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Functional Equations Solving Initial-Value Problems of Complex Burgers-Type Equations for One-Dimensional Log-Gases

Taiki ENDO [a], *Makoto KATORI* [a] *and Noriyoshi SAKUMA* [b]

[a] *Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan*
E-mail: *taiki@phys.chuo-u.ac.jp*, *katori@phys.chuo-u.ac.jp*

[b] *Graduate School of Natural Sciences, Nagoya City University, Mizuho-ku, Nagoya, Aichi 467-8501, Japan*
E-mail: *sakuma@nsc.nagoya-cu.ac.jp*

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Abstract. We study the hydrodynamic limits of three kinds of one-dimensional stochastic log-gases known as Dyson's Brownian motion model, its chiral version, and the Bru–Wishart process studied in dynamical random matrix theory. We define the measure-valued processes so that their Cauchy transforms solve the complex Burgers-type equations. We show that applications of the method of characteristic curves to these partial differential equations provide the functional equations relating the Cauchy transforms of measures at an arbitrary time with those at the initial time. We transform the functional equations for the Cauchy transforms to those for the R-transforms and the S-transforms of the measures, which play central roles in free probability theory. The obtained functional equations for the R-transforms and the S-transforms are simpler than those for the Cauchy transforms and useful for explicit calculations including the computation of free cumulant sequences. Some of the results are argued using the notion of free convolutions.

Key words: stochastic log-gases; complex Burgers-type equations; functional equations; Cauchy transforms; R-transforms; S-transforms; free probability and free convolutions

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1 Introduction and results

1.1 Transformations of measures and complex Burgers-type equations

Among a variety of recent developments in random matrix theory [1, 2, 35, 55], we study in this paper an intersection of two important topics; time-dependent random matrix models [19, 33, 46, 50, 52] and free probability theory [8, 11, 12, 56, 61, 69, 71].

The cases of the Gaussian unitary ensemble (GUE) and the chiral GUE are typical eigenvalue distributions on $\mathbb{R}$ of Hermitian random matrices, and their dynamical extensions are described by systems of stochastic differential equations (SDEs) called *Dyson's Brownian motion model* [33] and its chiral version [20, 50, 52]. (Compare the SDEs (2.4) for the chiral version with (2.1) for the original one. The two terms in the parentheses in the last term of the r.h.s. in (2.4) are *chiral* to each other.)

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For $S \subset \mathbb{R}$, let $\mathcal{P}^0(S)$ be a set of all Borel probability measures on S with bounded supports equipped with the weak topology. For an arbitrary but fixed $T > 0$, $\mathcal{C}\big([0,T] \to \mathcal{P}^0(S)\big)$ denotes the space of continuous processes defined in the time period $[0,T]$ realized in $\mathcal{P}^0(S)$.

In this manuscript, we consider the *hydrodynamic limits* of Dyson's Brownian motion model and its chiral version with an additional parameter $\lambda \in \mathbb{R}_+ := [0,\infty)$ as elements of $\mathcal{C}\big([0,\infty) \to \mathcal{P}^0(\mathbb{R})\big)$, which are denoted by $(\mathrm{w}_t)_{t\geq 0}$ and $(\mathrm{w}_{\lambda,t})_{t\geq 0}$, respectively. First, we assume that the initial probability measure w_0 and $\mathrm{w}_{\lambda,0}$ are in $\mathcal{P}^0(\mathbb{R})$ and the *Cauchy transforms* of measures,

$$G_\mu(z) := \int_S \frac{\mu(\mathrm{d}x)}{z-x}, \qquad z \in \mathbb{C}^+ := \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}, \tag{1.1}$$

are well defined satisfying the condition $\lim_{y\uparrow\infty} \sqrt{-1}yG_\mu\big(\sqrt{-1}y\big) = 1$ for $\mu = \mathrm{w}_t$ and $\mathrm{w}_{\lambda,t}$, $t \geq 0$. It is known that [14, 16, 17, 59, 60, 63] given G_{w_0} and $G_{\mathrm{w}_{\lambda,0}}$ obtained by w_0 and $\mathrm{w}_{\lambda,0}$, respectively, $(G_{\mathrm{w}_t})_{t\geq 0}$ and $(G_{\mathrm{w}_{\lambda,t}})_{t\geq 0}$ are uniquely determined by the solutions of the following partial differential equations (PDEs):

$$\frac{\partial G_{\mathrm{w}_t}(z)}{\partial t} + G_{\mathrm{w}_t}(z)\frac{\partial G_{\mathrm{w}_t}(z)}{\partial z} = 0, \qquad t \geq 0, \tag{1.2}$$

$$\frac{\partial G_{\mathrm{w}_{\lambda,t}}(z)}{\partial t} + \left(G_{\mathrm{w}_{\lambda,t}}(z) - \frac{1-\lambda}{2z}\right)\frac{\partial G_{\mathrm{w}_{\lambda,t}}(z)}{\partial z} + \frac{1-\lambda}{2z^2}G_{\mathrm{w}_{\lambda,t}}(z) = 0, \qquad t \geq 0. \tag{1.3}$$

Equation (1.2) is known as the *complex Burgers equation in the inviscid limit*. It is obvious that when $\lambda = 1$ (1.3) is reduced to (1.2) and hence, if $\mathrm{w}_0 = \mathrm{w}_{1,0}$, then

$$\mathrm{w}_t = \mathrm{w}_{1,t}, \qquad t \geq 0. \tag{1.4}$$

In other words, the process $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ is the one-parameter ($\lambda \in \mathbb{R}_+$) extension of $(\mathrm{w}_t)_{t\geq 0}$.

We denote by $\mathbf{1}_{(E)}$ the indicator function of an event E; $\mathbf{1}_{(E)} = 1$ if E occurs, and $\mathbf{1}_{(E)} = 0$ otherwise. As a special case, Kronecker's delta is defined by $\delta^{ij} := \mathbf{1}_{(i=j)}$, $i,j \in \mathbb{N}$. Moreover, the σ-algebra of Borel sets on S is denoted by $\mathcal{B}(S)$ and we define $1_B(x) := \mathbf{1}_{(x\in B)}$ for $B \in \mathcal{B}(S)$.

For $p > 0$ and $\mu \in \mathcal{P}^0(\mathbb{R})$, the p-th push-forward measure $\mu^{(p)}$ is defined by [62]

$$\mu^{(p)}(B) := \int_{\mathbb{R}} 1_B(|x|^p)\mu(\mathrm{d}x), \qquad B \in \mathcal{B}((0,\infty)). \tag{1.5}$$

We define

$$\mathrm{m}_{\lambda,t} := \mathrm{w}_{\lambda,t}^{(2)} \in \mathcal{P}^0(\mathbb{R}_+), \qquad t \geq 0, \tag{1.6}$$

provided the matching of initial measures $\mathrm{m}_{\lambda,0} = \mathrm{w}_{\lambda,0}^{(2)}$. Note that combining this definition with (1.4) we have the equality

$$\mathrm{w}_t^{(2)} = \mathrm{m}_{1,t}, \qquad t \geq 0. \tag{1.7}$$

We can show that by (1.6) the PDE (1.3) is transformed to the following equation for the Cauchy transform $G_{\mathrm{m}_{\lambda,t}}(z)$ of $\mathrm{m}_{\lambda,t}$, $\lambda \in \mathbb{R}_+$,

$$\frac{\partial G_{\mathrm{m}_{\lambda,t}}(z)}{\partial t} + \big\{2zG_{\mathrm{m}_{\lambda,t}}(z) - (1-\lambda)\big\}\frac{\partial G_{\mathrm{m}_{\lambda,t}}(z)}{\partial z} + G_{\mathrm{m}_{\lambda,t}}(z)^2 = 0, \qquad t \geq 0. \tag{1.8}$$

It should be noted that this PDE is obtained when we consider the hydrodynamic limit of the system of SDEs [21] known as the *Bru–Wishart process* in multivariate stochastic calculus [20, 50] and as the *Laguerre process* in dynamical random matrix theory [52]. The PDEs (1.3) and (1.8) describe the large-number limits of colors of the systems with the *chiral symmetry* in the *quantum chromodynamics* (QCD) in high energy physics [14, 15, 45, 53, 59, 60, 64, 67, 68]. (See also [1, 35].) We call (1.3) and (1.8) *complex Burgers-type equations* [16, 17, 34, 36], in which drift terms are modified and "external-force terms" are added compared with (1.2). (See Section 2.1 below.)

The simplest initial probability measure in $\mathcal{P}^0(S)$, $S=\mathbb{R}$ or $\mathbb{R}_+$, is the single delta measure δ_0 at the origin. We regard the solution of the complex Burgers-type equation starting from $G_{\delta_0}(z)=1/z$ as the *fundamental solution* and denote the obtained measure-valued process as $\big(\mu_t^0\big)_{t\ge 0}$ with a superscript 0. We can show that (see, for instance, [56])

$$G_{\mathrm{w}_t^0}(z)=\frac{1}{2t}\Big[z-\sqrt{z^2-4t}\Big], \qquad t\ge 0,$$
$$G_{\mathrm{m}_{\lambda,t}^0}(z)=\frac{1}{2tz}\Big[z+t(1-\lambda)-\sqrt{(z-x_{\lambda,t}^+)(z-x_{\lambda,t}^-)}\Big] \qquad \text{with} \quad x_{\lambda,t}^{\pm}=t\big(1\pm\sqrt{\lambda}\big)^2, \quad t\ge 0,$$

and they determine the time-dependent measures for $t\ge 0$ as

$$\mathrm{w}_t^0(\mathrm{d}x)=\frac{1}{2\pi t}\sqrt{4t-x^2}\,1_{[-2\sqrt{t},2\sqrt{t}]}(x)\,\mathrm{d}x, \tag{1.9}$$
$$\mathrm{m}_{\lambda,t}^0(\mathrm{d}x)=\max(0,1-\lambda)\delta_0(\mathrm{d}x)+\frac{1}{2\pi tx}\sqrt{(x-x_{\lambda,t}^-)(x_{\lambda,t}^+-x)}\,1_{[x_{\lambda,t}^-,x_{\lambda,t}^+]}(x)\,\mathrm{d}x, \tag{1.10}$$

respectively. The measure (1.9) is known as the centered *Wigner's semicircle distribution* with variance t and (1.10) is as the *two-parametric Marcenko–Pastur distribution* with parameters λ and t [16, 17, 34]. For the fundamental solutions explicitly given by (1.9) and (1.10), it is easy to verify the well-known equality $\big(\mathrm{w}_t^0\big)^{(2)}=\mathrm{m}_{1,t}^0$, $t\ge 0$. The equality (1.7) mentioned above generalizes it for any initial probability measure with bounded support satisfying $\mathrm{w}_0^{(2)}=\mathrm{m}_{1,0}$.

Let $\mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$ be the set of all symmetric Borel probability measures on $\mathbb{R}$ (i.e., $\mu(B)=\mu(-B)$, $B\in\mathcal{B}((0,\infty))$ for $\mu\in\mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$) and define the *symmetric Bernoulli delta measure* with displacement $2a>0$ as

$$\mathrm{d}_a:=\frac{1}{2}(\delta_{-a}+\delta_a)\in\mathcal{P}_{\mathrm{s}}^0(\mathbb{R}). \tag{1.11}$$

The processes $(\mathrm{w}_t)_{t\ge 0}$ and $(\mathrm{w}_{\lambda,t})_{t\ge 0}$ starting from d_a, which are defined as the time-dependent probability measures so that their Cauchy transforms solve the PDEs (1.2) and (1.3) under the initial condition $G_{\mathrm{d}_a}(z)=z/\big(z^2-a^2\big)$, were reported in [3, 57, 72] and in [34], respectively. These solutions show a dynamical phase transition, in which the positive parameter a controls the transition observed at a critical time. The singularity associated with this phase transition is very interesting and important, since it gives the mean-field description of the spontaneous chiral symmetry breaking [34]. But, due to this singularity, the solutions are much more complicated compared with the fundamental solutions (1.9) and $(\mathrm{w}_{\lambda,t}^0)_{t\ge 0}$ obtained from (1.10) by (1.6) (see [34, Remark 3]). It had seemed to be difficult to argue general properties of these measure-valued processes. (See [38] for the exact solutions for other initial probability measures.)

In the present paper, we solve the *initial-value problem* for these measure-valued processes $(\mu_t)_{t\ge 0}=(\mathrm{w}_t)_{t\ge 0}$, $(\mathrm{w}_{\lambda,t})_{t\ge 0}$, and $(\mathrm{m}_{\lambda,t})_{t\ge 0}$. We found that, by the method of characteristic curves, each of the PDEs of the Cauchy transforms $(G_{\mu_t})_{t\ge 0}$, given by (1.2), (1.3), and (1.8), is transformed to a *functional equation* which relates $G_{\mu_t}(z)$ at an arbitrary time $t>0$ with the initial function $G_{\mu_0}(z)$. The result for $(G_{\mathrm{w}_t})_{t\ge 0}$ given as Proposition 3.1(*i*) is well known and found in literature. The results for $(G_{\mathrm{m}_{\lambda,t}})_{t\ge 0}$ and $(G_{\mathrm{w}_{\lambda,t}})_{t\ge 0}$ given as Proposition 3.1(*ii*) and (*iii*), respectively, are new, but they are complicated and do not seem to be useful for explicit calculations. On the other hand, in free probability theory [4, 8, 11, 12, 40, 56, 61, 62, 69], we learn other importance transformations of probability measures different from the Cauchy transform; the R-transform and the S-transform. They define new types of convolutions of probability measures called *free convolutions*. (A brief review is given shortly.) We applied these transformations to our functional equations. The results for the R-transforms denoted by $(R_{\mu_t})_{t\geq 0}$ and the S-transforms by $(S_{\mu_t})_{t\geq 0}$ are given by Theorems 1.1 and 1.5, respectively. The obtained functional equations are expressed using the R-transforms and the S-transforms of the fundamental solutions and much simplified. In particular, as shown in Theorem 1.1(i), the functional equation (1.18) implies the decomposition formula (1.19) with respect to the free additive convolution (see also Remarks 1.2 and 1.3). Theorem 1.5(i) gives the simple but useful relationship among $(\mathrm{w}_{\lambda,t})_{t\geq 0}$, $(\mathrm{m}_{\lambda,t})_{t\geq 0}$, and d (see Remark 1.6). The functional equations are also useful for explicit calculations as demonstrated by Propositions 1.7 and 1.8. There the solutions for the processes $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ starting from d_a, $a>0$ and $(\mathrm{m}_{\lambda,t})_{t\geq 0}$ from δ_b, $b>0$ are shown, which are much simpler than the corresponding solutions of the Cauchy transformations reported in [3, 57, 72] and [34]. Comparing the results for $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ and those for $(\mathrm{m}_{\lambda,t})_{t\geq 0}$, the latter is simpler than the former. In the original complex Burgers-type equations, it is obvious at the PDE (1.3) for $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ that $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ is a one-parameter extension ($\lambda\in\mathbb{R}_+$) of $(\mathrm{w}_t)_{t\geq 0}$, but it is not at the PDE (1.8) for $(\mathrm{m}_{\lambda,t})_{t\geq 0}$. Our results shows, however, that the transformation from $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ to $(\mathrm{m}_{\lambda,t})_{t\geq 0}$ by the second push-forward transform is effective to make practical problems be simpler and solvable. Further study of the p-th push-forward transforms (1.5) will be important.

Given a probability measure $\mu\in\mathcal{P}^0(S)$, let $\tau_n(\mu)$ represent the n-th moment of μ, $n\in\mathbb{N}$ and define the *moment generating function* by

$$\Psi_\mu(z) := \int_S \frac{xz}{1-xz}\mu(\mathrm{d}x) = \sum_{n=1}^{\infty}\tau_n(\mu)z^n. \tag{1.12}$$

The Cauchy transform G_μ of μ is related to Ψ_μ by

$$\Psi_\mu(z) = \frac{G_\mu(1/z)}{z} - 1 \iff G_\mu(1/z) = z(\Psi_\mu(z)+1). \tag{1.13}$$

We write the inverse function of G_μ as $G_\mu^{\langle -1\rangle}$. The *R-transform* of μ is then defined by

$$R_\mu(z) := zG_\mu^{\langle -1\rangle}(z) - 1 \iff G_\mu^{\langle -1\rangle}(z) = \frac{R_\mu(z)+1}{z}. \tag{1.14}$$

This is the generating function of *free cumulants* $\kappa_n=\kappa_n(\mu)$, $n\in\mathbb{N}$,

$$R_\mu(z) = \sum_{n=1}^{\infty}\kappa_n(\mu)z^n.$$

(Notice that in the free probability literature the function $\mathcal{R}_\mu(z) := R_\mu(z)/z$ is also used and referred to as "the R-transform of μ". See, for instance, [61, equation (16.8)].) The relations between the moments $\{\tau_n(\mu)\}_{n\in\mathbb{N}}$ in (1.12) and the free cumulants $\{\kappa_n(\mu)\}_{n\in\mathbb{N}}$ are given by

$$\begin{aligned}
&\kappa_1(\mu) = \tau_1(\mu),\\
&\kappa_2(\mu) = \tau_2(\mu) - \tau_1(\mu)^2,\\
&\kappa_3(\mu) = \tau_3(\mu) - 3\tau_2(\mu)\tau_1(\mu) + 2\tau_1(\mu)^3,\\
&\kappa_4(\mu) = \tau_4(\mu) - 4\tau_3(\mu)\tau_1(\mu) - 2\tau_2(\mu)^2 + 10\tau_2(\mu)\tau_1(\mu)^2 - 5\tau_1(\mu)^4,\\
&\cdots\cdots\cdots\cdots,
\end{aligned}$$

which are generally different from the relations satisfied in the "classical probability theory". For two probability measures μ, ν, the *free additive convolution* of them is denoted by $\mu\boxplus\nu$ and defined by [8]

$$R_{\mu\boxplus\nu}(z) = R_\mu(z) + R_\nu(z),$$

which implies

$$\kappa_n(\mu \boxplus \nu) = \kappa_n(\mu) + \kappa_n(\nu), \qquad n \in \mathbb{N}.$$

For $\mu \in \mathcal{P}^0(\mathbb{R}_+)$ with $\mu(\{0\}) < 1$, the moment generating function $\Psi_\mu(z)$ defined by (1.13) has a unique inverse

$$\chi_\mu(z) := \Psi_\mu^{\langle -1\rangle}(z) \tag{1.15}$$

on the left-half plane $\sqrt{-1}\mathbb{C}^+$ [8]. In this case the S-*transform* of μ is defined by

$$S_\mu(z) := \frac{1+z}{z}\chi_\mu(z), \qquad z \in \Psi_\mu\big(\sqrt{-1}\mathbb{C}^+\big). \tag{1.16}$$

For two probability measures μ, ν having S-transforms, the *free multiplicative convolution* [8] is defined as the the probability measure $\mu \boxtimes \nu$ such that

$$S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z)$$

for z in a common region of $\Psi_\mu\big(\sqrt{-1}\mathbb{C}^+\big) \cup \Psi_\nu\big(\sqrt{-1}\mathbb{C}^+\big)$.

The definition of the S-transform can be extended for symmetric probability measures $\mu \in \mathcal{P}_{\rm s}^0(\mathbb{R})$ [8, 62]. When $\mu \in \mathcal{P}_{\rm s}^0(\mathbb{R})$ with $\mu(\{0\}) < 1$, $\Psi_\mu(z)$ has a unique inverse on $H := \{z \in \mathbb{C}^-\colon |\operatorname{Re} z| < |\operatorname{Im} z|\}$, $\chi_\mu\colon \Psi_\mu(H) \to H$ and a unique inverse on $\widetilde{H} := \{z \in \mathbb{C}^+\colon |\operatorname{Re} z| < |\operatorname{Im} z|\}$, $\widetilde{\chi}_\mu\colon \Psi_\mu(\widetilde{H}) \to \widetilde{H}$, where $\mathbb{C}^- := \{z \in \mathbb{C}\colon \operatorname{Im} z < 0\}$. Therefore, there are two S-transforms for μ given by

$$S_\mu(z) = \frac{1+z}{z}\chi_\mu(z) \qquad \text{and} \qquad \widetilde{S}_\mu(z) = \frac{1+z}{z}\widetilde{\chi}_\mu(z),$$

and they satisfy

$$S_\mu(z)^2 = \frac{1+z}{z}S_{\mu^{(2)}}(z) \qquad \text{and} \qquad \widetilde{S}_\mu(z)^2 = \frac{1+z}{z}S_{\mu^{(2)}}(z).$$

It is known that for a probability measure $\mu \in \mathcal{P}^0(\mathbb{R}_+)$, there exists a unique symmetric probability measure $\mu^{\rm s} \in \mathcal{P}_{\rm s}^0(\mathbb{R})$ such that

$$\int_{\mathbb{R}} f\big(x^2\big)\mu^{\rm s}({\rm d}x) = \int_{\mathbb{R}} f(x)\mu^{(1/2)}({\rm d}x)$$

for every compactly supported continuous function f. The probability measure $\mu^{\rm s}$ is called *symmetrization* of a probability measure μ. For details, see [41, p. 134]. We note that the S-transform of ${\rm d}_a$ defined by (1.11) is given by

$$S_{{\rm d}_a}(z) = \frac{1}{a}\sqrt{\frac{1+z}{z}}, \qquad a > 0. \tag{1.17}$$

If the parameter $a = 1$, we use the notation d for ${\rm d}_1$ [62].

The PDFs (1.2), (1.3) and (1.8) for the Cauchy transforms $(G_{\mu_t})_{t\geq 0}$ are transforms as follows: For $(R_{\mu_t})_{t\geq 0}$,

$$\begin{aligned}
&\frac{\partial R_{{\rm w}_t}(z)}{\partial t} - z^2 = 0,\\
&\frac{\partial R_{{\rm w}_{\lambda,t}}(z)}{\partial t} - \frac{(1-\lambda)z^3}{2(R_{{\rm w}_{\lambda,t}}(z)+1)^2}\frac{\partial R_{{\rm w}_{\lambda,t}}(z)}{\partial z} - z^2 + \frac{(1-\lambda)z^2}{R_{{\rm w}_{\lambda,t}}(z)+1} = 0,\\
&\frac{\partial R_{{\rm m}_{\lambda,t}}(z)}{\partial t} - z^2\frac{\partial R_{{\rm m}_{\lambda,t}}(z)}{\partial z} - z(R_{{\rm m}_{\lambda,t}}(z)+\lambda) = 0,
\end{aligned}$$

and for $(S_{\mu_t})_{t\geq 0}$,

$$\begin{aligned}
&\frac{\partial S_{\mathrm{w}_t}(z)}{\partial t} + z^2 S_{\mathrm{w}_t}(z)^2 \frac{\partial S_{\mathrm{w}_t}(z)}{\partial z} + z S_{\mathrm{w}_t}(z)^3 = 0, \\
&\frac{\partial S_{\mathrm{w}_{\lambda,t}}(z)}{\partial t} + z^2 \left(1 - \frac{1-\lambda}{1+z}\right) S_{\mathrm{w}_{\lambda,t}}(z)^2 \frac{\partial S_{\mathrm{w}_{\lambda,t}}(z)}{\partial z} + z\left\{1 - \frac{(1-\lambda)(z+2)}{2(1+z)^2}\right\} S_{\mathrm{w}_{\lambda,t}}(z)^3 = 0, \\
&\frac{\partial S_{\mathrm{m}_{\lambda,t}}(z)}{\partial t} + z(z+\lambda) S_{\mathrm{m}_{\lambda,t}}(z) \frac{\partial S_{\mathrm{m}_{\lambda,t}}(z)}{\partial z} + (2z+\lambda) S_{\mathrm{m}_{\lambda,t}}(z)^2 = 0.
\end{aligned}$$

We find that the equation of $(R_{\mathrm{w}_t})_{t\geq 0}$ is extremely simple and it corresponds to the asymptotic freeness of a random matrix in the GUE and a deterministic Hermitian matrix (see Theorem 1.1(i), Remarks 1.2 and 1.3 below). On the other hand, the "external-force terms" include both of the coordinate z and the "fields" in other equations and seem to be more complicated than the equation (1.8) for $(G_{\mathrm{m}_{\lambda,t}})_{t\geq 0}$ whose external-force term is simply given by $G_{m_{\lambda,t}}(z)^2$. Theorems 1.1 and 1.5 given below solve the initial-value problems for all of them.

1.2 Results for the *R*-transforms

The following are the results for the R-transforms.

Theorem 1.1.

(i) *Assume that* $\mathrm{w}_0 \in \mathcal{P}^0(\mathbb{R})$*. Then,*

$$R_{\mathrm{w}_t}(z) = R_{\mathrm{w}_0}(z) + R_{\mathrm{w}_t^0}(z), \qquad t \geq 0, \tag{1.18}$$

where $R_{\mathrm{w}_t^0}(z) = tz^2$, $t \geq 0$*. It means that* $(\mathrm{w}_t)_{t\geq 0}$ *is decomposed into the initial probability measure* w_0 *and the fundamental solution* $(\mathrm{w}_t^0)_{t\geq 0}$ *with respect to the free additive convolution as follows,*

$$\mathrm{w}_t = \mathrm{w}_0 \boxplus \mathrm{w}_t^0, \qquad t \geq 0. \tag{1.19}$$

(ii) *Assume that* $\mathrm{m}_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}_+)$, $\lambda \in \mathbb{R}_+$*. Then,*

$$R_{\mathrm{m}_{\lambda,t}}(z) = \frac{1}{1-tz} R_{\mathrm{m}_{\lambda,0}}\left(\frac{z}{1-tz}\right) + R_{\mathrm{m}_{\lambda,t}^0}(z), \qquad t \geq 0, \tag{1.20}$$

where $R_{\mathrm{m}_{\lambda,t}^0}(z) = \lambda tz/(1-tz)$, $t \geq 0$.

(iii) *Assume that* $\mathrm{w}_{\lambda,0} \in \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$, $\lambda \in \mathbb{R}_+$*. Then,*

$$\begin{aligned}
R_{\mathrm{w}_{\lambda,t}}(z) &+ \frac{(1-\lambda)tz^2}{R_{\mathrm{w}_{\lambda,t}}(z)+1} \\
&= R_{\mathrm{w}_{\lambda,t}^0}(z) + \frac{(1-\lambda)tz^2}{R_{\mathrm{w}_{\lambda,t}^0}(z)+1} \\
&\quad + R_{\mathrm{w}_{\lambda,0}}\left(z\sqrt{1-(1-\lambda)\left\{\frac{1}{R_{\mathrm{w}_{\lambda,t}}(z)+1} - \frac{1}{R_{\mathrm{w}_{\lambda,t}}(z)+1-tz^2}\right\}}\right), \qquad t \geq 0,
\end{aligned} \tag{1.21}$$

where $R_{\mathrm{w}_{\lambda,t}^0}(z) = \{-1 + tz^2 + \sqrt{1 + 2(2\lambda-1)tz^2 + t^2z^4}\}/2$.

Remark 1.2. Consider a matrix-valued Brownian motion $(M_t)_{t\ge 0}$ which is given by a time-evolution of a Hermitian $N\times N$ matrix starting from a Hermitian matrix M_0. When M_0 is a null matrix, we write this process as $\big(M_t^0\big)_{t\ge 0}$. Then we have

$$M_t = M_0 + M_t^0, \qquad t\ge 0. \tag{1.22}$$

We assume that the empirical eigenvalue distribution of M_0 converges to w_0 as $N\to\infty$. As an eigenvalue process of (1.22), we can obtain Dyson's Brownian motion model with $\beta=2$ starting from the eigenvalues of M_0. (See Section 2.1 below.) Moreover, for any $\beta>0$, we can obtain the same Cauchy transform. The process $(\mathrm{w}_t)_{t\ge 0}$ is obtained as the time-evolution of the limit empirical measure of Dyson's Brownian motion model. We can show that M_0 and M_t^0 are *asymptotically free* for any $t\ge 0$. Thus at each time $t\ge 0$, the limiting eigenvalue distribution w_t of M_t converges to $\mathrm{w}_0 \boxplus \mathrm{w}_t^0$. That is, the assertion (i) of Theorem 1.1 is consistent with the asymptotic freeness of a random matrix in the GUE and an arbitrary deterministic Hermitian matrix. Such an interpretation of the assertion (ii) of Theorem 1.1 is not yet known and is left as a challenging future problem.

Remark 1.3. With respect to the process $(\mathrm{w}_t)_{t\ge 0}$, it is pointed out that the functional equations (1.18) for the R-transform and (3.4) for the Cauchy transform given below are consequences of the *Markov property of freeness* and its relation to *analytic subordination* [5, 13, 70]. As shown in Sections 3 and 4.1, we will prove (1.18) from (3.4). In the context of analytic subordination, the functional equation (3.4) shall be considered to prove unique existence of the one-parameter $(t\in[0,\infty))$ family of functions ω_t such that

$$\lim_{y\uparrow\infty} \frac{\omega_t\big(\sqrt{-1}y\big)}{\sqrt{-1}y} = 1, \qquad \text{and} \qquad G_{\mathrm{w}_t} = G_{\mathrm{w}_0}\circ\omega_t, \quad t\in[0,\infty),$$

by explicitly showing that $\omega_t(z) = z - tG_{\mathrm{w}_t}(z)$. Here the unique existence of $(G_{\mathrm{w}_t}(z))_{t\ge 0}$ with appropriate properties is guaranteed by the fact that $(G_{\mathrm{w}_t}(z))_{t\ge 0}$ solves the complex Burgers equation (1.2). For the other processes $(\mathrm{m}_{\lambda,t})_{t\ge 0}$ and $(\mathrm{w}_{\lambda,t})_{t\ge 0}$, the connections of our results with analytic subordination properties should be studied in the future.

Remark 1.4. The process $\big(\mathrm{w}_t^0\big)_{t\ge 0}$ given by (1.9) is identified with the *free Brownian motion* studied in free probability theory [11, 65]. Here we would like to consider the process $(\mathrm{w}_t)_{t\ge 0}$ determined by the Burgers equation (1.2) as a generalization of the free Brownian motion, since initial probability measure w_0 is now arbitrary in $\mathcal{P}^0(\mathbb{R})$. See also [12, 69] for the important connection between the complex Burgers equation (1.2) and free probability theory. Capitaine and Donati-Martin [22] introduce the *free Wishart processes* based on the two-parametric Marcenko–Pastur distribution (1.10). Our process $(\mathrm{m}_{\lambda,t})_{t\ge 0}$ is defined as a solution of the complex Burgers-type equation (1.8) specified by its initial probability measure $\mathrm{m}_{\lambda,0}\in\mathcal{P}^0(\mathbb{R}_+)$. In other words, $(\mathrm{m}_{\lambda,t})_{t\ge 0}$ is a family of processes parameterized by an initial measure $\mathrm{m}_{\lambda,0}$, and hence it is different from the free Wishart process. In the functional equation (1.20) in Theorem 1.1(ii), the parameterization by an initial probability measure is realized by the first term in the r.h.s., which is added to the fundamental solution for the R-transform, $R_{\mathrm{m}^0_{\lambda,t}}(z)$, of the two-parametric Marcenko–Pastur distribution (1.10). The *rectangular free convolutions* studied by Benaych-Georges [6, 7] are very interesting and important extensions of the square free convolutions. They are based on the original Marcenko–Pastur distribution (i.e., the special case of (1.10) at $t=1$). By this reason, it is not easy to discuss the present study from the viewpoint of the rectangular free convolutions. We want to leave this topic as a future problem. The equality (1.21) in Theorem 1.1(iii) seems to be so complicated, but it clearly shows that if $\lambda=1$, this equality is reduced to (1.18) as expected from (1.4). To the best of our knowledge, the chiral GUE and its time evolution $(\mathrm{w}_{\lambda,t})_{t\ge 0}$ determined by (1.3) have not been systematically studied in free probability theory.

1.3 Results for the S-transforms

For $\left(\mathrm{w}_t^0\right)_{t\geq 0}$ starting from δ_0, it is easy to verify that (see, for instance, [62])

$$S_{\mathrm{w}_t^0}(z) = \sqrt{\frac{1}{tz}}, \qquad t \geq 0. \tag{1.23}$$

For the process $\left(\mathrm{m}_{\lambda,t}^0\right)_{t\geq 0}$ starting from δ_0, we have [62]

$$S_{\mathrm{m}_{\lambda,t}^0}(z) = \frac{1}{t(z+\lambda)}, \qquad t \geq 0.$$

Theorem 1.5.

(*i*) *Provided that* $\mathrm{w}_{\lambda,0} = \mathrm{m}_{\lambda,0}^{\mathbf{s}}$, $\lambda \in \mathbb{R}_+$, *the following equality holds,*

$$S_{\mathrm{w}_{\lambda,t}}(z) = S_{\mathrm{d}}(z)\sqrt{S_{\mathrm{m}_{\lambda,t}}(z)}, \qquad t \geq 0. \tag{1.24}$$

(*ii*) *Assume that the* S*-transform of the initial probability measure* $S_{\mathrm{w}_0}(z)$ *is well defined. Then,*

$$\frac{S_{\mathrm{w}_t}(z)}{1 - (S_{\mathrm{w}_t}(z)/S_{\mathrm{w}_t^0}(z))^2} = S_{\mathrm{w}_0}\left(z\left\{1 - \left(S_{\mathrm{w}_t}(z)/S_{\mathrm{w}_t^0}(z)\right)^2\right\}\right), \qquad t \geq 0. \tag{1.25}$$

(*iii*) *Assume that the* S*-transform of the initial measure* $S_{\mathrm{m}_{\lambda,0}}(z)$, $\lambda \in \mathbb{R}_+$ *is well defined. Then,*

$$\begin{aligned}\frac{S_{\mathrm{m}_{\lambda,t}}(z)}{\left\{1 - S_{\mathrm{m}_{\lambda,t}}(z)/S_{\mathrm{w}_t^0}(z)^2\right\}\left\{1 - S_{\mathrm{m}_{\lambda,t}}(z)/S_{\mathrm{m}_{\lambda,t}^0}(z)\right\}} \\ = S_{\mathrm{m}_{\lambda,0}}\left(z\left\{1 - S_{\mathrm{m}_{\lambda,t}}(z)/S_{\mathrm{m}_{\lambda,t}^0}(z)\right\}\right), \qquad t \geq 0.\end{aligned} \tag{1.26}$$

(*iv*) *Assume that the* S*-transform of the initial measure* $S_{\mathrm{w}_{\lambda,0}}(z)$, $\lambda \in \mathbb{R}_+$ *is well defined. Then,*

$$\begin{aligned}\sqrt{\frac{1 - \frac{z}{1+z}\left(S_{\mathrm{w}_{\lambda,t}}(z)/S_{\mathrm{w}_{\lambda,t}^0}(z)\right)^2}{1 - \frac{z}{1+z}\left(S_{\mathrm{w}_{\lambda,t}}(z)/S_{\mathrm{w}_t^0}(z)\right)^2}}\frac{S_{\mathrm{w}_{\lambda,t}}(z)}{1 - \left(S_{\mathrm{w}_{\lambda,t}}(z)/S_{\mathrm{w}_{\lambda,t}^0}(z)\right)^2} \\ = S_{\mathrm{w}_{\lambda,0}}\left(z\left\{1 - (S_{\mathrm{w}_{\lambda,t}}(z)/S_{\mathrm{w}_{\lambda,t}^0}(z))^2\right\}\right), \qquad t \geq 0.\end{aligned} \tag{1.27}$$

Since $\mathrm{w}_{1,t} = \mathrm{w}_t$, *if* $\lambda = 1$ (1.27) *is reduced to* (1.25).

Remark 1.6. The equality (1.24) in the assertion (i) of Theorem 1.5 can be regarded as a "push-back" representation of the equality (1.6) expressed using the second push-forward measure. This is very simple, but reveals an important role of the symmetric Bernoulli delta measure d_a with $a = 1$ defined by (1.11), whose S-transform is given by (1.17). If $\sqrt{S_{\mathrm{m}_{\lambda,t}}(z)}$, $t \geq 0$ is realized as an S-transform of a probability measure, say $(\nu_{\lambda,t})_{t\geq 0}$, then (1.24) implies the equality $\mathrm{w}_{\lambda,t} = \mathrm{d} \boxtimes \nu_{\lambda,t}$, $t \geq 0$.

1.4 Applications

First we apply the above theorems to the process $(\mathrm{w}_t)_{t\geq 0}$ starting from the symmetric Bernoulli delta measure d_a with displacement $2a > 0$ given by (1.11). Here we write this process as $(\mathrm{w}_t^a)_{t\geq 0}$.

Proposition 1.7. *The R-transform and the S-transform of $(\mathrm{w}_t^a)_{t\ge 0}$ are given by*

$$R_{\mathrm{w}_t^a}(z) = \frac{1}{2}\Big[2tz^2 - 1 + \sqrt{1+4a^2z^2}\Big], \qquad t \ge 0, \tag{1.28}$$

$$S_{\mathrm{w}_t^a}(z) = \frac{1}{(tz)^{1/2}}\left[1 + \frac{1}{2z} + \frac{a^2}{2tz} - \frac{1}{2z}\left(1+\frac{a^2}{t}\right)\sqrt{1+\frac{4a^2/t}{(1+a^2/t)^2}z}\,\right]^{1/2}, \qquad t \ge 0. \tag{1.29}$$

Note that (1.28) determines the free cumulants of $(\mathrm{w}_t^a)_{t\ge 0}$ as

$$\kappa_n(\mathrm{w}_t^a) = \begin{cases} t+a^2, & n=2, \\ -(-1)^{n/2}\dfrac{(n-3)!!}{(n/2)!}2^{n/2-1}a^n, & n \in \{4,6,8,\dots\}, \\ 0, & \text{otherwise}, \end{cases} \qquad t \ge 0.$$

As well known, for the fundamental solution $\left(\mathrm{w}_t^0\right)_{t\ge 0}$, $R_{\mathrm{w}_t^0}(z) = tz^2$, and hence $\kappa_n\left(\mathrm{w}_t^0\right) = t\delta_{n2}$, $n \in \mathbb{N}$, $t \ge 0$. The complexity of the solution $(\mathrm{w}_t^a)_{t\ge 0}$ with $a > 0$ reported in [3, 57, 72] is simply expressed here by the emergence of free cumulants $\kappa_n(\mathrm{w}_t^a)$ for all $n \in \mathbb{N}$, $t \ge 0$.

Next we apply the theorems to the process $(\mathrm{m}_{\lambda,t})_{t\ge 0}$ starting from δ_b with $b > 0$. Here we write this process as $\left(\mathrm{m}_{\lambda,t}^b\right)_{t\ge 0}$. If the variable x is replaced by x/λ, the parameters r by $1/\lambda$, and a by b/λ in the *three parametric Marcenko–Pastur measure* studied in [34], we obtain the present probability measure $\mathrm{m}_{\lambda,t}^b$, $t \ge 0$.

Proposition 1.8. *The R-transform and the S-transform of $\left(\mathrm{m}_{\lambda,t}^b\right)_{t\ge 0}$ are given by*

$$R_{\mathrm{m}_{\lambda,t}^b}(z) = \frac{z\big\{(\lambda t + b) - \lambda t^2 z\big\}}{(1-tz)^2}, \qquad t \ge 0, \tag{1.30}$$

$$S_{\mathrm{m}_{\lambda,t}^b}(z)\frac{1}{t(\lambda+z)}\left[1 + \frac{\lambda}{2z} + \frac{b}{2tz} - \frac{1}{2z}\left(\lambda+\frac{b}{t}\right)\sqrt{1+\frac{4b/t}{(\lambda+b/t)^2}z}\,\right], \qquad t \ge 0. \tag{1.31}$$

Comparing (1.29) and (1.31), we obtain the equality,

$$S_{\mathrm{w}_t^a}(z) = \sqrt{\frac{1+z}{z}}\sqrt{S_{\mathrm{m}_{1,t}^{a^2}}(z)}. \tag{1.32}$$

It is readily confirmed by the definition of the second push-forward measure (1.5) with $p = 2$ that $\mathrm{d}_a^{(2)} = \delta_{a^2}$, $a > 0$. Hence, the matching of initial measures is established, and as a special case of (1.7), $(\mathrm{w}_t^a)^{(2)} = m_{1,t}^{a^2}$, $t \ge 0$, $a > 0$. Therefore, (1.32) can be regarded as a special case of the assertion (i) of Theorem 1.5.

Note that (1.30) determines the free cumulants of $\mathrm{m}_{\lambda,t}^b$ as

$$\kappa_n\left(\mathrm{m}_{\lambda,t}^b\right) = (\lambda t + bn)t^{n-1}, \qquad n \in \mathbb{N}, \quad t \ge 0.$$

The complexity of $\left(\mathrm{m}_{\lambda,t}^b\right)_{t\ge 0}$ with $b > 0$ reported in [34] is simply expressed by a shift $\lambda t \to \lambda t + bn$ in the above formulas for $\kappa_n\left(\mathrm{m}_{\lambda,t}^b\right)$, $n \in \mathbb{N}$, $t \ge 0$. The dynamical phase transitions studied by [3, 34, 57, 72] seem to be hidden in the above solutions for the R-transforms and the S-transforms. Extracting the singularity at the transition point from the above results will be a future problem.

The present paper is organized as follows. In Section 2 we explain how the complex Burgers-type equations are derived in the hydrodynamic limits of stochastic log-gases. Then we prove fundamental relations between a measure $\mu \in \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$ and its second push-forward measure $\mu^{(2)} \in \mathcal{P}^0(\mathbb{R}_+)$. Section 3 is devoted to solving the present three kinds of complex Burgers equations (1.2), (1.3), and (1.8) by the method of characteristic curves [26, 29]. Proofs of Theorems 1.1, 1.5 and Propositions 1.7, 1.8 are given in Section 4. Concluding remarks are given in Section 5.

2 Preliminaries

2.1 From matrix-valued processes to complex Burgers-type equations through hydrodynamic limit

For $N \in \mathbb{N} := \{1, 2, \dots\}$, let H_N and U_N be the space of $N \times N$ Hermitian matrices and the group of $N \times N$ unitary matrices, respectively. We consider complex-valued continuous semi-martingale processes $\big(M_t^{ij}\big)_{t\ge 0}, 1 \le i, j \le N$ with the condition $\overline{M_t^{ji}} = M_t^{ij}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$, and define an H_N-valued process by $M_t = \big(M_t^{ij}\big)_{1\le i,j\le N}$. For $S = \mathbb{R}$ or $\mathbb{R}_+$, define the Weyl chambers as $\mathbb{W}_N(S) := \big\{\boldsymbol{x} = \big(x^1, \dots, x^N\big) \in S^N : x^1 < \dots < x^N\big\}$, and write their closures as $\overline{\mathbb{W}_N(S)} = \big\{\boldsymbol{x} \in \overline{S}^N : x^1 \le \dots \le x^N\big\}$. For each $t \ge 0$, there exists $U_t = \big(U_t^{ij}\big)_{1\le i,j\le N} \in \mathsf{U}_N$ such that it diagonalizes M_t as $U_t^\dagger M_t U_t = \operatorname{diag}\big(\Lambda_t^1, \dots, \Lambda_t^N\big)$ with the eigenvalues $\{\Lambda_t^i\}_{i=1}^N$ of M_t, where $U_t^\dagger$ is the Hermitian conjugate of U_t; $\big(U_t^{ij}\big)^\dagger = \overline{U_t^{ji}}$, $1 \le i, j \le N$, and we assume $\Lambda_t := \big(\Lambda_t^1, \dots, \Lambda_t^N\big) \in \overline{\mathbb{W}_N(\mathbb{R})}$, $t \ge 0$. For $\mathrm{d}M_t := \big(\mathrm{d}M_t^{ij}\big)_{1\le i,j\le N}$, define a set of quadratic variations,

$$\Gamma_t^{ij,k\ell} := \big\langle \big(U^\dagger \mathrm{d}MU\big)^{ij}, \big(U^\dagger \mathrm{d}MU\big)^{k\ell}\big\rangle_t, \qquad 1 \le i, j, k, \ell \le N, \quad t \ge 0.$$

The following is proved [19, 46, 50]. See [2, Section 4.3] for details of proof.

Proposition 2.1. *The eigenvalue process $(\Lambda_t)_{t\ge 0}$ satisfies the following system of SDEs,*

$$\mathrm{d}\Lambda_t^i = \mathrm{d}\mathcal{M}_t^i + \mathrm{d}J_t^i, \qquad t \ge 0, \quad 1 \le i \le N,$$

where $(\mathcal{M}_t^i)_{t\ge 0}, 1 \le i \le N$ are martingales with quadratic variations $\langle \mathrm{d}\mathcal{M}^i, \mathrm{d}\mathcal{M}^j\rangle_t = \Gamma_t^{ii,jj}\mathrm{d}t$, $t \ge 0$, and $(J_t^i)_{t\ge 0}, 1 \le i \le N$ are the processes with finite variations given by

$$\mathrm{d}J_t^i = \sum_{j=1}^N \frac{\mathbf{1}_{(\Lambda_t^i \ne \Lambda_t^j)}}{\Lambda_t^i - \Lambda_t^j} \Gamma_t^{ij,ji}\mathrm{d}t + \mathrm{d}\Upsilon_t^i.$$

Here $\mathrm{d}\Upsilon_t^i$ denotes the finite-variation part of $\big(U_t^\dagger \mathrm{d}M_t U_t\big)^{ii}$, $t \ge 0$, $1 \le i \le N$.

We will show two basic examples of $M_t \in \mathsf{H}_N, t \ge 0$ and applications of Proposition 2.1, see [50]. Let $\nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\big(B_t^{ij}\big)_{t\ge 0}$, $\big(\widetilde{B}_t^{ij}\big)_{t\ge 0}$, $1 \le i \le N + \nu$, $1 \le j \le N$ be independent one-dimensional standard Brownian motions. For $1 \le i \le j \le N$, put

$$S_t^{ij} = \begin{cases} B_t^{ij}/\sqrt{2}, & i < j, \\ B_t^{ii}, & i = j, \end{cases} \qquad A_t^{ij} = \begin{cases} \widetilde{B}_t^{ij}/\sqrt{2}, & i < j, \\ 0, & i = j, \end{cases}$$

and let $S_t^{ij} = S_t^{ji}$ (symmetric) and $A_t^{ij} = -A_t^{ji}$ (anti-symmetric), $t \ge 0$ for $1 \le j < i \le N$.

Example 2.2. Put

$$M_t = \big(M_t^{ij}\big) := \big(S_t^{ij} + \sqrt{-1}A_t^{ij}\big)_{1\le i,j\le N}, \qquad t \ge 0.$$

By definition $\langle \mathrm{d}M^{ij}, \mathrm{d}M^{k\ell}\rangle_t = \delta^{i\ell}\delta^{jk}\mathrm{d}t$, $t \ge 0$, $1 \le i, j, k, \ell \le N$. Hence, by unitarity of $U_t \in \mathsf{U}_N$, $t \ge 0$, we see that $\Gamma_t^{ij,k\ell} = \delta^{i\ell}\delta^{jk}$, which gives $\langle \mathrm{d}\mathcal{M}^i, \mathrm{d}\mathcal{M}^j\rangle_t = \Gamma_t^{ii,jj}\mathrm{d}t = \delta^{ij}\mathrm{d}t$ and $\Gamma_t^{ij,ji} \equiv 1$, $t \ge 0$, $1 \le i, j \le N$. Then Proposition 2.1 proves that the eigenvalue process $(\Lambda_t)_{t\ge 0}$, satisfies the following system of SDEs with $\beta = 2$,

$$\mathrm{d}\Lambda_t^i = \mathrm{d}B_t^i + \frac{\beta}{2} \sum_{1\le j\le N, j\ne i} \frac{\mathrm{d}t}{\Lambda_t^i - \Lambda_t^j}, \qquad t \ge 0, \quad 1 \le i \le N. \tag{2.1}$$

Here $\big(B^i_t\big)_{t\geq 0}, 1 \leq i \leq N$ are independent one-dimensional standard Brownian motions, which are different from $(B^{ij}_t)_{t\geq 0}$ and $\big(\widetilde{B}^{ij}_t\big)_{t\geq 0}$ used to define $\big(S^{ij}_t\big)_{t\geq 0}$ and $\big(A^{ij}_t\big)_{t\geq 0}$, $1 \leq i,j \leq N$. If $\beta = 2$ and the initial configuration is $N\delta_0$, that is, all N particles are at the origin, then at each time $t > 0$, $\Lambda_t = \big(\Lambda^1_t, \dots, \Lambda^N_t\big)$ gives a point process on $\mathbb{R}$ which is equal in distribution with the GUE eigenvalue point process with variance t [33, 50]. For $\beta > 0$, we call the solution of (2.1) the N-particle system of *Dyson's Brownian motion model* with parameter β [33], and write it as $\big(\Lambda^{\mathrm{D}(N,\beta)}_t\big)_{t\geq 0}$.

Example 2.3. Consider an $(N+\nu)\times N$ rectangular-matrix-valued process given by

$$K_t := \big(B^{ij}_t + \sqrt{-1}\widetilde{B}^{ij}_t\big)_{1\leq i\leq N+\nu, 1\leq j\leq N}, \qquad t \geq 0,$$

and define an H_N-valued process by

$$M_t = K^\dagger(t)K(t), \qquad t \geq 0.$$

The matrix M_t is positive semi-definite and hence the eigenvalues are non-negative; $\Lambda^i_t \in \mathbb{R}_+$, $t \geq 0$, $1 \leq i \leq N$. We see that the finite-variation part of dM^{ij}_t is equal to $2(N+\nu)\delta^{ij}\mathrm{d}t$, $t \geq 0$, and $\langle \mathrm{d}M^{ij}, \mathrm{d}M^{k\ell}\rangle_t = 2\big(M^{i\ell}_t\delta^{jk} + M^{kj}_t\delta^{i\ell}\big)\mathrm{d}t$, $t \geq 0$, $1 \leq i,j,k,\ell \leq N$, which implies that $\mathrm{d}\Upsilon^i_t = 2(N+\nu)\mathrm{d}t$, $\Gamma^{ij,ji}_t = 2\big(\Lambda^i_t + \Lambda^j_t\big)$, and $\langle \mathrm{d}\mathcal{M}^i, \mathrm{d}\mathcal{M}^j\rangle_t = \Gamma^{ii,jj}_t\mathrm{d}t = 4\Lambda^i_t\delta^{ij}\mathrm{d}t$, $t \geq 0$, $1 \leq i,j \leq N$. Then we have the following SDEs with $\beta = 2$ for the eigenvalue process of $(M_t)_{t\geq 0}$,

$$\mathrm{d}\Lambda^i_t = 2\sqrt{\Lambda^i_t}\mathrm{d}\widetilde{B}^i_t + \beta\left[(\nu+1) + 2\Lambda^i_t\sum_{1\leq j\leq N, j\neq i}\frac{1}{\Lambda^i_t - \Lambda^j_t}\right]\mathrm{d}t, \qquad t \geq 0, \quad 1 \leq i \leq N, \tag{2.2}$$

where $\big(\widetilde{B}^i_t\big)_{t\geq 0}, 1 \leq i \leq N$ are independent one-dimensional standard Brownian motions, which are different from $\big(B^{ij}_t\big)_{t\geq 0}$ and $\big(\widetilde{B}^{ij}_t\big)_{t\geq 0}$, $1 \leq i,j \leq N$, used above to define the rectangular-matrix-valued process $(K_t)_{t\geq 0}$. The parameter ν can be extended to $\nu > -1$, in which if $\nu \in (-1, 0)$, a reflecting wall is put at the origin [50]. We call the solution of (2.2) the N-particle system of the *Bru–Wishart process* with parameters (β, ν) [20], and write it as $\big(\Lambda^{\mathrm{BW}(N,\beta,\nu)}_t\big)_{t\geq 0}$.

The positive roots of eigenvalues of M_t give the *singular values* of the rectangular matrix K_t, which are denoted by

$$\mathcal{S}^i_t := \sqrt{\Lambda^i_t}, \qquad t \geq 0, \quad 1 \leq i \leq N. \tag{2.3}$$

The system of SDEs for them is readily obtained from (2.2) as

$$\mathrm{d}\mathcal{S}^i_t = \mathrm{d}\widetilde{B}^i_t + \frac{\beta(\nu+1)-1}{2\mathcal{S}^i_t}\mathrm{d}t + \frac{\beta}{2}\sum_{1\leq j\leq N, j\neq i}\left(\frac{1}{\mathcal{S}^i_t - \mathcal{S}^j_t} + \frac{1}{\mathcal{S}^i_t + \mathcal{S}^j_t}\right)\mathrm{d}t, \tag{2.4}$$

$t \geq 0, 1 \leq i \leq N$ with $\beta = 2$ and $\nu > -1$. If $\beta = 2$ and the initial configuration is $N\delta_0$, then at each time $t > 0$, $\mathcal{S}_t = \big(\mathcal{S}^1_t, \dots, \mathcal{S}^N_t\big)$ on $\mathbb{R}_+$ gives the *chiral GUE point process* with parameter ν and variance t studied in random matrix theory for high energy physics [1, 35, 64, 67, 68]. For $\beta > 0$, we call the solution of (2.4) the *chiral version of Dyson's Brownian motion model* with parameters (β, ν), and write it as $\big(\mathcal{S}^{\mathrm{chD}(N,\beta,\nu)}_t\big)_{t\geq 0}$.

At each time $t > 0$, the point processes $\Lambda^{\mathrm{D}(N,\beta)}_t$, $\Lambda^{\mathrm{BW}(N,\beta,\nu)}_t$, and $\mathcal{S}^{\mathrm{chD}(N,\beta,\nu)}_t$ are known as typical examples of one-dimensional *log-gases* [35]. Therefore, we will call the solutions of the SDEs (2.1), (2.2), and (2.4) *stochastic log-gases*. Note that, when $\beta = 2$, (2.2) and (2.4) can be regarded as the N-variable extensions of the $2(\nu+1)$-dimensional squared Bessel process and the Bessel process with parameter $\nu > -1$, respectively [46].

For $\Lambda_t^{\mathrm{D}(N,\beta)} = \big(\Lambda_t^{\mathrm{D}(N,\beta)\,1}, \dots, \Lambda_t^{\mathrm{D}(N,\beta)\,N}\big)$, $t \geq 0$, we regard the time evolution of empirical measures

$$\Xi_t^{\mathrm{D}(N,\beta)}(\cdot) := \frac{1}{N}\sum_{i=1}^{N} \delta_{\Lambda_t^{\mathrm{D}(N,\beta)\,i}}(\cdot), \qquad t \in [0,T],$$

as an element of $\mathcal{C}\big([0,T] \to \mathcal{P}^0(\mathbb{R})\big)$. For $\big(\Lambda_t^{\mathrm{BW}(N,\beta,\nu)}\big)_{t\geq 0}$ and $\big(\mathcal{S}_t^{\mathrm{D}(N,\beta,\nu)}\big)_{t\geq 0}$, let

$$\lambda := \frac{N+\nu}{N} \Longleftrightarrow \nu = (\lambda - 1)N,$$

and consider

$$\Xi_t^{\mathrm{BW}(N,\beta,(\lambda-1)N)}(\cdot) := \frac{1}{N}\sum_{i=1}^{N} \delta_{\Lambda_t^{\mathrm{BW}(N,\beta,(\lambda-1)N)\,i}}(\cdot), \qquad t \in [0,T]$$

as an element of $\mathcal{C}\big([0,T] \to \mathcal{P}^0(\mathbb{R}_+)\big)$, and with (2.3) consider

$$\Sigma_t^{\mathrm{chD}(N,\beta,(\lambda-1)N)}(\cdot) := \frac{1}{2N}\sum_{i=1}^{N} \big\{\delta_{\mathcal{S}_t^{\mathrm{chD}(N,\beta,(\lambda-1)N)\,i}}(\cdot) + \delta_{-\mathcal{S}_t^{\mathrm{chD}(N,\beta,(\lambda-1)N)\,i}}(\cdot)\big\}, \qquad t \in [0,T],$$

as an element of $\mathcal{C}\big([0,T] \to \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})\big)$. The following is proved [16, 21, 25, 63].

Theorem 2.4. *Assume that for any $N \in \mathbb{N}$, the initial measures $\Xi_0^{\mathrm{D}(N,\beta)}$, $\Xi_0^{\mathrm{BW}(N,\beta,(\lambda-1)N)}$, and $\Sigma_0^{\mathrm{chD}(N,\beta,(\lambda-1)N)}$ have bounded supports, where $\big(\Sigma_0^{\mathrm{chD}(N,\beta,(\lambda-1)N)}\big)^{(2)} = \Xi_0^{\mathrm{BW}(N,\beta,(\lambda-1)N)}$ is satisfied, and in $N \to \infty$ they converge weakly to the measures $\mathrm{w}_0 \in \mathcal{P}^0(\mathbb{R})$, $\mathrm{m}_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}_+)$, and $\mathrm{w}_{\lambda,0} \in \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$, respectively. Then for any fixed $T < \infty$,*

$$\begin{aligned}
&\big(\Xi_t^{\mathrm{D}(N,\beta)}(\cdot)\big)_{t\in[0,T]} \Longrightarrow (\mathrm{w}_t(\cdot))_{t\in[0,T]} \qquad \textit{a.s. in} \quad \mathcal{C}\big([0,T] \to \mathcal{P}^0(\mathbb{R})\big),\\
&\big(\Xi_t^{\mathrm{BW}(N,\beta,(\lambda-1)N)}(\cdot)\big)_{t\in[0,T]} \Longrightarrow (\mathrm{m}_{\lambda,t}(\cdot))_{t\in[0,T]} \qquad \textit{a.s. in} \quad \mathcal{C}\big([0,T] \to \mathcal{P}^0(\mathbb{R}_+)\big),\\
&\big(\Sigma_t^{\mathrm{chD}(N,\beta,(\lambda-1)N)}(\cdot)\big)_{t\in[0,T]} \Longrightarrow (\mathrm{w}_{\lambda,t}(\cdot))_{t\in[0,T]} \qquad \textit{a.s. in} \quad \mathcal{C}\big([0,T] \to \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})\big),
\end{aligned}$$

where $(\mathrm{w}_t)_{t\geq 0}$, $(\mathrm{m}_{\lambda,t})_{t\geq 0}$, and $(\mathrm{w}_{\lambda,t})_{t\geq 0}$ are the time-dependent probability measures defined so that their Cauchy transforms solve the PDEs (1.2), (1.8), and (1.3) under the initial probability measures w_0, $\mathrm{m}_{\lambda,0}$, and $\mathrm{w}_{\lambda,0}$, respectively.

Note that dependence on the parameter β vanishes in the limit $N \to \infty$.

By the construction mentioned above, the relation

$$\big(\Sigma_t^{\mathrm{chD}(N,\beta,(\lambda-1)N)}\big)^{(2)} \stackrel{\mathrm{d}}{=} \Xi_t^{\mathrm{BW}(N,\beta,(\lambda-1)N)}, \qquad t \geq 0,$$

holds, and then Theorem 2.4 proves the equality $\mathrm{w}_{\lambda,t}^{(2)} = \mathrm{m}_{\lambda,t}$, $t \geq 0$. This is consistent with the definition (1.6).

2.2 Expressions of second push-forward measures

We will prove the following.

Lemma 2.5. *For $\mu \in \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$ and $\nu \in \mathcal{P}^0(\mathbb{R}_+)$, the following four statements are equivalent with each other,*

$$(i) \quad \mu^{(2)} = \nu,$$

$$(ii) \quad G_\mu(z) = zG_\nu\big(z^2\big),$$

$$(iii) \quad R_\mu(z) = R_\nu\left(\frac{z^2}{R_\mu(z)+1}\right),$$

$$(iv) \quad S_\mu(z) = S_{\mathrm{d}}(z)\sqrt{S_\nu(z)}.$$

Proof. Assume that $\mu(\mathrm{d}x)$ (resp. $\nu(\mathrm{d}x)$) has a probability density function $\rho_\mu(x)$ (resp. $\rho_\nu(x)$). Let $B \in \mathcal{B}((0,\infty))$. Then

$$\nu(B) = \int_{\mathbb{R}} 1_B(x)\rho_\nu(x)\,\mathrm{d}x = \int_{\mathbb{R}_+} 1_B\big(x^2\big)\rho_\nu\big(x^2\big)\,\mathrm{d}x^2 = \int_{\mathbb{R}} 1_B\big(x^2\big)\rho_\nu\big(x^2\big)|x|\mathrm{d}x.$$

On the other hand, by definition (1.5), $\mu^{(2)}(B) = \int_{\mathbb{R}} 1_B\big(x^2\big)\rho_\mu(x)\,\mathrm{d}x$. Hence

$$(i) \iff \rho_\mu(x) = \rho_\nu\big(x^2\big)|x|, \qquad x \in \mathbb{R}. \tag{2.5}$$

Since (2.5) implies the symmetry $\rho_\mu(-x) = \rho_\mu(x)$, $x \in \mathbb{R}$, we see that

$$\begin{aligned} G_\mu(z) &= \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z-x}\mathrm{d}x = \frac{1}{2}\left\{\int_{\mathbb{R}} \frac{\rho_\mu(-x)}{z-x}\mathrm{d}x + \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z-x}\mathrm{d}x\right\} \\ &= \frac{1}{2}\left\{\int_{\mathbb{R}} \frac{\rho_\mu(x)}{z+x}\mathrm{d}x + \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z-x}\mathrm{d}x\right\} = z\int_{\mathbb{R}} \frac{\rho_\mu(x)}{z^2-x^2}\mathrm{d}x. \end{aligned}$$

Then when (2.5) is satisfied,

$$G_\mu(z) = z\int_{\mathbb{R}} \frac{\rho_\nu\big(x^2\big)|x|}{z^2-x^2}\mathrm{d}x = z\int_{\mathbb{R}_+} \frac{\rho_\nu\big(x^2\big)}{z^2-x^2}\mathrm{d}x^2 = zG_\nu\big(z^2\big) \iff (ii).$$

When (ii) is satisfied,

$$G_\mu\left(\frac{R_\mu(z)+1}{z}\right) = \frac{R_\mu(z)+1}{z}G_\nu\left(\left(\frac{R_\mu(z)+1}{z}\right)^2\right)$$

holds. By (1.14), this implies

$$\begin{aligned} z = \frac{R_\mu(z)+1}{z}G_\nu\left(\left(\frac{R_\mu(z)+1}{z}\right)^2\right) &\iff G_\nu\left(\left(\frac{R_\mu(z)+1}{z}\right)^2\right) = \frac{z^2}{R_\mu(z)+1} \\ \iff \left(\frac{R_\mu(z)+1}{z}\right)^2 = G_\nu^{\langle -1\rangle}\left(\frac{z^2}{R_\mu(z)+1}\right) &= \frac{R_\mu(z)+1}{z^2}\left\{R_\nu\left(\frac{z^2}{R_\mu(z)+1}\right)+1\right\}, \end{aligned}$$

where we used (1.14) again. This is equivalent with (iii).

When (ii) is satisfied,

$$G_\mu\left(\frac{z+1}{zS_\mu(z)}\right) = \frac{z+1}{zS_\mu(z)}G_\nu\left(\left(\frac{z+1}{zS_\mu(z)}\right)^2\right) \tag{2.6}$$

holds. By (1.13), (1.15), and (1.16), we can prove the equality

$$G_\mu\left(\frac{z+1}{zS_\mu(z)}\right) = zS_\mu(z).$$

Then (2.6) gives

$$G_\nu\left(\left(\frac{z+1}{zS_\mu(z)}\right)^2\right) = (z+1)\left(\frac{zS_\mu(z)}{z+1}\right)^2. \tag{2.7}$$

By (1.13), the l.h.s. of (2.7) is equal to $\left[\Psi_\nu\left(\{zS_\mu(z)/(z+1)\}^2\right)+1\right]\{zS_\mu(z)/(z+1)\}^2$. Hence we obtain the equalities $z = \Psi_\nu\left(\{zS_\mu(z)/(z+1)\}^2\right) \iff \chi_\nu(z) = \{zS_\mu(z)/(z+1)\}^2$. By (1.16), this gives

$$S_\mu(z)^2 = \frac{1+z}{z}S_\nu(z).$$

We use (1.17) with $a=1$ and then (iv) is obtained. Hence the proof is complete. ■

3 General solutions of complex Burgers-type equations

Let $t \in [0,\infty)$ and $z \in \mathbb{C}^+$ be independent variables and consider a PDE for a complex function $g = g(t,z) \in \mathbb{C}$ in the form,

$$A(t,z,g)\frac{\partial g}{\partial t} + B(t,z,g)\frac{\partial g}{\partial z} = C(t,z,g). \tag{3.1}$$

We regard the solution of (3.1) as a surface $g = g(t,z)$ in the space $[0,\infty)\times\mathbb{C}^+\times\mathbb{C}$. Then (3.1) is interpreted as a geometrical statement that the vector field $(A(t,z,g), B(t,z,g), C(t,z,g))$ is tangent to the surface at every point. This statement means that the graph of solution is given by a union of integral curves of this vector field. They are called the *characteristic curves* [26] of (3.1) and satisfy the *Lagrange–Charpit equatoin* (see [29] and references therein),

$$\frac{\mathrm{d}t}{A(t,z,g)} = \frac{\mathrm{d}z}{B(t,z,g)} = \frac{\mathrm{d}g}{C(t,z,g)}. \tag{3.2}$$

Here we consider the special case such that $A(t,z,g) \equiv 1$. Then (3.2) is written as

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = B(t,z,g), \\ \dfrac{\mathrm{d}g}{\mathrm{d}t} = C(t,z,g). \end{cases} \tag{3.3}$$

We will show that the solutions of (3.3) for (1.2), (1.3) and (1.8) are obtained in the forms of functional equations.

Proposition 3.1.

(i) *Given the Cauchy transform $G_{\mathrm{w}_0}(z)$ of the initial measure $\mathrm{w}_0 \in \mathcal{P}^0(\mathbb{R})$, the solution of* (1.2) *satisfies the functional equation,*

$$G_{\mathrm{w}_t}(z) = G_{\mathrm{w}_0}(z - tG_{\mathrm{w}_t}(z)), \qquad t \geq 0. \tag{3.4}$$

(ii) *Given the Cauchy transform $G_{\mathrm{m}_{\lambda,0}}(z)$ of the initial measure $\mathrm{m}_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}_+)$, the solution of* (1.8) *satisfies the functional equation,*

$$\frac{1}{G_{\mathrm{m}_{\lambda,t}}(z)} = t + \frac{1}{G_{\mathrm{m}_{\lambda,0}}((1-tG_{\mathrm{m}_{\lambda,t}}(z))\{(1-\lambda)t + (1-tG_{\mathrm{m}_{\lambda,t}}(z))z\})}, \quad t \geq 0. \tag{3.5}$$

(*iii*) *Given the Cauchy transform* $G_{\mathrm{w}_{\lambda,0}}(z)$ *of the initial measure* $\mathrm{w}_{\lambda,0} \in \mathcal{P}_{\mathrm{s}}^0(\mathbb{R})$, *the solution of* (1.3) *satisfies the functional equation,*

$$\frac{1}{G_{\mathrm{w}_{\lambda,t}}(z)} = \frac{t}{z} + \frac{\sqrt{(1-\frac{t}{z}G_{\mathrm{w}_{\lambda,t}}(z))\{(1-\lambda)t+(1-\frac{t}{z}G_{\mathrm{w}_{\lambda,t}}(z))z^2\}}}{zG_{\mathrm{w}_{\lambda,0}}\left(\sqrt{\left(1-\frac{t}{z}G_{\mathrm{w}_{\lambda,t}}(z)\right)\{(1-\lambda)t+\left(1-\frac{t}{z}G_{\mathrm{w}_{\lambda,t}}(z)\right)z^2\}}\right)}, \quad t \geq 0. \tag{3.6}$$

Proof. (*i*) Consider the PDE (1.2) for $g(t,z) = G_{\mathrm{w}_t}(z)$. In this case (3.3) becomes

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = g(t, z(t)), \tag{3.7}$$

$$\frac{\mathrm{d}g(t, z(t))}{\mathrm{d}t} = 0. \tag{3.8}$$

By (3.8), we can conclude that

$$g(t, z(t)) = g(0, z(0)) \qquad \forall t \geq 0. \tag{3.9}$$

Therefore, (3.7) is integrated as

$$z(t) = z(0) + tg(0, z(0)) = z(0) + tg(t, z(t)) \Longleftrightarrow z(0) = z(t) - tg(t, z(t)), \qquad t \geq 0.$$

Inserting this into (3.9), we obtain (3.4) for $g(t,z) = G_{\mathrm{w}_t}(z)$.

(*ii*) Consider the PDE (1.8) for $g(t,z) = G_{\mathrm{m}_{\lambda,t}}(z)$. In this case (3.3) becomes

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = 2zg(t, z(t)) - (1-\lambda), \tag{3.10}$$

$$\frac{\mathrm{d}g(t, z(t))}{\mathrm{d}t} = -g(t, z(t))^2. \tag{3.11}$$

The solution of (3.11) is given by

$$g(t, z(t)) = \frac{1}{t + 1/g(0, z(0))}. \tag{3.12}$$

Then (3.10) is written as

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{2z(t)}{t + 1/g(0, z(0))} - (1-\lambda).$$

This is integrated as

$$z(t) = \left(t + \frac{1}{g(0, z(0))}\right)\left\{1 - \lambda + C\left(t + \frac{1}{g(0, z(0))}\right)\right\}, \tag{3.13}$$

where C is an integral constant. By setting $t = 0$ in this equation, we see that

$$C = g(0, z(0))\{z(0)g(0, z(0)) - (1-\lambda)\}.$$

Using this and (3.12), (3.13) is rewritten as

$$z(t) = \frac{1}{g(t, z(t))} + \frac{\lambda tg(t, z(t)) - 1}{(1 - tg(t, z(t)))g(t, z(t))} + \frac{z(0)}{(1 - tg(t, z(t)))^2},$$

which gives

$$z(0) = (1 - tg(t, z(t)))\{(1-\lambda)t + (1 - tg(t, z(t)))z(t)\}, \qquad t \geq 0.$$

If we insert this expression of $z(0)$ into (3.12) and replace $z(t)$ by z, $g(t, z(t))$ by $G_{\mathrm{m}_{\lambda,t}}(z)$, and $g(0, \cdot)$ by $G_{\mathrm{m}_{\lambda,0}}(\cdot)$, then we obtain (3.5).

(*iii*) By Lemma 2.5(*ii*), (3.5) is transformed into (3.6). ■

4 Proofs

4.1 Proof of Theorem 1.1

(i) We put $z = G_{\mathrm{w}_t}^{\langle -1 \rangle}(\zeta)$ in (3.4). Then we have $\zeta = G_{\mathrm{w}_0}\big(G_{\mathrm{w}_t}^{\langle -1 \rangle}(\zeta) - t\zeta\big)$. Next we apply $G_{\mathrm{w}_0}^{\langle -1 \rangle}$ on both sides and obtain

$$G_{\mathrm{w}_0}^{\langle -1 \rangle}(\zeta) = G_{\mathrm{w}_t}^{\langle -1 \rangle}(\zeta) - t\zeta \iff \zeta G_{\mathrm{w}_0}^{\langle -1 \rangle}(\zeta) - 1 = \big(\zeta G_{\mathrm{w}_t}^{\langle -1 \rangle}(\zeta) - 1\big) - t\zeta^2.$$

By the definition (1.14), this implies the following equation between R-transforms

$$R_{\mathrm{w}_0}(\zeta) = R_{\mathrm{w}_t}(\zeta) - t\zeta^2, \qquad t \geq 0.$$

The assertion (i) of Theorem 1.1 is concluded by the well-known result [40], $R_{\mathrm{w}_t^0}(z) = tz^2$.

(ii) We put $z = G_{\mathrm{m}_{\lambda,t}}^{\langle -1 \rangle}(\zeta)$ in (3.5). Then we have

$$\begin{aligned} \frac{1}{\zeta} &= t + \frac{1}{G_{\mathrm{m}_{\lambda,0}}\big((1-t\zeta)\big\{(1-\lambda)t + (1-t\zeta)G_{\mathrm{m}_{\lambda,t}}^{\langle -1 \rangle}(\zeta)\big\}\big)} \\ &\iff G_{\mathrm{m}_{\lambda,0}}\big((1-t\zeta)\big\{(1-\lambda)t + (1-t\zeta)G_{\mathrm{m}_{\lambda,t}}^{\langle -1 \rangle}(\zeta)\big\}\big) = \frac{\zeta}{1-t\zeta}. \end{aligned}$$

We apply $G_{\mathrm{m}_{\lambda,0}}^{\langle -1 \rangle}$ on both sides and obtain

$$\begin{aligned} &(1-t\zeta)\big\{(1-\lambda)t + (1-t\zeta)G_{\mathrm{m}_{\lambda,t}}^{\langle -1 \rangle}(\zeta)\big\} = G_{\mathrm{m}_{\lambda,0}}^{\langle -1 \rangle}\left(\frac{\zeta}{1-t\zeta}\right) \\ &\qquad \iff -\lambda\zeta t + (1-t\zeta)\big[\zeta G_{\mathrm{m}_{\lambda,t}}^{\langle -1 \rangle}(\zeta) - 1\big] = \frac{\zeta}{1-t\zeta} G_{\mathrm{m}_{\lambda,0}}^{\langle -1 \rangle}\left(\frac{\zeta}{1-t\zeta}\right) - 1. \end{aligned}$$

By the definition (1.14), this implies the following equations between R-transforms,

$$\begin{aligned} &-\lambda tz + (1-tz)R_{\mathrm{m}_{\lambda,t}}(z) = R_{\mathrm{m}_{\lambda,0}}\left(\frac{z}{1-tz}\right) \\ &\qquad\qquad \iff R_{\mathrm{m}_{\lambda,t}}(z) = \frac{1}{1-tz} R_{\mathrm{m}_{\lambda,0}}\left(\frac{z}{1-tz}\right) + \frac{\lambda tz}{1-tz}. \end{aligned}$$

Since $R_{\mathrm{m}_{\lambda,t}^0}(z) = \lambda tz/(1-tz)$ [16, 40], the assertion (ii) is proved.

(iii) Applying Lemma 2.5(iii) to (1.6), (1.20) gives (1.21).

Hence the proof of Theorem 1.1 is complete.

4.2 Proof of Theorem 1.5

(i) Since $\mathrm{m}_{\lambda,t} \in \mathcal{P}^0(\mathbb{R}_+)$ is defined by (1.6), Lemma 2.5(iv) proves the assertion (i).

(ii) We start from (3.4) in Proposition 3.1(i). Replace z by $1/z$ and then apply (1.13). We have

$$z(\Psi_{\mathrm{w}_t}(z) + 1) = \frac{1}{1/z - tz(\Psi_{\mathrm{w}_t}(z)+1)} \left\{ \Psi_{\mathrm{w}_0}\left(\frac{1}{1/z - tz(\Psi_{\mathrm{w}_t}(z)+1)} \right) + 1 \right\}.$$

Put $z = \Psi_{\mathrm{w}_t}^{\langle -1 \rangle}(\zeta) =: \chi_{\mathrm{w}_t}(\zeta)$. Then

$$\chi_{\mathrm{w}_t}(\zeta)(\zeta+1) = \frac{1}{1/\chi_{\mathrm{w}_t}(\zeta) - t\chi_{\mathrm{w}_t}(\zeta)(\zeta+1)} \left\{ \Psi_{\mathrm{w}_0}\left(\frac{1}{1/\chi_{\mathrm{w}_t}(\zeta) - t\chi_{\mathrm{w}_t}(\zeta)(\zeta+1)} \right) + 1 \right\}.$$

By (1.16), the above is written as follows,

$$\zeta S_{\mathrm{w}_t}(\zeta) = \frac{1}{\frac{\zeta+1}{\zeta S_{\mathrm{w}_t}(\zeta)} - t\zeta S_{\mathrm{w}_t}(\zeta)} \left\{ \Psi_{\mathrm{w}_0}\left(\frac{1}{\frac{\zeta+1}{\zeta S_{\mathrm{w}_t}(\zeta)} - t\zeta S_{\mathrm{w}_t}(\zeta)} \right) + 1 \right\}$$

$$\iff \zeta\{1 - t\zeta S_{\mathrm{w}_t}(\zeta)^2\} = \Psi_{\mathrm{w}_0}\left(\frac{\zeta S_{\mathrm{w}_t}(\zeta)}{\zeta + 1 - t\zeta^2 S_{\mathrm{w}_t}(\zeta)^2} \right).$$

Now we apply χ_{w_0} on both sides and obtain

$$\chi_{\mathrm{w}_0}\big(\zeta\{1 - t\zeta S_{\mathrm{w}_t}(\zeta)^2\}\big) = \frac{\zeta S_{\mathrm{w}_t}(\zeta)}{\zeta + 1 - t\zeta^2 S_{\mathrm{w}_t}(\zeta)^2}.$$

Again we use (1.16) and replace the variable ζ by z. Then we obtain

$$\frac{S_{\mathrm{w}_t}(z)}{1 - tzS_{\mathrm{w}_t}(z)^2} = S_{\mathrm{w}_0}\big(z(1 - tzS_{\mathrm{w}_t}(z)^2)\big), \qquad t \geq 0.$$

which is written as (1.25) by (1.23).

(iii) We can prove (1.26) similarly to (1.25) as shown above.

(iv) By Lemma 2.5(iv), (1.26) is transformed to (1.27).

Hence the proof of Theorem 1.5 is complete.

4.3 Proof of Propisition 1.7

It is easy to verify that

$$R_{\mathrm{w}_t^0}(z) = tz^2,$$

$$R_{\mathrm{d}_a}(z) = \frac{1}{2}\Big[\sqrt{1 + 4a^2z^2} - 1\Big].$$

Then (1.28) is immediately concluded from Theorem 1.1(i). We have already obtained $S_{\mathrm{d}_a}(z)$ as (1.17) and $S_{\mathrm{w}_t^0}(z)$ as (1.23). Then (1.25) of Theorem 1.5(ii) gives

$$\frac{S_{\mathrm{w}_t}(z)}{1 - tzS_{\mathrm{w}_t}(z)^2} = \frac{1}{a}\sqrt{\frac{z + 1 - tz^2 S_{\mathrm{w}_t}(z)^2}{z - tz^2 S_{\mathrm{w}_t}(z)^2}}.$$

This is written as

$$t^2z^3 S_{\mathrm{w}_t}(z)^4 - z\{2tz + t + a^2\} S_{\mathrm{w}_t}(z)^2 + (z + 1) = 0,$$

which is solved by (1.29).

4.4 Proof of Proposition 1.8

It is easy to verify that

$$R_{\mathrm{m}_{\lambda,t}^0}(z) = \frac{\lambda tz}{1 - tz} = \lambda\left(\frac{1}{1 - tz} - 1\right),$$

$$R_{\delta_b}(z) = bz.$$

Then (1.30) is immediately concluded from Theorem 1.1(ii). We can see that $S_{\delta_b}(z) = 1/b$. Then (1.26) of Theorem 1.5(iii) gives

$$\frac{S_{\mathrm{m}_{\lambda,t}}(z)}{(1 - tzS_{\mathrm{m}_{\lambda,t}}(z))\{1 - t(\lambda + z)S_{\mathrm{m}_{\lambda,t}}(z)\}} = \frac{1}{b}.$$

This is written as

$$t^2 z(z + \lambda) S_{\mu_t}(z)^2 - (2tz + t\lambda + b) S_{\mu_t}(z) + 1 = 0.$$

which is solved by (1.31).

5 Concluding remarks

We list out some concluding remarks.

1. In addition to the free Brownian motion [11] and the free Wishart process [22], Demni introduced the *free Jacobi process* in [30]. This process has two parameters λ and θ. He derived the following PDE for the Cauchy transform of the measure-valued process $(\mathrm{k}_{\lambda,\theta,t})_{t\geq 0}$,

$$\frac{\partial G_{\mathrm{k}_{\lambda,\theta,t}}(z)}{\partial t} + [2\lambda\theta z(1-z)G_{\mathrm{k}_{\lambda,\theta,t}}(z) + \{(2\lambda\theta-1)z + \theta(1-\lambda)\}]\frac{\partial G_{\mathrm{k}_{\lambda,\theta,t}}(z)}{\partial z} + \{\lambda\theta(1-2z)G_{\mathrm{k}_{\lambda,\theta,t}}(z) + (2\lambda\theta-1)\}G_{\mathrm{k}_{\lambda,\theta,t}}(z) = 0. \tag{5.1}$$

Demni showed that this equation has the *stationary measure* given by

$$\mathrm{k}_{\lambda,\theta}(\mathrm{d}x) = \max\left(0, 1-\frac{1}{\lambda}\right)\delta_0(\mathrm{d}x) + \max\left(0, 1-\frac{1-\theta}{\lambda\theta}\right)\delta_1(\mathrm{d}x) + g(x)1_{[x_-,x_+]}(x)\,\mathrm{d}x,$$

where

$$g(x) = \frac{\sqrt{(x-x_-)(x_+-x)}}{2\lambda\theta\pi x(1-x)} \qquad \text{with} \quad x_\pm = \left(\sqrt{\theta(1-\lambda\theta)} \pm \sqrt{\lambda\theta(1-\theta)}\right)^2.$$

The distribution with the density $g(x)$ is known as the *Kesten–McKay law* [51, 54]. Is it possible to solve the initial-value problem for (5.1) as we did in this paper? When $\lambda = 1$ and $\theta = 1/2$, (5.1) is much simplified and given by

$$\frac{\partial G_{\mathrm{k}_{1,1/2,t}}(z)}{\partial t} + \frac{\partial}{\partial z}\left\{\frac{1}{2}z(1-z)G_{\mathrm{k}_{1,1/2,t}}(z)^2\right\} = 0.$$

This equation was solved by Demni, Hamdi, and Hmidi [32] when the initial probability measure is given by δ_1 and by Izumi and Ueda for general initial probability measure [44]. In these papers, the solutions are related with the *free unitary Brownian motion* [11]. See also [31, 39] for further study.

Another PDE for measure-valued process was reported in [66]

$$\frac{\partial G_{\mathrm{t}_{\alpha,c,t}}(z)}{\partial t} - \left\{2czG_{\mathrm{t}_{\alpha,c,t}}(z) + (z+2-\alpha)\right\}\frac{\partial G_{\mathrm{t}_{\alpha,c,t}}(z)}{\partial z} - z\frac{\partial^2 G_{\mathrm{t}_{\alpha,c,t}}(z)}{\partial z^2} - \left\{cG_{\mathrm{t}_{\alpha,c,t}}(z) + 1\right\}G_{\mathrm{t}_{\alpha,c,t}}(z) = 0,$$

where α and c are positive parameters. This describes the hydrodynamic limit of the Bru–Wishart (Laguerre) process in a high temperature regime (see also [58]). Notice that this equation involves a second-order derivative of $G_{\mathrm{t}_{\alpha,c,t}}(z)$ and hence it is regarded as a *viscous* Burgers-type equation. As proved by [25, 63], the hydrodynamic limit of Dyson's Brownian motion model is described by the *inviscid* Burgers equation (1.2). In this paper we have studied only such inviscid cases of Burgers-type equations (see Theorem 2.4 in Section 2.1). As shown by [18, 23, 24], however, if we consider the system of SDEs which have the same drift terms with Dyson's Brownian motion model (2.1) but the martingale terms are replaced as $\mathrm{d}B_t^i \to \sqrt{N}\mathrm{d}B_t^i$ (compare [23, equation (5.93)] with [63, equation (7)]), then we obtain the viscous Burgers equation in the hydrodynamic limit (see [23, equation (5.98)]). How can we solve such viscous Burgers-type equations?

2. Forrester and Grela [36] studied the hydrodynamic limits of the *circular ensemble* as well as the Jacobi ensemble. In the former case, they considered the following type of Cauchy transform,

$$G_\mu^\circ(z) = \frac{1}{2}\oint \cot\left(\frac{z-x}{2}\right)\mu(\mathrm{d}x).$$

This seems to be a *trigonometric extension* of the usual Cauchy transform (1.1), since if we introduce a parameter $r > 0$, then we see $(1/2r)\cot((z-x)/2r) \to 1/(z-x)$ as $r \to \infty$. See also [24, 43]. Some *elliptic extensions* of Cauchy-type transform have been also considered in a recent study of elliptic integrable systems [9, 10]. Is it meaningful to consider trigonometric and elliptic extensions of free probability theory?

3. In the present paper, we have studied the complex Burgers-type equations which are obtained in the hydrodynamic limits of the stochastic log-gases studied in random matrix theory [35, 46]. Recently, one of the present authors and Koshida proposed a new construction of the *multiple Schramm–Loewner evolutions* (SLEs) driven by stochastic log-gases using the notion of the *coupling* between the multiple SLEs and the *Gaussian free fields* [47, 48, 49]. Then the infinite-slit limits of the Loewner equations studied by [27, 28, 43] in the case that the slits are growing simultaneously can be interpreted as the hydrodynamic limits of the multiple SLEs [42] in the same context of stochastic processes as explained in Section 2.1. In the study of the hydrodynamic limits of the multiple SLEs, exact solutions of the complex Burgers-type equations for interesting initial conditions is very important [42]. On the other hand, new connections between the free probability theory and the Loewner chains have been reported [37, 43]. Moreover, an interesting discussion was given such that the complex Burgers-type equations themselves can be regarded as Loewner equations for certain subordination processes [43]. The method to solve the initial-value problems for the complex Burgers-type equations by deriving the functional equations reported in this paper shall be developed to analyze the hydrodynamic limits of multiple SLEs.

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