pmatrix} \delta_a^b \begin{pmatrix} \bar{c}_s & \\ & (\bar{m}_{\bar{s}})_i^j \end{pmatrix}_I^J & \\ & \delta_a^b \begin{pmatrix} \bar{c}_s & \\ & (\bar{m}_{\bar{s}})_i^j \end{pmatrix}_I^J \end{pmatrix}_\alpha^\beta.$$

In the first line, imposing the Kronecker δ 's in the a, b indices (on the right), and on the I, J indices (on the left) force to identify

$$c_s = \bar{c}_s := \lambda \in C^\infty(\mathcal{M}, \mathbb{R}) \quad \text{and} \quad m_{\bar{s}} = c_{\bar{s}} \mathbb{I}_3.$$

Then the equality between the left and the right hand terms yields $\lambda = \bar{c}_s$, hence

$$c_{\bar{s}} = \lambda, \quad m_{\bar{s}} = \lambda \mathbb{I}_3.$$

The second line of the matricial equation then gives

$$q_s = q_{\bar{s}} = \lambda \mathbb{I}_2.$$

Going back to (23), one thus obtains

$$a = (\lambda, \lambda, \lambda \mathbb{I}_2, \lambda \mathbb{I}_2, \lambda \mathbb{I}_3, \lambda \mathbb{I}_3), \quad (26)$$

hence

$$(\mathcal{A} \otimes \mathbb{C}^2)_J \simeq C^\infty(\mathcal{M}, \mathbb{R}).$$

Since the twist ρ leaves (26) invariant, the spectral triple (25) is actually non twisted. \blacksquare

The real part of the twist-by-grading of the standard model is the same as the non-twisted one (computed in [12, Section 14.1]). This is in agreement with Proposition 2: one has that $a = b^\circ$ for some $b = (R, N)$, if and only if $Q = \bar{N}$ and $M = \bar{R}$. By the same analysis as above, this shows that $a = b = \lambda \mathbb{I}$ that is, from (26), $a \in \mathcal{A}_J$. Hence $\mathcal{A} \cap \mathcal{A}^\circ = \mathcal{A}_J$.

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