

# Positive Scalar Curvature due to the Cokernel of the Classifying Map

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**Abstract.** This paper contributes to the classification of positive scalar curvature metrics up to bordism and up to concordance. Let  $M$  be a closed spin manifold of dimension  $\geq 5$  which admits a metric with positive scalar curvature. We give lower bounds on the rank of the group of psc metrics over  $M$  up to bordism in terms of the corank of the canonical map  $KO_*(M) \rightarrow KO_*(B\pi_1(M))$ , provided the rational analytic Novikov conjecture is true for  $\pi_1(M)$ .

*Key words:* positive scalar curvature; bordism; concordance; Stolz exact sequence; analytic surgery exact sequence; secondary index theory; higher index theory; K-theory

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## 1 Introduction

The study of metrics with positive scalar curvature is nowadays the focus of a very active area of research. The starting point typically will be a closed spin manifold  $M$ , and one would like to get suitable information about the possible Riemannian metrics on  $M$ .

Stephan Stolz introduced a long exact sequence for the systematic bordism classification of metrics of positive scalar curvature. For this, one has to fix an additional reference space  $X$ . Then this sequence is given by (ending with  $n = 5$  at the right)

$$\dots \longrightarrow \mathbf{R}_{n+1}^{\text{spin}}(\Gamma) \xrightarrow{\partial} \text{Pos}_n^{\text{spin}}(X) \longrightarrow \Omega_n^{\text{spin}}(X) \longrightarrow \mathbf{R}_n^{\text{spin}}(\Gamma) \xrightarrow{\partial} \dots \quad (1.1)$$

Here  $\Omega_n^{\text{spin}}(X)$  is the usual spin cobordism group, it consists of cobordism classes of cycles  $f: M \rightarrow X$ , with  $M$  a closed  $n$ -dimensional spin manifold;  $\text{Pos}_n^{\text{spin}}(X)$  is the group of bordism classes of metrics of positive scalar curvature on  $n$ -dimensional closed spin manifolds with reference map to  $X$ ; finally,  $\mathbf{R}_n^{\text{spin}}(\Gamma) := \mathbf{R}_n^{\text{spin}}(X)$  is a relative group discussed in more detail below, known to depend only on  $\Gamma := \pi_1(X)$ . The group structure in each of the three cases is given by disjoint union.

Because the starting point typically is a fixed manifold  $M$ , one has to make a suitable choice of  $X$ . The standard choice here is  $X = B\Gamma$  with  $\Gamma = \pi_1(M)$ . Note that with  $X = B\Gamma$  the Stolz sequence then contains in  $\text{Pos}_n^{\text{spin}}(B\Gamma)$  information for all spin manifolds with fundamental group  $\Gamma$  at once. This is the situation discussed in the majority of all the previous work.

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In the current article we change the paradigm a bit. We argue that, starting with  $M$ , the choice of  $X = M$  is even more canonical, and we study  $\text{Pos}_n^{\text{spin}}(M)$ . The usual applications to concordance classes of metrics of positive scalar curvature on  $M$  can still be made, and the theory is richer and more specific.

A very fruitful way to get information about the Stolz sequence (for arbitrary  $X$ ) uses the index theory of the spin Dirac operator. A systematic approach was given in [15], where the authors construct a mapping of (1.1) to the analytic surgery exact sequence of Higson and Roe (where  $\Gamma = \pi_1(X)$ )

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathbb{R}_{n+1}^{\text{spin}}(\Gamma) & \xrightarrow{\partial} & \text{Pos}_n^{\text{spin}}(X) & \longrightarrow & \Omega_n^{\text{spin}}(X) & \longrightarrow & \mathbb{R}_n^{\text{spin}}(\Gamma) & \xrightarrow{\partial} & \cdots \\
& & \downarrow \text{Ind}^\Gamma & & \downarrow \varrho & & \downarrow \beta & & \downarrow \text{Ind}^\Gamma & & \\
\cdots & \xrightarrow{\mu_X^\Gamma} & K_{n+1}(C^*\Gamma) & \longrightarrow & \mathcal{S}_n^\Gamma(\tilde{X}) & \longrightarrow & K_n(X) & \xrightarrow{\mu_X^\Gamma} & K_n(C^*\Gamma) & \longrightarrow & \cdots
\end{array} \tag{1.2}$$

A successful strategy for detecting non-trivial elements in  $\text{Pos}_n^{\text{spin}}(X)$  goes as follows: if one can construct a cycle  $\xi$  for  $\mathbb{R}_{n+1}^{\text{spin}}(\Gamma)$  such that  $\text{Ind}^\Gamma(\xi)$  maps to a nonzero element in the cokernel of  $\mu_X^\Gamma$ , then  $\partial(\xi)$  is non-zero in  $\text{Pos}_n^{\text{spin}}(X)$  because its image through  $\varrho$  does not vanish.

Indeed, this is the line followed by Weinberger and Yu in [23], where the authors define the so-called *finite part* of the K-theory of the maximal group C\*-algebra, which is proven to map injectively to the cokernel of the assembly map. Along with this they give the concrete construction of elements in  $\mathbb{R}_{n+1}^{\text{spin}}(\Gamma)$  whose higher index belongs to this finite part. Xie, Yu and Zeidler in [25] have systematized those constructions and corrected some mistakes, giving a more exhaustive description of the images of the vertical arrows in (1.2). These are complemented by a long line of results which instead make use of higher numerical invariants, such as [2, 13], where higher  $\eta$ -invariants are used, or [14], where Cheeger–Gromov  $L^2$ - $\varrho$ -invariants play an important role.

Another example of how K-theory methods could improve those which use numerical invariants can be appreciated by comparing [17] with [12], where  $\eta$ -invariants on end-periodic ends are used in order to study positive scalar curvature metrics on even dimensional manifolds. We don't want to repeat the results of this work in detail. The general pattern is: the invariants mentioned are constructed and shown to be invariants of classes in  $\text{Pos}_*^{\text{spin}}(X)$  or of the concordance classes, and then (many) elements are constructed which are distinguished by these invariants, giving rise to interesting lower bounds on the rank of  $\text{Pos}_*^{\text{spin}}(X)$ .

All the work described so far uses fundamentally that the group  $\Gamma$  contains non-trivial torsion. In particular, as it is explicitly explained in [25], the ultimate source of those constructions is the difference between  $\underline{E}\Gamma$ , the classifying space for proper actions, and  $E\Gamma$ , the classifying space for proper and free actions.

Now, if  $\Gamma$  is torsion-free, then  $\underline{E}\Gamma$  and  $E\Gamma$  coincide. Therefore one has to find an alternative source for non-trivial elements in  $\mathbb{R}_{n+1}^{\text{spin}}(\Gamma)$  whose higher index maps to non-trivial elements of the cokernel of the assembly map.

The main method of this paper is to use the homological difference between  $M$  and  $B\Gamma$  for this purpose. More generally, given  $X$  (which could be  $M$ ) with a classifying map  $u: X \rightarrow B\Gamma$  such that the homological difference between them is rich, we can construct non-trivial elements in  $\mathbb{R}_{n+1}^{\text{spin}}(\Gamma)$ . Although our main motivation was to obtain results for torsion-free fundamental groups, our constructions work for arbitrary  $\Gamma$ . In particular, we prove the following result.

**Theorem 1.1.** *Let  $X$  be a finite connected CW-complex and  $n \geq 5$ . Let  $u: X \rightarrow B\pi_1(X)$  be the classifying map of its universal covering. Let us assume that the (rational) strong Novikov conjecture holds for  $\Gamma := \pi_1(X)$ , i.e., the assembly map  $K_*(B\Gamma) \otimes \mathbb{Q} \rightarrow K_*(C^*\Gamma) \otimes \mathbb{Q}$  is injective.*

Set

$$k := \dim \left( \operatorname{coker} \left( \bigoplus_{j \geq 0} H_{n+1-4j}(X; \mathbb{Q}) \xrightarrow{u_*} \bigoplus_{j \geq 0} H_{n+1-4j}(B\Gamma; \mathbb{Q}) \right) \right),$$

and

$$k' := \dim \left( \operatorname{ker} \left( \bigoplus_{j \geq 0} H_{n-4j}(X; \mathbb{Q}) \xrightarrow{u_*} \bigoplus_{j \geq 0} H_{n-4j}(B\Gamma; \mathbb{Q}) \right) \right),$$

then

$$\operatorname{rk} \operatorname{Pos}_n^{\operatorname{spin}}(X) \geq k + k'.$$

The reason why in the statement of Theorem 1.1 we distinguish the elements coming from the kernel and those from the cokernel of  $u_*$  is that, if  $X$  is an  $n$ -dimensional closed spin manifold, those coming from the cokernel can be realized by cycles in  $\operatorname{Pos}_n^{\operatorname{spin}}(X)$  represented by the identity map. This fact is key for the main result of Section 3.

By standard surgery techniques we can refine the previous result if we look for metrics on a fixed manifold  $M$ . To formulate this, denote by  $P^+(M)$  the set of concordance classes of metrics with positive scalar curvature on an  $n$ -dimensional closed spin manifold  $M$ . In the proof of [20, Theorem 5.4], in order to construct a free and transitive action of  $\operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma)$  on  $P^+(M)$ , Stolz defines a “difference” map

$$i: P^+(M) \times P^+(M) \rightarrow \operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma) \tag{1.3}$$

such that

- $i(g, g) = 0$  and  $i(g, g') + i(g', g'') = i(g, g'')$  for all  $g, g', g'' \in P^+(M)$ ;
- the map  $i_g: P^+(M) \rightarrow \operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma)$ , which sends  $g'$  to  $i(g, g')$  is bijective for all  $g \in P^+(M)$ .

This induces on  $P^+(M)$  the structure of an  $\operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma)$ -torsor, or the structure of an affine space modelled on  $\operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma)$ . After picking any point  $g_0$  of  $P^+(M)$  as the identity,  $P^+(M)$  acquires a group structure isomorphic to  $\operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma)$ . This group structure is non-canonical as it depends on  $g_0$  and therefore seems only useful if there is a preferred  $g_0$  (e.g., one which bounds a metric of positive scalar curvature, as the standard metric on  $S^n$ ). This kind of structure is studied (and improved to an H-space structure on the space of metrics of positive scalar curvature) in [9]. We use the affine structure of  $P^+(M)$  in the last part of the following Theorem 1.2.

**Theorem 1.2.** *Let  $X$ ,  $n$ ,  $k$  and  $\Gamma$  be as in Theorem 1.1. Assume that there exists a cycle in  $\operatorname{Pos}_n^{\operatorname{spin}}(X)$ , given by  $(f: M \rightarrow X, g)$  such that  $f$  is 2-connected (i.e., inducing an isomorphism on  $\pi_0$  and  $\pi_1$  and a surjection on  $\pi_2$ ).*

*Then there are metrics with positive scalar curvature  $g, g_1, \dots, g_k$  on  $M$ , together with the fixed map  $f: M \rightarrow X$ , which*

- (1) *span an affine lattice of rank  $k$  in the abelian group  $\operatorname{Pos}_n^{\operatorname{spin}}(X)$  and hence an affine space of dimension  $k$  in  $\operatorname{Pos}_n^{\operatorname{spin}}(X) \otimes \mathbb{Q}$ ;*
- (2) *in particular, they span an affine lattice of rank  $k$  in  $\operatorname{Pos}_n^{\operatorname{spin}}(M)$  (with reference map the identity);*
- (3) *in particular, they span an affine lattice of rank  $k$  of concordance classes of positive scalar curvature metrics in  $P^+(M)$ . The lattice is with respect to the underlying structure of an affine space modelled on the abelian group  $\operatorname{R}_{n+1}^{\operatorname{spin}}(\Gamma)$ .*

Perhaps the first result which uses index methods to classify metrics of positive scalar curvature is obtained by Carr [4], where infinitely many concordance classes of metrics with positive scalar curvature are constructed even on simply connected manifolds  $M$  like the sphere (of the right dimension). This is different in spirit to our result: we prove in Remark 3.9 that the classes of Carr are all equal in  $\text{Pos}_n^{\text{spin}}(M)$ , i.e., although they are not concordant, they are all bordant.

Recently, Ebert and Randal-Williams in [8] developed a very sophisticated bordism category approach to study  $\mathcal{R}^+(M)$ , the space of the metrics with positive scalar curvature on  $M$ . Theorem C of [8] implies that, if  $M$  has even dimension  $2n$ , the fundamental group  $\Gamma$  verifies rationally the Baum–Connes conjecture and its homological dimension is less or equal to  $2n + 1$ , then the so-called index difference map is a rational surjection of  $\pi_0(\mathcal{R}^+(M))$  onto  $KO_{2n+1}(C^*\Gamma)$ .

In our results, we only assume the rational injectivity instead of bijectivity of the Baum–Connes assembly map for  $\Gamma$  and, as remarked in contrast with Carr, see [4], we obtain metrics which are not only non-isotopic, but also non-bordant. On the other hand, in [8] the authors are mainly interested in higher homotopy groups.

Finally, we provide a detailed and pedestrian proof how to pass from a bordism  $W \xrightarrow{F} X$  between  $M_0 \xrightarrow{f_0} X$  and  $M_1 \xrightarrow{f_1} X$  to a bordism  $W' \xrightarrow{F'} X$  with same ends which we call *Gromov–Lawson admissible*, meaning that it is built from  $M_0$  by attaching handles of codimension  $\geq 3$ , provided that  $f_1$  is 2-connected. This is certainly a well known and heavily used result, but does not seem treated well in a pedestrian way with all details, which we try to provide here.

The paper is organized as follows:

- In Section 2 we prove Theorem 1.1, which gives a lower bound for the rank of  $\text{Pos}_n^{\text{spin}}(X)$  in term of the difference between  $X$  and  $B\Gamma$ .
- In Section 3 we prove Theorem 1.2, which refines Theorem 1.1 to a result about concordance classes. In particular we give details how bordisms can be made Gromov–Lawson admissible in the sense mentioned above.

## 2 Mapping psc to analysis to detect bordism classes

In [15, Section 5] Piazza and Schick construct a map from the Stolz exact sequence to the Higson–Roe exact sequence (see also [24, 27] for different approaches). Instead of working with complex  $C^*$ -algebras as in [15], one can without extra effort adapt this construction to the setting of real  $C^*$ -algebras (compare [26]). All of the constructions are natural. As a result, for a finite connected CW-complex  $X$  with  $\Gamma = \pi_1(X)$  and classifying map  $u: X \rightarrow B\Gamma$  for its universal covering we obtain the following commuting diagram of Stolz exact sequences

$$\begin{array}{ccccccc}
\longrightarrow & \text{Pos}_{n+1}^{\text{spin}}(X) & \longrightarrow & \Omega_{n+1}^{\text{spin}}(X) & \xrightarrow{j_X} & \mathbb{R}_{n+1}^{\text{spin}}(X) & \xrightarrow{\partial_X} & \text{Pos}_n^{\text{spin}}(X) & \longrightarrow \\
& \downarrow u_* & & \downarrow u_*^\Omega & & \downarrow \cong & & \downarrow u_* & \\
\longrightarrow & \text{Pos}_{n+1}^{\text{spin}}(B\Gamma) & \longrightarrow & \Omega_{n+1}^{\text{spin}}(B\Gamma) & \xrightarrow{j} & \mathbb{R}_{n+1}^{\text{spin}}(\Gamma) & \xrightarrow{\partial} & \text{Pos}_n^{\text{spin}}(B\Gamma) & \longrightarrow,
\end{array} \tag{2.1}$$

which is mapped to the corresponding diagram of Higson–Roe sequences

$$\begin{array}{ccccccc}
\longrightarrow & SO_{n+1}^\Gamma(\tilde{X}) & \longrightarrow & KO_{n+1}(X) & \xrightarrow{\mu_X^\Gamma} & KO_{n+1}(C^*\Gamma) & \xrightarrow{\iota_X} & SO_n^\Gamma(\tilde{X}) & \longrightarrow \\
& \downarrow u_* & & \downarrow u_*^{KO} & & \downarrow = & & \downarrow u_* & \\
\longrightarrow & SO_{n+1}^\Gamma & \longrightarrow & KO_{n+1}(B\Gamma) & \xrightarrow{\mu_{B\Gamma}^\Gamma} & KO_{n+1}(C^*\Gamma) & \xrightarrow{\iota} & SO_n^\Gamma & \longrightarrow.
\end{array} \tag{2.2}$$

The relevant maps are the transformation of homology theories  $\beta: \Omega^{\text{spin}} \rightarrow KO$ , the APS-index map  $\text{Ind}_\Gamma: R^{\text{spin}} \rightarrow KO(C^*\Gamma)$  and the secondary index map  $\rho: \text{Pos}^{\text{spin}} \rightarrow SO^\Gamma$ . Moreover, we

set  $SO_n^\Gamma(\tilde{X}) := KO_n(D_{\mathbb{R}}^*(\tilde{X})^\Gamma)$ , the K-theory of the Roe's  $D^*$ -algebra. Finally, the universal analytic structure group  $SO_n^\Gamma$  is the limit of  $SO_n^\Gamma(Z)$  over all  $\Gamma$ -compact subspaces  $Z$  of  $E\Gamma$ .

Recall that the Pontrjagin character  $\text{Ph}: KO_*(X) \rightarrow \bigoplus_{j \in \mathbb{Z}} H_{*+4j}(X; \mathbb{Q})$  is defined as the composition of the complexification map in K-homology  $KO_*(X) \xrightarrow{\otimes \mathbb{C}} K_*(X)$  and the Chern character  $\text{Ch}: K_*(X) \rightarrow \bigoplus_{k \in \mathbb{Z}} H_{*+2k}(X; \mathbb{Q})$ . It so happens that  $\text{Ph}$  takes values in the subgroup  $\bigoplus_{j \in \mathbb{Z}} H_{*+4j}(X; \mathbb{Q})$  and is a rational isomorphism onto that subgroup. Rationally, connective real K-homology is a subspace of periodic real K-homology,  $ko_*(X) \otimes \mathbb{Q} \subset KO_*(X) \otimes \mathbb{Q}$ , under the Pontryagin character isomorphism  $ko_*(X) \otimes \mathbb{Q} \xrightarrow[\cong]{\text{Ph}} \bigoplus_{j \geq 0} H_{*-4j}(X; \mathbb{Q})$ .

**Lemma 2.1.** *Let  $X$  be a space and  $n \geq 0$ . Then the composition*

$$\Omega_n^{\text{spin}}(X) \otimes \mathbb{Q} \xrightarrow{\beta} ko_n(X) \otimes \mathbb{Q} \xrightarrow[\cong]{\text{Ph}} \bigoplus_{j \geq 0} H_{n-4j}(X; \mathbb{Q}),$$

which assigns to a cobordism class  $[M \xrightarrow{f} X]$  the class  $f_*(\hat{A}(M) \cap [M])$ , is a natural surjection. The same holds for the relative groups of a map  $u: X \rightarrow Y$ .

**Proof.** If  $x \in H_{n-4j}(X; \mathbb{Q})$  then by [25, Proposition 3.1] there exists a spin manifold  $M$  of dimension  $n - 4j$  and a map  $f: M \rightarrow X$  such that a non-zero multiple of  $x$  is the Pontrjagin character of  $\beta([f: M \rightarrow X]) = f_*[\not{D}_M] \in KO_{n-4j}(X) \otimes \mathbb{Q}$ . Finally, recall the Kummer surface  $V$ , a spin manifold whose index generates  $KO_4(*) \otimes \mathbb{Q}$ . Observe that the cartesian product of  $f: M \rightarrow X$  with  $V^j \rightarrow *$  is  $n$ -dimensional with a map to  $X$  such that the push-forward of its Pontrjagin character is still a non-zero multiple  $x$ , thanks to the multiplicativity of the Pontrjagin character with respect to Cartesian products.

The relative generalized homology groups of  $u$  are the (reduced) generalized homology groups of the mapping cone of  $u$ . Therefore, the absolute version immediately implies the relative version.  $\blacksquare$

We are now able to prove the first main result of this paper.

**Proof of Theorem 1.1.** It is well known that the natural map  $KO_*(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_*^\Gamma(E\Gamma) \otimes \mathbb{Q}$  is injective, compare, e.g., [1, Section 7]. Secondly, the rational strong Novikov conjecture for real and complex K-theory are equivalent, compare, e.g., [19]. Therefore, if the strong Novikov conjecture holds for  $\Gamma$ , it follows that  $\mu_{B\Gamma}^\Gamma: KO_*(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_*(C^*\Gamma) \otimes \mathbb{Q}$  is injective.

As  $\Omega_{n+1}^{\text{spin}}(X)$  is mapped to  $KO_{n+1}(X)$  under the map  $\beta$  from the Stolz sequences (2.1) to the Higson–Roe sequences (2.2) of [15, Section 5], after factoring out their images in the appropriate places, the following maps are well-defined:  $\beta_*: \text{coker}_{n+1}(u_*^\Omega) \rightarrow \text{coker}_{n+1}(u_*^{KO})$  and  $(\text{Ind}_\Gamma)_*: \text{coker}_{n+1}(j_X) \rightarrow \text{coker}_{n+1}(\mu_X^\Gamma)$ . Hence we obtain the following commuting diagram (using along the way the inverses of the third vertical arrow in (2.1) or (2.2), which are isomorphisms)

$$\begin{array}{ccccccc} \Omega_{n+1}^{\text{spin}}(B\Gamma) & \longrightarrow & \text{coker}_{n+1}(u_*^\Omega) & \xrightarrow{j_*} & \text{coker}_{n+1}(j_X) & \xrightarrow{(\partial_X)_*} & \text{Pos}_n^{\text{spin}}(X) \\ \downarrow \beta & & \downarrow \beta_* & & \downarrow (\text{Ind}_\Gamma)_* & & \downarrow \varrho \\ KO_{n+1}(B\Gamma) & \longrightarrow & \text{coker}_{n+1}(u_*^{KO}) & \xrightarrow{(\mu_{B\Gamma}^\Gamma)_*} & \text{coker}_{n+1}(\mu_X^\Gamma) & \xrightarrow{(\iota_X)_*} & SO_n^\Gamma(\tilde{X}). \end{array} \quad (2.3)$$

The two rightmost horizontal arrows are injective by the exactness of the top rows of (2.1) and (2.2). The middle lower arrow  $(\mu_{B\Gamma}^\Gamma)_*$  becomes injective after tensoring with  $\mathbb{Q}$ , due to the assumption that  $\Gamma$  satisfies the (rational) strong Novikov conjecture, i.e., that  $\mu_{B\Gamma}^\Gamma$  is injective

after tensoring with  $\mathbb{Q}$ , and an injective map remains injective if we quotient out the images of the same group (here  $KO_{n+1}(X) \otimes \mathbb{Q}$ ) in source and target.

By using Lemma 2.1 and that

$$k = \dim \left( \operatorname{coker} \left( \bigoplus_{j \geq 0} H_{n+1-4j}(X; \mathbb{Q}) \xrightarrow{u_*^H} \bigoplus_{j \geq 0} H_{n+1-4j}(B\Gamma; \mathbb{Q}) \right) \right),$$

we pick  $x_1, \dots, x_k \in \Omega_{n+1}^{\operatorname{spin}}(B\Gamma)$  such that their images span a  $k$ -dimensional subspace in

$$\operatorname{coker} \left( \bigoplus_{j \geq 0} u_{n+1-4j}^H \right) \cong \operatorname{coker}_{n+1}(u_*^{ko}) \otimes \mathbb{Q} \subset \operatorname{coker}_{n+1}(u_*^{KO}) \otimes \mathbb{Q}.$$

By commutativity of (2.3) and the injectivity of the lower row after tensoring with  $\mathbb{Q}$ , the images of  $x_1, \dots, x_k$  in  $\operatorname{Pos}_n^{\operatorname{spin}}(X)$  under the composition of the top horizontal arrows then span a free abelian subgroup  $W$  of  $\operatorname{Pos}_n^{\operatorname{spin}}(X)$  of rank  $k$ .

Let  $\Omega_*^{\operatorname{spin}}(u)$  be the relative generalized homology group (here spin bordism) for the map  $u: X \rightarrow B\Gamma$ . In the following commutative diagram

$$\begin{array}{ccccccc} \Omega_{n+1}^{\operatorname{spin}}(u) \otimes \mathbb{Q} & \longrightarrow & \ker_n(u_*^\Omega) \otimes \mathbb{Q} & \longrightarrow & \Omega_n^{\operatorname{spin}}(X) \otimes \mathbb{Q} & \xrightarrow{u_*^\Omega} & \Omega_n^{\operatorname{spin}}(B\Gamma) \otimes \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{j \geq 0} H_{n+1-4j}(u; \mathbb{Q}) & \longrightarrow & \bigoplus_{j \geq 0} \ker_{n-4j}(u_*) & \longrightarrow & \bigoplus_{j \geq 0} H_{n-4j}(X; \mathbb{Q}) & \xrightarrow{u_*} & \bigoplus_{j \geq 0} H_{n-4j}(B\Gamma; \mathbb{Q}) \end{array}$$

the leftmost vertical arrow is surjective by the relative version of Lemma 2.1 and the left horizontal arrows are surjective by the exactness of the pair sequence. It follows immediately that the second vertical arrow is also surjective.

Now observe that  $\ker_n(u_*^\Omega) \otimes \mathbb{Q}$ , by the commutativity of the middle square in (2.1), is also contained in the kernel of  $j_X: \Omega_n^{\operatorname{spin}}(X) \otimes \mathbb{Q} \rightarrow \operatorname{R}_n^{\operatorname{spin}}(X) \otimes \mathbb{Q}$ . Since  $\dim(\ker_n(u_*^\Omega) \otimes \mathbb{Q}) \geq k'$  and by exactness of the Stolz sequence, it lifts to a subspace  $W'$  of  $\operatorname{Pos}_n^{\operatorname{spin}}(X) \otimes \mathbb{Q}$  of dimension greater or equal to  $k'$ . It is a general fact that  $W'$  is generated by a free abelian subgroup of  $\operatorname{Pos}_n^{\operatorname{spin}}(X)$  whose rank is  $\dim(W')$ . Finally, the exactness of the Stolz sequence implies that  $W \otimes \mathbb{Q} \cap W' = \{0\}$  in  $\operatorname{Pos}_n^{\operatorname{spin}}(X) \otimes \mathbb{Q}$  and the result is proven.  $\blacksquare$

**Remark 2.2.** We are thankful to a referee for stressing a more conceptual approach to the proof of this estimation, using more directly the relative generalized homology groups. By [5, Proposition 2.4] and [5, Corollary 2.6] we have the following long exact sequence of groups

$$\dots \longrightarrow \operatorname{Pos}_{n+1}^{\operatorname{spin}}(B\Gamma) \longrightarrow \Omega_{n+1}^{\operatorname{spin}}(u) \longrightarrow \operatorname{Pos}_n^{\operatorname{spin}}(X) \xrightarrow{u_*} \operatorname{Pos}_n^{\operatorname{spin}}(B\Gamma) \longrightarrow \dots \quad (2.4)$$

Moreover in the following commutative diagram

$$\begin{array}{ccccccc} \operatorname{Pos}_{n+1}^{\operatorname{spin}}(B\Gamma) \otimes \mathbb{Q} & \longrightarrow & ko_{n+1}(B\Gamma) \otimes \mathbb{Q} & \xrightarrow{\mu_{B\Gamma}^\Gamma} & KO_{n+1}(B\Gamma) \otimes \mathbb{Q} & \xrightarrow{\mu_{B\Gamma}^\Gamma} & KO_{n+1}(C_{\mathbb{R}}^*\Gamma) \otimes \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega_{n+1}^{\operatorname{spin}}(u) \otimes \mathbb{Q} & \longrightarrow & ko_{n+1}(u) \otimes \mathbb{Q} & \longrightarrow & KO_{n+1}(u) \otimes \mathbb{Q} & & \end{array}$$

the composition of the horizontal arrows in the first row is zero as a consequence of the exactness of (2.1) and (2.2). As  $\mu_{B\Gamma}^\Gamma$  is injective by assumption, it follows that the map  $\operatorname{Pos}_{n+1}^{\operatorname{spin}}(B\Gamma) \otimes \mathbb{Q} \rightarrow ko_{n+1}(B\Gamma)$  is zero, too. Thus, we get the following factorization of the bottom line

$$\Omega_{n+1}^{\operatorname{spin}}(u) \otimes \mathbb{Q} \rightarrow \operatorname{coker}(\operatorname{Pos}_{n+1}^{\operatorname{spin}}(B\Gamma) \rightarrow \Omega_{n+1}^{\operatorname{spin}}(u)) \otimes \mathbb{Q} \rightarrow ko_{n+1}(u) \otimes \mathbb{Q}.$$

Since by Lemma 2.1 we know that  $\Omega_{n+1}^{\text{spin}}(u) \otimes \mathbb{Q} \rightarrow ko_{n+1}(u) \otimes \mathbb{Q}$  is surjective, we have that  $\text{coker}(\text{Pos}_{n+1}^{\text{spin}}(B\Gamma) \rightarrow \Omega_{n+1}^{\text{spin}}(u) \otimes \mathbb{Q} \rightarrow ko_{n+1}(u) \otimes \mathbb{Q})$  is surjective, too. Then, since by the relative Stolz exact sequence (2.4)  $\text{coker}(\text{Pos}_{n+1}^{\text{spin}}(B\Gamma) \rightarrow \Omega_{n+1}^{\text{spin}}(u))$  is a subspace of  $\text{Pos}_n^{\text{spin}}(X)$ , we conclude that

$$\dim \text{Pos}_n^{\text{spin}}(X) \otimes \mathbb{Q} \geq \dim ko_{n+1}(u) \otimes \mathbb{Q} = \dim \bigoplus_{j \geq 0} H_{n+1-4j}(u; \mathbb{Q}) = k + k'.$$

### 3 Concordance classes

The basis of most constructions of positive scalar curvature metrics is the surgery theorem of Gromov and Lawson, see [10] or [7] for full details. It says that, given a bordism  $W$  from  $M_0$  to  $M_1$  such that  $W$  is obtained from  $M_0$  attaching handles only of codimension  $\geq 3$ , then a metric of positive scalar curvature on  $M_0$  can be extended to a metric of positive scalar curvature on  $W$  with product structure near the boundary. In particular, one obtains a “transported” positive scalar curvature metric on  $M_1$ . We call bordisms satisfying the codimension condition *Gromov–Lawson admissible*.

In the following, we discuss the details how Gromov–Lawson admissible bordism  $W$  can be obtained, focusing on the not quite so obvious question why finitely many surgery steps suffice. The result appears also, e.g., as [18, Theorem 2.2] where the finiteness questions are not discussed or in a much more general setup in [11, Appendix 2].

**Proposition 3.1.** *Let  $X$  be a CW-complex. Let  $F: W \rightarrow X$  be a spin bordism with reference map between  $f_0: M_0 \rightarrow X$  and  $f_1: M_1 \rightarrow X$ . Assume  $f_1$  is 2-connected ( $\pi_j(f_1)$  is an isomorphism for  $j < 2$  and an epimorphism for  $j = 2$ ) and  $n := \dim(M_1) \geq 5$ . Then we can change  $W$  in the interior to  $F': W' \rightarrow X$  such that  $W'$  is a Gromov–Lawson admissible bordism from  $M_0$  to  $M_1$ .*

**Proof.** By standard results from surgery theory (compare [22]), the desired bordism  $W'$  is Gromov–Lawson admissible if the inclusion  $M_1 \hookrightarrow W'$  is 2-connected. We perform surgeries in the interior of  $W$  to achieve this.

**Isomorphism on  $\pi_0$ .** We treat each component  $A$  of  $X$  (or equivalently of  $M_1$ ) at a time. We have then to modify  $W_A := F^{-1}(A)$  so that it becomes connected. This is achieved by (interior) connected sum of the finitely many components of  $W'$ . Because also  $X$  is path-connected, the map  $F: W \rightarrow X$  can be extended over the connected sum of its components.

**Isomorphism on  $\pi_1$ .** The composition  $\pi_1(M_1) \rightarrow \pi_1(W) \rightarrow \pi_1(X)$  is an isomorphism, therefore the map  $\pi_1(W) \rightarrow \pi_1(X)$  is surjective. We want to modify  $W$  with further surgeries which eliminate its kernel, then  $\pi_1(W') \rightarrow \pi_1(X)$  and consequently also  $\pi_1(M_1) \rightarrow \pi_1(W')$  is an isomorphism. As  $\pi_1(M_1) \cong \pi_1(X)$  is finitely presented (as fundamental group of a smooth compact manifold) and  $\pi_1(W)$  is finitely generated, this kernel is finitely generated as a normal subgroup, see Lemma 3.2 below. So we have to do a finite number of surgeries along embedded circles (in the interior of  $W$ ). Because  $W$  is oriented, these automatically have trivial normal bundle, so surgery is possible. It is also well-known that, possibly after changing the trivialization by the non-trivial element of  $\pi_1(SO(n)) \cong \mathbb{Z}/2$ , one can equip the result of the surgeries with a spin structure. The fact that we do surgery along elements which lie in the kernel of  $\pi_1(F)$  means precisely that  $F$  can be extended over the disks and thus over the new bordism, which we continue to denote  $W$  by small abuse of notation.

**Epimorphism on  $\pi_2$ .** We finally have to perform surgeries so that  $\iota_*: \pi_2(M_1) \rightarrow \pi_2(W)$  becomes surjective, where  $\iota: M_1 \hookrightarrow W$  is the inclusion. We follow the proof of [20, Lemma 5.6] adapted to our situation. Since  $M_1$  and  $W$  are compact manifolds, the relative 2-skeleton  $(W, M_1)^{(2)}$  of  $W$  is obtained by attaching a finite number of 2-cells to  $M_1$ . To see this one

starts with a handlebody decomposition of  $W$  relative to  $M_1$  and then uses handle cancellation (the standard results from surgery theory, [22], alluded to above) to get rid of 0-handles and 1-handles using the fact that  $M_1 \hookrightarrow W$  now is 1-connected. The 2-skeleton  $(W, M_1)^{(2)}$  of the CW-decomposition of  $W$  relative  $M_1$  arising from this new handlebody is then homotopy equivalent to  $M_1 \vee (\bigvee_{j \in J} S^2)$ .<sup>1</sup>

Now, the cokernel of  $\iota_*: \pi_2(M_1) \rightarrow \pi_2(W)$  is finitely generated by these spheres  $x_j$ , for  $j \in J$ . Since  $(f_1)_*: \pi_2(M_1) \rightarrow \pi_2(X)$  is surjective, there exist elements  $\{y_j \in \pi_2(M_1)\}_{j \in J}$  such that  $(f_1)_*(y_j) = F_*(x_j)$ . It follows that the alternative generators of the cokernel given by  $\iota_*(y_j^{-1})x_j$  satisfy

$$F_*(\iota_*(y_j^{-1})x_j) = (\iota \circ F)_*(y_j^{-1})F_*(x_j) = (f_1)_*(y_j^{-1})F_*(x_j) = 0 \quad \forall j.$$

These generators we can assume embedded because  $n \geq 5$ . Since  $W$  is spin, the normal bundle of these embedded spheres is automatically trivial and surgery along them is possible.

Because of this, we can extend  $F$  over the surgeries along the alternative generators  $\iota_*(y_j^{-1})x_j$  and we obtain the desired cobordism  $F': W' \rightarrow X$  such that the inclusion of  $M_1$  into  $W'$  is a 2-equivalence.

We conclude by explaining why this now is a Gromov–Lawson admissible bordism. Start with an arbitrary handle decomposition of  $W$  relative to  $M_1$ . Now we are in the situation to apply the handle cancellation method [22] again: because the map is 2-connected, we find an alternative handle decomposition without 0-, 1-, or 2-handles. Turning this upside-down this handle decomposition can be interpreted as a handle decomposition of  $W$  relative to  $M_0$ . In this interpretation, what was previously the dimension of the handle now becomes the codimension. As the result, we obtain  $W$  from  $M_0$  attaching handles only of codimension  $\geq 3$ , as desired. ■

**Lemma 3.2.** *Let  $\alpha: \Gamma' \rightarrow \Gamma$  be a surjective group homomorphism between finitely generated groups. Assume in addition that  $\Gamma$  is finitely presented. Then the kernel of  $\alpha$  is finitely generated as a normal subgroup of  $\Gamma'$ .*

**Proof.** Let us fix a finite presentation  $\Gamma = \langle x_1, \dots, x_h; r_1, \dots, r_k \rangle$ , where the relations  $r_j$  are given by fixed words  $w_j(x_1^{\pm 1}, \dots, x_h^{\pm 1})$ . Let us fix also a finite set of generators  $\{y_1, \dots, y_n\}$  for  $\Gamma'$ . Pick  $a_1, \dots, a_h \in \Gamma'$  such that  $\alpha(a_j) = x_j$  for all  $j$  and set  $w'_l(x_1^{\pm 1}, \dots, x_h^{\pm 1}) := \alpha(y_l)$ . Then it follows that

$$\{w_1(a_1^{\pm 1}, \dots, a_h^{\pm 1}), \dots, w_k(x_1^{\pm 1}, \dots, x_h^{\pm 1}), y_1^{-1}w'_1(x_1^{\pm 1}, \dots, x_h^{\pm 1}), \dots, y_n^{-1}w'_n(x_1^{\pm 1}, \dots, x_h^{\pm 1})\}$$

is a finite set of generators as a normal subgroup for  $\ker \alpha$ . ■

Now we are ready for the proof of the main result of this section.

**Proof of Theorem 1.2.** Let us consider again the situation of Theorem 1.1, where we have classes  $x_1 = [M_1 \xrightarrow{f_1} X, h_1], \dots, x_k = [M_k \xrightarrow{f_k} X, h_k]$  in  $\text{Pos}_n^{\text{spin}}(X)$  which span a subgroup of rank  $k$ , but are trivial when mapped to  $\Omega_n^{\text{spin}}(X)$  (and a fortiori to  $\Omega_n^{\text{spin}}(B\Gamma)$ ), in particular they are null-bordant. Let us pick such null-bordisms  $F^i: Y_i \rightarrow X$ , so that  $M_i$  is the boundary of  $Y_i$  and  $f_i$  is the restriction of  $F^i$  to the boundary.

For  $i \in \{1, \dots, k\}$ , the disjoint union of  $M$  and  $M_i$  is spin bordant to  $M$ , with bordism  $G_i: W_i \rightarrow X$  given by the disjoint union of  $f \times id: M \times [0, 1] \rightarrow X$  and  $F_i: Y_i \rightarrow X$ . By Proposition 3.1 we can modify these bordisms in the interior and then assume that  $W_i$  is Gromov–Lawson admissible.

<sup>1</sup>Note that this implies a special case of the following result of Wall [21, proof of Lemma 1.1]: if  $X, Y$  are finite connected CW-complexes and  $f: X \rightarrow Y$  induces an isomorphism on  $\pi_1$ , then  $\pi_2(f)$ , here isomorphic to  $\text{coker}(\pi_2(f): \pi_2(X) \rightarrow \pi_2(Y))$  is finitely generated as a module over the (common)  $\pi_1$ .



Now we can use the Gromov–Lawson surgery theorem to “push” the given metrics  $g \amalg h_i$  of positive scalar curvature from  $M \amalg M_i$  through the new bordism to positive scalar curvature metrics  $g_i$  on  $M$ . We obtain new representatives  $[M \xrightarrow{f} X, g_i]$  of  $x_i$ . As the  $x_i$  span an affine lattice of rank  $k$  in  $\text{Pos}^{\text{spin}}(X)$ , this finishes the proof.  $\blacksquare$

Let us spell out the special case  $X = M$  of Theorem 1.2:

**Corollary 3.3.** *Let  $(M, g)$  be an  $n$ -dimensional connected spin manifold of positive scalar curvature with  $\pi_1(M) = \Gamma$ ,  $n \geq 5$ . Let  $u: M \rightarrow B\Gamma$  be an isomorphism on fundamental groups. Set*

$$k := \sum_{\substack{0 \leq j \leq n+1 \\ j-n \equiv 1 \pmod{4}}} \dim(\text{coker}(u_*: H_j(M; \mathbb{Q}) \rightarrow H_j(B\Gamma; \mathbb{Q}))).$$

*Then  $M$  admits metrics  $g_1, \dots, g_k$  of positive scalar curvature such that the elements  $(M, g), (M, g_1), \dots, (M, g_k)$  span a  $k$ -dimensional affine subspace of  $\text{Pos}_n^{\text{spin}}(M) \otimes \mathbb{Q}$ . In particular, these metrics form a  $k$ -dimensional lattice of non-concordant metrics of positive scalar curvature on  $M$ .*

**Example 3.4.** Let  $(M, g)$  be a connected  $n$ -dimensional spin manifold of positive scalar curvature such that  $\dim(H_{n+1}(B\pi_1(M); \mathbb{Q})) = k$ . Then the cokernel of the map induced by the inclusion in homology of degree  $n+1$  is  $H_{n+1}(B\pi_1(M))$ , as  $H_{n+1}(M) = 0$  for degree reasons. Therefore there exist  $k$  metrics  $g_1, \dots, g_k$  of positive scalar curvature on  $M$  which span together with  $g$  an affine space of rank  $k$  in  $\text{Pos}_n^{\text{spin}}(M)$  and, in particular, give rise to a lattice of rank  $k$  of concordance classes of positive scalar curvature metrics on  $M$ .

For example, if  $\pi_1(M) \cong \mathbb{Z}^N$  then we have  $\dim(H_{n+1}(\mathbb{Z}^N; \mathbb{Q})) = \binom{N}{n+1}$ .

**Example 3.5.** Assume that  $n \geq 5$  and  $\Gamma = \langle x_1, \dots, x_k \mid r_1, \dots, r_h \rangle$  is finitely presented. Then there exists a closed spin manifold  $M$  of dimension  $n$  with fundamental group  $\Gamma$  which admits a metric  $g$  of positive scalar curvature.

Indeed, take the wedge of  $k$  circles and, for each relation  $r_i$ , attach a two cell. Denote by  $X$  this 2-dimensional CW-complex. Finally embed  $X$  into  $\mathbb{R}^{n+1}$  and consider a tubular neighbourhood  $\mathcal{N}$  of  $X$ . Then  $M := \partial\mathcal{N}$  is an  $n$  dimensional spin manifold with fundamental group  $\Gamma$ . Observe that  $\mathcal{N}$  is a spin  $B\Gamma$  null-bordism for  $M$  or, after cutting out a disk, a  $B\Gamma$  bordism from  $S^n$  to  $M$ . By Proposition 3.1 we can assume that this bordism is Gromov–Lawson admissible. Then  $M$  admits a metric  $g$  of positive scalar curvature by the Gromov–Lawson surgery theorem.

In addition, observe that for the manifold we constructed we have a factorization  $u: M \rightarrow \mathcal{N} \rightarrow B\Gamma$  and  $\mathcal{N}$  is homotopy equivalent to a 2-dimensional CW-complex. Therefore

$$\text{im}(u_*: H_j(M) \rightarrow H_j(B\Gamma)) = \{0\} \quad \forall j \geq 3.$$

By Corollary 3.3, with  $k = \sum_{3 \leq j \leq n+1, j \equiv n+1 \pmod{4}} \dim(H_j(B\Gamma; \mathbb{Q}))$  we find metrics  $g_1, \dots, g_k$  of positive scalar curvature on  $M$  such that the  $(M, g_i)$  together with  $(M, g)$  span a  $k$ -dimensional affine subspace of  $\text{Pos}_n^{\text{spin}}(M) \otimes \mathbb{Q}$ . Note that this includes many examples where  $k = +\infty$ , whenever the rational homology of  $\Gamma$  is not finitely generated in the appropriate degrees.

**Remark 3.6.** In Example 3.5 we focused on  $\text{Pos}_n^{\text{spin}}(M) \otimes \mathbb{Q}$ . Nevertheless, as one of the referees suggested, if we consider  $P^+(M)$  we can obtain the better estimate of its rank as follows:

$$\dim P^+(M) \otimes \mathbb{Q} = \dim R_n^{\text{spin}}(M) \otimes \mathbb{Q} \geq \dim(\text{im}(\Omega_{n+1}^{\text{spin}}(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_{n+1}(B\Gamma) \otimes \mathbb{Q})),$$

where the last inequality is obtained by combining the mapping from the Stolz exact sequence to the Higson–Roe exact sequence and the assumption that  $\Gamma$  verifies the (rational) strong Novikov conjecture. Then we deduce from Lemma 2.1 that

$$\dim P^+(M) \otimes \mathbb{Q} \geq \sum_{j \geq 0} \dim H_{n+1-4j}(B\Gamma; \mathbb{Q}). \quad (3.1)$$

The difference between this lower bound and the one obtained in Theorem 1.2 is due to the fact that there can be metrics which are not concordant (hence different in  $P^+(M)$ ) but bordant (hence equal in  $\text{Pos}_n^{\text{spin}}(M)$ ). Indeed, to understand where the larger number of linearly independent metrics predicted by (3.1) compared to Theorem 1.2 come from, let us consider

$$\text{im} \left( \bigoplus_{j \geq 0} H_{n+1-4j}(M; \mathbb{Q}) \xrightarrow{u_*} \bigoplus_{j \geq 0} H_{n+1-4j}(B\Gamma; \mathbb{Q}) \right).$$

Let  $V' \subset \bigoplus_{j \geq 0} H_{n+1-4j}(M; \mathbb{Q})$  be a subspace of maximal dimension on which  $u_*$  is injective. By Lemma 2.1 we can lift it to a subspace  $V$  of  $\Omega_n^{\text{spin}}(M) \otimes \mathbb{Q}$  and, because of the assumption about the (rational) strong Novikov conjecture,  $V$  injects into  $R_n^{\text{spin}}(M) \otimes \mathbb{Q}$  and provides an additional subspace of  $P^+(M) \otimes \mathbb{Q}$ . It is immediate to see that, by the exactness of the Stolz sequence,  $V$  is mapped to 0 in  $\text{Pos}_n^{\text{spin}}(M) \otimes \mathbb{Q}$  and therefore all the metrics given by  $V$  in  $R^+(M)$  are null-bordant. We are going to see examples of this kind of metrics in Remark 3.9.

**Remark 3.7.** In special situations, the different metrics constructed in Theorem 1.2, Corollary 3.3 and the examples remain different also in the moduli space of Riemannian metrics of positive scalar curvature on  $M$ , the quotient by the action of the diffeomorphisms group. This is worked out in detail in [16]. As indicated in the introduction, this is based on the use of higher numeric rho invariants, whose behavior under the action of the diffeomorphism group can be controlled.

**Remark 3.8.** Consider the map  $i$  from (1.3). If we compose it with the map  $\text{Ind}^\Gamma$  in (1.2), it is easy to see that we obtain the map

$$\text{Ind diff}^\Gamma : P^+(M) \times P^+(M) \rightarrow KO_{n+1}(C^*\Gamma)$$

used in [8, Section 5.3]. More precisely, in [8] the map is defined on the space of isotopy classes of metrics with positive scalar curvature, but it descends to  $P^+(M)$ , using that the different definitions of  $\text{Ind diff}^\Gamma$  all coincide, proved in detail in [6] and for non-trivial  $\Gamma$  in the Münster dissertation of Buggisch [3].

It is straightforward to see that, rationally, the affine subspace generated by the lattice of  $P^+(M)$  in Theorem 1.2, (3) is mapped surjectively onto the image of the rational assembly map  $ko_{n+1}(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_{n+1}(C^*\Gamma) \otimes \mathbb{Q}$ .

**Remark 3.9.** As a predecessor construction of concordance classes which does not make use of non-trivial torsion, let us recall the construction of Carr, see [4].

First, consider the sphere  $S^{4n-1}$ . Carr takes a 2-connected  $4n$  dimensional spin manifold  $B$  with  $\hat{A}(B) = 1$  and removes two disks to obtain a bordism  $W$  from  $S^{4n-1}$  to  $S^{4n-1}$ . Positive scalar curvature surgery produces a metric of positive scalar curvature on  $W$  starting with the canonical metric on  $S^{4n-1}$  and ending with a non-concordant new metric of positive scalar curvature on  $S^{4n-1}$ .

However, these metrics are equal in  $\text{Pos}_{4n-1}^{\text{spin}}(S^{4n-1})$ . To see this, we have just to construct the reference map  $F : W \rightarrow S^{4n-1}$  which restricts to the identity on the boundary components. For this, choose a path which is a clean embedding of the closed interval into  $W$ , joining two points

in the two boundary spheres. Choose then a tubular neighbourhood of this one dimensional submanifold of  $W$ , which is necessarily trivial. Now a trivialization of the tubular neighbourhood defines a collapse map from  $W$  to  $S^{4n-1}$ , whose restriction to the boundary components is homotopic to the identity. Putting these homotopies on collar neighbourhoods of the boundary components, we obtain the desired map  $F$ .

More generally, given an arbitrary closed spin manifold  $M$  of dimension  $4n - 1$  with positive scalar curvature metric  $g$ , Carr makes a connected sum of  $M \times [0, 1]$  with  $W$  along a path parallel to the previously chosen one, to obtain a psc bordism  $V$  from  $(M, g)$  to  $(M, g')$ . These two metrics have non-zero index difference and therefore they are not concordant. Nevertheless, they are equal in  $\text{Pos}_{4n-1}^{\text{spin}}(M)$ . We obtain the desired reference map from  $V$  to  $M$  by connected sum of the previous map with the projection from  $M \times [0, 1]$  to  $M$ .

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## References

- [1] Baum P., Connes A., Higson N., Classifying space for proper actions and  $K$ -theory of group  $C^*$ -algebras, in  $C^*$ -Algebras: 1943–1993 (San Antonio, TX, 1993), *Contemp. Math.*, Vol. 167, *Amer. Math. Soc.*, Providence, RI, 1994, 240–291.
- [2] Botvinnik B., Gilkey P.B., The eta invariant and metrics of positive scalar curvature, *Math. Ann.* **302** (1995), 507–517.
- [3] Buggisch L., The spectral flow theorem for families of twisted Dirac operators, Ph.D. Thesis, Westfälische Wilhelms-Universität Münster, 2019.
- [4] Carr R., Construction of manifolds of positive scalar curvature, *Trans. Amer. Math. Soc.* **307** (1988), 63–74.
- [5] Cecchini S., Seyedhosseini M., Zenobi V.F., Relative torsion and bordism classes of positive scalar metrics on manifolds with boundary, [arXiv:2009.13718](https://arxiv.org/abs/2009.13718).
- [6] Ebert J., The two definitions of the index difference, *Trans. Amer. Math. Soc.* **369** (2017), 7469–7507, [arXiv:1308.4998](https://arxiv.org/abs/1308.4998).
- [7] Ebert J., Frenck G., The Gromov–Lawson–Chernysh surgery theorem, [arXiv:1807.06311](https://arxiv.org/abs/1807.06311).
- [8] Ebert J., Randal-Williams O., Infinite loop spaces and positive scalar curvature in the presence of a fundamental group, *Geom. Topol.* **23** (2019), 1549–1610, [arXiv:1711.11363](https://arxiv.org/abs/1711.11363).
- [9] Frenck G., H-Space structures on spaces of metrics of positive scalar curvature, [arXiv:2004.01033](https://arxiv.org/abs/2004.01033).
- [10] Gromov M., Lawson Jr. H.B., The classification of simply connected manifolds of positive scalar curvature, *Ann. of Math.* **111** (1980), 423–434.
- [11] Hebestreit F., Joachim M., Twisted spin cobordism and positive scalar curvature, *J. Topol.* **13** (2020), 1–58, [arXiv:1311.3164](https://arxiv.org/abs/1311.3164).
- [12] Kazaras D., Ruberman D., Saveliev N., On positive scalar curvature cobordisms and the conformal Laplacian on end-periodic manifolds, *Comm. Anal. Geom.*, to appear, [arXiv:1902.00443](https://arxiv.org/abs/1902.00443).
- [13] Leichtnam E., Piazza P., On higher eta-invariants and metrics of positive scalar curvature, *K-Theory* **24** (2001), 341–359.
- [14] Piazza P., Schick T., Groups with torsion, bordism and rho invariants, *Pacific J. Math.* **232** (2007), 355–378, [arXiv:math.GN/0604319](https://arxiv.org/abs/math.GN/0604319).
- [15] Piazza P., Schick T., Rho-classes, index theory and Stolz’ positive scalar curvature sequence, *J. Topol.* **7** (2014), 965–1004, [arXiv:1210.6892](https://arxiv.org/abs/1210.6892).
- [16] Piazza P., Schick T., Zenobi V.F., Higher rho numbers and the mapping of analytic surgery to homology, [arXiv:1905.11861](https://arxiv.org/abs/1905.11861).

- 
- [17] Piazza P., Schick T., Zenobi V.F., On positive scalar curvature bordism, *Comm. Anal. Geom.*, to appear, [arXiv:1912.09168](https://arxiv.org/abs/1912.09168).
  - [18] Rosenberg J.,  $C^*$ -algebras, positive scalar curvature and the Novikov conjecture. II, in Geometric Methods in Operator Algebras (Kyoto, 1983), *Pitman Res. Notes Math. Ser.*, Vol. 123, Longman Sci. Tech., Harlow, 1986, 341–374.
  - [19] Schick T., Real versus complex  $K$ -theory using Kasparov’s bivariant  $KK$ -theory, *Algebr. Geom. Topol.* **4** (2004), 333–346, [arXiv:math.KT/0311295](https://arxiv.org/abs/math/0311295).
  - [20] Stolz S., Concordance classes of positive scalar curvature metrics, Preprint, 1998, available at <http://www3.nd.edu/~stolz/concordance.ps>.
  - [21] Wall C.T.C., Finiteness conditions for CW-complexes, *Ann. of Math.* **81** (1965), 56–69.
  - [22] Wall C.T.C., Geometrical connectivity. I, *J. London Math. Soc.* **3** (1971), 597–604.
  - [23] Weinberger S., Yu G., Finite part of operator  $K$ -theory for groups finitely embeddable into Hilbert space and the degree of nonrigidity of manifolds, *Geom. Topol.* **19** (2015), 2767–2799, [arXiv:1308.4744](https://arxiv.org/abs/1308.4744).
  - [24] Xie Z., Yu G., Positive scalar curvature, higher rho invariants and localization algebras, *Adv. Math.* **262** (2014), 823–866, [arXiv:1302.4418](https://arxiv.org/abs/1302.4418).
  - [25] Xie Z., Yu G., Zeidler R., On the range of the relative higher index and the higher rho-invariant for positive scalar curvature, [arXiv:1712.03722](https://arxiv.org/abs/1712.03722).
  - [26] Zeidler R., Positive scalar curvature and product formulas for secondary index invariants, *J. Topol.* **9** (2016), 687–724, [arXiv:1412.0685](https://arxiv.org/abs/1412.0685).
  - [27] Zenobi V.F., Adiabatic groupoid and secondary invariants in K-theory, *Adv. Math.* **347** (2019), 940–1001, [arXiv:1609.08015](https://arxiv.org/abs/1609.08015).