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From pp-Waves to Galilean Spacetimes

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Abstract. We exhibit all spatially isotropic homogeneous Galilean spacetimes of dimension $(n+1) \geq 4$, including the novel torsional ones, as null reductions of homogeneous pp-wave spacetimes. We also show that the pp-waves are sourced by pure radiation fields and analyse their global properties.

Key words: pp-waves; Galilean spacetimes; null reduction

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1 Motivation and introduction

Maximally symmetric Lorentzian spacetimes provide a natural arena for many areas of physics ranging from high energy physics (Minkowski space) to cosmology (de Sitter space) and quantum gravity (anti-de Sitter space). Beyond these well-known Lorentzian spacetimes, there is a plethora of equally interesting Galilean, Carrollian and Aristotelian spacetimes. These spacetimes were classified in [14], completing the pioneering work of Bacry and Lévy-Leblond [1]. One of the salient features of the classification in [14] is the emergence of two novel families of generically torsional[1] Galilean spacetimes, containing the (A)dS–Galilei spacetimes as special (non-torsional) points, see Figure 1.

Unlike many of the spacetimes in the classification, very little is known about these torsional Galilean spacetimes; however, there are various lines of investigation in which they may play interesting roles.

Holography. Particles in (A)dS–Galilei spacetimes have recently appeared as instructive toy models in the holography literature [4, 20, 22]. It is, therefore, natural to ask how these torsional geometries, which interpolate between AdS–Galilei and dS–Galilei, could appear in these models.

Intrinsically Galilean. The torsional Galilei algebras are a purely Galilean feature with no Lorentzian or Carrollian counterpart [14]. They are the only one-parameter family of non-equivalent maximally symmetric spaces, and they do not arise naturally as limits of Lorentzian spaces. Therefore, it could well be that effects due exclusively to the presence of torsion may be intrinsically Galilean, in some sense.

[1] These reductive homogeneous spacetimes are torsional in the sense that their canonical invariant connections have nonzero torsion. This should not be confused with the *intrinsic* torsion [9, 19] of the Galilean geometry, which is given by the exterior derivative of the clock one-form. The intrinsic torsion vanishes in all cases considered, but that does not mean that every connection has vanishing torsion, just that there is at least one connection which does.

This paper and the accompanying work [10] aim to deepen our understanding of these novel geometries and demystify them.

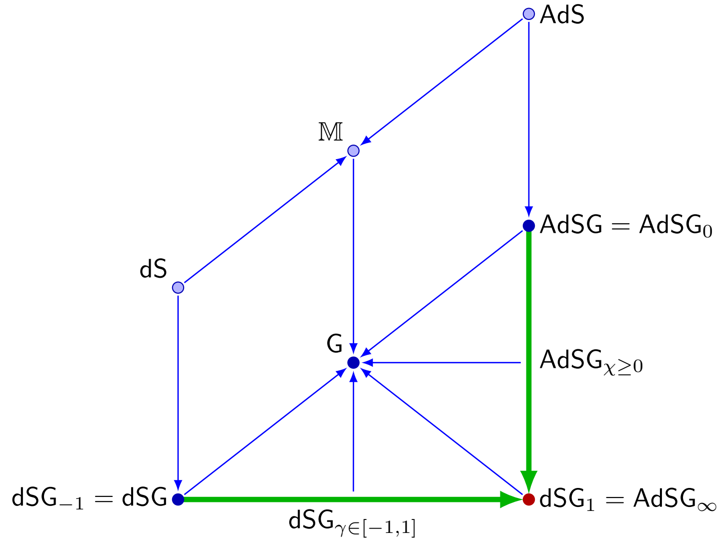

Figure 1. Spatially isotropic homogeneous Galilean spacetimes in dimension $n+1 \geq 4$. The blue arrows denote limits arising from Lie algebra contractions and show that (A)dS–Galilei arise naturally as limits of (A)dS, while the families of torsional Galilean spacetimes (green arrows) do not.

It is well known that we can obtain non-relativistic spacetimes by null reductions of Bargmann spacetimes in one dimension higher [7, 18]; namely, Lorentzian spacetimes admitting a nowhere-vanishing null vector field. In Duval, Burdet, Künzle and Perrin [7] this null vector field is assumed to be parallel with respect to the Levi-Civita connection; in other words, the Bargmann spacetime is a Brinkmann spacetime [5] or, equivalently, a pp-wave [8]. Starting with Minkowski spacetime, the null reduction along a parallel null vector field gives Galilei spacetime. In [18], the null vector field need not be parallel, but only Killing, and hence the Lorentzian manifolds go beyond the class of pp-waves. Later, Gibbons and Patricot [17] extended these ideas to obtain the non-relativistic limits of (anti) de Sitter spaces, which they called Newton–Hooke spacetimes, as null reductions of homogeneous pp-waves. These (A)dS–Galilei spacetimes can be interpreted as non-relativistic spacetimes with a non-vanishing cosmological constant. More recently Bekaert and Morand [3] studied conditions under which the null reduction by proper and free, but not necessarily isometric, action of the additive reals on a Lorentzian manifold results in a Galilean spacetime. We shall see examples of both the Julia–Nicolai null reduction and the more general null reduction of Bekaert–Morand in this paper.

Before we start, a word of nomenclature. From now on, by the word 'spacetime' we shall always mean a reductive homogeneous spacetime. Any reductive homogeneous space carries a canonical invariant connection (defined by the vanishing of the Nomizu map, as explained in the current context, for example, in [11]), as well as a number of homogeneous tensor fields arising by the holonomy principle from any invariant tensors of the linear isotropy representation at any chosen "origin" of the spacetime, as will be recalled below. By reduction we shall mean first and foremost reduction of homogeneous spaces, but in some cases this reduction also results directly in a reduction of the additional structure: connection and/or homogeneous tensor fields.

The symmetric Galilean spacetimes – namely, Galilei and (anti) de Sitter–Galilei – are homogeneous spaces of the Galilei and Newton–Hooke groups. They are known to arise as null reductions of certain homogeneous pp-wave spacetimes and hence it may be expected that the torsional Galilean spacetimes also admit such a description. In this paper, we show that this expectation is correct and exhibit explicit homogeneous pp-wave spacetimes whose null reductions agree with all the spatially isotropic homogeneous Galilean spacetimes in [14], including the torsional cases. We also show that these pp-waves are solutions of the Einstein field equations sourced by pure radiation fields. For convenience, we collect Galilean spacetimes in Table 1, whose notation we now briefly explain.

Table 1. Spatially isotropic homogeneous Galilean spacetimes ($n > 2$).

Spacetime	Additional nonzero Lie brackets	Name
G		Galilei
dSG	$[H,\boldsymbol{P}] = -\boldsymbol{B}$	de Sitter–Galilei (also $\mathsf{dSG}_{\gamma=-1}$)
dSG_γ	$[H,\boldsymbol{P}] = \gamma\boldsymbol{B} + (1+\gamma)\boldsymbol{P}$	torsional de Sitter–Galilei ($\gamma \in (-1,1)$)
dSG_1	$[H,\boldsymbol{P}] = \boldsymbol{B} + 2\boldsymbol{P}$	torsional de Sitter–Galilei ($\gamma = 1$)
AdSG	$[H,\boldsymbol{P}] = \boldsymbol{B}$	anti de Sitter–Galilei (also $\mathsf{AdSG}_{\chi=0}$)
AdSG_χ	$[H,\boldsymbol{P}] = \left(1+\chi^2\right)\boldsymbol{B} + 2\chi\boldsymbol{P}$	torsional anti de Sitter–Galilei ($\chi > 0$)

In this table we provide the nonzero Lie brackets in addition to $[\boldsymbol{J},\boldsymbol{J}] = \boldsymbol{J}$, $[\boldsymbol{J},\boldsymbol{B}] = \boldsymbol{B}$, $[\boldsymbol{J},\boldsymbol{P}] = \boldsymbol{P}$ and $[\boldsymbol{B},H] = \boldsymbol{P}$, in a basis where the stabiliser subalgebra $\mathfrak{h}$ is spanned by $\langle \boldsymbol{J},\boldsymbol{B}\rangle$.

A kinematical Lie algebra (in spatial dimension n) is a real Lie algebra with $\mathfrak{so}(n)$ rotations accompanied by two vectors (boosts and spatial translations) and one scalar (temporal translation). More concretely, $\mathfrak{g}$ is spanned by $J_{ij} = -J_{ji}$, B_i, P_i and H, where $i,j = 1,\dots,n$ and the Lie brackets include the following

$$\begin{aligned}
[J_{ij}, J_{k\ell}] &= \delta_{jk}J_{i\ell} - \delta_{ik}J_{j\ell} - \delta_{j\ell}J_{ik} + \delta_{i\ell}J_{jk},\\
[J_{ij}, B_k] &= \delta_{jk}B_i - \delta_{ik}B_j,\\
[J_{ij}, P_k] &= \delta_{jk}P_i - \delta_{ik}P_j,\\
[J_{ij}, H] &= 0,
\end{aligned} \tag{1.1}$$

and any other brackets consistent with the Jacobi identities. It is convenient to use an abbreviated notation in which we avoid the indices and write the above brackets as

$$[\boldsymbol{J},\boldsymbol{J}] = \boldsymbol{J}, \qquad [\boldsymbol{J},\boldsymbol{B}] = \boldsymbol{B}, \qquad [\boldsymbol{J},\boldsymbol{P}] = \boldsymbol{P}, \qquad [\boldsymbol{J},H] = 0.$$

In this work we focus on homogeneous kinematical spacetimes; that is, homogeneous spacetimes of a kinematical Lie group $\mathcal{G}$. Every such spacetime is described infinitesimally by a Klein pair[2] $(\mathfrak{g},\mathfrak{h})$, where $\mathfrak{g}$, the Lie algebra of $\mathcal{G}$, is the transitive Lie algebra, and $\mathfrak{h}$, the isotropy subalgebra, is the Lie algebra of the subgroup $\mathcal{H}$ of $\mathcal{G}$ which fixes a chosen origin for the homogeneous spacetime. It is this choice of homogeneous spacetime that provides a physical interpretation to the abstract Lie group $\mathcal{G}$ and gives meaning to what we call a boost (whose generator belongs to $\mathfrak{h}$ along with the rotations) and what we call a translation (which is in the complement). For example, the Poincaré group gives rise to a plethora of interesting inequivalent homogeneous spacetimes with different physical interpretations, see, e.g., [13].

In Table 1, we have chosen a basis for $\mathfrak{g}$ in such a way that the Lie subalgebra $\mathfrak{h}$ is spanned by J_{ij} and B_i, the rotations and (Galilean) boosts. Every row in the table lists the additional (i.e., not included in equation (1.1) or $[\boldsymbol{B},H] = \boldsymbol{P}$, which is common to Galilean transitive Lie algebras) nonzero Lie brackets in such a basis in the above abbreviated notation. As shown in [14], the Klein pairs in the table are in bijective correspondence with (isomorphism classes of) simply-connected spatially isotropic homogeneous Galilean spacetimes. These spacetimes form

[2] Originally, and unwisely, called a Lie pair in [14].

two continua dSG_γ, for $-1 \leq \gamma < 1$, and AdSG_χ, for $\chi \geq 0$. They both have a common limit $\lim_{\gamma\to 1} \mathsf{dSG}_\gamma = \lim_{\chi\to\infty} \mathsf{AdSG}_\chi$, which is no contraction and is represented by the red dot in Figure 1. All torsional Galilean spacetimes share a common geometric limit (i.e., a contraction at the level of the kinematical Lie algebras) to the Galilei spacetime G, as illustrated by the blue arrows that leave the green arrows in Figure 1. That figure also contains the maximally symmetric Lorentzian spacetimes and their Galilean (non-relativistic) and zero-curvature limits.

The spacetime denoted dSG_1 is more properly thought of as a limit $\lim_{\gamma\to 1} \mathsf{dSG}_\gamma$ and it coincides with the limit $\lim_{\chi\to\infty} \mathsf{AdSG}_\chi$. All Galilean spacetimes are reductive and G, dSG and AdSG are symmetric. They are the ones which can be obtained as non-relativistic limits of Minkowski, de Sitter and anti de Sitter spacetimes, respectively.

This note is organised as follows. In Section 2, we introduce some Klein pairs corresponding to homogeneous pp-wave spacetimes; that is, homogeneous Lorentzian manifolds admitting a parallel (relative to the Levi-Civita connection) null vector field. In Section 3, we describe the invariant Lorentzian metrics relative to (modified) exponential coordinates. In Section 4, we calculate the curvature of the pp-wave metrics and show that they are solutions of the Einstein field equations sourced by pure radiation fields. In Section 5, we exhibit explicit expressions for the Killing vector fields. In Section 6, we discuss the null reductions along the parallel null vector and show that they give the homogeneous Galilean spacetimes described in the Introduction. The Galilean structure induced by the reduction only partially agrees with the invariant Galilean structure: the spatial cometric is the invariant one, but the clock one-form is only homothetic to the invariant one; that is conformal to it but with a constant conformal factor. We then introduce a different null reduction resulting in the same homogeneous Galilean spacetime, but inducing the invariant Galilean structure on the nose. Two of the pp-wave metrics in Section 3 are flat and hence locally isometric, but their reductions result in non-isomorphic homogeneous Galilean spacetimes. This is discussed in Section 7. Finally, in Section 8, we offer some conclusions. In Appendix A, we provide a complementary set of coordinates and study global properties of the homogeneous pp-waves.

2 Lorentzian Klein pairs of the pp-waves

In this section, we introduce a number of Klein pairs corresponding to homogeneous Lorentzian manifolds admitting a parallel null vector field; that is, homogeneous pp-wave spacetimes. We will later show that their associated null reductions give all the homogeneous Galilean spacetimes in Table 1.

Before describing them, it is perhaps useful to say something about where they come from. They arose initially in a forthcoming follow-up paper to [12] in which we discuss geometries associated to Lie algebras obtained from Lifshitz Lie algebras by the addition of boosts. The homogeneous pp-wave spacetimes in question are geometric realisations of effective Klein pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a deformation of the centrally extended static kinematical Lie algebra and $\mathfrak{h}$ is spanned by what could be interpreted as spatial rotations and boosts. The Lie algebras $\mathfrak{g}$ had already appeared in [16, Table 2] for spatial dimension $n = 3$ and in [15, Table 18] for $n > 3$. These Lie algebras also arise naturally in the description of particle dynamics [10] on the homogeneous Galilean spacetimes in Table 1.

Let us now describe the Lorentzian Klein pairs $(\mathfrak{g}, \mathfrak{h})$ in question. The Lie algebra $\mathfrak{g}$ is spanned by $J_{ij} = -J_{ji}$, B_i, P_i, H and Z; although despite the notation Z is not necessarily central. The Lie brackets are written in the abbreviated form and we list only the nonzero Lie brackets in $\mathfrak{g}$ in addition to $[\boldsymbol{J}, \boldsymbol{J}] = \boldsymbol{J}$, $[\boldsymbol{J}, \boldsymbol{B}] = \boldsymbol{B}$, $[\boldsymbol{J}, \boldsymbol{P}] = \boldsymbol{P}$, $[\boldsymbol{B}, \boldsymbol{P}] = Z$ and $[\boldsymbol{B}, H] = \boldsymbol{P}$, in a basis where the stabiliser subalgebra $\mathfrak{h}$ is spanned by J_{ij} and B_i. They are listed in Table 2, where the spacetimes have been labelled as PX with X a Galilean spacetime, foreshadowing their interpretation as a principal (right) $\mathbb{R}$-bundle over X.

Table 2. Homogeneous Lorentzian spacetimes.

Spacetime	Additional nonzero Lie brackets		Comments
PG			
PdSG	$[H,\boldsymbol{P}] = -\boldsymbol{B}$		
PdSG_γ	$[H,\boldsymbol{P}] = \gamma\boldsymbol{B} + (1+\gamma)\boldsymbol{P}$	$[H,Z] = (1+\gamma)Z$	$\gamma \in (-1,1)$
PdSG_1	$[H,\boldsymbol{P}] = \boldsymbol{B} + 2\boldsymbol{P}$	$[H,Z] = 2Z$	
PAdSG	$[H,\boldsymbol{P}] = \boldsymbol{B}$		
PAdSG_χ	$[H,\boldsymbol{P}] = \left(1+\chi^2\right)\boldsymbol{B} + 2\chi\boldsymbol{P}$	$[H,Z] = 2\chi Z$	$\chi > 0$

In this table we provide the nonzero Lie brackets in addition to $[\boldsymbol{J},\boldsymbol{J}] = \boldsymbol{J}$, $[\boldsymbol{J},\boldsymbol{B}] = \boldsymbol{B}$, $[\boldsymbol{J},\boldsymbol{P}] = \boldsymbol{P}$, $[\boldsymbol{B},\boldsymbol{P}] = Z$ and $[\boldsymbol{B},H] = \boldsymbol{P}$, in a basis where the stabiliser subalgebra $\mathfrak{h}$ is spanned by $\langle\boldsymbol{J},\boldsymbol{B}\rangle$.

For each pair $(\mathfrak{g},\mathfrak{h})$ above we have a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ with $\mathfrak{m} = \langle\boldsymbol{P}, H, Z\rangle$. Let π^a, η, ζ be the canonically dual basis for $\mathfrak{m}^*$, i.e., the nonzero relations are $\langle\pi^a, P_b\rangle = \delta^a_b$, $\langle\eta, H\rangle = 1$ and $\langle Z,\zeta\rangle = 1$. Then for all Klein pairs in Table 2, there is an $\mathcal{H}$-invariant Lorentzian inner product on $\mathfrak{m}$ given by $\pi^2 - 2\eta\zeta$ and an $\mathcal{H}$-invariant vector given by Z. The holonomy principle associates to the inner product a $\mathcal{G}$-invariant Lorentzian metric g on the homogeneous space with Klein pair $(\mathfrak{g},\mathfrak{h})$ and to the invariant vector a nowhere-vanishing null vector field ζ. Both the metric and the null vector field are parallel with respect to the canonical invariant connection on the homogeneous space, whose holonomy representation is the linear isotropy representation of $\mathcal{H}$. It is only in the case of a symmetric space ($[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$) that the canonical invariant connection agrees with the Levi-Civita connection of g. In that case, the null vector field ζ is parallel with respect to the Levi-Civita connection and indeed agrees with the parallel null vector of the pp-wave. In this case, and this case alone, is ζ also, in particular, Killing.

The above reductive split lets us introduce the coset parametrisation[3] $\sigma\colon \mathfrak{m} \to \mathcal{G}$

$$\sigma(u,v,\boldsymbol{x}) = \exp(vZ)\exp(\boldsymbol{x}\cdot\boldsymbol{P})\exp(uH). \tag{2.1}$$

This parametrisation is of course not unique and we provide a complementary set of coordinates in Appendix A. The pull-back $\sigma^*\vartheta = \sigma^{-1}\mathrm{d}\sigma$ of the Maurer–Cartan one-form on $\mathcal{G}$ decomposes into $\sigma^*\vartheta = \theta + \omega$, where θ is a local coframe, in terms of which the $\mathcal{G}$-invariant Lorentzian metric is given by $g = \left(\pi^2 - 2\eta\zeta\right)(\theta,\theta)$. The one-form ω is the canonical invariant connection and it only agrees with the Levi-Civita connection of the invariant metric in the symmetric case; that is, when $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. This is only the case for PG, PdSG and PAdSG. Not coincidentally, these are precisely the cases where Z is central and hence $\mathfrak{g}$ is a central extension of a Galilean kinematical Lie algebra: Bargmann in the case of PG and (anti) de Sitter–Bargmann in the cases of PdSG and PAdSG which are sometimes also called Bargmann–Newton–Hooke.

3 The pp-wave metrics and the invariant vector fields

We now construct the explicit pp-wave metrics g and the invariant vector fields ζ in the modified exponential coordinates $\left(u,v,x^i\right)$ in equation (2.1). We calculate the coframe θ and then we evaluate the metric $g = \left(\pi^2 - 2\eta\zeta\right)(\theta,\theta)$. The value at $p \in M$ of the invariant vector field ζ is the inverse image of Z under the isomorphism $T_pM \to \mathfrak{m}$ defined by the coframe.

For PG we find that

$$\sigma^*\vartheta = \mathrm{d}uH + \mathrm{d}vZ + \mathrm{d}\boldsymbol{x}\cdot\boldsymbol{P} = \theta,$$

[3] We refer to $(u,v,\boldsymbol{x})$ as modified exponential coordinates, to distinguish them from the truly exponential coordinates which would be defined by $\sigma(u,v,\boldsymbol{x}) = \exp(vZ + uH + \boldsymbol{x}\cdot\boldsymbol{P})$.

so that $\omega = 0$. We recognise the metric

$$g = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} - 2\mathrm{d}u\mathrm{d}v \tag{3.1}$$

as Minkowski spacetime in light-cone coordinates. This metric is of course invariant under the full Poincaré group, but here we are reducing the structure to the Bargmann subgroup singled out by the distinguished null vector $\zeta = \partial_v$ corresponding to Z.

For PdSG_γ, we have

$$\begin{aligned}\sigma^*\vartheta &= \mathrm{d}uH + \exp(-u\,\mathrm{ad}_H)(\mathrm{d}vZ + \mathrm{d}\boldsymbol{x}\cdot\boldsymbol{P}) \\ &= \mathrm{d}uH + \mathrm{e}^{-u(1+\gamma)}\mathrm{d}vZ + (\exp(-u\,\mathrm{ad}_H)\boldsymbol{P})\cdot\mathrm{d}\boldsymbol{x}.\end{aligned}$$

From

$$-u\,\mathrm{ad}_H\begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix} = \begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix}\begin{pmatrix}0 & -\gamma u \\ u & -(1+\gamma)u\end{pmatrix}$$

we find that for $\gamma \in [-1,1)$,

$$\exp(-u\,\mathrm{ad}_H)\begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix} = \begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix}\frac{1}{1-\gamma}\begin{pmatrix}\mathrm{e}^{-u\gamma} - \gamma\mathrm{e}^{-u} & \gamma\left(\mathrm{e}^{-u} - \mathrm{e}^{-u\gamma}\right) \\ \mathrm{e}^{-u\gamma} - \mathrm{e}^{-u} & \mathrm{e}^{-u} - \gamma\mathrm{e}^{-u\gamma}\end{pmatrix}$$

and for $\gamma = 1$,

$$\exp(-u\,\mathrm{ad}_H)\begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix} = \begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-u}(1+u) & -u\mathrm{e}^{-u} \\ u\mathrm{e}^{-u} & \mathrm{e}^{-u}(1-u)\end{pmatrix},$$

from where we read off

$$(\exp(-u\,\mathrm{ad}_H)\boldsymbol{P}) = \begin{cases}\dfrac{\gamma}{1-\gamma}\left(\mathrm{e}^{-u} - \mathrm{e}^{-u\gamma}\right)\boldsymbol{B} + \dfrac{1}{1-\gamma}\left(\mathrm{e}^{-u} - \gamma\mathrm{e}^{-u\gamma}\right)\boldsymbol{P}, & \gamma \in [-1,1), \\ -u\mathrm{e}^{-u}\boldsymbol{B} + (1-u)\mathrm{e}^{-u}\boldsymbol{P}, & \gamma = 1.\end{cases}$$

The local coframe is then given by

$$\theta = \mathrm{d}uH + \begin{cases}\mathrm{e}^{-u(1+\gamma)}\mathrm{d}vZ + \dfrac{1}{1-\gamma}\left(\mathrm{e}^{-u} - \gamma\mathrm{e}^{-u\gamma}\right)\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{x}, & \gamma \in [-1,1), \\ \mathrm{e}^{-2u}\mathrm{d}vZ + (1-u)\mathrm{e}^{-u}\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{x}, & \gamma = 1\end{cases} \tag{3.2}$$

and the connection one form is given by

$$\omega = \begin{cases}\dfrac{\gamma}{1-\gamma}\left(\mathrm{e}^{-u} - \mathrm{e}^{-u\gamma}\right)\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{x}, & \gamma \in [-1,1), \\ -u\mathrm{e}^{-u}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{x} & \gamma = 1.\end{cases}$$

For $\gamma \leq 0$, the coframe is invertible for any $u \in \mathbb{R}$ whereas for $\gamma \in (0,1)$ we have the restriction $u < \frac{\ln\gamma}{\gamma-1}$. From equation (3.2) we can determine the invariant vector field ζ as well as the resulting metric g. With $\gamma \in (-1,1)$ they are given by (we have singled out the two extreme points):

$$\begin{aligned}&(\mathsf{PdSG}) && g = -2\mathrm{d}u\mathrm{d}v + (\cosh u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}, && \zeta = \partial_v, \\ &(\mathsf{PdSG}_\gamma) && g = -2\mathrm{e}^{-u(1+\gamma)}\mathrm{d}u\mathrm{d}v + \frac{1}{(1-\gamma)^2}\left(\mathrm{e}^{-u} - \gamma\mathrm{e}^{-u\gamma}\right)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}, && \zeta = \mathrm{e}^{(1+\gamma)u}\partial_v, \\ &(\mathsf{PdSG}_1) && g = -2\mathrm{e}^{-2u}\mathrm{d}u\mathrm{d}v + \mathrm{e}^{-2u}(1-u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}, && \zeta = \mathrm{e}^{2u}\partial_v.\end{aligned}$$

Lastly, for PAdSG_χ we have

$$\begin{aligned}\sigma^*\vartheta &= \mathrm{d}uH + \exp(-u\,\mathrm{ad}_H)(\mathrm{d}vZ + \mathrm{d}\boldsymbol{x}\cdot\boldsymbol{P})\\ &= \mathrm{d}uH + \mathrm{e}^{-2u\chi}\mathrm{d}vZ + (\exp(-u\,\mathrm{ad}_H)\boldsymbol{P})\cdot\mathrm{d}\boldsymbol{x}.\end{aligned}$$

From

$$-u\,\mathrm{ad}_H\begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix} = \begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix}\begin{pmatrix}0 & -u(1+\chi^2)\\ u & -2u\chi\end{pmatrix}$$

we find that

$$\exp(-u\,\mathrm{ad}_H)\begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix} = \begin{pmatrix}\boldsymbol{B} & \boldsymbol{P}\end{pmatrix}\mathrm{e}^{-u\chi}\begin{pmatrix}\cos u + \chi\sin u & -(1+\chi^2)\sin u\\ \sin u & \cos u - \chi\sin u,\end{pmatrix}$$

from where we read off

$$(\exp(-u\,\mathrm{ad}_H)\boldsymbol{P}) = \mathrm{e}^{-u\chi}(\cos u - \chi\sin u)\boldsymbol{P} - \mathrm{e}^{-u\chi}(1+\chi^2)\sin u\boldsymbol{B}.$$

The local coframe is given by

$$\theta = \mathrm{d}uH + \mathrm{e}^{-2u\chi}\mathrm{d}vZ + \mathrm{e}^{-u\chi}(\cos u - \chi\sin u)\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{x}$$

and the connection one-form is given by

$$\omega = -\mathrm{e}^{-u\chi}(1+\chi^2)\sin u\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{x}.$$

The invariant metrics and vector fields, where now $\chi > 0$ and we have singled out the case $\chi = 0$, are given by

$$\begin{aligned}&(\mathsf{PAdSG}) \quad g = -2\mathrm{d}u\mathrm{d}v + (\cos u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}, \qquad \zeta = \partial_v\\ &(\mathsf{PAdSG}_\chi) \quad g = -2\mathrm{e}^{-2u\chi}\mathrm{d}u\mathrm{d}v + \mathrm{e}^{-2u\chi}(\cos u - \chi\sin u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}, \qquad \zeta = \mathrm{e}^{2\chi u}\partial_v.\end{aligned}$$

For $\chi > 0$ the coframe is invertible in the range $-\frac{\pi}{2} < u < \arctan\left(\frac{1}{\chi}\right)$ and for $\chi = 0$ in $-\pi/2 < u < \pi/2$.

In Appendix A, using a different set of (global) coordinates, we show that the simply-connected Lorentzian spacetimes are actually diffeomorphic to $\mathbb{R}^{n+2}$.

We summarise the results of this section in Table 3.

Table 3. Invariant metrics and vector fields.

Spacetime	Metric g	Vector field ζ	Comments
PG	$-2\mathrm{d}u\mathrm{d}v + d\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$	∂_v	
PdSG	$-2\mathrm{d}u\mathrm{d}v + (\cosh u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$	∂_v	$\mathsf{PdSG}_{\gamma=-1}$
PdSG_γ	$-2\mathrm{e}^{-(1+\gamma)u}\mathrm{d}u\mathrm{d}v + \left(\frac{\mathrm{e}^{-u}-\gamma\mathrm{e}^{-\gamma u}}{1-\gamma}\right)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$	$\mathrm{e}^{(1+\gamma)u}\partial_v$	$\gamma\in(-1,1)$
PdSG_1	$-2\mathrm{e}^{-2u}\mathrm{d}u\mathrm{d}v + \mathrm{e}^{-2u}(1-u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$	$\mathrm{e}^{2u}\partial_v$	
PAdSG	$-2\mathrm{d}u\mathrm{d}v + (\cos u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$	∂_v	$\mathsf{PAdSG}_{\chi=0}$
PAdSG_χ	$-2\mathrm{e}^{-2\chi u}\mathrm{d}u\mathrm{d}v + \mathrm{e}^{-2\chi u}(\cos u - \chi\sin u)^2\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$	$\mathrm{e}^{2\chi u}\partial_v$	$\chi > 0$

4 Curvature tensors

Every metric in Table 3 is of the following form

$$g = -2a(u)\mathrm{d}u\mathrm{d}v + b(u)\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x},$$

for some functions $a(u)$ and $b(u)$, which can easily be read off. Our first observation is that such a metric is conformally flat. Indeed, away from the set of points where $b(u) = 0$, we may factor out $b(u)$ and write

$$g = b(u)\bigl(-2\tfrac{a(u)}{b(u)}\mathrm{d}u\mathrm{d}v + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}\bigr)$$

and then simply define a new coordinate t by $\mathrm{d}t = -a(u)/b(u)\mathrm{d}u$ so that the metric becomes

$$g = b(u)(2\mathrm{d}t\mathrm{d}v + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}),$$

which is manifestly conformally flat. The Weyl tensor vanishes outside the set of points where $b(u) = 0$, which for the metrics in Table 3 is either the whole space (when b never vanishes, as in PG and PdSG) or a dense open subset (in the remaining metrics). In either case, if the Weyl tensor vanishes in an open dense subset, it vanishes everywhere.

The nonzero components of the Riemann and Ricci tensor of the Levi-Civita connection of g are given by

$$\begin{aligned}
&R_{ux^iux^i} = \tfrac{1}{4}\bigl(\tfrac{2a'b'}{a} + \tfrac{b'^2}{b} - 2b''\bigr),\\
&R_{uu} = \tfrac{n}{b}R_{ux^iux^i}
\end{aligned}$$

and any other components related to these by the symmetries of the Riemann tensor, where $i = 1,\dots,n$ and where we do not sum over the i indices.

For the case at hand this reduces to

$$\begin{aligned}
&(\mathsf{PdSG}) \quad R_{ux^iux^i} = -(\cosh u)^2, \qquad R_{uu} = -n,\\
&(\mathsf{PdSG}_\gamma) \quad R_{ux^iux^i} = \frac{\gamma}{(1-\gamma)^2}\bigl(\mathrm{e}^{-u} - \gamma\mathrm{e}^{-u\gamma}\bigr)^2, \qquad R_{uu} = n\gamma,\\
&(\mathsf{PdSG}_1) \quad R_{ux^iux^i} = \mathrm{e}^{-2u}(1-u)^2, \qquad R_{uu} = n
\end{aligned}$$

and

$$\begin{aligned}
&(\mathsf{PAdSG}) \quad R_{ux^iux^i} = (\cos u)^2, \qquad R_{uu} = n,\\
&(\mathsf{PAdSG}_\chi) \quad R_{ux^iux^i} = \bigl(1+\chi^2\bigr)\mathrm{e}^{-2u\chi}(\cos u - \chi\sin u)^2, \qquad R_{uu} = n\bigl(1+\chi^2\bigr).
\end{aligned}$$

Notice that for $\gamma = 0$, the Riemann tensor of the metric in PdSG_γ vanishes and thus $\mathsf{PdSG}_{\gamma=0}$ and PG are locally isometric. We shall contrast these two cases in more detail in Section 7.

By inspection we also see that the Ricci scalar vanishes in all cases. This means that the only non-vanishing components of the Einstein tensor are the null Ricci tensor components R_{uu}. This suggests that these metrics can be understood as solutions of the Einstein field equations which are sourced by pure radiation fields (null dust) for which the energy momentum tensor satisfies (see, e.g., [21, Section 5.2])

$$T_{\mu\nu} = \phi^2 k_\mu k_\nu, \qquad k_\mu k^\mu = 0.$$

For our metrics this means that $k = du$ and $\phi^2 = n\gamma$ or $\phi^2 = n\bigl(1+\chi^2\bigr)$, which would seem to require $\gamma \geq 0$. The physical interpretation of the metrics $\mathsf{PdSG}_{\gamma<0}$ is not so clear.

5 The Killing vectors

We will now exhibit explicit expressions for the Killing vector fields of the metrics in Table 3. For each generator $X \in \mathfrak{g}$, we will exhibit vector fields ξ_X such that $[\xi_X, \xi_Y] = -\xi_{[X,Y]}$ for all $X, Y \in \mathfrak{g}$.

In all cases $\xi_{J_{ij}} = -x_i\partial_j + x_j\partial_i$, $\xi_{P_i} = \partial_i$ and $\xi_Z = \partial_v$. It then remains to give expressions for ξ_{B_i} and ξ_H. We find that

$$\begin{aligned} \xi_{B_i} &= x_i\partial_v + f(u)\partial_i, \\ \xi_H &= \partial_u + h(u)x^i\partial_i + \left(\lambda v + \tfrac{1}{2}\mu x^2\right)\partial_v, \end{aligned} \tag{5.1}$$

for functions f, h of u and constants λ, μ. The value of λ is determined from the $[\xi_H, \xi_Z]$ bracket and that of μ by $[\xi_H, \xi_{P_i}]$, which also gives an algebraic relation allowing us to solve for h in terms of f. Finally, the bracket $[\xi_{B_i}, \xi_H]$ gives a first-order ODE for f, which we can solve in each case. The results of these calculations are summarised in Table 4. As a check on the calculations, one can show that the invariant vector fields in Table 3 commute with all the Killing vectors.

Table 4. Data for ξ_{B_i} and ξ_H in equation (5.1).

Spacetime	λ	μ	$f(u)$	$h(u)$
PG	0	0	u	0
PdSG	0	-1	$\tanh u$	$-\tanh u$
$\mathsf{PdSG}_{\gamma\in(-1,1)}$	$1+\gamma$	γ	$-\frac{e^{-u}-e^{-\gamma u}}{e^{-u}-\gamma e^{-\gamma u}}$	$\frac{e^{-u}-\gamma^2 e^{-\gamma u}}{e^{-u}-\gamma e^{-\gamma u}}$
PdSG_1	2	1	$\frac{u}{1-u}$	$\frac{2-u}{1-u}$
PAdSG	0	1	$\tan u$	$\tan u$
$\mathsf{PAdSG}_{\chi>0}$	2χ	$1+\chi^2$	$\frac{\sin u}{\cos u - \chi\sin u}$	$2\chi + \left(1+\chi^2\right)\frac{\sin u}{\cos u-\chi\sin u}$

6 The null reductions

We now show that the homogeneous pp-waves null reduce to the torsional Galilean spaces as homogeneous spaces (spacetimes). We do this first via Killing reduction, which is always guaranteed to result in a Galilean structure in the quotient. Doing so, however, we find that the reduced invariant structure matches the clock one-form of the torsional spaces only up to scale. This is then remedied by performing a reduction by the invariant vector field ζ. This results in the same homogeneous quotient (since ζ is invariant) but now with the invariant Galilean structure.

6.1 Killing reduction of spacetime

Each of the Lorentzian metrics in Table 3 possesses a nowhere vanishing null Killing vector field $\xi = \partial_v$ in the modified exponential coordinates employed above. This Killing vector field generates a one-parameter subgroup Γ of the isometry group of (M, g): the one generated by $Z \in \mathfrak{g}$. The space of orbits M/Γ can be given the structure of a smooth manifold in such a way that the canonical projection $\pi\colon M \to M/\Gamma$ taking a point to its orbit under Γ is a smooth map. This allows us to pull back functions and differential forms from M/Γ to M and sets up an isomorphism of $C^\infty(M/\Gamma)$-modules between the forms on M/Γ and the basic forms on M, where we remind the reader that basic forms are those forms α which are horizontal, so that $\imath_\xi\alpha = 0$, and invariant, so that $\mathscr{L}_\xi\alpha = 0$. Equivalently, α is basic if both $\imath_\xi\alpha = 0$ and $\imath_\xi d\alpha = 0$.

Let $\xi^\flat = g(\xi, -)$ be the one-form metrically dual to ξ. Then $\imath_\xi \xi^\flat = g(\xi,\xi) = 0$ because ξ is null and $\mathscr{L}_\xi \xi^\flat = (\mathscr{L}_\xi \xi)^\flat = 0$, where we have used that ξ is a Killing vector field, so that it commutes with the musical isomorphisms. This shows that $\xi^\flat = \pi^*\tau$, for some one-form $\tau \in \Omega^1(M/\Gamma)$.

If $\alpha \in \Omega^1(M/\Gamma)$, we may construct a vector field $(\pi^*\alpha)^\sharp$ on M by pulling α back to M and then applying the musical isomorphism. Given $\alpha, \beta \in \Omega^1(M/\Gamma)$ we get a function

$$g\big((\pi^*\alpha)^\sharp, (\pi^*\beta)^\sharp\big) = (\pi^*\alpha)\big((\pi^*\beta)^\sharp\big) = \pi^*\alpha\big(\pi_*(\pi^*\beta)^\sharp\big).$$

This defines $\psi \in \Gamma\big(\odot^2 T(M/\Gamma)\big)$ by

$$\psi(\alpha,\beta) = \alpha\big(\pi_*(\pi^*\beta)^\sharp\big).$$

Clearly, if $\beta = \tau$, then $(\pi^*\beta)^\sharp = \xi$ and since $\pi_*\xi = 0$, we see that $\psi(\alpha,\tau) = 0$ for all $\alpha \in \Omega^1(M)$. The pair (τ,ψ) defines a Galilean structure on M. Galilean structures can be classified into several types according to their intrinsic torsion $\mathrm{d}\tau$ [2, 6, 9, 19]. In all examples in this section, the null killing vector $\xi = \partial_v$ is actually parallel, so that $\mathrm{d}\xi^\flat = 0$ and hence $\mathrm{d}\tau = 0$. So that the intrinsic torsion of the Galilean spacetimes vanishes in all cases.

Below we will calculate local coordinate expressions for τ and ψ, so let us unpack the previous discussion and re-express everything in a local chart. It is often convenient to work in local coordinates which are adapted to the reduction. This means choosing local coordinates $x^\mu = (x^a, v)$ for M such that x^a are local coordinates for the base M/Γ. Since functions on the base lift to Γ-invariant functions on M, it follows that $\xi x^a = 0$, so that $\xi \propto \partial_v$. It is convenient to choose the coordinate v to be adapted to ξ, so that $\xi = \partial_v$. Nevertheless we will write $\xi = \xi^\mu \partial_\mu$, with the tacit understanding that $\xi^a = 0$ and $\xi^v = 1$. The metric has a local expression $g = g_{\lambda\rho}\mathrm{d}x^\lambda \mathrm{d}x^\rho$ and $g_{vv} = 0$ since ξ is null. The dual one-form $\xi^\flat$ has a local expression $\xi^\flat = \xi_\mu \mathrm{d}x^\mu$, where $\xi_\mu = g_{\mu\rho}\xi^\rho$. Since ξ is null, it follows that $\xi_v = 0$ and the only nonzero components are $\xi_a =: \tau_a$, the clock one-form. It is locally a one-form on M/Γ since ξ is also a Killing vector. To obtain a local coordinate expression for ψ, let α, β be two one-forms on the base, whose local expressions are $\alpha = \alpha_a \mathrm{d}x^a$ and similarly for β. Then $\psi(\alpha,\beta) = \psi^{ab}\alpha_a\beta_b = g_{\lambda\rho}g^{\lambda a}g^{\rho b}\alpha_a\beta_b$, so that $\psi^{ab} = g^{ab}$. Below we will actually choose local coordinates $x^a = \big(u, x^i\big)$ and we will see that the only nonzero components of ψ are ψ^{ij} and that τ is proportional to $\mathrm{d}u$.

Let $(\mathfrak{g},\mathfrak{h})$ be one of the pp-wave Klein pairs in Table 2. It is clear that in all cases, the generator $Z \in \mathfrak{g}$ spans an ideal $\langle Z\rangle$ of $\mathfrak{g}$. Quotienting by the action of the one-parameter subgroup generated by Z gives rise to a homogeneous space with Klein pair $\big(\overline{\mathfrak{g}},\overline{\mathfrak{h}}\big)$, where $\overline{\mathfrak{g}} = \mathfrak{g}/\langle Z\rangle$ and $\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h}\cap\langle Z\rangle \cong \mathfrak{h}$. It is clear by inspection of Table 2 that for each such Klein pair $(\mathfrak{g},\mathfrak{h})$, the quotient Klein pair $\big(\overline{\mathfrak{g}},\overline{\mathfrak{h}}\big)$ is the corresponding one in Table 1. In other words, the homogeneous pp-wave spacetimes in Table 2 can be realised as the total space of principal (right) $\mathbb{R}$-bundles over the homogeneous Galilean spacetimes in Table 1. In summary, this exhibits the Galilean spatially isotropic homogeneous spacetimes in Table 1 as null reductions of homogeneous pp-wave spacetimes with metrics given by Table 3.

6.2 Reduction of invariant structure

A natural question is whether we recover them not just as homogeneous spaces, but whether the induced Galilean structure is the invariant one. The clock one-form τ pulls back to the one-form metrically dual to ξ_Z. It follows that if $X \in \mathfrak{g}$ does not commute with Z, the fundamental vector field $\overline{\xi}_X$ in the quotient does not preserve τ. Indeed, we have that

$$\pi^*\mathscr{L}_{\overline{\xi}_X}\tau = \mathscr{L}_{\xi_X}\pi^*\tau = \mathscr{L}_{\xi_X}\xi_Z^\flat = [\xi_X,\xi_Z]^\flat = \xi^\flat_{[Z,X]}.$$

It is clear from the Lie brackets in Table 2, that H is the only generator of $\mathfrak{g}$ which may fail to commute with Z and, when this happens, it simply rescales Z by a constant. Suppose that

$[Z, H] = wZ$ for some weight w. Then it follows from the above calculation that $\mathscr{L}_{\overline{\xi}_H}\tau = w\tau$, so that it acts homothetically on the clock one-form.

It is a simple matter to read off the Galilean structure (τ, ψ) relative to the coordinates $\left(t = -u, x^i\right)$ for each of the above spacetimes. These are listed in Table 5. That table also lists the homogeneous Galilean spacetimes corresponding to the null reductions, using the notation of [14]. They cannot be immediately compared with the ones in [11] since we are using a different coordinate system, but they can be recognised by their Klein pair as explained above.

Table 5. Galilean structures of null reductions.

Spacetime	τ	ψ
G	$\mathrm{d}t$	$\delta^{ij}\partial_i \otimes \partial_j$
dSG	$\mathrm{d}t$	$(\cosh t)^{-2}\delta^{ij}\partial_i \otimes \partial_j$
dSG_γ	$\mathrm{e}^{(1+\gamma)t}\mathrm{d}t$	$\left(\frac{\mathrm{e}^t - \gamma\mathrm{e}^{\gamma t}}{1-\gamma}\right)^{-2}\delta^{ij}\partial_i \otimes \partial_j$
dSG_1	$\mathrm{e}^{2t}\mathrm{d}t$	$\mathrm{e}^{-2t}(1+t)^{-2}\delta^{ij}\partial_i \otimes \partial_j$
AdSG	$\mathrm{d}t$	$(\cos t)^{-2}\delta^{ij}\partial_i \otimes \partial_j$
AdSG_χ	$\mathrm{e}^{2\chi t}\mathrm{d}t$	$\mathrm{e}^{-2\chi t}(\cos t + \chi \sin t)^{-2}\delta^{ij}\partial_i \otimes \partial_j$

The fundamental vector fields $\overline{\xi}_X$ in the quotient are easy to determine from the expressions of the Killing vector fields: all we need to do is drop any component along $\xi_Z = \partial_v$. Doing so, we see that $\overline{\xi}_{J_{ij}}$ and $\overline{\xi}_{P_i}$ are given formally by the same expression as the Killing vector fields, whereas $\overline{\xi}_{B_i} = f(u)\partial_i$ and $\overline{\xi}_H = \partial_u + h(u)x^i\partial_i$, where $f(u)$ and $h(u)$ can be read off from Table 4. It is easy to check that these fundamental vector fields define an anti-representation of the Galilean kinematical Lie algebra $\overline{\mathfrak{g}}$. It follows from an explicit calculation that the Galilean structure (τ, ψ) in the quotient is invariant under the first derived ideal $[\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]$, which is spanned by J_{ij}, B_i, P_i. On the other hand, the generator H leaves invariant ψ, but acts homothetically on τ, as explained above.

Indeed, it is a simple calculation to check that for G, dSG and AdSG, the Galilean structure induced from the null reduction is the invariant one, whereas for dSG_γ, one finds $\mathscr{L}_{\overline{\xi}_H}\tau = (1+\gamma)\tau$ and for AdSG_χ, one finds $\mathscr{L}_{\overline{\xi}_H}\tau = 2\chi\tau$. It follows that the invariant clock one-form is conformal to the one induced by the null-reduction. Indeed, for dSG_γ, it is $\mathrm{e}^{-(1+\gamma)t}\tau$ which is invariant, whereas for AdSG_χ, the invariant clock one-form is $\mathrm{e}^{-2\chi t}\tau$. In [10, Appendix D], it is shown that there exists a conformal rescaling of the Lorentzian metric such that the vector field ξ_Z remains Killing and the corresponding null reduction does give the invariant clock-one form on the nose; although one now pays the price that the induced spatial cometric is only homothetic to the invariant one.

6.3 Reduction by invariant vector field

We may solve the above problems by reducing via the action of the reals generated by the invariant vector field ζ. This vector field is parallel with respect to the canonical invariant connection, which is compatible with the invariant metric g, but has typically nonzero torsion. It is therefore not Killing in the torsional cases, but since it commutes with Killing vector fields, its flow commutes with the G-action. This means that the quotient M/Γ_ζ, with Γ_ζ the one-dimensional subgroup of diffeomorphisms of M generated by ζ, admits an action of G. However the normal subgroup $\Gamma \subset G$ generated by Z acts trivially on M/Γ_ζ. Indeed, the unparametrized integral curves of ζ and ξ_Z agree. To see this, notice from Table 3, that $\zeta = \mathrm{e}^{\beta u}\partial_v$ in all cases, for some $\beta \in \mathbb{R}$. The integral curves of ζ are given by

$$c(s) = \left(u_0, v_0 + \mathrm{e}^{\beta u_0}s, \boldsymbol{x}_0\right),$$

whereas the integral curves of $\xi_Z = \partial_v$ are given by

$$c(s) = (u_0, v_0 + s, \boldsymbol{x}_0).$$

As a consequence, $\overline{M} := M/\Gamma_\zeta$ is a homogeneous space of $\overline{G} = G/\Gamma$ and provides a geometric realisation of the Klein pair $(\overline{\mathfrak{g}}, \overline{\mathfrak{h}})$, where $\overline{\mathfrak{g}} = \mathfrak{g}/\langle Z\rangle$ and $\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h} \cap \langle Z\rangle \cong \mathfrak{h}$, as above.

One can check that the induced Galilean structure agrees with the invariant one, but there is no need to do this explicitly, since the fact that the action of Γ_ζ on M commutes with the G-action, any projectable G-invariant tensor field on M (i.e., one descending to $\overline{M}$) is automatically $\overline{G}$-invariant.

For example, it follows from the explicit expressions of ζ in Table 3, that the metrically dual one-form ζ^β is $-\mathrm{d}u$ in all cases, which agrees with the invariant clock one-forms on the Galilean spacetimes in the coordinates $(t = -u, x^i)$, as can be checked from the formulae in Section 3, but ignoring the Z-component of the soldering forms. The expressions for the induced spatial cometric ψ are precisely those in Table 5.

7 Is $\mathsf{G} = \mathsf{dSG}_0$?

As we saw above, the invariant Lorentzian metric in $\mathsf{PdSG}_{\gamma=0}$ is flat. This means that the spacetime $\mathsf{PdSG}_{\gamma=0}$ is (perhaps only a portion of) Minkowski spacetime. The coordinates u, v, $\boldsymbol{x}$ are unconstrained, but they are not flat coordinates. Indeed, relative to these coordinates, the metric takes the form

$$g = -2\mathrm{e}^{-u}\mathrm{d}u\mathrm{d}v + \mathrm{e}^{-2u}\mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x}.$$

This suggests first of all defining a new coordinate $t = \mathrm{e}^{-u}$, which takes only positive real values. Relative to (t, v, x^i) the metric becomes

$$g = 2\mathrm{d}t\mathrm{d}v + t^2\mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x}.$$

We wish to find flat coordinates for this metric. Flat coordinates are characterised by the fact that the coordinate vector fields are parallel and the metrically dual one-forms are exact. Hence to find them, we must first determine the parallel vectors. A vector field $\xi = \xi^\mu\partial_\mu$ is parallel if

$$\partial_\mu\xi^\rho + \Gamma_{\mu\nu}{}^\rho\xi^\nu = 0.$$

We may find the Christoffel symbols by comparing the Euler–Lagrange equations of the lagrangian

$$L = \dot{t}\dot{v} + \tfrac{1}{2}t^2\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}$$

with the geodesic equation

$$\ddot{x}^\rho + \Gamma_{\mu\nu}{}^\rho\dot{x}^\mu\dot{x}^\nu = 0.$$

Doing so, we find the following nonzero Christoffel symbols

$$\Gamma_{ij}{}^v = -t\delta_{ij} \qquad \text{and} \qquad \Gamma_{ti}{}^j = \Gamma_{it}{}^j = \tfrac{1}{t}\delta_i^j.$$

The vector field $\xi = \xi^\mu\partial_\mu$ is parallel if and only if the components ξ^μ satisfy the following partial differential relations:

$$\begin{aligned}
&\partial_\mu\xi^t = 0, \qquad &&\partial_t\xi^i + \tfrac{1}{t}\xi^i = 0, \qquad &&\partial_t\xi^v = 0,\\
&\partial_v\xi^\mu = 0, \qquad &&\partial_i\xi^j + \tfrac{1}{t}\delta_i^j\xi^t = 0, \qquad &&\partial_i\xi^v - t\delta_{ij}\xi^j = 0.
\end{aligned}$$

Solving these relations, we find the following parallel vector fields

$$\partial_v, \qquad x^i\partial_v + \tfrac{1}{t}\partial_a, \qquad \partial_t - \tfrac{1}{t}x^i\partial_i - \tfrac{1}{2}x^2\partial_v.$$

The metrically dual one-forms are, respectively,

$$\mathrm{d}t, \qquad \mathrm{d}\big(tx^i\big) \qquad \text{and} \qquad \mathrm{d}\big(v - \tfrac{1}{2}tx^2\big),$$

from where we can read off the flat coordinates

$$T = t, \qquad X^i = tx^i \qquad \text{and} \qquad V = v - \tfrac{1}{2}tx^2,$$

which we may invert to write

$$t = T, \qquad x^i = \tfrac{X^i}{T} \qquad \text{and} \qquad v = V + \tfrac{1}{2}\tfrac{X^2}{T}.$$

Expressing the metric relative to these coordinates, we find

$$g = 2\mathrm{d}T\mathrm{d}V + \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X},$$

which is indeed manifestly flat. Notice that since $t > 0$, it is also the case that $T > 0$ so the original metric covers half of Minkowski spacetime and the new coordinates allow us to extend the metric to the whole of Minkowski spacetime.

The null vector $\xi_Z = \partial_v$ becomes

$$\partial_v = \frac{\partial T}{\partial v}\partial_T + \frac{\partial V}{\partial v}\partial_V + \frac{\partial X^i}{\partial v}\partial_{X^i} = \partial_V$$

in the new coordinates. The null reduction of this metric along ∂_V is clearly the same as the null reduction of the metric in equation (3.1) along ∂_v: simply change coordinates to $t = -u$ and notice that the metrics agree under $\big(t, v, x^i\big) \mapsto \big(T, V, X^i\big)$.

This does not mean, however, that the homogeneous Galilean spacetimes dSG_0 and G are the same. One way to see this is to notice that under the improved reduction by the invariant vector field ζ, which results in the invariant Galilean structure, the reductions are not the same. For G, it is still the case that $\zeta = \partial_v$, but for dSG_0, $\zeta = \mathrm{e}^u\partial_v$ or, after the change of coordinates, $\zeta = \frac{1}{T}\partial_V$, which does not extend smoothly to the full Minkowski spacetime in $(T, V, \boldsymbol{X})$ coordinates.

8 Discussion and conclusions

In this note we have exhibited the spatially isotropic homogeneous Galilean spacetimes (of spatial dimension $n > 2$) classified in [14] as null reductions of homogeneous pp-wave spacetimes. We have performed two null reductions with identical quotients as homogeneous spacetimes: one by a Killing vector field and one by an invariant vector field. These two reductions agree for the non-torsional (i.e., symmetric) homogeneous spacetimes, but are different for the torsional ones. In the former reduction, we find that although the null reduction gives the homogeneous Galilean spacetimes as homogeneous spaces, the Galilean structure induced from the null reduction is not the invariant one in the torsional cases (dSG_γ for $-1 < \gamma \leq 1$ and AdSG_χ for $\chi > 0$), but only homothetic to it. This is a reflection of the fact that the null Killing vector is not central in the torsional cases. In contrast, the null reduction by the invariant vector field results in the invariant Galilean structure by design.

This work was focused on Galilean spacetimes and their intricate properties. As final note let us advertise their Carrollian curved counterparts (A)dS–Carroll, which arise as a ultra-relativistic limit of (A)dS. They are complementary equally interesting candidates for the study of holography in a possibly more tractable setup (besides the intriguing relation of AdS-Carroll to time-like infinity [13]). They can be seen as null hypersurfaces of (A)dS [14] and AdS–Carroll shares the box-like (its spatial metric is hyperbolic) and dS–Carroll the cosmological character (its spatial metric is the sphere) of their Lorentzian parents [11].

A Global properties of the Lorentzian spacetimes

In this appendix we show that every simply-connected homogeneous Lorentzian spacetime M whose Klein pair is listed in Table 2 is diffeomorphic to $\mathbb{R}^{n+2}$.

To do this we introduce another set of coordinates (u, v, x^a) and define the parametrisation

$$\sigma(u, v, \boldsymbol{x}) = \exp(vZ)\exp(uH)\exp(\boldsymbol{x}\cdot\boldsymbol{P}),$$

which is closer in spirit to the one used in [11, Appendix A.2] for the Galilean spacetimes. With $[H, \boldsymbol{P}] = \alpha\boldsymbol{B} + \beta\boldsymbol{P}$ and $[H, Z] = \beta Z$ and using

$$\mathrm{e}^{-\boldsymbol{x}\cdot\boldsymbol{P}} H \mathrm{e}^{\boldsymbol{x}\cdot\boldsymbol{P}} = H + \alpha\boldsymbol{x}\cdot\boldsymbol{B} + \beta\boldsymbol{x}\cdot\boldsymbol{P} + \tfrac{1}{2}\alpha x^2 Z,$$

we obtain

$$\sigma^*\vartheta = \mathrm{d}uH + \big(\mathrm{e}^{-\beta u}\mathrm{d}v + \tfrac{1}{2}\alpha x^2\mathrm{d}u\big)Z + (\mathrm{d}\boldsymbol{x} + \beta\mathrm{d}u\boldsymbol{x})\cdot\boldsymbol{P} + \alpha\mathrm{d}u\boldsymbol{x}\cdot\boldsymbol{B}. \tag{A.1}$$

The soldering form

$$\theta = \mathrm{d}uH + \big(\mathrm{e}^{-\beta u}\mathrm{d}v + \tfrac{1}{2}\alpha x^2\mathrm{d}u\big)Z + (\mathrm{d}\boldsymbol{x} + \beta\mathrm{d}u\boldsymbol{x})\cdot\boldsymbol{P}$$

is everywhere invertible.

Let us define a map $j\colon \mathbb{R}^{n+2} \to M$ by $j(u, v, \boldsymbol{x}) = \sigma(u, v, \boldsymbol{x})\cdot o$, where $o \in M$ is a choice of origin with stabiliser the subgroup $\mathcal{H}$ generated by spatial rotations and boosts. The first thing to remark is that we may ignore the rotations and think of M simply as a homogeneous space of the solvable Lie group $\mathcal{K}$ generated by $\boldsymbol{B}$, $\boldsymbol{P}$, H, Z. The stabiliser of the origin is then the subgroup $\mathcal{B}$ spanned by $\boldsymbol{B}$. We define an action of $\mathcal{K}$ on $\mathbb{R}^{n+2}$ by requiring j to be equivariant. Introducing the shorthands

$$\varpi := \tfrac{1}{2}\sqrt{4\alpha - \beta^2}, \qquad s := \frac{\sin(\varpi u)}{2\varpi} \qquad \text{and} \qquad f_\pm = \cos(\varpi u) \pm \beta s,$$

a short calculation reveals that

$$\begin{aligned}
\mathrm{e}^{aZ}\cdot(u, v, \boldsymbol{x}) &= (u, v + a, \boldsymbol{x}),\\
\mathrm{e}^{bH}\cdot(u, v, \boldsymbol{x}) &= \big(u + b, \mathrm{e}^{b\beta}v, \boldsymbol{x}\big),\\
\mathrm{e}^{\boldsymbol{w}\cdot\boldsymbol{B}}\cdot(u, v, \boldsymbol{x}) &= \big(u, v + sf_+w^2 + \mathrm{e}^{u\beta/2}f_+\boldsymbol{w}\cdot\boldsymbol{x}, \boldsymbol{x} + 2\mathrm{e}^{-u\beta/2}s\boldsymbol{w}\big),\\
\mathrm{e}^{\boldsymbol{y}\cdot\boldsymbol{P}}\cdot(u, v, \boldsymbol{x}) &= \big(u, v - \alpha sf_-y^2 - 2\alpha u^{u\beta/2}s\boldsymbol{y}\cdot\boldsymbol{x}, \boldsymbol{x} + \mathrm{e}^{-u\beta/2}f_-\boldsymbol{y}\big),
\end{aligned}$$

from where we may read the fundamental vector fields:

$$\begin{aligned}
\xi_Z &= \partial_v,\\
\xi_H &= \partial_u + \beta v\partial_v,\\
\xi_{B_a} &= \mathrm{e}^{u\beta/2}f_+x^a\partial_v + 2\mathrm{e}^{-u\beta/2}s\partial_a,\\
\xi_{P_a} &= -2\alpha\mathrm{e}^{u\beta/2}sx^a\partial_v + \mathrm{e}^{-u\beta/2}f_-\partial_a.
\end{aligned}$$

One can check that these form an anti-representation of the Lie algebra $\mathfrak{k}$ of $\mathcal{K}$.

Our first observation is that the action of $\mathcal{K}$ on $\mathbb{R}^{n+2}$ is transitive. Indeed, taking as origin the point with coordinates $(0, 0, \boldsymbol{0})$, we see that we can reach any other point by the action of $\sigma(u, v, \boldsymbol{x}) \in \mathcal{K}$:

$$(u, v, \boldsymbol{x}) = \sigma(u, v, \boldsymbol{x})\cdot(0, 0, \boldsymbol{0}),$$

which follows essentially by definition. The smooth map $j\colon \mathbb{R}^{n+2} \to M$ is a local diffeomorphism, since the soldering form is everywhere invertible, and it is $\mathcal{K}$-equivariant. Together with transitivity, this says that j is a branched covering map, but the branched locus has to be empty, otherwise it would not be homogeneous. Therefore j is a covering which, since $\mathbb{R}^{n+2}$ is simply connected, is universal. But M was assumed to be the simply-connected homogeneous spacetime realising the given Klein pair, hence j is an isomorphism of $\mathcal{K}$-homogeneous spacetimes and, in particular, M is diffeomorphic to $\mathbb{R}^{n+2}$.

These coordinates have other advantages. For example, the connection one-form is uniformly given by $\omega = \alpha \mathrm{d}u\boldsymbol{x} \cdot \boldsymbol{B}$, see the last term in (A.1), and the invariant Lorentzian metric g and vector field ζ can be brought to a more uniform form. To see this, let us write the invariant Lorentzian metric $g = \big(\pi^2 - 2\eta\zeta\big)(\theta,\theta)$ as

$$g = -2\big(\mathrm{e}^{-\beta u}\mathrm{d}v + \tfrac{\alpha}{2}x^2\mathrm{d}u\big)\mathrm{d}u + (\mathrm{d}\boldsymbol{x} + \beta\mathrm{d}u\boldsymbol{x})^2,$$

and the invariant vector field ζ as

$$\zeta = \mathrm{e}^{\beta u}\partial_v.$$

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