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Accessory Parameters for Four-Punctured Spheres

Gabriele BOGO

Fachbereich Mathematik, Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany
E-mail: *bogo@mathematik.tu-darmstadt.de*

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Abstract. We study the accessory parameter problem for four-punctured spheres from the point of view of modular forms. The value of the accessory parameter giving the uniformization is characterized as the unique zero of a system of equations. This gives an effective method to compute the uniformizing differential equation. As an application, we compute numerically and study the local expansion of the real-analytic function associating to a four-punctured sphere the value of its uniformizing parameter, and make some observations on its coefficients.

Key words: accessory parameters; Fuchsian uniformization; modular forms

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1 Introduction

Classically, the uniformization of a genus g Riemann surface X with n punctures, $2g-2+n>0$, was related to second-order linear differential equations depending on $3g-3+n$ parameters called *accessory parameters*. Poincaré [17] conjectured the existence of a unique choice of the accessory parameters with the following property: the ratio of linearly independent solutions of the associated differential equation lifts to a biholomorphism between the universal covering $\widetilde{X}$ of X and the upper half-plane $\mathbb{H}$. This identification would give an explicit universal covering map for X. Despite many efforts, nobody could determine in general this choice of parameters, and the uniformization theorem was eventually proved with different techniques. The determination of the special choice of parameters is known as the *accessory parameter problem*.

Even if other approaches have proved to be better suited for the classical problem of uniformization, the accessory parameter problem is still of interest both in mathematics and physics. J. Thompson [21] discussed the algebraicity of accessory parameters of spheres with algebraic punctures in relation with Belyi's theorem; D. Chudnovsky and G. Chudnovsky [7] computed numerically the accessory parameter for genus one curves with one puncture in their numerical investigations on the Grothendiek–Katz's p-curvature conjecture; L. Takhtajan and P. Zograf [23] related the accessory parameters to the Weil–Petersson metric on the Teichmüller space of n-punctured spheres, their discoveries being stimulated by a conjecture of Polyakov's in string theory [18]. Because of the relation with Liouville theory, the computation of the accessory parameters is an active field of research in mathematical physics (see [20] for a general introduction). Finally, a relation between local deformations and extensions of symmetric tensor representations, via accessory parameters, has been investigated in [4].

In this paper we are concerned with the simplest case of the accessory parameter problem, that of a four-punctured sphere $X_\alpha := \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$ where $\alpha \in \mathbb{C} \smallsetminus \{0,1\}$. The associated family of differential equations is of the form

$$P(t)\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + P'(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + (t-\rho)y(t) = 0, \qquad P(t) := t(t-1)\big(t-\alpha^{-1}\big), \tag{1.1}$$

where $\rho \in \mathbb{C}$ is the *accessory parameter*. We call the unique choice ρ_{F} of the accessory parameter inducing the identification $\widetilde{X}_\alpha \to \mathbb{H}$ the *Fuchsian value*.

Apart from very special cases, e.g., when the differential equation (1.1) is a Picard–Fuchs differential equation [5, 22], it is still not known how to determine the Fuchsian parameter even in this simplest case. Several papers in the literature deal with the numerical computation of the Fuchsian parameter for four-punctured spheres or, equivalently, elliptic curves with one puncture. Other to the above mentioned work of Chudnovsky and Chudnovsky, one should mention the work of L. Keen, H. Rauch, and A. Vasquez [13], and J. Hoffman's Ph.D. Thesis [12]. These works are based on the observation that the monodromy group of the uniformizing differential equation, which is the Deck group of the universal covering $\mathbb{H} \to X$, is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. To require the monodromy of (1.1) to have real coefficients imposes constraints on the choice of the accessory parameters that can be used to numerically compute them. However, since (1.1) can have a monodromy group with real coefficients without being the uniformizing differential equation (in fact, this happens for a discrete set of accessory parameters), a further analysis to determine the Fuchsian one is needed. A new idea, based on the theory of Painlevé VI equation and isomonodromy deformations, leading to the numerical computation of the Fuchsian parameter has recently appeared in [1].

In this note a different approach, based on the modularity of the solution of the uniformizing differential equation, is described. As an application, we get an efficient way to compute numerically the value of the Fuchsian parameter of a given four-punctured sphere X_α in terms of α. The main result can be stated as follows

Theorem (Theorem 4.4)**.** *The Fuchsian value for the punctured sphere X_α is the unique zero of a system of infinitely many equations constructed from the differential equation* (1.1)*.*

What makes the accessory parameter problem hard is that the dependence of the uniformization data (monodromy, covering map) on the location of the punctures is quite obscure. Theorem 4.4 shows that in the case of four-punctured spheres X_α it is possible to construct a system of equations solved by the Fuchsian parameter using only basic properties of X_α, namely the existence of non-trivial automorphisms. Nehari [16], using different ideas, also characterized the Fuchsian parameter as a zero of a system of infinitely many equations in the case all the punctures lie on the real line.

We describe the main ideas of the proof of Theorem 4.4.

1. For every choice of $\alpha \in \mathbb{C} \setminus \{0,1\}$, the punctured sphere $X_\alpha \simeq \mathbb{H}/\Gamma_\alpha$ has a Klein group of automorphisms permuting the punctures. The fixed points of these automorphisms are in correspondence with cusp representatives of the uniformizing group Γ_α. It turns out that a set of generators of Γ_α can be described purely in terms of these cusp representatives and then, via the covering map, in terms of fixed points of X_α. This is discussed in Section 3. From the point of view of the differential equation (1.1) we have the following description. If $\rho = \rho_{\mathrm{F}}$ is the Fuchsian parameter, the ratio $\eta_{\rho_{\mathrm{F}}}$ of independent solutions is an inverse of the covering map; the images of the fixed points of X_α via $\eta_{\rho_{\mathrm{F}}}$ can be used to construct the cusp representatives of Γ_α and finally the uniformizing group itself. It follows that the group constructed in this way is the monodromy group of (1.1) for $\rho = \rho_{\mathrm{F}}$.
2. For a generic choice of the accessory parameter ρ, the ratio η_ρ of independent solutions of (1.1) is not an injective map. However, there is an open set $\mathcal{B}$ of parameters such that η_ρ is injective if $\rho \in \mathcal{B}$ (such η_ρ gives a quasi-Fuchsian uniformization of X_α). Of course $\rho_{\mathrm{F}} \in \mathcal{B}$. The idea is to mimic the construction of the previous paragraph for the accessory parameters in $\mathcal{B}$. More precisely, we attach a Fuchsian group $\Gamma(\rho)$ to every ρ in an open subset of $\mathcal{B}$ by defining real numbers $c_1(\rho)$, $c_2(\rho)$ ("potential" cusp representatives) from the images of the fixed points of X_α via η_ρ. The group $\Gamma(\rho)$ is constructed from $c_1(\rho)$, $c_2(\rho)$

by using Poincaré's theorem. This is discussed in Section 4. We remark that contrary to the case $\rho = \rho_{\mathrm{F}}$, the group $\Gamma(\rho)$ is not the monodromy group of (1.1) which, for $\rho_{\mathrm{F}} \neq \rho \in \mathcal{B}$ is not a Fuchsian group.

3. When $\rho = \rho_{\mathrm{F}}$, a holomorphic solution of (1.1) lifts to a holomorphic modular form $f(\tau)$ for the uniformizing group Γ_α; its q-expansion is easily computed from the solutions of (1.1). Similarly, for any ρ we can construct a Q-expansion $f_\rho(Q)$ whose coefficients depend on ρ. For ρ in a subset of $\mathcal{B}$ we can test the modularity of $f_\rho(Q)$ with respect to the Fuchsian group $\Gamma(\rho)$ constructed in the previous step. It turns out that $f_\rho(Q)$ is modular for $\Gamma(\rho)$ if and only if $\rho = \rho_{\mathrm{F}}$ (Section 4.3). The equations describing the modular transformations of $f_\rho(Q)$ with respect to the generators of $\Gamma(\rho)$ give the system in Theorem 4.4.

As mentioned above, this construction gives an efficient method to compute numerically the Fuchsian parameter by approximating a solution of the system of equations in Theorem 4.4. Section 5 presents an application of this method to the study of the analytic properties of the Fuchsian accessory parameter function. This map associates to the four-punctured sphere X_α its Fuchsian value $\rho_{\mathrm{F}}(X_\alpha)$; we can see this as a map $\rho_{\mathrm{F}} \colon \mathbb{C} \smallsetminus \{0,1\} \to \mathbb{C}$. It is known that this map is real-analytic and not holomorphic. By using the method presented above we computed the coefficients of its local expansion for different values of α. An interesting phenomenon we can observe from the numerical data (tables at page 15) concerns the size of the coefficients of this expansion. It appears that the holomorphic part of the Fuchsian parameter function ρ_{F} is much larger than the rest. This suggests that the function ρ_{F} may have nice analytic properties, for instance be a quasiregular map.

2 Uniformization, modular forms, and differential equations

2.1 Uniformization and differential equations

We recall the classical theory in the case of hyperbolic Riemann surfaces of genus zero. A good reference for the general theory is the book [8]. Let $X := \mathbb{P}^1 \smallsetminus \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_{n-1} = 0, \alpha_n = \infty\}$, where $\alpha_i \in \mathbb{C} \smallsetminus \{0\}$ and $\alpha_i \neq \alpha_j$ if $i \neq j$, be an n-punctured sphere. Consider a second-order linear differential equation on X with holomorphic coefficients:

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + q(t)y(t) = 0.$$

Let $y_1(t)$ and $y_2(t)$ be linearly independent solutions. The ratio $\eta(t) := y_2(t)/y_1(t)$ can be analytically continued to the Riemann surface X and induces a non-constant function $\widetilde{\eta}(t) \colon \widetilde{X} \to \mathbb{C}$ on the universal covering $\widetilde{X}$ of X. It is easy to verify that $\widetilde{\eta}$ is a local biholomorphism. Conversely, every local biholomorphism $\widetilde{X} \to \mathbb{C}$ arises in this way. In particular, every global biholomorphism between $\widetilde{X}$ and a subdomain of $\mathbb{C}$, if any, arises from the ratio of linearly independent solutions of differential equations of the form [8, 9]

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \left(\frac{1}{4}\sum_{j=1}^{n-1} \frac{1}{(t-\alpha_j)^2} + \frac{1}{2}\sum_{j=1}^{n-1} \frac{m_j}{(t-\alpha_j)} \right) y(t) = 0, \tag{2.1}$$

where $m_0, \ldots, m_{n-1}$ are complex parameters, called *accessory parameters*, subject to the following relations[1]

$$\sum_{j=1}^{n-1} m_j = 0, \qquad \sum_{j=1}^{n-1} \alpha_j m_j = 1 - \frac{n}{2}. \tag{2.2}$$

[1] In the literature often appears another parameter m_n associated to the puncture at ∞; it is defined from the asymptotic expansion of the rational function in (2.1) as $t \to \infty$. It turns out that m_n can be expressed in terms of $m_1, \ldots, m_{n-1}$ and of the punctures $\alpha_1, \ldots, \alpha_{n-1}$ as $m_n = \sum_{j=1}^{n-1} \alpha_j (1 + m_j \alpha_j)$.

More precisely, for certain choices of the accessory parameters $m_1, \dots, m_{n-1}$, the ratio of linearly independent solutions induce a biholomorphic map $\widetilde{\eta}\colon \widetilde{X} \to \widetilde{\eta}(\widetilde{X}) \subset \mathbb{C}$.

The name "accessory parameters" is due to the fact that the choice of $m_1, \dots, m_{n-1}$ does not affect the local behaviour of solutions of (2.1) near the singular points (but of course influences the global behaviour of the solutions).

Nowadays it is well known that the space of accessory parameters inducing a biholomorphism is a non-empty open connected set $\mathcal{B}$ in $\mathbb{C}^{n-3}$ called the *Bers slice* [3]. The image of the map $\widetilde{\eta}\colon \widetilde{X} \to \mathbb{C}$ is in general a *quasidisk*, i.e., the image of a disk under a quasiconformal transformation, with a nowhere-smooth boundary of Hausdorff dimension > 1. However, for a special choice of the accessory parameters the universal covering $\widetilde{X}$ is identified, via $\widetilde{\eta}$, with the upper half-plane $\mathbb{H}$. It is known since Poincaré [17] that this choice is unique; we call the unique value of the accessory parameters giving the above identification the *Fuchsian value*, and the corresponding differential equation the *uniformizing differential equation*. The monodromy group of the uniformizing differential equation is the Deck group of transformations of the covering; it follows that it is conjugated to a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (but the converse is not true: there exists infinitely many choices of the accessory parameters such that the monodromy group is discrete in $\mathrm{SL}_2(\mathbb{R})$, but the ratio of solutions does not induce a biholomorphic map on the universal covering [10].)

The *accessory parameter problem* consists in finding the Fuchsian value for a given punctured sphere X. This problem turned out to be very hard and only partial or numerical solutions for spheres with a low number of punctures (in fact, only four punctures) have been found. It is worth noting that even the existence of the Fuchsian value has never been proved directly, i.e., without referring to the uniformization theorem; the only exception is the case of four-punctured spheres with real punctures, which was solved by V. Smirnov [19].

2.2 Modular forms and differential equations

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a cofinite discrete group, let $t\colon \mathbb{H}/\Gamma \to \mathbb{C}$ be a modular function, and let $f \in M_k(\Gamma)$ be a modular form of weight k on Γ. If we express locally f as a function of t, i.e., $\phi(t(\tau)) = f(\tau)$, then the function $\phi(t)$ satisfies a linear differential equation of order $k+1$ with algebraic coefficients. Similarly, a k-th root of f, if expressed as a function of t, satisfies a linear differential equation of order 2 with algebraic coefficients; a local basis of solutions is given by $\{\phi(t), \hat{\phi}(t)\}$ where $\phi(t(\tau)) = f^{1/k}(\tau)$, $\hat{\phi}(t(\tau)) = \tau f^{1/k}(\tau)$ (see Chapter 5 of the first part of [6] for details).

Now let Γ be of genus zero and torsion free, and let t be a Hauptmodul, i.e., a modular function that extends to an isomorphism between the compactification of $\mathbb{H}/\Gamma$ and $\mathbb{P}^1(\mathbb{C})$. In this setting the linear differential equation satisfied by $f^{1/k}$ is defined on the punctured sphere $t(\mathbb{H}/\Gamma)$ and its coefficients are rational functions of t. Since the ratio of the independent solutions $\phi(t)$, $\hat{\phi}(t)$ lifts to the coordinate $\tau \in \mathbb{H}$, we see that the differential equation satisfied by a k-th root of (every) $f \in M_k(\Gamma)$ with respect to t is the uniformizing differential equation of $t(\mathbb{H}/\Gamma)$ in the sense of the previous section. We can then reformulate the accessory parameter problem as follows.

Proposition 2.1. *The Fuchsian value is the unique choice of accessory parameters such that the holomorphic solution of the associated differential equation lifts to a k-th root of a modular form $f \in M_k(\Gamma)$ with respect to the monodromy group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$.*

If the Hauptmodul t is fixed, different choices of $f \in M_*(\Gamma)$ yield different differential equations; however, the ratio of independent solutions always lift to $\tau \in \mathbb{H}$, that is, differential equations associated to different choices of f are projectively equivalent. In particular, the equation in (2.1) correspond to the choice of the meromorphic modular form $f = \mathrm{d}t/\mathrm{d}\tau$. A way to see this is to describe the coefficients of the differential equation in terms of *Rankin–Cohen brackets*: if $f \in M_k(\Gamma)$ and $g \in M_l(\Gamma)$ these are defined as follows

$$[f,g]_1 := kfg' - lgf', \qquad [f,g]_2 := \frac{k(k+1)}{2} fg'' - 2(k+1)(l+1)f'g' + \frac{l(l+1)}{2} gf'',$$

where $' = (2\pi \mathrm{i})^{-1}\mathrm{d}/\mathrm{d}\tau$. The brackets $[f,g]_1$ and $[f,g]_2$ are modular forms of weight $k+l+2$ and $k+l+4$ respectively. If we set $g = t'$, which is a meromorphic modular form of weight 2, the quotients

$$A(t) := \frac{[f,t']}{kft'^2}, \qquad B(t) := -\frac{[f,f]_2}{k^2(k+1)f^2t'^2}$$

are modular forms of weight zero, so in particular they are rational functions of the Hauptmodul t. It is easy to verify that the differential equation satisfied by $\phi(t) = f^{1/k}(\tau)$ is given by $\mathrm{d}^2\phi(t)/\mathrm{d}t^2 + A(t)\mathrm{d}\phi(t)/\mathrm{d}t + B(t)\phi(t) = 0$. In the case also $f = t'$ a simple computation reveals that $A(t) = 0$ and $B(t)$ is the Schwarzian derivative $B(t) = \{\tau, t\}/2$, where $\tau \in \mathbb{H}$. We can then recall the classical identity (see for example the first section of [23])

$$\{\tau, t\} = \frac{1}{2}\sum_{j=1}^{n-1} \frac{1}{(t-\alpha_j)^2} + \sum_{j=1}^{n-1} \frac{m_j}{(t-\alpha_j)}$$

to conclude that the differential equation satisfied by $\phi(t) = \sqrt{t'}$ is precisely (2.1).

In Appendix A we compute the differential equation associated to (the square root of) a special choice of $f \in M_2(\Gamma)$. In the case $n = 4$ this reduces to the well-known Heun equation.

3 Four-punctured spheres

In this section we show how the generators of a torsion-free Fuchsian group with four cusps and the automorphisms of the corresponding punctured sphere are related.

Let $\alpha \neq 0, 1$ be a complex number and consider the four-punctured sphere $X_\alpha := \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$. We are going to choose a uniformizing group Γ_α and a Hauptmodul t for X_α. A priori they are not uniquely defined: the uniformizing group is determined only up to conjugacy in $\mathrm{SL}_2(\mathbb{R})$, and the composition of a given Hauptmodul with any automorphism of X still yields a Hauptmodul. In any case, the group Γ_α is torsion free and has four non-equivalent cusps; we denote the equivalence classes of cusps by $[c_1]$, $[c_2]$, $[c_3]$, $[c_4]$ (later we will fix $[c_3] = [\infty]$ an $c_4 = [0]$). The cusps are in bijection, via t, with the punctures of X_α. A picture of a fundamental domain for the action of Γ_α on $\mathbb{H}$ in the special case Γ is the congruence subgroup $\Gamma_1(5)$ is given in Figure 1. The next lemma describes the normalization of Γ_α and t we choose.

Lemma 3.1. *Let $X_\alpha = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$, $\alpha \in \mathbb{C} \smallsetminus \{0,1\}$. There exists a pair (Γ_α, t) with $t \colon \overline{\mathbb{H}/\Gamma_\alpha} \xrightarrow{\sim} \overline{X_\alpha} = \mathbb{P}^1(\mathbb{C})$ and such that*

1) *the group Γ_α has inequivalent cusps $[\infty]$, $[0]$ and the stabilizer $\Gamma_{\alpha\infty}$ of ∞ in Γ_α is generated by*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_\alpha;$$

2) *the values of t at the inequivalent cusps $\infty, 0$ are $t(\infty) = 0$ and $t(0) = \alpha^{-1}$.*

These choices uniquely determine Γ and t.

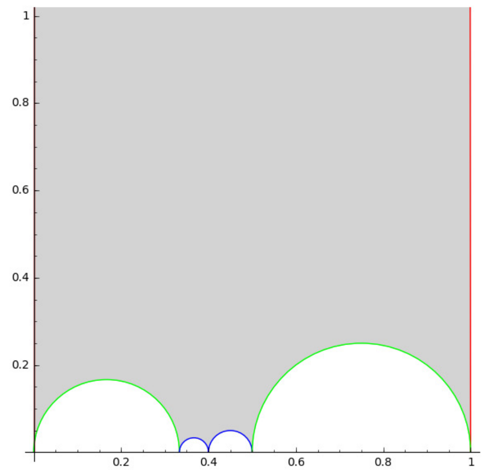

Figure 1. Fundamental domain of $\Gamma_1(5)$. The cusp representatives are $0, 1/3, 2/5$ and ∞.

Proof. Let $\hat\Gamma$ and $\hat t$ be such that $X_\alpha = \hat t\big(\mathbb{H}/\hat\Gamma\big)$. If $\hat\Gamma$ and $\hat t$ are as in the statement we are done. If not, compose $\hat t$ with an automorphism ϕ of X_α in such a way that $\hat t_1 := \phi\circ\hat t$ maps the cusp ∞ to the puncture 0 (such automorphism of X_α always exists, see the next section). For nonzero real numbers a, b consider the matrix $\sigma := \left(\begin{smallmatrix} a & b \\ 0 & 1/a\end{smallmatrix}\right)$ and define

$$\Gamma(a,b) := \sigma\hat\Gamma\sigma^{-1}, \qquad t(\tau) := \hat t_1\big(\sigma^{-1}\tau\big).$$

The map t is by construction a Hauptmodul for $\Gamma(a,b)$; since $\sigma\infty = \infty$ and $\hat t_1(\infty) = 0$ we also have $t(\infty) = 0$. The conjugation of $\hat\Gamma$ by σ amounts to determine the coordinate τ on $\mathbb{H}$ given by the uniformizing differential equation up to a linear map $\tau \mapsto a\tau + b$, where $a, b \in \mathbb{R}$. To fix it uniquely we only need to choose a and b. A simple computation shows that the generator of $\Gamma(a,b)_\infty$, the stabilizer of ∞ in $\Gamma(a,b)$, only depends on the choice of a; we can choose $a = \bar a$ such that $\Gamma(\bar a, b)_\infty = \langle T\rangle$. Finally, we see that $t(0) = \hat t_1\big(\sigma^{-1}0\big) = \hat t_1(b)$, and we can pick any $\bar b \in \mathbb{R}$ such that $\hat t_1(\bar b) = \alpha^{-1}$. Then $\big(\Gamma_\alpha := \Gamma(\bar a, \bar b), t\big)$ is the desired pair. ■

In the following, Γ_α and t will always be normalized as in the above lemma. In this case, the Fourier expansion of t at ∞ starts

$$t = rq + \cdots, \qquad q = \mathrm{e}^{2\pi\mathrm{i}\tau}, \quad \tau \in \mathbb{H}, \tag{3.1}$$

for some $r \in \mathbb{C} \smallsetminus \{0\}$.

3.1 Generators of the uniformizing group

The goal of this section is to write a set of parabolic generators of a torsion-free genus zero Fuchsian group Γ with four cusps only in terms of cusp representatives. By a set of parabolic generators we mean a set of matrices $\{M_1, \ldots, M_4\}$ with $\mathrm{Tr}^2(M_i) = 4$ that generate Γ with the relation $\prod_{i=1}^4 M_i = \mathrm{Id}$.

It follows from the existence of non-trivial automorphisms of $\mathbb{H}/\Gamma$ that the cusps of Γ are all regular or irregular. In the next lemma (and in the rest of the paper) we assume that the cusps are regular; the case of irregular cusps can be handled analogously.

Lemma 3.2. *Let Γ be a torsion free Fuchsian group of genus zero with four cusps. Assume that $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right) \in \Gamma$, and let $0 < c_1 < c_2$ be representatives of the non-equivalent finite cusps. Then $\Gamma = \big\langle T, S_0, S_{c_1}, S_{c_2} \,|\, S_{c_2}S_{c_1}S_0T^{-1} = \mathrm{Id}\big\rangle$ where*

$$S_0 = \begin{pmatrix} 1 & 0 \\ D_0 & 1\end{pmatrix}, \qquad S_{c_1} = \begin{pmatrix} 1 + c_1D_{c_1} & -c_1^2D_{c_1} \\ D_{c_1} & 1 - c_1D_{c_1}\end{pmatrix}, \qquad S_{c_2} = \begin{pmatrix} 1 + c_2D_{c_2} & -c_2^2D_{c_2} \\ D_{c_2} & 1 - c_2D_{c_2}\end{pmatrix},$$

and the constants D_0, D_{c_1}, D_{c_2} are given by

$$D_0 = \frac{1}{c_1(1-c_2)}, \qquad D_{c_1} = \frac{1}{c_1(c_2-c_1)}, \qquad D_{c_2} = \frac{1}{(c_2-c_1)(1-c_2)}. \tag{3.2}$$

Proof. It is well known that Γ is generated by the stabilizers of its cusps, and that the stabilizer of the finite (regular) cusp c_i is of the form

$$S_{c_i} = \begin{pmatrix} 1+c_iD_{c_i} & -c_i^2D_{c_i} \\ D_{c_i} & 1-c_iD_{c_i} \end{pmatrix} \tag{3.3}$$

for some positive $D_{c_i} \in \mathbb{R}$. The only thing to prove are the formulae in (3.2).

The choice of cusp representatives in the statement fixes a fundamental domain $\mathcal{F}$ for the action of Γ. It is well known that a free generating set for Γ is given by the Möbius transformations which pairs the boundary geodesics of $\mathcal{F}$. Among these transformations there is one that fixes one of the finite cusp representatives (see for instance Figure 1, where this cusp representative is $2/5$). In our case, the fixed cusp representative is c_2, since we have set $0 < c_1 < c_2$. Call S_{c_2} the transformation that fixes c_2 and pairs the relative boundary geodesics.

The transformation S_{c_2} also exchanges c_1 with its equivalent $c_1' > c_2$. There is a transformation that exchanges c_1 with c_1', and sends 0 to 1; call it P_{0,c_1}. It follows that

$$S_{c_1} := S_{c_2}^{-1}P_{0,c_1}$$

fixes c_1. In the same way $S_0 := P_{0,c_1}^{-1}T$ fixes 0. The matrices S_*, $* = 0, c_1, c_2$ generate the stabilizer of the cusp $*$ and satisfy the parabolic relation $S_{c_2}S_{c_1}S_0T^{-1} = \mathrm{Id}$. It follows that every S_*, is of the form (3.3) and then one can compute the real numbers D_*, $* = 0, c_1, c_2$, by solving the system given by the parabolic relation

$$\begin{pmatrix} 1+c_2D_{c_2} & -c_2^2D_{c_2} \\ D_{c_2} & 1-c_2D_{c_2} \end{pmatrix} \begin{pmatrix} 1+c_1D_{c_1} & -c_1^2D_{c_1} \\ D_{c_1} & 1-c_1D_{c_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D_0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The formulae in (3.2) follow after an easy computation. ■

3.2 Automorphisms of four-punctured spheres and cusp representatives

For every choice of $\alpha \neq 0, 1$, the surface X_α admits a Klein four-group of automorphisms $\mathrm{Aut}_0(X_\alpha)$ generated by any two of the involutions

$$\phi_0\colon\ t \mapsto \frac{1-\alpha t}{\alpha(1-t)}, \qquad \phi_1\colon\ t \mapsto \frac{t-1}{\alpha t-1}, \qquad \phi_2\colon\ t \mapsto \frac{1}{\alpha t}, \tag{3.4}$$

where $\phi_0 = \phi_1 \circ \phi_2$. In general $\mathrm{Aut}(X_\alpha) = \mathrm{Aut}_0(X_\alpha)$, but for exceptional choices of α, the automorphism group of X_α is larger. If $\alpha = -1, 1/2, 2$ then $\mathrm{Aut}(X_\alpha)$ has order 8; if $\alpha = 1/2 \pm \mathrm{i}\sqrt{3}/2$ then $\mathrm{Aut}(X_\alpha)$ has order 12. In these exceptional cases, the Fuchsian parameter and the uniformization of X_α can be easily computed (see [11]).

Let $t\colon \mathbb{H}/\Gamma \to X_\alpha$ be normalized as in Lemma 3.1. Every automorphism $\phi \in \mathrm{Aut}(X_\alpha)$ lifts to an automorphism $\tilde{\phi}$ of the universal covering $\mathbb{H}$. Every such automorphism $\tilde{\phi}$ can be represented by a matrix $W_\phi \in \mathrm{SL}_2(\mathbb{R})$ and belongs to the normalizer $N(\Gamma)$ of Γ. This, together with Lemma 3.1, implies that $W_\phi T W_\phi^{-1} \in \Gamma$. In particular, if $\phi \in \mathrm{Aut}_0(X_\alpha)$ and W_ϕ sends the cusp $[c]$ to the cusp ∞, the element $S_c := W_\phi T W_\phi^{-1}$ sends the cusp $[c]$ to itself, so it belongs to the stabilizer Γ_c. Actually, more is true.

Lemma 3.3. *Let W_ϕ be an involution of $\mathbb{H}$ obtained by lifting $\phi \in \mathrm{Aut}_0(X_\alpha)$ and such that $W_\phi(c) = \infty$ for $[c]$ a finite cusp of Γ. Then the transformation $S_c = W_\phi T W_\phi^{-1}$ generates the stabilizer Γ_c of $[c]$ in Γ.*

From Lemma 3.3 and (3.3) it follows that W_ϕ is of the form

$$W_\phi = \sqrt{D_c}\begin{pmatrix} c & \dfrac{-1-c^2D_c}{D_c} \\ 1 & -c \end{pmatrix}$$

for some representative c of $[c]$ and some positive constant D_c. In other words, we can describe every lift W_ϕ of $\phi \in \mathrm{Aut}_0(X_\alpha)$ in terms of the cusp representatives c_1, c_2 and the positive real constants D_0, D_{c_1}, D_{c_2} in (3.2). The transformation W_ϕ has a unique fixed point in $\mathbb{H}$ given by

$$\tau_\phi = c + \mathrm{i}/\sqrt{D_c}; \tag{3.5}$$

its image via t is a fixed point of the involution $\phi \in \mathrm{Aut}_0(X_\alpha)$. The next lemma establish which fixed points of the automorphisms $\phi_j \in \mathrm{Aut}_0(X_\alpha)$, $j = 0, 1, 2$, are images of the fixed points τ_j of W_{ϕ_j}. In the next lemma, and in the rest of the paper, we assume that α satisfies

$$\left|\alpha^{-1}\right| \leq 1 \qquad \text{and} \qquad \mathrm{Re}\left(\alpha^{-1}\right), \mathrm{Im}\left(\alpha^{-1}\right) \geq 0. \tag{3.6}$$

One can always reduce to this case via conformal transformations and complex conjugation. The behavior of the accessory parameters with respect to these transformations is known (see [11]).

Lemma 3.4. *Let α^{-1} be as in (3.6) and let $t\colon \mathbb{H}/\Gamma \to X_\alpha$ be as in Lemma 3.1. Let W_{ϕ_j} be the lifting of $\phi_j \in \mathrm{Aut}_0(X_\alpha)$, $j = 0, 1, 2$. Then the image $z_j \in X_\alpha$ of the unique fixed point τ_j of W_{ϕ_j} on $\mathbb{H}$ is given, in terms of α, by*

$$z_0 = \frac{\alpha - \sqrt{\alpha(\alpha-1)}}{\alpha}, \qquad z_1 = \frac{1+\sqrt{1-\alpha}}{\alpha}, \qquad z_2 = \frac{1}{\sqrt{\alpha}}.$$

Proof. The fixed points of ϕ_0 are the solutions of

$$\alpha t^2 - 2\alpha t + 1 = 0. \tag{3.7}$$

Since $\phi_0(0) = \alpha^{-1}$ and $t(\infty) = 0$, $t(0) = \alpha^{-1}$, the lift W_{ϕ_0} sends the cusp ∞ to 0. It follows that the fixed point of W_{ϕ_0} on $\mathbb{H}$ is $\tau_0 = \mathrm{i}/\sqrt{D_0}$. As τ_0 lies on the imaginary axis, its image on X_α belongs to the geodesic (determined by t) joining the punctures $t(\infty) = 0$ and $t(0) = \alpha^{-1}$. Looking at the two roots of (3.7) and considering the constraints (3.6) on α^{-1} it follows that

$$t(\tau_0) = z_0 = 1 - \frac{\sqrt{\alpha(\alpha-1)}}{\alpha}.$$

Now consider the involution $\phi_1 \in \mathrm{Aut}_0(X_\alpha)$ defined in (3.4). The fixed points of ϕ_1 are $\alpha^{-1} \pm \left(\sqrt{1-\alpha}\right)/\alpha$. Since $|\alpha| > 1$, none of these roots is real; one lies above the real axis and the other below. The fundamental domain for Γ that we fixed in Lemma 3.2 lies at the left of the boundary geodesic going from $\tau = \mathrm{i}\infty$ to $\tau = 0$. This implies that the image, via t, of the fundamental domain lies above the curve on X_α that joins the punctures 0, α^{-1}. This implies that the root we have to choose is the one with positive imaginary part. If $\tau_1 = \hat{c}_{\phi_1} + 1/\sqrt{D_1}$ is the fixed point on $\mathbb{H}$ of the lift of ϕ_1, we have

$$t(\tau_1) = z_1 = \frac{1}{\alpha} + \frac{\sqrt{1-\alpha}}{\alpha}.$$

Similar considerations apply to the choice of the fixed points z_2 of the third non-trivial involution of X_α. ■

4 Finding the Fuchsian value

4.1 "Potential" modular forms

The family of differential equations associated to X_α, determined from the general one in (2.1) by using the relations (2.2), is given by

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}+\left(\frac{P(t)'^2-P(t)P(t)''}{4P(t)^2}-\frac{t-m_0\alpha^{-1}}{2P(t)}\right)y(t)=0,\qquad P(t):=t(t-1)\big(t-\alpha^1\big).$$

In the following, we will not consider the above differential equation, but the projectively equivalent one

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}+\frac{P'(t)}{P(t)}\frac{\mathrm{d}y(t)}{\mathrm{d}t}+\frac{(t-\rho)}{P(t)}y(t)=0, \tag{4.1}$$

which is known as the *Heun equation.* The new accessory parameter ρ is related to m_0 by $m_0 = 1+\alpha-2\rho$. As it will be clear, our results do not depend on the choice of the differential equation. We work with the Heun equation because its solutions are better behaved from the modular point of view; this will be relevant in numerical applications of our main result. It can be shown in fact that when $\rho=\rho_{\mathrm{F}}$ is the Fuchsian value, the holomorphic solution lifts to (the square root of) a weight two modular form with a double zero in the cusp where the Hauptmodul t has its unique pole (see Appendix A for the details).

The differential equation (4.1) has at every finite singularity a holomorphic solution and a solution with a logarithmic singularity. In particular, near the regular singular point $t=0$ a basis of solutions is given by

$$\begin{aligned}
y_{\alpha,\rho}(t)&=\sum_{n\geq 0}a_n(\alpha,\rho)t^n=1+\alpha\rho t+\frac{\alpha^2}{4}\big(\rho^2-2\rho(\alpha+1)-\alpha\big)t^2+\cdots,\\
\hat{y}_{\alpha,\rho}(t)&=\log(t)y_{\alpha,\rho}(t)+\sum_{n\geq 0}b_n(\alpha,\rho)t^n=\log(t)y_{\alpha,\rho}(t)+\alpha(-2\rho+\alpha+1)t+\cdots,
\end{aligned}$$

where the coefficients $a_n(\alpha,\rho)$, $b_n(\alpha,\rho)$ are polynomials in ρ of degree n and satisfy the following linear recursions (Frobenius method)

$$\begin{aligned}
&\alpha n^2a_{n-1}(\rho)-\big((\alpha+1)\big(n^2+n\big)+\rho\big)a_n(\rho)+(n+1)^2a_{n+1}(\rho)=0,\\
&\alpha n^2b_{n-1}(\rho)-\big((\alpha+1)\big(n^2+n\big)+\rho\big)b_n(\rho)+(n+1)^2b_{n+1}(\rho)\\
&\qquad+2\alpha na_{n-1}(\rho)-(2n+1)(\alpha+1)a_n(\rho)+2(n+1)a_{n+1}(\rho)=0,
\end{aligned}$$

with initial data $a_n=0$ if $n<0$, $a_0=1$ and $b_n=0$ if $n\leq 0$.

The relevant function for the uniformization of X_α is the ratio of the two solutions y_ρ, $\hat{y}_\rho$. However, due to the logarithmic term, using power series it is more appropriate to work with the exponential of this ratio

$$Q_{\alpha,\rho}(t):=\exp(\hat{y}_{\alpha,\rho}(t)/y_{\alpha,\rho}(t))=\sum_{n\geq 1}Q_n(\alpha,\rho)t^n=t+\alpha(-2\rho+\alpha+1)t^2+\cdots. \tag{4.2}$$

The function $Q_{\alpha,\rho}(t)$ is a local biholomorphism as a function of t; inverting the series (4.2) we find the Q-expansion of its local inverse $t_{\alpha,\rho}(Q)$ around $Q=0$:

$$t_{\alpha,\rho}(Q)=\sum_{n\geq 1}t_n(\alpha,\rho)Q^n=Q-\alpha(-2\rho+\alpha+1)Q^2+\cdots. \tag{4.3}$$

Finally, substituting the above series $t_{\alpha,\rho}(Q)$ into the holomorphic solution $y_{\alpha,\rho}(t)$, we get a new power series in Q:

$$f_{\alpha,\rho}(Q) := y_{\alpha,\rho}(t_{\alpha,\rho}(Q)) = \sum_{n\geq 0} f_n(\alpha,\rho)Q^n$$
$$= 1 + \alpha\rho Q + \frac{\alpha^2}{4}\big(9\rho^2 - 2\rho(\alpha+1) - \alpha\big)Q^2 + \cdots . \tag{4.4}$$

When the accessory parameter specializes to the Fuchsian value ρ_{F} the ratio $\hat{y}_{\alpha,\rho_{\mathrm{F}}}(t)/y_{\alpha,\rho_{\mathrm{F}}}(t)$ gives a coordinate on the universal covering $\mathbb{H}$ of X_α. It follows from (4.2) that $Q_{\alpha,\rho_{\mathrm{F}}}(t)$ is a local parameter at the cusp ∞ and that $t_{\alpha,\rho_{\mathrm{F}}}(Q)$ is the local expansion of the Hauptmodul $t\colon \mathbb{H} \to X_\alpha$ in the parameter Q. A comparison between the expressions (4.3) and (3.1) gives

$$Q = rq, \qquad \text{where} \quad q = \mathrm{e}^{2\pi\mathrm{i}\tau}, \quad \tau \in \mathbb{H}, \tag{4.5}$$

for some non-zero $r \in \mathbb{C}$. It follows that the Q-expansions (4.3), (4.4) of $t_{\rho_{\mathrm{F}}}(Q)$ and $f_{\rho_{\mathrm{F}}}(Q)$ can be turned into q-expansions, which eventually make them holomorphic functions on $\mathbb{H}$:

$$t(\tau) := t_{\alpha,\rho_{\mathrm{F}}}(rq) = \sum_{n\geq 1} \hat{t}_n q^n, \qquad \hat{t}_n = t_n(\alpha,\rho_{\mathrm{F}})r^n,$$
$$f(\tau) := f_{\alpha,\rho_{\mathrm{F}}}(rq) = \sum_{n\geq 0} \hat{f}_n q^n, \qquad \hat{f}_n = f_n(\alpha,\rho_{\mathrm{F}})r^n.$$

From the discussion following (4.1) we conclude that f^2 is a weight two modular form with respect to the uniformizing group of X_α. On the contrary, the expansions $t_{\alpha,\rho}(Q)$ and $f_{\alpha,\rho}(Q)$ are *"potential" modular forms* in the sense that they extends to holomorphic functions on $\mathbb{H}$ with modular properties only for the correct value ρ_{F} of ρ. In the following we see them as functions depending on the parameter ρ.

4.2 "Potential" cusp representatives

Consider a four-punctured sphere X_α where α^{-1} is as in (3.6) and let z_j, $j = 0,1,2$ be the fixed points of the automorphisms of X_α specified in Lemma 3.4. We are going to consider a subset $\mathcal{P}$ of the set of accessory parameters with a special property.

Definition 4.1. For every ρ consider the power series $t_{\alpha,\rho}(Q)$ defined in (4.3) and let D_ρ denote its disk of convergence centered in $Q = 0$. We say that $\rho \in \mathcal{P}$ if, for every $j = 0,1,2$, there exist $Q_j \in D_\rho$ such that $t_{\alpha,\rho}(Q_j) = z_j$.

This condition is not satisfied by most accessory parameters ρ, but it is certainly satisfied by the Fuchsian parameter ρ_{F} and, consequently, by an open subset of the set $\mathcal{B}$ of parameters realizing a quasifuchsian uniformization of X_α. For $\rho \in \mathcal{P}$ and for $j = 0,1,2$ the function

$$\frac{1}{t_{\alpha,\rho}(Q) - z_j} = \sum_{n\geq 0} T_{j,n}(\rho)Q^n, \qquad j = 0,1,2,$$

has a simple pole in Q_j and is holomorphic in a punctured domain containing Q_j. It follows that the limits

$$Q_j(\rho) := \lim_{n\to\infty} \frac{T_{j,n}(\rho)}{T_{j,n+1}(\rho)}, \qquad j = 0,1,2,$$

exist for every fixed value of $\rho \in \mathcal{P}$ and in fact define complex-valued functions of ρ. Finally, define the following real-valued functions of $\rho \in \mathcal{P}$

$$c_j(\rho) := \mathrm{Re}\left(\frac{1}{2\pi\mathrm{i}} \log\left(\frac{Q_j(\rho)}{Q_0(\rho)}\right)\right), \qquad j = 0,1,2,$$

where $\log(z) := \log|z| + \mathrm{i}\,\mathrm{Arg}(z)$ and $\mathrm{Arg}(z) \in (-\pi, \pi]$. The basic properties of the functions $c_j(\rho)$ are given in the following lemma.

Lemma 4.2.

1. *$c_0(\rho) = 0$, and $c_j(\rho) \in (-1/2, 1/2]$ for every $\rho \in \mathcal{P}$.*
2. *For every $\rho \in \mathcal{P}$ $c_i(\rho) \neq c_j(\rho)$ if $i \neq j$.*

In the next fundamental proposition we attach a Fuchsian group to every differential equation (4.1) with $\rho \in \mathcal{P}$. We remark that the Fuchsian group $\Gamma(\rho)$ attached to $\rho \in \mathcal{P}$ is in general not the monodromy group of the associated differential equation, which is a Kleinian non Fuchsian group, but a group constructed by considering the automorphisms of the four-punctured sphere. The group $\Gamma(\rho)$ is the monodromy group only when $\rho = \rho_{\mathrm{F}}$ is the Fuchsian parameter (point 3 of the proposition).

Proposition 4.3.

1. *For every $\rho \in \mathcal{P}$ there exist a unique torsion-free Fuchsian group $\Gamma(\rho)$ of genus zero with four cusps and nonequivalent cusp representatives 0, $c_1(\rho)$, $c_2(\rho)$, ∞.*
2. *For every fixed $\rho \in \mathcal{P}$ let x_0, x_1, x_2 be real numbers such that*

$$\{x_0, x_1, x_2\} = \{0, c_1(\rho), c_2(\rho)\} \qquad \textit{and} \qquad x_0 < x_1 < x_2.$$

Define, for $j = 0, 1, 2$,

$$S_j = S_j(\rho) := \begin{pmatrix} 1 + x_j A_j & -x_j^2 A_j \\ A_j & 1 - x_j A_j \end{pmatrix},$$

where $A_j = A_j(\rho) := (x_j - x_{j-1})^{-1}(x_{j+1} - x_j)^{-1}$, and $x_{-1} := x_2 - 1$, $x_3 := x_0 + 1$. Then

$$\Gamma(\rho) = \langle T, S_0(\rho), S_1(\rho), S_2(\rho) \rangle \qquad \textit{and} \qquad S_2 S_1 S_0 T^{-1} = 1.$$

3. *When $\rho = \rho_{\mathrm{F}}$ is the Fuchsian value, the group $\Gamma(\rho_{\mathrm{F}})$ is the uniformizing group of X_α.*

Proof. Consider three real numbers $x_0, x_1, x_2 \in (-1/2, 1/2]$ such that $x_0 < x_1 < x_2$. We shall associate to the triple (x_0, x_1, x_2) a torsion-free Fuchsian group with four cusps whose representatives are x_0, x_1, x_2 and ∞ by using Poincaré's theorem.

Let $x_1' := \frac{x_2 + x_1(x_2 - 1) - x_0 x_1}{x_2 - x_0}$. Using the properties $-1/2 < x_j \leq 1/2$ and $x_0 < x_1 < x_2$, it is easy to verify that $x_2 < x_1' < x_0 + 1$. Consider $\mathbb{H}$ as a model of the hyperbolic plane, and let $\mathcal{F} \subset \mathbb{H}$ be the hyperbolic geodesic polygon with vertices $\{x_0, x_1, x_2, x_1', x_0 + 1, \infty\}$. A simple calculation shows that the set of transformations $G := \left\{T, S_2, S_2 S_1^{-1}\right\}$ is a side-pairing for the geodesic boundary of $\mathcal{F}$ and $S_2 S_1 S_0 T^{-1} = 1$. We can conclude by Poincaré's theorem (see [2]) that the group generated by the transformations in G is a Fuchsian group of genus zero with no torsion and four cusps and with fundamental domain $\mathcal{F}$. The first two points of the proposition follow by choosing $x_j = x_j(\rho)$ as in the statement. We denote by $\Gamma(\rho)$ the Fuchsian group obtained in this way.

We prove point 3. When $\rho = \rho_{\mathrm{F}}$ we know by (4.5) that $Q = r e^{2\pi \mathrm{i} \tau}$ for some non-zero $r \in \mathbb{C}$. It follows that $Q_j = r e^{2\pi \mathrm{i} \tau_j}$, $j = 0, 1, 2$, where τ_j is the fixed point in $\mathbb{H}$ of the lifting of the automorphism ϕ_j of X_α (see Section 3.2). Using the description of τ_j in (3.5) we see that

$$\log(Q_j / Q_0)/(2\pi \mathrm{i}) = \tau_j - \tau_0 = c_j + \mathrm{i}\big(1/\sqrt{D_j} - 1/\sqrt{D_0}\big), \qquad j = 1, 2,$$

where c_1, c_2 are inequivalent cusps of the uniformizing group of X_α. Since $c_0(\rho_{\mathrm{F}}) = 0$ and $c_j(\rho_{\mathrm{F}}) = c_j$, $j = 1, 2$, we conclude that the group $\Gamma(\rho_{\mathrm{F}})$ constructed in point 2 is the uniformizing group of X_α. ∎

Similarly to the "potential" modular forms of Section 4.1, the functions $c_j(\rho)$ are cusps of the uniformizing group of X_α for the value ρ_{F} of the accessory parameter. For this reason, we call the $c_j(\rho)$ *"potential" cusp representatives*, even though they are actually cusps for the group $\Gamma(\rho)$ for every $\rho \in \mathcal{P}$.

4.3 Finding the Fuchsian value

In the previous section we constructed a Fuchsian group $\Gamma(\rho)$ from the differential equation (4.1) attached to the four-punctured sphere X_α if $\rho \in \mathcal{P}$. When $\rho = \rho_{\mathrm{F}}$ the group $\Gamma(\rho_{\mathrm{F}})$ is the uniformizing group of X_α and the function $t_{\alpha,\rho_{\mathrm{F}}}(rq)$ obtained by inverting the exponential of the ratio of independent solutions of (4.1) is a modular function with respect to $\Gamma(\rho_{\mathrm{F}})$.

The idea is to mimic this situation in the case $\rho \in \mathcal{P}$ in order to check whether $\rho = \rho_{\mathrm{F}}$ by checking the modularity of $t_{\alpha,\rho}(Q)$ with respect to $\Gamma(\rho)$. To do this, we need to make $t_{\alpha,\rho}(Q)$ a function on $\mathbb{H}$ for every $\rho \in \mathcal{P}$. We do it as follows.

In the proof of Proposition 4.3 we showed that when $\rho = \rho_{\mathrm{F}}$ we have $Q_0 = r\mathrm{e}^{2\pi \mathrm{i}\tau_0}$ for some non-zero $r \in \mathbb{C}$. Moreover $\tau_0 = \mathrm{i}/\sqrt{D_0} = \mathrm{i}(c_1(1-c_2))^{1/2}$ as follows from (3.5) and (3.2). We can then easily determine $r = \log(Q_0)/\exp\left(-2\pi/\sqrt{D_0}\right)$. For a generic $\rho \in \mathcal{P}$ then it makes sense to define

$$r(\rho) := \frac{Q_0(\rho)}{\exp\left(-2\pi/\sqrt{D_0(\rho)}\right)}, \qquad D_0(\rho) = \frac{1}{c_1(\rho)(1-c_2(\rho))},$$

and make $t_{\alpha,\rho}(Q)$ into a holomorphic function on $\mathbb{H}$ by setting

$$t_{\alpha,\rho}(\tau) := t_{\alpha,\rho}\big(r(\rho)\mathrm{e}^{2\pi \mathrm{i}\tau}\big), \qquad \tau \in \mathbb{H}.$$

The function $t_{\alpha,\rho}(\tau)$ is modular with respect to the group $\Gamma(\rho)$ if the following functions

$$E_{\alpha,j}(\rho,\tau) := t_{\alpha,\rho}\big(r(\rho)\mathrm{e}^{2\pi \mathrm{i} S_j(\rho)\tau}\big) - t_{\alpha,\rho}\big(r(\rho)\mathrm{e}^{2\pi \mathrm{i}\tau}\big), \qquad j = 0,1,2, \tag{4.6}$$

where $S_j(\rho)$ are the generators of $\Gamma(\rho)$, are zero for every τ in a fundamental domain for $\Gamma(\rho)$.

Theorem 4.4. *Let X_α be a four-punctured sphere and let $E_{\alpha,j}(\rho,\tau)$, $j = 0,1,2$ be as in (4.6). The Fuchsian value ρ_{F} for the uniformization of X_α is the unique zero of the system of equations*

$$E_{\alpha,j}(\rho,\tau) = 0, \qquad j = 0,1,2,$$

for every τ in a fundamental domain of $\Gamma(\rho)$.

Proof. It is clear that ρ_{F} is a zero of $E_{\alpha,j}(\rho,\tau)$ for $j = 0,1,2$ and every $\tau \in \mathbb{H}$.

Let $\rho_1 \in \mathcal{P}$ be such that the identity in the statement holds for every $j = 0,1,2$. Since the function $t_{\alpha,\rho_1}(Q)$ is univalent in Q and $t_{\alpha,\rho_1}(0) = 0$ it follows that $t_{\alpha,\rho_1}(\tau)$ is never zero on $\mathbb{H}$ and then, being holomorphic, it is a Hauptmodul for the group $\Gamma(\rho_1)$. The only thing to check is that $\Gamma(\rho_1) = \Gamma(\rho_{\mathrm{F}})$ and that $t_{\alpha,\rho_1}(\tau)$ is a covering map for X_α.

Since $t_{\alpha,\rho_1}(\tau)$ has a simple pole at one cusp it maps the Riemann surface $\mathbb{H}/\Gamma(\rho_1)$ to the punctured sphere $\mathbb{P}^1 \smallsetminus \left\{\infty, 0, a, b^{-1}\right\}$ for some $a, b \in \mathbb{C} \smallsetminus \{0\}$, $a \neq b^{-1}$. We can assume that $t_{\alpha,\rho_1}(\tau)$ maps the cusp 0 to b^{-1}. It follows that $a^{-1}t_{\alpha,\rho_1}(\tau)$ is a Hauptmodul for the punctured sphere $X_{ab} = \mathbb{P}^1 \smallsetminus \left\{\infty, 0, 1, a^{-1}b^{-1}\right\}$ and that $a^{-1}t_{\alpha,\rho_1}(\tau)$ is normalized as in Lemma 3.1. We can then obtain the expansion at ∞ of $a^{-1}t_{\alpha,\rho_1}(\tau)$ from a basis of solutions $\{y_{ab,\hat\rho}(t), \hat y_{ab,\hat\rho}(t)\}$ of the uniformizing differential equation of X_{ab} as in Section 4.1, i.e.,

$$a^{-1}t_{\alpha,\rho_1}(\tau) = t_{ab,\hat\rho}(\tau) = \sum_{n\geq 0} t_n(ab,\hat\rho)r(\hat\rho)^n q^n,$$

where $\hat{\rho}$ is the Fuchsian parameter associated to the uniformizing differential equation of X_{ab}. On the other hand, we can describe $t_{\alpha,\rho_1}(\tau)$ at ∞ with a power series constructed from a basis $\{y_{\alpha,\rho_1}(t), \hat{y}_{\alpha,\rho_1}(t)\}$ of solutions of the differential equation on X_α with accessory parameter ρ_1

$$t_{\alpha,\rho_1}(\tau) = \sum_{n\geq 0} t_n(\alpha,\rho_1) r(\rho_1)^n q^n.$$

Finally, since $\{y_{ab,\hat{\rho}}(t), \hat{y}_{ab,\hat{\rho}}(t)\}$ is a basis of solutions of the uniformizing equation for X_{ab}, and by comparing the power series representations of t_{α,ρ_1} we get

$$\begin{aligned} a\exp\left(\frac{\hat{y}_{ab,\hat{\rho}}}{y_{ab,\hat{\rho}}}\right) &= \left(a\sum_{n=1}^{\infty} t_n(ab,\hat{\rho}) r(\hat{\rho})^n q^n\right)^{-1} = \left(\sum_{n=1}^{\infty} t_n(\alpha,\rho_1)\frac{r(\rho_1)^n}{r(\hat{\rho})^n} r(\hat{\rho})^n q^n\right)^{-1} \\ &= \frac{r(\rho_1)}{r(\hat{\rho})}\exp\left(\frac{\hat{y}_{\alpha,\rho_1}}{y_{\alpha,\rho_1}}\right), \end{aligned}$$

where $^{-1}$ denotes the compositional inverse. It follows that the ratio of solutions of the differential equation for X_α with accessory parameter ρ_1 and of the uniformizing one of X_{ab} differ only by a constant factor. This implies that the ratio $\hat{y}_{\alpha,\rho_1}/y_{\alpha,\rho_1}$ induces a biholomorphism $\widetilde{X}_\alpha \to \mathbb{H}$, i.e., that ρ_1 is the Fuchsian parameter. By the uniqueness of the Fuchsian parameter we can conclude that $\rho_1 = \rho_{\mathrm{F}}$. ∎

5 Example: local expansion of the Fuchsian value function

5.1 Numerical computation of the Fuchsian value

We first explain how to use Theorem 4.4 to approximate numerically the Fuchsian value for a given four-punctured sphere $X_\alpha = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$. The behavior of the Fuchsian value under the action of the anharmonic group (the group of order six generated by $z \to z^{-1}$ and $z \to 1-z$) and the action of complex conjugation is known [13]; we need then to consider only the case

$$\alpha^{-1} \in \{z \in \mathbb{C}\colon 0 \leq \mathrm{Re}(z) \leq 1/2,\ |z| \leq 1\} \smallsetminus \{0\}.$$

It follows from Theorem 4.4 that to compute the Fuchsian value for X_α is equivalent to compute the common zero of the equations $E_{\alpha,j}(\rho,\tau) = 0$ in (4.6). Notice that all the quantities involved in the definition of $E_{\alpha,j}(\rho,\tau)$ are computable, as functions of ρ, from the Frobenius solutions of the differential equation (4.1). We proceed as follows. Fix $\tau_0 \in \mathbb{H}$ and consider, for $j = 0$, the equation $E_{\alpha,0}(\rho,\tau_0) = 0$. We use Newton's method to find the zero of this equation that is the Fuchsian value. As it is well known, the Newton method works if we are able to give an initial guess for the zero that it is close enough to it. In other words, to start the iteration we should choose a value of ρ that is close to the Fuchsian value. Since the function associating to a four-punctured sphere its Fuchsian value is continuous, a good choice for the initial value is the Fuchsian value of a four-punctured sphere X_β with β^{-1} close to α^{-1}. There are four exceptional choices of β^{-1} for which is possible to determine exactly the value of the Fuchsian parameter via symmetries (see [11, Section 7] and [22]; the uniformizing groups in these cases are conjugated to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$). These choices and their Fuchsian value are displayed in the following table

β^{-1}	$\frac{1}{2}$	$\frac{1}{9}$	$\frac{25-11\sqrt{5}}{50}$	$\frac{1+i\sqrt{3}}{2}$
ρ_{F}	1	3	$\frac{35+15\sqrt{5}}{2}$	$\frac{3-i\sqrt{3}}{6}$

Assume for now that the value of α^{-1} is close enough to one of the special values β^{-1}. In this case we start the Newton's method with the Fuchsian value $\rho_{\mathrm{F}}(\beta^{-1})$. The iteration gives the approximation of a zero of $E_{\alpha,0}(\rho,\tau_0)$ that is close to the Fuchsian parameter. To verify if it really is the Fuchsian parameter, we check whether it is a zero of other equations $E_{\alpha,j}(\rho,\tau)=0$ for different choices of j and $\tau \in \mathbb{H}$.

In the case α^{-1} is not close enough to one of the special values, one can gradually approach the computation of the Fuchsian value of X_α by computing the Fuchsian value of some points between one of the special values β^{-1} and α^{-1}.

5.2 Application: local expansion of the Fuchsian value function

As an application, we compute numerically the local expansion of the function that associates to a four-punctured sphere X_α the Fuchsian value $\rho_{\mathrm{F}}(X_\alpha)$. We can define this function in full generality for an n-punctured sphere X. Let $W_n = \{(w_1,\dots,w_{n-3}) \,|\, w_i \neq w_j \text{ if } i \neq j,\ w_i \neq 0,1\}$ and consider the function

$$\rho_{\mathrm{F}}\colon\ W_n \to \mathbb{C}^{n-3}, \qquad w=(w_1,\dots,w_{n-3}) \mapsto \rho_{\mathrm{F}}(w)=(\rho_1,\dots,\rho_{n-3}),$$

where $\rho_{\mathrm{F}}=(\rho_1,\dots,\rho_{n-3})$ is the Fuchsian value for the n-punctured sphere

$$X=\mathbb{P}^1 \smallsetminus \{w_1,w_2,\dots,w_{n-3},0,1,\infty\}.$$

Kra [14] proved that the function ρ_{F} is real-analytic, but non complex-analytic; in particular, if z is a local parameter on W_n, the function ρ_{F} has a local expansion near the point $z_0 \in W_n$ of the form

$$\rho_{\mathrm{F}}(z_0+z)=\sum_{j,k\geq 0} a_{j,k} z^j \bar{z}^k, \qquad a_{j,k}\in\mathbb{C}. \tag{5.1}$$

In the following we concentrate on the case $n=4$ and study the expansion of the function

$$\rho_{\mathrm{F}}\colon\ \mathbb{C}\setminus\{0,1\}\to\mathbb{C}, \qquad \alpha^{-1}\mapsto \rho_{\mathrm{F}}(X_\alpha),$$

around the points $\alpha^{-1}=1/2, 1/3$, and $2/5+3\mathrm{i}/10$.

The computation of the coefficients of (5.1) goes as follows. Fix $\alpha^{-1}\in\mathbb{C}\smallsetminus\{0,1\}$ and consider for every $m\in\mathbb{N}$ the line L_m: $\mathrm{Im}(z)=\mathrm{Re}(z)/m-\alpha^{-1}/m$. The expansion of ρ_{F} on each line L_m depends only on one real variable x, since $z\in L_m$ if $z=\alpha^{-1}+x(1+\mathrm{i}/m)$ and then

$$\rho_{\mathrm{F}}\left(\alpha^{-1}+z\right)=\sum_{j,k\geq 0} a_{j,k}(1+\mathrm{i}/m)^j(1-\mathrm{i}/m)^k x^{k+j}=\sum_{s=0}^{\infty} b_s(m)x^m, \qquad b_s(m)\in\mathbb{C}. \tag{5.2}$$

The coefficients $b_s(m)$ are easily computed once we know enough values of the function ρ_{F} on the line L_m near α^{-1}. These can be computed by using the method illustrated in Section 5.1 and the computer algebra system PARI. The coefficients $a_{j,k}$ in the expansion of (5.1) are then obtained by computing the expansion of ρ_{F} along the lines L_m for different values of $m\in\mathbb{N}$ (the number of those depends on the number of $a_{j,k}$ one wants to compute) and exploiting the relation between $a_{j,k}$ and $b_s(m)$ in (5.2). The result of the computations for $\alpha^{-1}=1/2, 1/3, 2/5+3\mathrm{i}/10$ are given in Tables 1, 2, and 3 respectively (the numbers in the tables are approximations of the actual values).

A first interesting observation we can make from the numerical data is about the size of the coefficients $a_{j,k}$. For every $s=1,\dots,5$ we have that $|a_{s,0}|>|a_{j,k}|$ with $j+k=s$. In other words,

Table 1. $\alpha^{-1} = 1/2$, $\rho_{\mathrm{F}} = 1$.

$a_{1,0} = -1.2311296972$	$a_{0,1} = 0.0638754899$
$a_{2,0} = 2.4622593944$	$a_{1,1} = -0.1277509798$
$a_{0,2} = 0$	
$a_{3,0} = -4.8236918585$	$a_{2,1} = 0.1890620793$
$a_{1,2} = 0.0117490877$	$a_{0,3} = 0.0630206931$
$a_{4,0} = 9.6473837171$	$a_{3,1} = -0.3781241587$
$a_{2,2} = -0.0234981755$	$a_{1,3} = -0.1260413862$
$a_{0,4} = 0$	
$a_{5,0} = -19.094665845$	$a_{4,1} = 0.6673769276$
$a_{3,2} = 0.0466379026$	$a_{2,3} = 0.1888625099$
$a_{1,4} = 0.0233189513$	$a_{0,5} = 0.1334753855$

Table 2. $\alpha^{-1} = 1/3$, $\rho_{\mathrm{F}} = 1.29101$.

$a_{1,0} = -2.711485382$	$a_{0,1} = 0.1025201219$
$a_{2,0} = 8.0641055547$	$a_{1,1} = -0.2750330946$
$a_{0,2} = -0.0606264558$	
$a_{3,0} = -23.9531822161$	$a_{2,1} = 0.6879078089$
$a_{1,2} = 0.1854950416$	$a_{0,3} = 0.1686761471$
$a_{4,0} = 71.4914941489$	$a_{3,1} = -1.9281043695$
$a_{2,2} = -0.4821058340$	$a_{1,3} = 0.4798627522$
$a_{0,4} = 0.2922654268$	
$a_{5,0} = -213.5180837271$	$a_{4,1} = 5.4375565883$
$a_{3,2} = 1.3699401215$	$a_{2,3} = 1.2274288636$
$a_{1,4} = 0.8681638848$	$a_{0,5} = 0.7367927177$

Table 3. $\alpha^{-1} = \frac{4+3\mathrm{i}}{10}$, $\rho_{\mathrm{F}} = 0.86175 - 0.38528\mathrm{i}$

$a_{1,0} = -0.3328603817 + 1.2004121803\mathrm{i}$	$a_{0,1} = 0.0512782931 - 0.0256391465\mathrm{i}$
$a_{2,0} = -0.8635602303 - 2.2690958807\mathrm{i}$	$a_{1,1} = -0.0638839260 + 0.0719692197\mathrm{i}$
$a_{0,2} = -0.0223137191 - 0.03022272424\mathrm{i}$	
$a_{3,0} = 4.0817033346 + 2.5852542694\mathrm{i}$	$a_{2,1} = -0.0024129061 - 0.1684860963\mathrm{i}$
$a_{1,2} = 0.0599341643 + 0.0300072676\mathrm{i}$	$a_{0,3} = 0.01118030554 + 0.0173932256\mathrm{i}$
$a_{4,0} = -9.5604869979 + 0.7791704955\mathrm{i}$	$a_{3,1} = 0.1800042196 + 0.2360936333\mathrm{i}$
$a_{2,2} = -0.1150333973 + 0.0337939365\mathrm{i}$	$a_{1,3} = -0.0337943677 - 0.0275496197\mathrm{i}$
$a_{0,4} = -0.0003411753 - 0.0596118160\mathrm{i}$	
$a_{5,0} = 14.2865112529 - 12.6279958566\mathrm{i}$	$a_{4,1} = -0.5419356177 - 0.1625218437\mathrm{i}$
$a_{3,2} = 0.1261663672 - 0.1720412035\mathrm{i}$	$a_{2,3} = 0.0798727692 - 0.0023875661\mathrm{i}$
$a_{1,4} = 0.0384447463 + 0.0991919592\mathrm{i}$	$a_{0,5} = -0.0711856680 + 0.0428083431\mathrm{i}$

the holomorphic part of the expansion of $\rho(\alpha^{-1})$ seems to be larger than the rest. This suggests that the Fuchsian parameter function may be quasiregular, i.e., it may satisfy the inequality

$$\frac{\partial \rho_{\mathrm{F}}}{\partial \bar{z}} \leq k \frac{\partial \rho_{\mathrm{F}}}{\partial z}$$

for some $k < 1$. This would in particular imply that the Fuchsian parameter map is open.

We further notice the following relations among the coefficients in Table 1:

$$a_{2,0} = -2a_{1,0}, \qquad a_{1,1} = -2a_{0,1}, \tag{5.3}$$

$$a_{0,2} = 0 = a_{0,4}, \qquad a_{2,1} = 3a_{0,3}, \tag{5.4}$$

$$a_{3,2} = 2a_{1,4}, \qquad a_{4,1} = 5a_{0,5}. \tag{5.5}$$

These numerical identities can be proven by using the symmetry of ρ_{F} near $1/2$, and a result of Takhtajan and Zograf [23]. Analogous identities in Table 2 or Table 3 can be proved similarly. The point $z_0 = 1/2$ is the fixed point of the involution $z \mapsto 1 - z$. It is known (see [13]) that the following identity holds

$$\rho_{\mathrm{F}}(1-z) = \frac{z\rho_{\mathrm{F}}(z) - 1}{z-1}.$$

It follows that, near the point $z_0 = 1/2$, one has

$$(z - 1/2)\rho_{\mathrm{F}}(1/2 - z) = (1/2 + z)\rho_{\mathrm{F}}(1/2 + z) - 1,$$

which gives

$$\sum_{j,k\geq 0} a_{j,k}\left[1 + (-1)^{j+k}\right] z^j \overline{z}^k + 2\sum_{j,k\geq 0} a_{j,k}\left[1 - (-1)^{j+k}\right] z^{j+1}\overline{z}^k - 2 = 0. \tag{5.6}$$

The above relation implies that

$$a_{0,2k} = 0 \qquad \text{if } k \geq 1,$$

$$a_{j+1,k} = -2a_{j,k} \qquad \text{if } k + j \text{ is odd}.$$

This explains why $a_{0,2} = a_{0,4} = 0$. The result of Takhtajan and Zograf [23, formula (4.1)] reduces to the following identity in the case of four-punctured spheres[2]

$$(1 - 2z)\overline{\left(\frac{\partial \rho_{\mathrm{F}}}{\partial \overline{z}}\right)} = (1 - 2\overline{z})\frac{\partial \rho_{\mathrm{F}}}{\partial \overline{z}}. \tag{5.7}$$

The differential equation (5.7) implies the following relations between the coefficients of the local expansion of ρ_{F}:

$$(k+1)a_{j,k+1} - 2ka_{j,k} = (j+1)a_{k,j+1} - 2ja_{k,j}, \qquad j,k \geq 0.$$

It is easy to check that the relations (5.3)–(5.5) come from this one and from (5.6). For instance, by choosing $(j,k) = (0,1)$ in the identity above we get

$$2a_{0,2} - 2a_{0,1} = a_{1,1}.$$

This, together with $a_{0,2} = 0$, gives the first identity in (5.3).

A Modular derivation of the uniformizing differential equation

Denote by $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a genus zero Fuchsian group with no torsion and with $n \geq 3$ inequivalent cusps. Normalize it by assuming that one of its cusps is at ∞, and that this cusp has width one. Let t be a Hauptmodul and, without loss of generality, assume that its unique pole is at

[2] this equation is formulated in [23] in terms of the accessory parameters m_i appearing in the Schwarzian differential equation (2.1); here we express it in terms of the accessory parameter of the Heun equation (4.1).

a cusp c_0 and its unique zero is at ∞. Finally, let X_Γ be the n-punctured sphere isomorphic to $\mathbb{H}/\Gamma$ via t

$$t\colon\ \mathbb{H}/\Gamma \xrightarrow{\sim} X_\Gamma = \mathbb{P}^1 \smallsetminus \{\alpha_1, \alpha_2, \dots, \alpha_{n-1} = 0, \alpha_n = \infty\},$$

where $\alpha_i \in \mathbb{C} \smallsetminus \{0, 1\}$, $i = 1, \dots, n-2$, and $\alpha_i \neq \alpha_j$ if $i \neq j$.

In the following we compute the differential equation satisfied by a certain modular form f with respect to t. This differential equation is projectively equivalent to the differential equation (2.1) associated to the uniformization of X_Γ, and in the case $n = 4$ reduces to the Heun equation (4.1) considered in Section 4. In particular, this gives a purely modular definition of the accessory parameters, as it will be clear from the proof of Proposition A.3.

Since the differential equation in (2.1) has order two it would be natural, according to Section 2.2, to consider a weight one modular form on Γ, which satisfies a second order differential equation. It is known however that not every group Γ admits weight one modular forms (for now we are only assuming that Γ is of genus zero and torsion free). It makes sense then to consider a square root of a modular form of weight two, since $\dim M_2(\Gamma) = n-1$ for every torsion-free genus zero group Γ with n cusps. We choose to work with a weight two modular form whose zeros are concentrated in a certain cusp; as the next lemma shows, this choice is always possible.

Lemma A.1. *Let Γ be torsion free and of genus zero, let t be an Hauptmodul, and denote by c_0 the cusp of Γ where t has its unique pole. There exists a modular form $f \in M_2(\Gamma)$, unique up to scalar multiplication, with all its zeros in c_0. In particular, f has no zeros in $\mathbb{H}$.*

Proof. Let $g \in M_2(\Gamma)$ and let $\sigma \in \mathrm{SL}(2, \mathbb{R})$ be such that $\sigma c_0 = \infty$. Let

$$\big(g\big|_2 \sigma^{-1}\big)(\tau) = \sum_{m \geq 0} g_m q^m$$

denote the Fourier expansion of g at c_0, where $q = \mathrm{e}^{2\pi \mathrm{i}\tau/h}, \tau \in \mathbb{H}$, is a local parameter. It is known that the degree of the divisor associated to any $g \in M_2(\Gamma)$ is $d = n-2$. Let ϕ be the map

$$\phi\colon\ M_2(\Gamma) \to \mathbb{C}^d, \qquad g \mapsto (g_0, g_1, \dots, g_{d-1}).$$

that sends a modular form of weight 2 to the vector defined by its first d Fourier coefficients at the cusp c_0. This map is linear.

The dimension of $M_2(\Gamma)$ is $n-1 = d+1$, so the map ϕ has a non-trivial kernel of dimension ≥ 1. Let $f \in \mathrm{Ker}(\phi)$. Such f can have at most d zeros in $\mathbb{H} \cup \{\text{cusps}\}$, and they are all in c_0 by construction. Finally, let $f, g \in \mathrm{Ker}(\phi)$ be linearly independent. The ratio f/g is a weight zero modular form holomorphic in $\mathbb{H}$ and in all the cusps, since f and g have all their zeros at the same cusp c_0. This implies that f/g is a constant, i.e., $\dim \mathrm{Ker}(\phi) = 1$. ■

Given f and t as in Lemma A.1 we can construct all the modular forms of even weight on Γ.

Lemma A.2. *Let $k \geq 0$ be an integer, and let f and t be as in Lemma A.1. The functions*

$$f^k t^i, \qquad i = 0, \dots, k(n-2),$$

form a basis of the space $M_{2k}(\Gamma)$.

Proof. By construction, the weight $2k$ modular form f^k has $k(n-2)$ zeros at the cusp c_0 where t has a simple pole, and these are the only zeros of f^k. It follows that $f^k t^i$ is a holomorphic modular form for every $i = 0, \dots, k(n-2)$, and meromorphic for every other value of i. By looking at the location of the zeros, we can prove that the holomorphic functions in the statement are linearly independent. From the dimension formula for $M_{2k}(\Gamma)$ (see for example [15, Chapter 2]) we conclude that they form a basis. ■

The second order linear differential operator associated to a square root of the modular form $f \in M_2(\Gamma)$ in Lemma A.1 and to the Hauptmodul t is given in the next proposition.

Proposition A.3. *Let Γ be a genus zero torsion-free Fuchsian group with $n \geq 3$ cusps, and let t be a Hauptmodul such that $t\colon \mathbb{H}/\Gamma \xrightarrow{\sim} X_\Gamma = \mathbb{P}^1 \smallsetminus \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1} = 0, \alpha_n = \infty\}$. Denote by c_0 the cusp of Γ where t has its unique pole, and let $f \in M_2(\Gamma)$ be such that all its zeros are at the cusp c_0. Then the differential operator L associated to a square root of f and to t is given by*

$$L = \frac{\mathrm{d}}{\mathrm{d}t}\left(P(t)\frac{\mathrm{d}}{\mathrm{d}t}\right) + \sum_{i=0}^{n-3} \rho_i t^i, \tag{A.1}$$

where $P(t) = \prod_{j=1}^{n-1}(t - \alpha_j)$, $\rho_{n-3} = (n/2-1)^2$, and $\rho_0, \dots, \rho_{n-4} \in \mathbb{C}$ are uniquely determined by f, t.

Proof. Recall from Section 2.2 that L can be computed in terms of Rankin–Cohen brackets of f and t by

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{[f,t']_1}{2ft'^2}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{[f,f]_2}{12f^2t'^2}. \tag{A.2}$$

We have to write the coefficients of L as rational functions of t. First we prove that

$$(-1)^n\left(\prod_{j=1}^{n-2}\alpha_j\right)t' = fP(t).$$

The ratio t'/f is a meromorphic modular function, so it is a rational function of t. From the assumption on the zeros of f it follows that the modular function t'/f has a simple zero at every cusp different from c_0, i.e., $n-1$ simple zeros (since these are the zeros of t'). It has also a unique pole of order $n-1$ at c_0, since f has $n-2$ zeros there and t' a simple pole. The rational functions of t with these zeros and poles are given by the polynomials $\kappa^{-1}P(t)$, $\kappa \in \mathbb{C}^*$, where $P(t)$ is as in the statement. Looking at the first coefficient of the q-expansion of t'/f at ∞, we find the correct factor $\kappa = (-1)^n \prod_{j=1}^{n-2}\alpha_j$.

Next, we compute the brackets $[f,t']$, and $[f,f]_2$. The first one is very easy

$$\begin{aligned}[f,t'] &= 2ft'' - 2f't' = f(fP(t)\kappa^{-1})' - 2f't' \\ &= 2f't' + 2\kappa^{-1}f^2P'(t)t' - 2f't' = 2\kappa^{-1}f^2t'P'(t).\end{aligned}$$

Dividing $[f,t']$ by $2ft'^2 = 2\kappa f^2P(t)t'$ we see that the coefficient of $\mathrm{d}/\mathrm{d}t$ in (A.2) is given by the rational function $P'(t)/P(t)$, as in the statement (A.1).

The computation of the bracket $[f,f]_2$ needs a little more work. From the definition of RC brackets, we see that $[f,f]_2$ is a cusp form of weight eight. Moreover, it has a zero of order $2n-4$ where f is zero, so it is necessarily divisible by f^2. There exists then an element $h_4 \in M_4(\Gamma)$ such that $[f,f]_2 = f^2h_4$. By Lemma A.2 we know that h_4 is of the form

$$h_4 = f^2Q(t),$$

where $Q(t)$ is a polynomial in t of degree $\dim M_4(\Gamma) = 2n-3$. Since $[f,f]_2$ is a cusp form, $[f,f]_2/f^2$ has a zero in every cusp different from c_0, and these zeros are simple. This means that the polynomial $Q(t)$ is divisible by $P(t)$. We have then

$$h_4 = f^2P(t)\big(\hat\rho_{n-3}t^{n-3} + \hat\rho_{n-2}t^{n-2} + \cdots + \hat\rho_0\big),$$

for some $\hat{\rho}_0, \dots, \hat{\rho}_{n-3} \in \mathbb{C}$. We can determine $\hat{\rho}_{n-3}$ by considering the expansion of f at the cusp c_0. If q_0 denotes a local parameter at c_0, the expansions of f and t are given by

$$f = cq_0^{n-2} + \cdots, \qquad t = sq_0^{-1} + \cdots,$$

for some non-zero $c, s \in \mathbb{C}$. In c_0 the bracket $[f,f]_2$ has expansion

$$[f,f]_2 = 6ff'' - 9f'^2 = -3c^2(n-2)^2 q_0^{2n-4} + \cdots,$$

while h_4 is given by

$$h_4 = \left(cq_0^{n-2} + \cdots\right)^2 \left(\hat{\rho}_{n-3} s^{n-3} q_0^{3-n} + \cdots\right)\left(s^{n-1} q_0^{1-n} + \cdots\right) = \hat{\rho}_{n-3} c^2 s^{2n-4} q_0^0 + \cdots.$$

The above expansions and the equality $[f,f]_2 = f^2 h_4$ imply that

$$\hat{\rho}_{n-3} c^2 s^{2n-4} = -(n-2)^2.$$

From the relation $t' = \kappa^{-1} P(t) f$ we can compute the constant κ in terms of the coefficients appearing in the expansions at c_0, obtaining $\kappa = -cs^{n-2}$. This implies that

$$\hat{\rho}_{n-3} = -3\kappa^{-2}(n-2)^2.$$

It finally follows that

$$-\frac{[f,f]_2}{12 f^2 t'^2} = \frac{-f^4 P(t)\left(-3\kappa^{-2}(n-2)^2 t^{n-3} + \hat{\rho}_4 t^{n-4} + \cdots + \hat{\rho}_0\right)}{12\kappa^{-2} f^4 P(t)^2}$$
$$= \frac{(n/2-1)^2 t^{n-3} + \rho_{n-4} t^{n-4} + \cdots + \rho_0}{P(t)},$$

where $\rho_i = -\hat{\rho}_i \kappa^2/12$. ■

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