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Rigidity and Non-Rigidity of $\mathbb{H}^n/\mathbb{Z}^{n-2}$ with Scalar Curvature Bounded from Below

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Abstract. We show that the hyperbolic manifold $\mathbb{H}^n/\mathbb{Z}^{n-2}$ is not rigid under all compactly supported deformations that preserve the scalar curvature lower bound $-n(n-1)$, and that it is rigid under deformations that are further constrained by certain topological conditions. In addition, we prove two related splitting results.

Key words: scalar curvature; rigidity; ALH manifolds; μ-bubbles

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Dedicated to Jean-Pierre Bourguignon on the occasion of his 75th birthday

1 Introduction

In [21, Section 3] and [22, p. 240], Gromov stated the following generalization of Min-Oo's hyperbolic rigidity theorem [30].

Statement 1.1 ("generalised Min-Oo rigidity theorem")**.** *Parabolic quotients $Z = \mathbb{H}^n/\Gamma$ of the hyperbolic n-space admit no non-trivial, compactly supported 'deformation' with scalar curvature $R \geq -n(n-1)$.*

According to [21], a *deformation* can change not only the metric, but also the topology of a compact region in Z. If the deformation is topologically a connected sum with a closed n-manifold, Statement 1.1 is known to be true for (at least) $Z = \mathbb{H}^n/\mathbb{Z}^{n-1}$, with idea of proof already outlined by [19, Section $5\frac{5}{6}$] (for a detailed treatment, see also [2, Theorem 1.1]). The situation turns out to be more subtle if broader types of deformations are considered, allowing, for example, surgeries along an embedded, non-contractible loop. In this latter case we construct a counterexample to Statement 1.1, which, more precisely, demonstrates the following.

Theorem 1.2. *For $n \geq 3$, let $\mathbb{H}^n/\mathbb{Z}^{n-2}$ be equipped with the standard hyperbolic metric. There exists a complete Riemannian manifold (M^n, g), not (globally) hyperbolic, and compact subsets $K \subset M$ and $K' \subset \mathbb{H}^n/\mathbb{Z}^{n-2}$, such that* (1) *$R_g \geq -n(n-1)$ and* (2) *$M \setminus K$ is isometric to $\left(\mathbb{H}^n/\mathbb{Z}^{n-2}\right) \setminus K'$.*

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Remark 1.3.

(1) While the theorem above concerns non-rigidity of $\mathbb{H}^n/\mathbb{Z}^{n-2}$, it is also interesting to ask whether its statement still holds if $\mathbb{H}^n/\mathbb{Z}^{n-2}$ is replaced by $\mathbb{H}^n/\mathbb{Z}^{n-1}$; this will be answered in the affirmative in Section 2.4.1. Thus, we obtain counterexamples to the "weak rigidity of $\mathbb{H}^n/\mathbb{Z}^{n-1}$" mentioned in [20, p. 678].

(2) En route to proving Theorem 1.2, we obtain counterexamples (see Proposition 2.6) to the following statement in [21, p. 12]: *Represent $\mathbb{H}^n/\mathbb{Z}^{n-2}$ as a warped product $\big(\mathbb{H}^2\times\mathbb{T}^{n-2}, g_H\big)$ (see formula (2.1)), and, for a geodesic 2-disk $\mathbb{D}^2\subset\mathbb{H}^2$, let $X=\mathbb{D}^2\times\mathbb{T}^{n-2}\subset\mathbb{H}^2\times\mathbb{T}^{n-2}$ with the restricted metric $g_H|_X$; then no Riemannian manifold $\big(M^n,g\big)$ with boundary isometric to ∂X can have scalar curvature $R_g\geq -n(n-1)$ and mean curvature*[1] *of ∂M greater than that of ∂X.*

(3) Our proof of Theorem 1.2 is constructive, which, a little to our surprise, shows that M can be chosen to be homeomorphic to $\mathbb{H}^n/\mathbb{Z}^{n-2}$ (see Section 2.4.2); moreover, $R_g>-n(n-1)$ for some points in K.

From the perspective of our construction, the non-rigidity of $\mathbb{H}^n/\mathbb{Z}^{n-2}$ seems closely related to the fact: *A deformation supported in a compact subset K can 'break' the incompressibility*[2] *of some submanifold that is disjoint from K.* On the other hand, rigidity does hold if one only considers deformations that preserve such incompressibility, as the next theorem shows (cf. [11, Theorem 1.8]).

Theorem 1.4. *For $3\leq n\leq 7$, let (M^n,g) be a complete Riemannian manifold*[3] *with scalar curvature $R_g\geq -n(n-1)$. Suppose that there exist compact subsets $K\subset M$, $K'\subset\mathbb{H}^n/\mathbb{Z}^{n-2}$, and an isometry $f\colon M\setminus K\to\big(\mathbb{H}^n/\mathbb{Z}^{n-2}\big)\setminus K'$. Representing $\mathbb{H}^n/\mathbb{Z}^{n-2}$ topologically as $\mathbb{R}^2_+\times\mathbb{T}^{n-2}$, let $p\in\mathbb{R}^2_+$ be such that $T=\{p\}\times\mathbb{T}^{n-2}$ is disjoint from K', and suppose that the map $f^{-1}|_T\colon T\to M$ is incompressible. Then (M,g) is isometric to $\mathbb{H}^n/\mathbb{Z}^{n-2}$.*

Technically, we will derive Theorem 1.4 as a consequence of Theorem 1.5 below. The latter can be regarded as a kind of positive mass type theorem for manifolds with an ALH end; its statement relies on a gluing construction, which we now describe.

Gluing construction: Let N^n be a smooth manifold, and suppose that $\phi\colon\mathbb{T}^k\to N$ ($1\leq k\leq n-2$) is an embedding with trivial normal bundle. Moreover, write $\mathbb{H}^n/\mathbb{Z}^{n-1}$ (topologically) as the product $\mathbb{R}\times\mathbb{T}^{n-k-1}\times\mathbb{T}^k$, and define

$$\psi\colon\ \mathbb{T}^k\to\mathbb{R}\times\mathbb{T}^{n-k-1}\times\mathbb{T}^k\cong\mathbb{H}^n/\mathbb{Z}^{n-1}\qquad\text{by}\quad\psi(p)=(t,q,p)$$

for some fixed $t\in\mathbb{R}$ and $q\in\mathbb{T}^{n-k-1}$. By removing tubular neighborhoods of $\phi\big(\mathbb{T}^k\big)\subset N$ and $\psi\big(\mathbb{T}^k\big)\subset\mathbb{H}^n/\mathbb{Z}^{n-1}$ and then identifying the respective boundaries in the obvious way, we obtain a manifold M. For brevity, M will be referred to as obtained by *gluing N and $\mathbb{H}^n/\mathbb{Z}^{n-1}$ along $\mathbb{T}^k$ via (ϕ,ψ)*. In particular, for c sufficiently large, $(c,\infty)\times\mathbb{T}^{n-1}\subset\mathbb{H}^n/\mathbb{Z}^{n-1}$ remains an 'end' of M, and this end is denoted by $\mathcal{E}$.

Theorem 1.5. *For $3\leq n\leq 7$, let N^n be a smooth manifold that is either closed or non-compact without boundary, and let M^n be obtained by gluing N with $\mathbb{H}^n/\mathbb{Z}^{n-1}$ along $\mathbb{T}^k$ via (ϕ,ψ) (see description above). Suppose that*

(a) *the map $\phi\colon\mathbb{T}^k\to N$ is incompressible;*

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[1] Unless specified otherwise, in this article the mean curvature along a boundary will always be computed with respect to the *outward* unit normal.

[2] A continuous map $f\colon X\to Y$ between topological spaces is said to be *incompressible* if the induced map $f_*\colon\pi_1(X)\to\pi_1(Y)$ is injective; when f is an inclusion, we say '*X is incompressible in Y*'.

[3] In this article, all manifolds are assumed to be orientable, and all hypersurfaces 2-sided.

(*b*) *g is a complete Riemannian metric on M with $R_g \geq -n(n-1)$;*

(*c*) *$(\mathcal{E}, g)$ is asymptotically locally hyperbolic (ALH) (see Definition 3.1).*

Then $\bar{m}_{\mathcal{E},g} \geq 0$ (see Definition 3.2). In addition, suppose that

(*d*) *the curvature tensor of (M, g) and its first covariant derivatives are bounded;*

(*e*) *there exists some $\alpha > 0$ such that $R_g \leq -\alpha$ outside a compact set.*

Then $\kappa = 0$ (see (3.1) for the definition of κ) only if (M, g) is Einstein.

Readers familiar with positive mass theorems may have noticed that the second half of Theorem 1.5 is not in an ideal form; in other words, one wants to know whether the vanishing of $\bar{m}_{\mathcal{E},g}$, and not just κ, implies that (M, g) is isometric to $\mathbb{H}^n/\mathbb{Z}^{n-1}$, even without the assumptions (d) and (e). In our proof, these assumptions play a role in making sure that the normalized Ricci flow (NRF) starting at g has desired properties (see Lemma 3.4); on the other hand, it seems subtle to prove hyperbolicity from (M, g) being Einstein and the assumed ALH decay rate. Thus we decide to leave the stronger statement for future investigation.

Theorem 1.5 has the following corollary.

Corollary 1.6. *For $3 \leq n \leq 7$, let N^n be a closed manifold, and suppose that M^n is obtained by gluing N with $\mathbb{H}^n/\mathbb{Z}^{n-1}$ along $\mathbb{T}^k$ via (ϕ, ψ). Suppose that g is a complete metric on M such that (M, g) is isometric to the hyperbolic manifold $\mathbb{H}^n/\mathbb{Z}^{n-1}$ outside a compact set,*[4] *and suppose that*

(*a*) *the map $\phi \colon \mathbb{T}^k \to N$ is incompressible;*

(*b*) *$R_g \geq -n(n-1)$.*

Then (M, g) is isometric to $\mathbb{H}^n/\mathbb{Z}^{n-1}$.

In fact, Corollary 1.6 remains true if N is allowed to be non-compact, which can be deduced as a corollary of Theorem 1.10 below (see Remark 4.5).

Besides rigidity problems modeled on complete manifolds, it is often natural to consider similar problems for manifolds with boundary and scalar/mean curvature bounds. In this regard, we present a splitting result of 'cuspidal-boundary' type (see [21, Section 4, last paragraph]). Our proof relies on an approximation scheme developed in [38] involving μ-bubbles.

Theorem 1.7. *Let $\left(M^4, g\right)$ be a complete, non-compact Riemannian 4-manifold with compact, connected boundary ∂M. Suppose that $\pi_2(M) = \pi_3(M) = 0$ and that the scalar curvature $R_g \geq -12$. Then*

$$\inf_{\partial M} H \leq 3,$$

where H is the mean curvature of ∂M. Moreover, if

$$\inf_{\partial M} H = 3,$$

then (M, g) is isometric to $\left((-\infty, 0] \times \Sigma, \mathrm{d}t^2 + \mathrm{e}^{2t} g_0\right)$, where t is the coordinate on $(-\infty, 0]$ and (Σ, g_0) is a closed 3-manifold with a flat metric.

[4] That is, there exists an isometry $f \colon M \setminus K \to (\mathbb{H}^n/\mathbb{Z}^{n-1}) \setminus K'$ for some compact sets $K \subset M$ and $K' \subset \mathbb{H}^n/\mathbb{Z}^{n-1}$.

Remark 1.8. Theorem 1.7 would fail if one allows M to be compact. Indeed, take

$$M = \mathbb{S}^1 \times \mathbb{B}^3, \qquad g = \cosh^2 \rho \mathrm{d}\theta^2 + \mathrm{d}\rho^2 + \sinh^2 \rho g_{\mathbb{S}^2}, \qquad \rho \le \rho_0,$$

where $\theta \in \mathbb{S}^1$, ρ is the radial coordinate on $\mathbb{B}^3$, and $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$. In this example, M has contractible universal cover, so both its π_2 and π_3 vanish. Moreover, since g is hyperbolic, $R_g = -12$, but the mean curvature $H_{\partial M} = 2\coth\rho_0 + \tanh\rho_0 > 3$.

Counterexamples also exist if one drops the assumption on $\pi_2(M)$ and $\pi_3(M)$. In fact, let us take the manifold $(\boldsymbol{M}', \boldsymbol{g}')$ in Section 2.4.1 and then, for sufficiently small $z_0 > 0$, remove the subset $\{0 < z < z_0\}$ from $\boldsymbol{M}'$; the result is a manifold $\boldsymbol{M}''$ with

$$\pi_2(\boldsymbol{M}'') \neq 0, \qquad H_{\partial \boldsymbol{M}''} = 3 \qquad \text{and} \qquad R \ge -12.$$

Clearly, $\boldsymbol{M}'' \ncong [c, \infty) \times \partial \boldsymbol{M}'' \cong [c, \infty) \times \mathbb{T}^3$.

Finally, we present an analogue of Theorem 1.7 in more general dimensions.

Definition 1.9 (cf. [11])**.** We say that a closed, connected manifold Σ *belongs to the class* $\mathcal{C}_{\mathrm{deg}}$, if

- Σ is aspherical,[5] and
- any compact manifold Σ' that admits a map to Σ of nonzero degree cannot be endowed with a PSC metric (i.e., metric with positive scalar curvature).

It is well known that $\mathbb{T}^n \in \mathcal{C}_{\mathrm{deg}}$ for $n \le 7$; also note that the second item in Definition 1.9 is redundant when $\dim \Sigma \le 5$, according to [13].

Theorem 1.10. *For $3 \le n \le 7$, let (M^n, g) be a complete and non-compact Riemannian manifold with compact, connected boundary ∂M. Suppose that*

(*a*) *∂M is incompressible in M,*

(*b*) *$\partial M \in \mathcal{C}_{\mathrm{deg}}$,*

(*c*) *$R_g \ge -n(n-1)$,*

then

$$\inf_{\partial M} H \le n - 1,$$

where H is the the mean curvature of ∂M.

Moreover, if

$$\inf_{\partial M} H = n - 1,$$

then (M, g) is isometric to $((-\infty, 0] \times \Sigma, \mathrm{d}t^2 + \mathrm{e}^{2t} g_0)$ where t is the coordinate on $(-\infty, 0]$ and (Σ, g_0) is a closed $(n-1)$-manifold with a flat metric.

Additional notes on the literature. (*a*) All our main theorems are fundamentally related to Gromov's fill-in problems (e.g., [21, Problems A and B]; [22, p. 234, Question (c)]). (*b*) Theorem 1.10 can be viewed as a generalization of [36, Theorem 3.2]. (*c*) It is a classical theme to relate incompressibility conditions with scalar curvature (see [23, Section 11]). (*d*) To adapt to modern language, our Theorem 1.5 considers manifolds with a prescribed end and some 'arbitrary ends'; the study of positive-mass type theorems on such manifolds has generated considerable interest recently (see, for example, [10, 12, 29, 40]). (*e*) While in this paper we focus

[5] A closed, connected manifold is said to be *aspherical* if it has contractible universal cover.

on rigidity results for complete, non-compact manifolds with boundary and scalar curvature lower bounds, similar results in the compact case (with boundary) are obtained by Gromov in [21, Section 4]. In both cases, the proofs rely on the μ-bubble technique. (f) It would be interesting to compare Theorem 1.5 with some recent progress in proving positive mass and rigidity results for ALH manifolds (see [1, 16, 17, 26]); in this latter development, manifolds are often assumed to have nonempty inner boundary with the mean curvature bound $H \leq n-1$ (now H is computed with respect to the *inner* unit normal); such mean curvature bounds serve as barrier conditions in the method of 'marginally outer trapped surfaces' (MOTS), which can be viewed as a generalization of the μ-bubble technique.

Organization of this article. The proof of Theorem 1.2 is technically independent from the rest of the work and is included in Section 2. Section 3 serves as a preliminary to proving Theorem 1.5, presenting results concerning NRF and conformal deformations. In Section 4, we prove Theorem 1.5, followed by proofs of Corollary 1.6 and Theorem 1.4. In Section 5, we prove Theorem 1.7 and Theorem 1.10. Several of the proofs rely on the so-called 'μ-bubble' technique, a brief discussion of which is included in Appendix A. Appendix B includes two topological lemmas.

2 Non-rigidity of $\mathbb{H}^n/\mathbb{Z}^{n-2}$

Let the hyperbolic n-space $\mathbb{H}^n$ be represented by the upper half-space model $\mathbb{R}^n_+ = \{(x, \boldsymbol{y}, z) \colon x \in \mathbb{R}, \boldsymbol{y} \in \mathbb{R}^{n-2}, z > 0\}$, and let $\mathbb{Z}^{n-2}$ act by translating along the orthogonal lattice $2\pi\mathbb{Z}^{n-2} \subset \mathbb{R}^{n-2}$ while keeping the x, z-coordinates fixed. The quotient space is denoted by $\mathbb{H}^n/\mathbb{Z}^{n-2}$ and has the hyperbolic metric

$$g_H = z^{-2}\big(\mathrm{d}z^2 + \mathrm{d}x^2\big) + z^{-2} g_{\mathbb{T}^{n-2}}, \tag{2.1}$$

where the subscript 'H' stands for 'hyperbolic', and $g_{\mathbb{T}^{n-2}}$ is the associated flat metric on $\mathbb{T}^{n-2}$. Henceforth, we will regard (x, z) as coordinates on the hyperbolic plane $\mathbb{H}^2$; manifestly that $\big(\mathbb{H}^n/\mathbb{Z}^{n-2}, g_H\big)$ is a warped product of $\mathbb{H}^2$ and $\big(\mathbb{T}^{n-2}, g_{\mathbb{T}^{n-2}}\big)$.

The following lemma is easily verified by standard computation, so we omit its proof.

Lemma 2.1. *Let ∇, ∇^2 denote the gradient and Hessian with respect to g_H (same below). We have*

- *(a)* $\nabla z = z^2 \partial/\partial z$,
- *(b)* $\nabla^2 z(\partial/\partial x, \partial/\partial x) = -\nabla^2 z(\partial/\partial z, \partial/\partial z) = -1/z$,
- *(c)* $\nabla^2 z(\partial/\partial z, \partial/\partial x) = 0$.

Next, we proceed to prove Theorem 1.2 by constructing an example that satisfies all its conditions. The idea is to remove a suitable compact subset, $X_{p,r}$, from $\mathbb{H}^n/\mathbb{Z}^{n-2}$ and then 'glue' the result with a compact manifold, $\bar{X}_r$, along their boundaries; $X_{p,r}$ and $\bar{X}_r$ will be defined in Sections 2.1 and 2.2 respectively, and then we handle the gluing step in Section 2.3.

2.1 First preliminary construction

Let $p \in \mathbb{H}^2$, and define

$$X_{p,r} := \mathbb{D}_r(p) \times \mathbb{T}^{n-2} \subset \mathbb{H}^n/\mathbb{Z}^{n-2} \qquad \text{and} \qquad Y_{p,r} := \partial X_{p,r}, \tag{2.2}$$

where $\mathbb{D}_r(p) \subset \mathbb{H}^2$ is the geodesic disc, centered at p, of radius $r > 0$; the inclusion in (2.2) makes sense since $\mathbb{H}^n/\mathbb{Z}^{n-2}$ is a warped product of $\mathbb{H}^2$ and $\mathbb{T}^{n-2}$, as we already noted.

Now we have two sets of coordinates for $\mathbb{H}^2$: (x, z) and the polar coordinates (ϱ, θ) centered at p. In terms of the polar coordinates, the metric on $\mathbb{H}^2$ reads

$$g_{\mathbb{H}^2} = \mathrm{d}\varrho^2 + \sinh^2 \varrho \, \mathrm{d}\theta^2.$$

Lemma 2.2. *The boundary $Y_{p,r} \subset (X_{p,r}, g_H)$ has the mean curvature*

$$H_{p,r} = \coth r - (n-2) z^{-1} \frac{\partial z}{\partial \varrho}. \tag{2.3}$$

Moreover,

(a) $|H_{p,r} - \coth r| \leq n - 2$;

(b) *There exists a constant $r_0 > 0$ such that $H_{p,r} > 0$ for all $r \leq r_0$.*

Proof. The formula (2.3) is straightforward to check by using the representation

$$g_H = \mathrm{d}\varrho^2 + \sinh^2 \varrho \, \mathrm{d}\theta^2 + z^{-2} g_{\mathbb{T}^{n-2}}.$$

Moreover, since both $z^{-1}\nabla z$ and $\nabla \varrho$ have unit norm with respect to g_H,

$$\left|\frac{\partial z}{\partial \varrho}\right| = \left|\left\langle \nabla z, \nabla \varrho \right\rangle\right| = \left| z \left\langle z^{-1} \nabla z, \nabla \varrho \right\rangle\right| \leq z. \tag{2.4}$$

This implies (a), and (b) follows since $\coth r \to \infty$ as $r \to 0$. ∎

Lemma 2.3. *There exists a constant $C_r > 0$, depending only on r, such that on $\partial \mathbb{D}_r(p)$ we have*

$$|\partial_\theta z(r, \theta)| \leq z \sinh r \qquad \text{and} \qquad \left|\partial_\theta^2 z(r, \theta)\right| \leq C_r z.$$

Proof. Since both $z^{-1}\nabla z$ and $(\sinh r)^{-1}(\partial/\partial\theta)$ have unit norm with respect to g_H, we have

$$|\partial_\theta z(r, \theta)| = \left|\left\langle \nabla z, \partial/\partial\theta \right\rangle\right| \leq z \sinh r.$$

Moreover, a calculation shows that

$$\nabla^2 z(\partial/\partial\theta, \partial/\partial\theta) = \partial_\theta^2 z + (\partial_\varrho z) \sinh \varrho \cosh \varrho. \tag{2.5}$$

By Lemma 2.1 $(b), (c)$, the left-hand side of (2.5) has its magnitude bounded by $(\sinh^2 \rho) z$; thus, using (2.4) and evaluating (2.5) at $\varrho = r$, we get

$$\left|\partial_\theta^2 z(r, \theta)\right| \leq \sinh r (\sinh r + \cosh r) z.$$

Taking $C_r = \sinh r(\sinh r + \cosh r)$ finishes the proof. ∎

2.2 Second preliminary construction

Let $\mathcal{D}$ be a 2-disc with polar coordinates $(\bar{\varrho}, \bar{\theta})$, where

$$0 \leq \bar{\varrho} \leq \pi/3 \qquad \text{and} \qquad 0 \leq \bar{\theta} < 2\pi.$$

Equip $\mathcal{D}$ with the metric

$$g_{\mathcal{D}} = \mathrm{d}\bar{\varrho}^2 + 4 \sin^2(\bar{\varrho}/2) \mathrm{d}\bar{\theta}^2.$$

Thus, $(\mathcal{D}, g_{\mathcal{D}})$ is isometric to a 'cap' in the round sphere of radius 2.

Now let $r > 0$ and $z(\varrho, \theta)$ be as in Section 2.1 above. Consider

$$\bar{X}_r := \mathbb{S}^1 \times \mathcal{D} \times \mathbb{T}^{n-3}$$

equipped with the metric

$$\bar{g} = \sinh^2 r \, \mathrm{d}\theta^2 + \big(z(r,\theta)\big)^{-2} g_{\mathcal{D}} + \big(z(r,\theta)\big)^{-2} g_{\mathbb{T}^{n-3}}, \tag{2.6}$$

and let $\bar{Y}_r := \partial \bar{X}_r$. By construction, the boundaries $(Y_{p,r}, g_H|_{Y_{p,r}})$ and $\big(\bar{Y}_r, \bar{g}|_{\bar{Y}_r}\big)$ are isometric under the obvious identification.

Lemma 2.4. *The boundary $\bar{Y}_r \subset \big(\bar{X}_r, \bar{g}\big)$ has the mean curvature*

$$\bar{H}_r = \frac{\sqrt{3}}{2} z(r,\theta). \tag{2.7}$$

Proof. Standard computation by using (2.6). ∎

Regarding the scalar curvature of a warped-product metric, the following is well-known.

Lemma 2.5 (cf. [23, Proposition 7.33])**.** *Let $\big(N^{n-1}, h\big)$ be an $(n-1)$-dimensional Riemannian manifold with scalar curvature R_h. Given any smooth function $\phi(\theta)$ defined on an interval I and a constant $a > 0$, the warped product metric $g = a^2 \mathrm{d}\theta^2 + \phi(\theta)^2 h$ defined on $I \times N$ has the scalar curvature*

$$R_g = \frac{n-1}{a^2} \left[-2 \left(\frac{\phi'}{\phi} \right)' - n \left(\frac{\phi'}{\phi} \right)^2 \right] + \phi^{-2} R_h. \tag{2.8}$$

In our case, to compute the scalar curvature of $\bar{g}$, it suffices to substitute $h = g_{\mathcal{D}} + g_{\mathbb{T}^{n-3}}$, $\phi(\theta) = 1/z(r,\theta)$ and $a = \sinh r$ into (2.8). Noting that $R_h = 1/2$, we have

$$\begin{aligned} R_{\bar{g}} &= (n-1)(\sinh r)^{-2} \big\{ -2\partial_\theta [z \partial_\theta (1/z)] - n[z\partial_\theta(1/z)]^2 \big\} + z^2/2 \\ &= (n-1)(\sinh r)^{-2} \big\{ 2\big(\partial_\theta^2 z\big)/z - (n+2)[(\partial_\theta z)/z]^2 \big\} + z^2/2, \end{aligned} \tag{2.9}$$

where z, $\partial_\theta z$ and $\partial_\theta^2 z$ are evaluated at (r, θ).

Now we are ready to observe the following.

Proposition 2.6. *For fixed $r > 0$, the manifold $(\bar{X}_r, \bar{g})$ satisfies:*

(*a*) *The scalar curvature*

$$R_{\bar{g}} \geq \frac{1}{2} [z(r,\theta)]^2 - C_{n,r}$$

for a constant $C_{n,r} > 0$ depending only on n and r. In particular, we have $R_{\bar{g}} > -n(n-1)$ provided that $p \in \mathbb{H}^2$ is chosen to have a large enough z-coordinate;

(*b*) *Under the obvious identification (isometry) between $Y_{p,r}$ and $\bar{Y}_r$, we have $\bar{H}_r > H_{p,r}$ provided that the z-coordinate of p is large enough.*

Proof. (*a*) follows from (2.9) and Lemma 2.3; (*b*) follows from Lemma 2.2 (*a*) and (2.7). ∎

2.3 The gluing step

Lemma 2.7 ([8, Theorem 5]). *Let Ω be a compact n-manifold with boundary $\partial\Omega$, and let g and $\tilde{g}$ be two smooth Riemannian metrics on Ω such that*

(a) *$g - \tilde{g} = 0$ at each point on $\partial\Omega$;*

(b) *the mean curvatures satisfy $H_{\tilde{g}} - H_g > 0$ at each point on $\partial\Omega$.*

Then, given any $\epsilon > 0$ and any neighborhood U of $\partial\Omega$, there exists a smooth metric $\hat{g}$ on Ω with the following properties:

(1) *$R_{\hat{g}} \geq \min\{R_g, R_{\tilde{g}}\} - \epsilon$ in Ω;*

(2) *$\hat{g} = \tilde{g}$ in $\Omega \setminus U$;*

(3) *$\hat{g} = g$ in a neighborhood of $\partial\Omega$.*

Remark 2.8. By an arbitrary extension, in Lemma 2.7 it suffices to assume that g is defined only in a neighborhood of $\partial\Omega$.

To prove Theorem 1.2, a basic idea is to apply Lemma 2.7 to obtain a metric $\hat{g}$ on $\bar{X}_r$ which agrees with g_H in a neighborhood of $\partial\bar{X}_r \cong \partial X_{p,r}$, so $\hat{g}$ extends smoothly into $\left(\mathbb{H}^n/\mathbb{Z}^{n-2}\right) \setminus X_{p,r}$ by g_H. A compromise is the ϵ-cost to the scalar curvature estimate. Thus, one would like to have a bit more scalar curvature to begin with, so that the cost can be absorbed, maintaining the desired lower bound $R_{\hat{g}} \geq -n(n-1)$. This can be achieved by a suitable deformation of g_H in a neighborhood of $Y_{p,r} \subset \mathbb{H}^n/\mathbb{Z}^{n-2}$, as the following lemma shows.

Lemma 2.9. *Let*

$$u(\varrho) = \begin{cases} 1 - \mathrm{e}^{\frac{1}{\varrho - r_0}}, & \varrho < r_0, \\ 1, & \varrho \geq r_0, \end{cases} \tag{2.10}$$

and define

$$g'_H := \left[u(\varrho)\right]^2 \mathrm{d}\varrho^2 + \sinh^2\varrho\, \mathrm{d}\theta^2 + \left[z(\varrho,\theta)\right]^{-2} g_{\mathbb{T}^{n-2}}.$$

As long as $r_0 > 0$ is small enough, we can find $\delta > 0$ such that

$$R_{g'_H} + n(n-1) > 0 \qquad \text{for } \varrho \in [r_0 - 2\delta, r_0).$$

Proof. By [34, Claim 2.1], we have

$$R_{g'_H} = R_{g_H} + \left(1 - u^{-2}\right)\left(R_{\gamma(\varrho)} - R_{g_H}\right) + 2u^{-3}u'(\varrho)H_{p,\varrho}, \tag{2.11}$$

where $\gamma(\varrho) = \sinh^2\varrho\, \mathrm{d}\theta^2 + \left[z(\varrho,\theta)\right]^{-2} g_{\mathbb{T}^{n-2}}$ and $R_{g_H} = -n(n-1)$.

We want to estimate the right-hand side of formula (2.11). To start with, by Lemma 2.5,

$$R_{\gamma(\varrho)} = (n-2)(\sinh\varrho)^{-2}\left\{2\left(\partial_\theta^2 z\right)/z - (n+1)[(\partial_\theta z)/z]^2\right\}.$$

Thus, by the proof of Lemma 2.3, there exists a constant C_{n,r_0}, depending on n, r_0 only, such that

$$|R_{\gamma(\varrho)}| \leq C_{n,r_0} \qquad \text{for } \varrho \in [r_0/2, r_0]. \tag{2.12}$$

Next, by the definition of u, we have, for $\varrho \leq r_0$,

$$0 \geq 1 - u^{-2} = u^{-2}\left(-2\mathrm{e}^{\frac{1}{\varrho - r_0}} + \mathrm{e}^{\frac{2}{\varrho - r_0}}\right) \geq -2u^{-2}\mathrm{e}^{\frac{1}{\varrho - r_0}} \geq -2u^{-3}\mathrm{e}^{\frac{1}{\varrho - r_0}}. \tag{2.13}$$

Moreover, for sufficiently small r_0, we have $H_{p,\varrho} \geq 1$ for any $\varrho \leq r_0$ (Lemma 2.2 (b)), and so

$$2u^{-3}u'(\varrho)H_{p,\varrho} \geq 2u^{-3}\mathrm{e}^{\frac{1}{\varrho-r_0}}(\varrho-r_0)^{-2}. \tag{2.14}$$

On combining (2.11), (2.12), (2.13) and (2.14), we obtain that

$$R_{g'_H} - R_{g_H} \geq 2u^{-3}\mathrm{e}^{\frac{1}{\varrho-r_0}}\left[(r_0-\varrho)^{-2} - C_{n,r_0} - n(n-1)\right] \qquad \text{for } \varrho \in [r_0/2, r_0].$$

Clearly, we can choose a small $\delta > 0$ such that

$$R_{g'_H} - R_{g_H} > 0 \qquad \text{for } \varrho \in [r_0 - 2\delta, r_0).$$

This completes the proof. ■

Proof of Theorem 1.2. Let r_0 be small enough, and let $u(\varrho)$, g'_H and δ be as in Lemma 2.9. Define

$$c := \min_{\varrho\in[r_0-2\delta, r_0-\delta]} R_{g'_H} + n(n-1) > 0.$$

Take $r := r_0 - \delta$, and note that we still have the freedom of choosing $p \in \mathbb{H}^2$.

Suppose that the isometry between $Y_{p,r}$ and $\bar{Y}_r$ maps $\boldsymbol{q} \in Y_{p,r}$ to $\bar{\boldsymbol{q}} \in \bar{Y}_r$. Furthermore, by using Fermi coordinates, any point in a small neighborhood of $Y_{p,r} \subset X_{p,r}$ is uniquely represented by a pair $(\boldsymbol{q}, d')$, where d' is the g'_H-distance to $Y_{p,r}$. Similarly, $(\bar{\boldsymbol{q}}, \bar{d})$ coordinatizes a neighborhood of $\bar{Y}_r \subset \bar{X}_r$. By identifying $(\boldsymbol{q}, d)$ with $(\bar{\boldsymbol{q}}, d)$, we have arranged that $g'_H = \bar{g}$ along $\bar{Y}_r$.

To apply Lemma 2.7, assign $\Omega = \bar{X}_r$, $g = g'_H$ (defined in a neighborhood U of $\bar{Y}_r \subset \bar{X}_r$, via the identification above) and $\tilde{g} = \bar{g}$ (defined on $\bar{X}_r$). As noted above, Lemma 2.7 (a) is satisfied. Furthermore, the mean curvature of $Y_{p,r} \subset X_{p,r}$ with respect to g'_H is $H'_{p,r} := H_{p,r}/u(r) \geq H_{p,r}$, but by choosing p to have large z-coordinate, we can still arrange that $\bar{H}_r > H'_{p,r}$ (see the proof of Proposition 2.6 (b)). Next, by shrinking U if needed, we can assume that $R_{g'_H} \geq c - n(n-1)$ on U, and we can assume the same lower bound for $R_{\bar{g}}$ by choosing p suitably (Proposition 2.6 (a)). Finally, take $\epsilon = c/2$.

With the above setting, Lemma 2.7 applies and yields a metric $\hat{g}$ defined on $\bar{X}_r$, satisfying

- $R_{\hat{g}} \geq -n(n-1) + c/2$;
- $\hat{g} = \bar{g}$ on $\bar{X}_r \setminus U$;
- $\hat{g} = g'_H$ in a neighborhood of $\bar{Y}_r \subset \bar{X}_r$.

Thus, $\hat{g}$ and g'_H piece together to become a smooth metric $\boldsymbol{g}$ defined on

$$\boldsymbol{M} := \left[\left(\mathbb{H}^n/\mathbb{Z}^{n-2}\right) \setminus X_{p,r}\right] \cup \bar{X}_r/\!\sim,$$

where $\sim$ indicates boundary identification, with (non-constant) scalar curvature $R_{\boldsymbol{g}} \geq -n(n-1)$. (For the reader's convenience, Figure 1 includes a schematic, 1-dimensional illustration of the construction.)

In the statement of Theorem 1.2, take $(M, g) = (\boldsymbol{M}, \boldsymbol{g})$, $K = \bar{X}_r \cup (X_{p,r_0} \backslash X_{p,r}) \subset \boldsymbol{M}$ and $K' = X_{p,r_0}$, and the proof is complete. ■

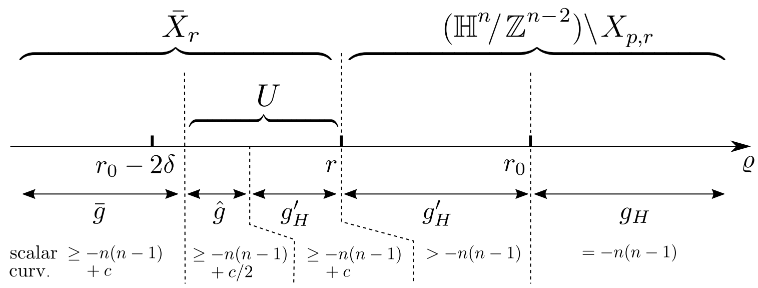

Figure 1. A schematic picture of $(\boldsymbol{M}, \boldsymbol{g})$.

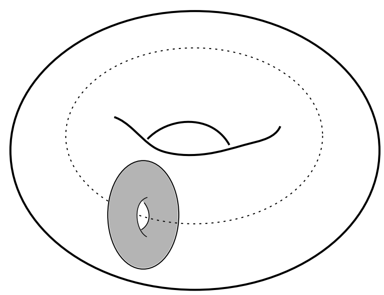

Figure 2. An illustration of $\left(\mathbb{S}^1 \times \mathbb{R}^2\right) \setminus \left(\mathbb{D}^2_\epsilon \times \mathbb{S}^1\right)$, where $\mathbb{D}^2_\epsilon \times \mathbb{S}^1$ is shaded.

2.4 Further remarks

2.4.1 Surgery applied to $\mathbb{H}^n/\mathbb{Z}^{n-1}$

The construction above only modifies a portion of $\mathbb{H}^n/\mathbb{Z}^{n-2}$ that is contained in between $x_0 < x < x_1$ for some $x_0, x_1 \in \mathbb{R}$. By a translation, we can always arrange that $x_0 = 0$. Now, $T\colon (x, \boldsymbol{y}, z) \mapsto (x + x_1, \boldsymbol{y}, z)$ maps a neighborhood of $\{x = 0\}$ isometrically to a neighborhood of $\{x = x_1\}$. Thus, by removing the subsets $\{x < 0\}$ and $\{x > x_1\}$ from $\boldsymbol{M}$ and then identifying $\{x = 0\}$ and $\{x = x_1\}$ via T, we obtain a smooth Riemannian manifold $(\boldsymbol{M'}, \boldsymbol{g'})$ that satisfies $R_{\boldsymbol{g'}} \geq -n(n-1)$. In fact, $(\boldsymbol{M'}, \boldsymbol{g'})$ can be viewed as a compactly supported 'deformation' of a hyperbolic cusp $\mathbb{H}^n/\mathbb{Z}^{n-1}$, where $\mathbb{Z}^{n-1}$ acts on $(x, \boldsymbol{y}) \in \mathbb{R}^{n-1}$ by translating along the lattice $x_1\mathbb{Z} \times 2\pi\mathbb{Z}^{n-2}$. This serves as yet another counterexample to Gromov's Statement 1.1.

2.4.2 A note on topology

It is interesting to determine the topology of both $\boldsymbol{M}$ and $\boldsymbol{M'}$ above.

Topologically, $\boldsymbol{M}$ is obtained by a *surgery* along $\{p\} \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{S}^1$ and then taking product with $\mathbb{T}^{n-3}$. The result of that surgery is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. To see this, view $\mathbb{S}^3$ as the union of $\mathbb{D}^2 \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{D}^2$ with the boundaries identified. Then $\mathbb{R}^2 \times \mathbb{S}^1$ is simply $\mathbb{S}^3$ with the core circle $\mathcal{C} = \mathbb{S}^1 \times \{q\}$ removed. Surgery of $\mathbb{S}^3$ along $\{p\} \times \mathbb{S}^1$ yields $\mathbb{S}^1 \times \mathbb{S}^2$. Then removing $\mathcal{C}$ from $\mathbb{S}^1 \times \mathbb{S}^2$ gives $\mathbb{S}^1 \times \mathbb{R}^2$. In conclusion, $\boldsymbol{M} \cong \mathbb{S}^1 \times \mathbb{R}^2 \times \mathbb{T}^{n-3}$, which is homeomorphic to $\mathbb{H}^n/\mathbb{Z}^{n-2} \cong \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{T}^{n-3}$ via a map that switches the first two factors.

Regarding $\boldsymbol{M'}$, note that by identifying $x = 0$ and $x = x_1$ in $\{0 \leq x \leq x_1\} \subset \mathbb{H}^2$, one obtains an open annulus, or equivalently $\mathbb{R}^2 \backslash \{\boldsymbol{0}\}$. Thus, $\boldsymbol{M'}$ is obtained by a *surgery* along $\{p\} \times \mathbb{S}^1 \subset \left(\mathbb{R}^2 \backslash \{\boldsymbol{0}\}\right) \times \mathbb{S}^1$ and then taking product with $\mathbb{T}^{n-3}$. In this case, a similar argument as the above applies, and the result of the surgery is homeomorphic to $\left(\mathbb{S}^1 \times \mathbb{R}^2\right) \setminus \left(\mathbb{D}^2_\epsilon \times \mathbb{S}^1\right)$, i.e., the result of removing a solid torus that is contained in a 3-ball $\mathbb{B}^3 \subset \mathbb{S}^1 \times \mathbb{R}^2$ (see Figure 2). Thus $\boldsymbol{M'} \cong \left[\left(\mathbb{S}^1 \times \mathbb{R}^2\right) \setminus \left(\mathbb{D}^2_\epsilon \times \mathbb{S}^1\right)\right] \times \mathbb{T}^{n-3}$. In particular, the two ends of $\boldsymbol{M'}$ are separated by a hypersurface with the topology $\mathbb{S}^2 \times \mathbb{T}^{n-3}$; the same is *not* true for $\mathbb{H}^n/\mathbb{Z}^{n-1}$.

3 ALH manifolds, mass and deformations

This section includes basic notions and results concerning ALH manifolds, possibly with arbitrary ends, and their NRF and conformal deformations. These results will be used in proving Theorem 1.5.

3.1 ALH manifolds and mass

Definition 3.1. Let (M^n, g) be a complete Riemannian manifold without boundary. Suppose that

(1) for some (sufficiently large) compact set $K \subset M$, $M \setminus K$ has a connected component $\mathcal{E}$ that is diffeomorphic to $(0,1) \times \mathbb{T}^{n-1}$, and

(2) restricted to $\mathcal{E}$, the metric g admits an asymptotic expansion of the form

$$g = \frac{1}{\tau^2}\left[\mathrm{d}\tau^2 + h + \frac{\tau^n}{n}\kappa + \mathcal{O}\big(\tau^{n+1}\big)\right], \tag{3.1}$$

where τ is the coordinate on the interval $(0,1)$; h denotes a flat metric on $\mathbb{T}^{n-1}$, which represents metric at the conformal infinity $\mathcal{E}_0$ (i.e., when $\tau = 0$); $\kappa = \kappa_{AB}\,\mathrm{d}y^A \mathrm{d}y^B$ is a symmetric tensor defined on $\mathbb{T}^{n-1}$, where $\big(y^A\big)$ are flat coordinates on $\mathbb{T}^{n-1}$; finally, $\mathcal{O}\big(\tau^{n+1}\big)$ stands for a remainder $Q = Q_{AB}\,\mathrm{d}y^A \mathrm{d}y^B$ with the asymptotics

$$\big|Q_{AB}\big| + \sum_{|\alpha|+k\le 2} \big|\tau^k \partial_y^\alpha \partial_\tau^k \, Q_{AB}\big| \le C\tau^{n+1} \qquad \text{as } \tau \to 0,$$

for some constant C, where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ are multi-indices.

Such an (M, g) is called *asymptotically locally hyperbolic* (*ALH*), and $\mathcal{E}$ an *ALH end*. Moreover, if $M \setminus \mathcal{E}$ is non-compact, we say that (M, g) is *ALH with arbitrary ends*.

Definition 3.2 (cf. [28, Definition 1.1])**.** Given a Riemannian manifold (M, g) with an ALH end $\mathcal{E}$ on which g admits the expansion (3.1), we call

$$m_{\mathcal{E},g} := \mathrm{tr}_h \kappa = h^{AB}\kappa_{AB}$$

the *mass aspect function* associated to the pair $(\mathcal{E}, g)$. Furthermore, define

$$\bar{m}_{\mathcal{E},g} := \sup_{\mathbb{T}^{n-1}} m_{\mathcal{E},g}.$$

Throughout, let each τ-level set in $\mathcal{E}$ be denoted by $\mathcal{E}_\tau$. The following lemma shows how $\bar{m}_{\mathcal{E},g}$ is related to the mean curvature of $\mathcal{E}_\tau \subset \mathcal{E}$.

Lemma 3.3. *Let $\big(M^n, g\big)$ be a Riemannian manifold with an ALH end $\mathcal{E}$. If $\bar{m}_{\mathcal{E},g} < 0$, then there exist constants $\tau_0, C > 0$ such that*

$$H_{\mathcal{E}_\tau} \ge (n-1) + C\tau^n \qquad \textit{for } \tau \le \tau_0, \tag{3.2}$$

where $H_{\mathcal{E}_\tau}$ is the mean curvature of $\mathcal{E}_\tau$ computed with respect to the 'outward normal' $-\partial/\partial\tau$.

Proof. Before making any assumption about $\bar{m}_{\mathcal{E},g}$, we have

$$H_{\mathcal{E}_\tau} = (n-1) - \frac{n-2}{2n} m_{\mathcal{E},g}\tau^n + \mathcal{O}\big(\tau^{n+1}\big). \tag{3.3}$$

For $\bar{m}_{\mathcal{E},g} < 0$, let us take $C = -\bar{m}_{\mathcal{E},g}/10$, and clearly (3.2) holds for some $\tau_0 > 0$. ■

3.2 NRF deformations

Given a Riemannian n-manifold (M^n, g_0), the *normalized Ricci flow* (NRF), with initial metric g_0, is by definition a smooth family of Riemannian metrics $g(t)$ on M satisfying the evolution equation

$$\partial_t g = -2[\mathrm{Ric}_g + (n-1)g], \qquad g(0) = g_0. \tag{3.4}$$

Lemma 3.4. *Suppose that (M^n, g_0) is a complete Riemannian manifold with an ALH end $\mathcal{E}$ that satisfies $R_{g_0} \geq -n(n-1)$ as well as the assumptions (d) and (e) in Theorem 1.5. Then there exists a $T > 0$ such that, for $t \in (0, T]$, $g(t)$ is complete and satisfies (3.4) along with the following properties:*

(*i*) *$(\mathcal{E}, g(t)|_{\mathcal{E}})$ remains ALH, and $g(t)$ has the expansion (see equation (3.1))*

$$g(t) = \frac{1}{\tau^2}\left[\mathrm{d}\tau^2 + h + \frac{\tau^n}{n}\kappa(t) + \mathcal{O}\big(\tau^{n+1}\big)\right];$$

(*ii*) *on M, $R_{g(t)} \geq -n(n-1)$ for all $t \in (0, T]$;*

(*iii*) *if g_0 is not Einstein, then $R_{g(t)} > -n(n-1)$ for all $t \in (0, T]$;*

(*iv*) *outside a compact subset in M, $R_{g(t)} \leq -\alpha/2$ for $t \in (0, T]$;*

(*v*) *if $\kappa(0) = 0$, then $\kappa(t) = 0$ for all $t \in (0, T]$;*

(*vi*) *if $\kappa(0) = 0$, then for any $t \in (0, T]$ we have $R_{g(t)} + n(n-1) = \mathcal{O}(\tau^{n+1})$ as $\tau \to 0$.*

Proof. The existence of $g(t)$, $t \in (0, T]$, satisfying (3.4) follows from the existence of a solution $\tilde{g}(t)$, $t \in (0, \widetilde{T}]$, of the Ricci flow initiated at g_0. They are related by a time-transformation:

$$g(t) := \mathrm{e}^{-2(n-1)t}\tilde{g}\left(\Phi(t)\right), \qquad \text{where } \Phi(t) = \frac{\mathrm{e}^{2(n-1)t} - 1}{2(n-1)}.$$

Thus, up to constant factors, the curvature tensor $\mathrm{Rm}(t)$ of $g(t)$ satisfies the same estimates as $\widetilde{\mathrm{Rm}}(\Phi(t))$ of $\tilde{g}(\Phi(t))$. In particular, it follows from [33] that, for all $t \in (0, T]$, $g(t)$ is complete, and $|\mathrm{Rm}(t)|$ is uniformly bounded.

Now we turn to proving the properties. (i) follows from [5, Proposition 3.1]. (ii) can be verified by applying the maximum principle (see [15, Theorem 7.42]) to the evolution equation[6] satisfied by $\mathrm{e}^{2(n-1)t}(R_{g(t)} + n(n-1))$; to prove ($iii$), invoke the strong maximum principle on the domain $\Omega \times [0, t]$, where $\Omega \subset M$ is compact on which g_0 is not Einstein, and then let Ω exhaust M. (iv) would follow once we show that the integral

$$\int_0^t \partial_{t'}\widetilde{\mathrm{Rm}}\,\mathrm{d}t', \qquad t \in \big(0, \widetilde{T}\big] \tag{3.5}$$

is uniformly bounded; to see this, note that the first covariant derivatives of $\widetilde{\mathrm{Rm}}(0)$ are assumed to be bounded (assumption (d) in Theorem 1.5), by [14, Theorem 14.16], we have

$$\big|\nabla^2_{\tilde{g}(t)}\widetilde{\mathrm{Rm}}(t)\big| \leq \frac{C}{\sqrt{t}}$$

for some constant $C > 0$; in addition, the evolution equation of $\widetilde{\mathrm{Rm}}$ reads[7]

$$\partial_t\widetilde{\mathrm{Rm}} = \Delta_{\tilde{g}(t)}\widetilde{\mathrm{Rm}} + \widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}};$$

of course, $1/\sqrt{t}$ is integrable; combining these, it is easy to see that (3.5) is uniformly bounded for small enough $\widetilde{T}$; since, by assumption (e) in Theorem 1.5, $R_g \leq -\alpha$ outside a compact set, (iv) follows. (v) follows from [5, Proposition 4.3]. Finally, (vi) follows from (v) and [5, formulas (3.19)–(3.21)] (note that $g^{ij}(\tau)$ provides an extra factor of τ^2). ∎

[6] For the evolution equation satisfied by $R_{g(t)}$, see [5, formula (5.1)].

[7] $\widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}}$ indicates a specific linear combination of the traces of $\widetilde{\mathrm{Rm}} \otimes \widetilde{\mathrm{Rm}}$.

3.3 Conformal deformations

Throughout this section, $c_n := 4(n-1)/(n-2)$.

Lemma 3.5. *Let (M, g) be complete with an ALH end $\mathcal{E}$, and let $f \in C^\infty(M)$ be a non-negative function that satisfies*

(a) $\operatorname{supp} f \subset K \cup \mathcal{E}$ for some compact subset $K \subset M$;

(b) $f \in \mathcal{O}(\tau^n)$ as $\tau \to 0$ where τ is the function occurring in the expansion (3.1).

Then there exist a function $v \in C^\infty(M)$ and a constant δ_0 such that $0 < \delta_0 \le v \le 1$ and

$$-c_n \Delta_g v + f v = 0 \qquad \text{in } M. \tag{3.6}$$

Proof. Let $\{\Omega_i\}_{i=0}^\infty$ be a sequence of smooth, bounded domains satisfying $\Omega_i \Subset \Omega_{i+1}$ and $\bigcup_i \Omega_i = M$. For each i, the Dirichlet problem

$$\begin{aligned} -c_n \Delta_g v_i + f v_i &= 0 && \text{in } \Omega_i, \\ v_i &= 1 && \text{on } \partial\Omega_i, \end{aligned} \tag{3.7}$$

has a positive solution v_i. By the maximum principle, $0 < v_i \le 1$. Thus, $v := \lim_{i\to\infty} v_i$ is well defined on M, satisfying $0 \le v \le 1$ and (3.6). It remains to show that v has a positive lower bound.

Without loss of generality, assume that $\Sigma_i \subset \partial\Omega_i$ is the only component of $\partial\Omega_i$ that is contained in $\mathcal{E}$; in fact, let us assume that each Σ_i is a τ-level set. Denote $\tau_0 := \tau|_{\Sigma_0}$.

To refine the estimate of v_i, we construct an auxiliary function ξ and compare it with v_i via the maximum principle. Indeed, let $\alpha \in (0, n-1)$ be any constant, and define

$$\xi = 1 - (\tau/\tau_0)^\alpha, \qquad \tau \le \tau_0.$$

Using the fact that $-\ln\tau$ is, up to adding a constant, the distance function to Σ_0, one easily computes that

$$\Delta_g \xi = \alpha(H_{\mathcal{E}_\tau} - \alpha)(\tau/\tau_0)^\alpha. \tag{3.8}$$

Thus, by (3.3), for sufficiently small τ_0, there exists a constant $C_{n,\alpha,\tau_0} > 0$ such that

$$\Delta_g \xi \ge C_{n,\alpha,\tau_0} \tau^\alpha \qquad \text{for any } \tau \le \tau_0.$$

Now, (3.7), the fact that $v_i \le 1$, and the assumption that $f \in \mathcal{O}(\tau^n)$ together imply

$$\begin{aligned} \Delta_g v_i &\le C'_{f,n} \tau^n && \text{in } (\Omega_i \backslash \Omega_0) \cap \mathcal{E}, \\ v_i &> 0 && \text{on } \Sigma_0, \\ v_i &= 1 && \text{on } \Sigma_i, \end{aligned}$$

where $C'_{f,n}$ is a constant depending only on f and n. In comparison,

$$\begin{aligned} \Delta_g \xi &\ge C_{n,\alpha,\tau_0} \tau^\alpha && \text{in } \mathcal{E} \backslash \Omega_0, \\ \xi &= 0 && \text{on } \Sigma_0, \\ \xi &< 1 && \text{on } \Sigma_i. \end{aligned}$$

Thus, for sufficiently small τ_0, the maximum principle implies that $v_i \ge \xi$ in $(\Omega_i \backslash \Omega_0) \cap \mathcal{E}$. Upon taking limit, $v \ge \xi > 0$ on $\mathcal{E} \backslash \Omega_1$. Since $v \ge 0$, the strong maximum principle, applied to (3.6), implies that $v > 0$ on M.

When $M \setminus \mathcal{E}$ is compact (i.e., M having no arbitrary end), the above already implies that v has a positive lower bound. When $M \setminus \mathcal{E}$ is non-compact, since f is supported in $K \cup \mathcal{E}$, by choosing Ω_0 to include K, we have that each v_i $(i \geq 1)$ is harmonic on $\Omega_i \setminus (\Omega_0 \cup \mathcal{E})$; using the maximum principle again, we get

$$\min_{\Omega_i \setminus (\Omega_0 \cup \mathcal{E})} v_i = \min_{\partial\Omega_0 \setminus \mathcal{E}} v_i \xrightarrow{i\to\infty} \min_{\partial\Omega_0 \setminus \mathcal{E}} v =: \delta_{\mathrm{arb}} > 0.$$

To finish the proof, it suffices to take $\delta_0 = \min\{\delta_{\mathrm{arb}}, \inf_{\Omega_1} v, \inf_{\mathcal{E}\setminus\Omega_1} \xi\}$. ■

Proposition 3.6. *Let (M^n, g) be complete, with an ALH end $\mathcal{E}$ and with $R_g \geq -n(n-1)$ on M. Let $\bar{R} \in C^\infty(M)$ be a function that satisfies*

(a) $-n(n-1) \leq \bar{R} \leq \min\{R_g, 0\}$*;*

(b) $\operatorname{supp}(R_g - \bar{R}) \subset \mathcal{E} \cup K$ *for some compact subset* $K \subset M$*;*

(c) $\bar{R} \equiv -n(n-1)$ *on* $\mathcal{E} \setminus K'$ *for some compact subset* $K' \subset \mathcal{E}$*.*

Then the Yamabe equation

$$\begin{aligned} &-c_n\Delta_g u + R_g u - \bar{R}u^{\frac{n+2}{n-2}} = 0 && \text{in } M,\\ &u \to 1 && \text{towards } \mathcal{E}_0 \end{aligned} \tag{3.9}$$

has a solution u with $0 < \delta_0 \leq u \leq 1$ for some constant δ_0. In particular, the metric $u^{4/(n-2)}g$ is complete and has the scalar curvature $\bar{R}$.

Proof. The proof follows a super/sub-solution argument. To start with, define L_g by

$$L_g u = -c_n \Delta_g u + R_g u - \bar{R}u^{\frac{n+2}{n-2}}.$$

Note that $L_g 1 = R_g - \bar{R} \geq 0$ by assumption. Thus, 1 is a super-solution of (3.9).

To find a sub-solution to (3.9), take $f := R_g - \bar{R} \geq 0$. Note that $R_g = -n(n-1) + \mathcal{O}(\tau^n)$ in $\mathcal{E}$. Thus, Lemma 3.5 applies and yields a solution v to (3.6), satisfying $0 < \delta_0 \leq v \leq 1$ for some constant δ_0. Now we compute

$$-c_n\Delta_g v + R_g v - \bar{R}v^{\frac{n+2}{n-2}} = -c_n\Delta_g v + fv + \bar{R}\big(1 - v^{\frac{4}{n-2}}\big)v = \bar{R}\big(1 - v^{\frac{4}{n-2}}\big)v \leq 0,$$

where the inequality follows from the assumption that $\bar{R} \leq 0$ and the bounds for v. Thus, v is a sub-solution of (3.9).

Then one finishes the proof by following the argument of [4, Proposition 2.1]. ■

Next, we will focus on the behavior of u towards the ALH infinity $\mathcal{E}_0$.

Lemma 3.7. *Let u be as in Proposition 3.6. Given any $\alpha \in (0, n-1)$, there exists a constant $\tau_0 > 0$ such that*

$$1 - (\tau/\tau_0)^\alpha \leq u \leq 1 \qquad \text{for any } \tau \leq \tau_0.$$

Proof. Let $\xi := 1 - (\tau/\tau_0)^\alpha$. By (3.8), we have

$$c_n\Delta_g \xi - \big(R_g - \bar{R}\big)\xi = c_n\alpha(H_{\mathcal{E}_\tau} - \alpha)(\tau/\tau_0)^\alpha - \big(R_g - \bar{R}\big)\xi. \tag{3.10}$$

Since $R_g - \bar{R} \in \mathcal{O}(\tau^n)$ in $\mathcal{E}$, the right-hand side of (3.10) is positive for $\tau \leq \tau_0$, provided that τ_0 is sufficiently small. On the other hand, since $\bar{R} \leq 0$ and $0 < u \leq 1$, (3.9) implies that

$$c_n\Delta_g u - \big(R_g - \bar{R}\big)u \leq 0.$$

Regarding boundary data,

$$u - \xi \geq 0 \qquad \text{along } \tau = \tau_0 \qquad \text{and} \qquad \lim_{\tau\to 0} u = \lim_{\tau \to 0} \xi = 1.$$

Now the maximum principle implies that $u \geq \xi$ for $\tau \in (0, \tau_0]$. ■

Proposition 3.8. *Let (M^n, g), $\bar{R}$ and u be as in Proposition 3.6. Additionally, suppose that $R_g + n(n-1) \in \mathcal{O}(\tau^{n+1})$ and that, however small τ_0 is, $R_g > -n(n-1)$ at some point in $\{\tau \le \tau_0\} \subset \mathcal{E}$. Then u must have the following asymptotic expansion near $\tau = 0$:*

$$u = 1 + u_{n0}\tau^n + \mathcal{O}(\tau^{n+1-\epsilon}),$$

where $u_{n0} < 0$ is a smooth function defined on the conformal infinity $\mathcal{E}_0 \cong \mathbb{T}^{n-1}$ and $\epsilon > 0$ is an arbitrary small constant.

Proof. Let us take $w := u - 1 \le 0$. By [3, Theorem 1.3], w has the expansion

$$w = \sum_{i=1}^{\infty}\sum_{j=0}^{N_i} u_{ij}\tau^i(\log\tau)^j,$$

where $u_{ij} \in C^\infty(\mathcal{E}_0)$. Clearly, the proof would be complete once we verify the conditions:

$(C1)$ $u_{ij} = 0$ for $i < n$;

$(C2)$ $u_{nj} = 0$ for $j > 0$;

$(C3)$ $u_{n0} < 0$.

Verification of $(C1)$. By (3.9), w satisfies

$$\Delta_g w - nw = \frac{1}{c_n}\left[R_g(w+1) - \bar{R}(w+1)^{\frac{n+2}{n-2}}\right] - nw.$$

Since only a neighborhood of $\mathcal{E}_0$ is concerned, we can simply substitute $\bar{R} = -n(n-1)$; by rearranging terms, we get

$$\Delta_g w - nw = \frac{1}{c_n}[R_g + n(n-1)]u + \frac{n(n-1)}{c_n}\left[(w+1)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}w\right] =: A + B.$$

Since $\lim_{\tau\to 0} u = 1$ and $0 \le R_g + n(n-1) \in \mathcal{O}(\tau^{n+1})$, we have $A \ge 0$ and $A \in \mathcal{O}(\tau^{n+1})$. On the other hand, B is the remainder of a Taylor expansion truncated at the linear term, so $B = \mathcal{O}(w^2)$ as $\tau \to 0$. By Lemma 3.7, $w = \mathcal{O}(\tau^\alpha)$ for any $\alpha < n-1$. Of course, we can choose $\alpha > (n+1)/2$, and thus $B = \mathcal{O}(\tau^{2\alpha}) = o(\tau^{n+1})$. In summary, for sufficiently small τ_0, we have

$$0 \le \Delta_g w - nw \in \mathcal{O}(\tau^{n+1}) \qquad \text{for } \tau \le \tau_0. \tag{3.11}$$

Now consider any $\beta \in (n-1, n)$. Using (3.3), it is easy to verify that

$$\Delta_g \tau^\beta - n\tau^\beta = -(\beta+1)(n-\beta)\tau^\beta + \mathcal{O}(\tau^{n+2}).$$

Clearly, there exists $\tau_0 > 0$ such that

$$(\Delta_g - n)(w + \lambda\tau^\beta) \le 0 \qquad \text{for all } \tau \le \tau_0 \text{ and constants } \lambda \ge 1.$$

Fix such a τ_0, and let us choose $\lambda \ge 1$ such that $w|_{\{\tau=\tau_0\}} + \lambda\tau_0^\beta \ge 0$; moreover, we have $\lim_{\tau\to 0}(w + \lambda\tau^\beta) = 0$. Thus, by the maximum principle,

$$w \ge -\lambda\tau^\beta \qquad \text{for } \tau \le \tau_0.$$

Since $\beta \in (n-1, n)$ is arbitrary and $w \le 0$, this verifies $(C1)$.

Verification of $(C2)$. By $(C1)$, we have

$$w = \sum_{j=0}^{N_n} u_{nj}\tau^n(\log\tau)^j + \mathcal{O}\big(\tau^{n+1-\epsilon}\big).$$

Further information about u_{nj} is obtainable by computing $(\Delta_g - n)w$ using this expansion and then comparing the result with (3.11). In fact, direct computation and (3.3) yield:

$$\begin{aligned}
&(\Delta_g - n)\tau^n = \mathcal{O}\big(\tau^{n+2}\big),\\
&(\Delta_g - n)[\tau^n(\log\tau)^j] = \big[(n+1)j(\log\tau)^{j-1} + j(j-1)(\log\tau)^{j-2}\big]\,\tau^n + \mathcal{O}\big(\tau^{n+2}\big)
\end{aligned}$$

with $1 \le j \le N_i$. Now, since u_{nj} are all defined on $\mathcal{E}_0$, we have $\Delta_g u_{nj} \in \mathcal{O}\big(\tau^2\big)$; and since the remainder $\mathcal{O}(\tau^{n+1-\epsilon})$ does not contribute to the coefficients s_j of $\tau^n(\log\tau)^j$ in $(\Delta_g - n)w$, we have that s_j equals to

$$\begin{aligned}
&(n+1)u_{n1} + 2u_{n2} && \text{for } j = 0,\\
&2(n+1)u_{n2} + 6u_{n3} && \text{for } j = 1,\\
&\vdots\\
&N_n(n+1)u_{nN_n} && \text{for } j = N_n - 1,\\
&0 && \text{for } j = N_n.
\end{aligned}$$

By (3.11), all s_j must vanish, which implies that

$$u_{nj} \equiv 0 \qquad \text{for } j = 1, \dots, N_n.$$

This verifies $(C2)$.

Verification of $(C3)$. Consider an auxiliary function $\zeta := -\delta\big(\tau^n + \tau^{n+1}\big)$ where $\delta > 0$ remains to be chosen. Now

$$(\Delta_g - n)\zeta = -\delta\big[(n+2)\tau^{n+1} + \mathcal{O}\big(\tau^{n+2}\big)\big],$$

so $(\Delta_g - n)\zeta \le 0$ provided that τ is small, and let us choose τ_0 accordingly (note: this is independent of the choice of δ). By comparison, recall from (3.11) that $(\Delta_g - n)w \ge 0$ for $\tau \le \tau_0$.

Regarding boundary data, first note that the assumption about R_g implies that w cannot be identically zero for $\tau \in (0, \tau_0]$; thus, the strong maximum principle implies, in particular, that $w < 0$ along $\tau = \tau_0$. This allows us to choose δ such that $w \le \zeta$ along $\tau = \tau_0$. Moreover, both $w, \zeta \to 0$ as $\tau \to 0$. Now, by the maximum principle, we get

$$w \le \zeta = -\delta\big(\tau^n + \tau^{n+1}\big) \qquad \text{for } \tau \le \tau_0.$$

This proves that $u_{n0} < 0$, verifying $(C3)$. ∎

Lemma 3.9. *Let (M^n, g) be a Riemannian manifold with an ALH end $\mathcal{E}$, on which the asymptotic expansion (3.1) applies. Suppose that $u = 1 + \varphi\tau^n + \mathcal{O}(\tau^{n+1})$ is a function defined on $\mathcal{E}$, where $\varphi \in C^\infty(\mathcal{E}_0)$. Then, up to a diffeomorphism that restricts to be the identity on $\mathcal{E}_0$, the deformed metric $\bar{g} = u^{\frac{4}{n-2}}g$ on $\mathcal{E}$ has the expansion*

$$\bar{g} = \frac{1}{\bar{\tau}^2}\left[\mathrm{d}\bar{\tau}^2 + \bar{h} + \frac{\bar{\tau}^n}{n}\bar{\kappa} + \mathcal{O}\big(\bar{\tau}^{n+1}\big)\right],$$

where

$$\bar{h} = h \qquad \textit{and} \qquad \bar{\kappa} = \kappa + \frac{4(n+1)}{n-2}\varphi h.$$

Proof. A standard argument following the proof of [6, Lemma 6.5]. ∎

4 Two rigidity results

The goal of this section is to prove Theorem 1.5, Corollary 1.6 and Theorem 1.4. The reader may consult Appendix A before proceeding.

Proposition 4.1 (cf. [11, Theorem 1.1])**.** *For $3 \le n \le 7$, let M^n be a (connected) non-compact manifold with connected, compact boundary Σ. Let $\iota\colon \Sigma \hookrightarrow M$ be the inclusion map. Suppose that $\Sigma \in \mathcal{C}_{\deg}$ (see Definition 1.9) and that the map ι is incompressible. Then M admits no complete metric g with $R_g \ge -n(n-1)$ and $H_\Sigma > n-1$.*

Proof. To begin with, by the classification of covering spaces, there exists a covering of M, say $p\colon \hat{M} \to M$, that satisfies

$$p_*\big(\pi_1(\hat{M})\big) = \iota_*(\pi_1(\Sigma)) \subset \pi_1(M), \tag{4.1}$$

where base points for the fundamental groups are omitted. Moreover, by the homotopy lifting property, there exists an embedding $\hat{\iota}\colon \Sigma \to \hat{M}$ such that $\iota = p \circ \hat{\iota}$.

By (4.1) and the incompressibility of ι, the composition

$$J := \left({\iota_*}^{-1}\big|_{\iota_*(\pi_1(\Sigma))}\right) \circ p_* \colon\ \pi_1(\hat{M}) \to \pi_1(\Sigma)$$

is a well-defined group homomorphism. Since Σ is aspherical, by [24, Proposition 1B.9], there exists a map $j\colon \hat{M} \to \Sigma$ such that $j_*\colon \pi_1(\hat{M}) \to \pi_1(\Sigma)$ is equal to J; in particular, $j_* \circ \hat{\iota}_* = \mathrm{id}_{\pi_1(\Sigma)}$; then, by applying the uniqueness part of [24, Proposition 1B.9] to Σ, it is easy to see that $j \circ \hat{\iota}$ is in fact homotopic to id_Σ.

Since ι is an embedding, each boundary component of $\hat{M}$, which is a lifting of Σ, must be diffeomorphic to Σ. In particular, denote $\hat{\Sigma} = \hat{\iota}(\Sigma)$. Since $j \circ \hat{\iota}$ is homotopic to id_Σ, we have $\big[\hat{\Sigma}\big] = \hat{\iota}_*[\Sigma] \ne 0 \in H_{n-1}(\hat{M};\mathbb{Z})$.

Now, for the sake of deriving a contradiction, suppose that g is a complete metric on M with $R_g \ge -n(n-1)$ and $H_\Sigma \ge (n-1)(1+\delta)$ for some constant $\delta > 0$. Let $\hat{g} := p^* g$ be the pull-back metric on $\hat{M}$, and define $\rho(x) := \mathrm{dist}_{\hat{g}}(x, \hat{\Sigma})$ for $x \in \hat{M}$.

For an arbitrarily large $T > 0$, let

$$\mathcal{D}_T := \big\{x \in \hat{M} \colon \rho(x) \le T\big\},$$

and let $\hat{\Sigma}_i$ $(0 \le i \le k)$ be those components of $\partial \hat{M}$ that satisfy

$$\hat{\Sigma}_i \cap \mathcal{D}_T \ne \varnothing,$$

where $\hat{\Sigma}_0 = \hat{\Sigma}$. Define (see Figure 3 below)

$$\mathcal{U}_T = \mathcal{D}_T \cup \bigcup_{0 \le i \le k} \hat{\Sigma}_i \qquad \text{and} \qquad \mathcal{U}_{T,\epsilon} = \big\{x \in \hat{M} \colon \mathrm{dist}_{\hat{g}}(x, \mathcal{U}_T) < \epsilon\big\}.$$

Since M is complete, connected and non-compact, so is $\hat{M}$, and we have $\bar{\mathcal{U}}_T \Subset \mathcal{U}_{T,\epsilon}$. Moreover, for small enough ϵ,

$$\mathcal{U}_{T,\epsilon} \cap \left(\partial \hat{M} - \bigcup_{0 \le i \le k} \hat{\Sigma}_i\right) = \varnothing.$$

Thus, by the smooth Urysohn lemma, there exists a function $\eta \in C^\infty(\hat{M})$ with

$$\eta(x) = \begin{cases} 0, & x \in \mathcal{U}_T, \\ 1, & x \in \hat{M} \setminus \mathcal{U}_{T,\epsilon}. \end{cases}$$

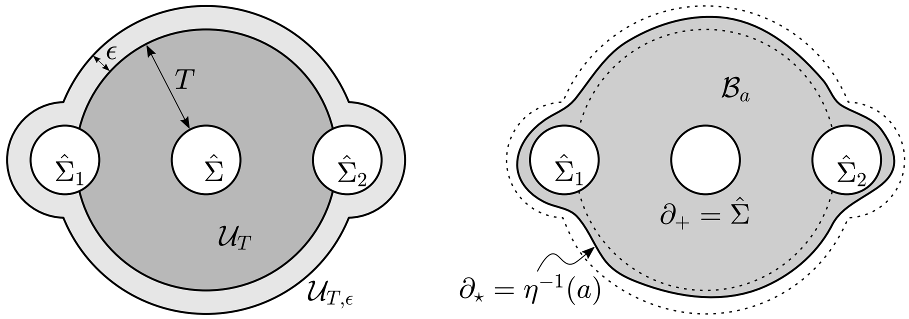

Figure 3. A schematic picture showing $\mathcal{U}_T$, $\mathcal{U}_{T,\epsilon}$ (left figure) and $\mathcal{B}_a$ (right figure). The complement of $\mathcal{U}_{T,\epsilon}$, which may include more boundary components of $\hat{M}$, is not displayed.

Let $a \in (0,1)$ be a regular value of η. Automatically, $\eta^{-1}(a)$ is a smooth, closed hypersurface of $\hat{M}$, and $\eta^{-1}(a) \cap \partial\hat{M} = \varnothing$.

By the above arrangement, $\mathcal{B}_a := \eta^{-1}\big([0,a]\big)$, equipped with the restriction of the metric $\hat{g}$, is a Riemannian band with

$$\partial_+ = \hat{\Sigma} \qquad \text{and} \qquad \partial_- = \partial\mathcal{B}_a \setminus \hat{\Sigma} = \eta^{-1}(a) \cup \bigcup_{1\le i\le k} \hat{\Sigma}_i.$$

By letting $f = j|_{\mathcal{B}_a}$ and using Lemma A.8, one easily sees that $\mathcal{B}_a$ satisfies the $\mathbf{NSep}^+$property (see Definition A.6). Then take $\partial_\star = \eta^{-1}(a)$.

With these choices, all assumptions of Lemma A.9 are satisfied for $(\mathcal{B}_a, \hat{g}|_{\mathcal{B}_a}; \partial_-, \partial_+)$ and $\partial_\star$. Since $\big(\hat{M}, \hat{g}\big)$ is complete and non-compact, the distance $\mathrm{dist}_{\hat{g}}(\partial_\star, \partial_+)$ can get arbitrarily large as one chooses large T. This contradicts Lemma A.9. ∎

Remark 4.2. Proposition 4.1 still holds if M is allowed to be compact. In fact, proceeding along the same proof, we still have $\big[\hat{\Sigma}\big] \neq 0 \in H_{n-1}\big(\hat{M};\mathbb{Z}\big)$, so $\hat{M}$ cannot be compact with a single boundary component. Hence, either (1) $\hat{M}$ is non-compact, and the previous proof applies verbatim; or (2) $\hat{M}$ is itself a Riemannian band with $\partial_+ = \hat{\Sigma}$ that satisfies the $\mathbf{NSep}^+$property and the curvature bounds $R_{\hat{g}} \ge -n(n-1)$, $H_{\partial\hat{M}} \ge (n-1)(1+\delta)$; however, by Remark A.10 (A), such a band cannot exist, reaching a contradiction.

Proposition 4.3. *For $3 \le n \le 7$, let $\big(M^n, g\big)$ be a complete Riemannian manifold without boundary, with an ALH end $\mathcal{E} \cong (0,1) \times \mathbb{T}^{n-1}$, and satisfying $R_g \ge -n(n-1)$. Suppose that $Y := M \setminus \mathcal{E}$ is non-compact and that $\partial Y \cong \mathbb{T}^{n-1}$ is incompressible in M. Then $\bar{m}_{\mathcal{E},g} \ge 0$. In addition, if the assumptions (d), (e) in Theorem 1.5 hold, then $\kappa = 0$ only if (M,g) is Einstein.*

Proof. Suppose, on the contrary, that $\bar{m}_{\mathcal{E},g} < 0$. Let τ be a defining function compatible with the ALH structure of $\mathcal{E}$ (see (3.1)). Then by Lemma 3.3, there exists a small $\tau_0 > 0$ such that the mean curvature of the level set $\mathcal{E}_{\tau_0}$ satisfies

$$H_{\mathcal{E}_{\tau_0}} \ge (n-1) + \delta_0$$

for some $\delta_0 > 0$.

Now, remove $\{0 < \tau < \tau_0\}$, a subset of $\mathcal{E}$, from M and denote the resulting manifold by M'. By using the assumptions, it is easy to see that $\partial M' = \mathcal{E}_{\tau_0} \cong \mathbb{T}^{n-1}$ is incompressible in M'. Clearly, $\partial M' \in \mathcal{C}_{\deg}$. By Proposition 4.1, we get a contradiction. This proves the inequality $\bar{m}_{\mathcal{E},g} \ge 0$.

Next we turn to the second part of the proposition. Again we argue by contradiction. Assume that $\kappa = 0$ without (M, g) being Einstein. Let $g(t)$ be the NRF initiated at g. Then by Lemma 3.4, for some small t_0, we have

(*i*) $R_{g(t_0)} > -n(n-1)$ on M;

(*ii*) $R_{g(t_0)} \leq -\alpha/2 < 0$ outside a compact subset of M;

(*iii*) $R_{g(t_0)} = -n(n-1) + \mathcal{O}\big(\tau^{n+1}\big)$ on $\mathcal{E}$;

(*iv*) $\big(\mathcal{E}, g(t_0)|_{\mathcal{E}}\big)$ remains ALH with $\kappa(t_0) = 0$.

It is easy to check that a function $\bar{R}$ as described in Proposition 3.6 exists; thus, there is a positive function $u \in C^\infty(M)$ such that $\bar{g} := u^{4/(n-2)} g(t_0)$ is complete with $R_{\bar{g}} = \bar{R} \geq -n(n-1)$. Furthermore, thanks to (*i*) and (*iii*) above, both Proposition 3.8 and Lemma 3.9 apply. As a consequence, $\big(\mathcal{E}, \bar{g}|_{\mathcal{E}}\big)$ remains ALH and satisfies

$$\bar{\kappa} = \frac{4(n+1)}{(n-2)} u_{n0} h,$$

where $u_{n0} < 0$, and h is a flat metric on $\mathbb{T}^{n-1}$. Clearly, $\bar{m}_{\mathcal{E},\bar{g}} = \mathrm{tr}_h \bar{\kappa} < 0$. This contradicts the first part of the proposition. ■

Proof of Theorem 1.5. For convenience, let $\mathcal{N}_o$ (resp., $\mathcal{H}_o$) denote the result of removing a tubular neighborhood of $\phi\big(\mathbb{T}^k\big)$ from N (resp., $\psi\big(\mathbb{T}^k\big)$ from $\mathbb{H}^n/\mathbb{Z}^{n-1}$). Both $\partial\mathcal{N}_o$ and $\partial\mathcal{H}_o$ inherit the product structure $\mathbb{S}^{n-k-1} \times \mathbb{T}^k$, which are identified to form M. In symbols, $M = \mathcal{H}_o \sqcup_\Phi \mathcal{N}_o$, where $\Phi\colon \partial\mathcal{H}_o \to \partial\mathcal{N}_o$ is the identification map.

By Proposition 4.3, to prove the theorem it suffices to show that the boundary Σ of $M \setminus \mathcal{E}$ is incompressible in M.

To show this, it in turn suffices to show that the $\mathbb{T}^k$-factor of $\partial\mathcal{N}_o$ is incompressible in M, according to Lemma B.2.

If this was not the case, let L be a non-contractible loop in $\{x\} \times \mathbb{T}^k \subset \partial\mathcal{N}_o$ that is contractible in M.

Now consider

$$\mathcal{H}' := \big(\mathbb{S}^1 \times \mathbb{T}^{n-k-1} - \mathbf{B}\big) \times \mathbb{T}^k,$$

where $\mathbf{B}$ is an $(n-k)$-ball embedded in $\mathbb{S}^1 \times \mathbb{T}^{n-k-1}$. Topologically, M can be viewed as a subset of $M' := \mathcal{H}' \sqcup_\Phi \mathcal{N}_o$, so L is also contractible in M'. By [11, Lemma A.3], $\mathcal{H}'$ satisfies the 'lifting property' (see [11, Definition A.2]). Thus, [11, Lemma A.4] applies, showing that L is contractible in $\mathcal{N}_o$ and hence in N; since $\{x\} \times \mathbb{T}^k$ and $\phi(\mathbb{T}^k)$ are homotopic in N, ϕ cannot be incompressible, violating the assumption (*a*). ■

Remark 4.4. The proof above can be made more direct if one assumes that $k < n-2$. In this case, both $\pi_1(\partial\mathcal{N}_o)$ and $\pi_1(\partial\mathcal{H}_o)$ are isomorphic to $\pi_1\big(\mathbb{T}^k\big)$, and it is easy to see that the maps $\pi_1(\partial\mathcal{N}_o) \to \pi_1(\mathcal{N}_o)$ and $\pi_1(\partial\mathcal{H}_o) \to \pi_1(\mathcal{H}_o)$ are both injective. By van Kampen's theorem, we have $\pi_1(M) \cong \pi_1(\mathcal{H}_o) *_{\pi_1(\partial\mathcal{N}_o)} \pi_1(\mathcal{N}_o)$. Thus, a direct application of [31, Theorem 11.67 (*i*)] shows that $\partial\mathcal{N}_o$ is incompressible in M, and it follows that the $\mathbb{T}^k$-factor of $\partial\mathcal{N}_o$ is also incompressible in M.

Proof of Corollary 1.6. In this setting, the assumptions $(a-e)$ in Theorem 1.5 are satisfied. Since κ automatically vanishes, we conclude that g is Einstein. Write the metric on $\mathbb{H}^n/\mathbb{Z}^{n-1}$ as $\mathrm{d}t^2 + \mathrm{e}^{2t} g_0$ where g_0 is a flat metric on $\mathbb{T}^{n-1}$. Since $\mathbb{H}^n/\mathbb{Z}^{n-1}$ is isometric to (M, g) outside a compact set, one can remove the corresponding cusp (i.e., $\{t < -a\}$ for some $a \gg 0$) from M and obtain a complete, non-compact manifold (M', g') with boundary $\partial M' \cong \mathbb{T}^{n-1}$, satisfying

$H_{\partial M'} \equiv n-1$, where the mean curvature is computed with respect to the *inward* normal. By [18, Theorem 2], (M', g') is isometric to $[-a, \infty) \times \mathbb{T}^{n-1}$ with the warped product metric $\mathrm{d}t'^2 + \mathrm{e}^{2t'} g_0$; by using this fact and the respective distance functions to $\partial M' \subset M$ and $\{-a\} \times \mathbb{T}^{n-1} \subset \mathbb{H}^n/\mathbb{Z}^{n-1}$, it is easy to construct an isometry between (M, g) and $\mathbb{H}^n/\mathbb{Z}^{n-1}$. ∎

Remark 4.5. The statement of Corollary 1.6 remains true when N is non-compact without boundary. In fact, one only needs to prove the incompressibility of a $\mathbb{T}^{n-1}$-slice located far into the ALH infinity of M, and this is handled by a corresponding step in the proof of Theorem 1.5. Then the result follows directly from Theorem 1.10.

Proof of Theorem 1.4. Let (x, z) be the standard coordinates on $\mathbb{R}^2_+$, a topological factor of $\mathbb{H}^n/\mathbb{Z}^{n-2}$. Since K is compact, via the isometry f, both x and z can be regarded as coordinate functions on $M \setminus K$. Thus, for a large enough $x_0 > 0$, we can remove $\{|x| > x_0\}$ from M and then identify $\{x = \pm x_0\}$ in the same way as we did in Section 2.4.1. The result is a complete Riemannian manifold (M^*, g^*) with an ALH end $\mathcal{E}$, satisfying $R_{g^*} \geq -n(n-1)$. Moreover, (M^*, g^*) is isometric to $\mathbb{H}^n/\mathbb{Z}^{n-1}$ outside a compact set; thus, the assumptions (*d*), (*e*) in Theorem 1.5 hold automatically, and $\kappa = 0$ for $(\mathcal{E}, g^*|_{\mathcal{E}})$.

It is easy to see that M^* is of the form $M_1 \sqcup_\Phi M_2$ as described in Lemma B.2 with $k = n-2$. In particular, M_2 can be viewed as a subset of M. By assumption, $f^{-1}(T)$ is incompressible in M and hence in M_2. Using the proof of Theorem 1.5, one can show that $f^{-1}(T)$ is incompressible in M^*; then by Lemma B.2, $\partial(M \setminus \mathcal{E}) \cong \mathbb{T}^{n-1}$ is incompressible in M^*.

Thus, all conditions in Proposition 4.3 are verified for (M^*, g^*), and we conclude that g^* is Einstein. The proof of Corollary 1.6 shows that there is an isometry $\tilde{f} \colon (M^*, g^*) \to \mathbb{H}^n/\mathbb{Z}^{n-1}$ that uniquely extends the isometry, induced by f, between the 'cuspidal ends' in M^* and $\mathbb{H}^n/\mathbb{Z}^{n-1}$. Let $z_0 > 0$ be sufficiently small; then by using distance functions to the hypersurfaces $\{z = z_0\}$ in both M and $\mathbb{H}^n/\mathbb{Z}^{n-2}$, it is easy to construct an isometry between (M, g) and $\mathbb{H}^n/\mathbb{Z}^{n-2}$; details are left to the interested reader. ∎

5 Two splitting results of 'cuspidal-boundary' type

The bulk of this section is dedicated to proving Theorem 1.7. The proof of Theorem 1.10, which largely depends on those of Proposition 4.1 and Theorem 1.7, will be sketched at the end of the section.

Now we begin our proof of Theorem 1.7.

In addition to its hypothesis, let us assume that $H_{\partial M} \geq 3$. Under this assumption, the proof would be complete once we show that (M, g) is isometric to $\big((-\infty, 0] \times \Sigma, \mathrm{d}t^2 + \mathrm{e}^{2t} g_0\big)$ for some closed 3-manifold Σ carrying a flat metric g_0. In fact, Σ will occur as a hypersurface in M, obtained by an approximation scheme involving μ-bubbles (Sections 5.1 and 5.2); then we show that Σ must be compact and that (M, g) is isometric to the desired warped product (Section 5.3).

The reader is recommended to consult Appendix A before proceeding.

5.1 Specification of μ_k and E_k

Since M is non-compact with compact boundary, there exists a smooth, proper map $\rho \colon M \to (-\infty, 0]$ (see [38, Lemma 2.1]) such that

$$\rho^{-1}(0) = \partial M, \qquad |\mathrm{d}\rho|_g < 1.$$

Fix a smooth function $\eta \in C^\infty((-\infty, 0])$ satisfying

$$\eta(t) = 0 \quad \text{for any } t \leq -1 \qquad \text{and} \qquad \eta(0) = 2;$$

define τ_k by $3\coth(2\tau_k) = 3 + k^{-1}$, and then define $\hat{\mu}_k : (-\tau_k, 0] \to \mathbb{R}$ by

$$\hat{\mu}_k(t) = 3\coth\big(2(t+\tau_k)\big) - k^{-1}\eta(t).$$

Thus, $\{\tau_k\}_{k=1}^{\infty}$ is increasing and tends to infinity, and

$$\hat{\mu}_k(-\tau_k) = +\infty \qquad \text{and} \qquad \hat{\mu}_k(0) = 3 - k^{-1}.$$

Now, choose a_k, regular values of ρ, such that $\tau_k \le a_k < \min\{\tau_{k+1}, \tau_k + 1\}$, and then define $E_k := \rho^{-1}\big([-a_k, 0]\big) \subset M$. Denote $\partial_k^- := \rho^{-1}(-a_k)$, which are smooth hypersurfaces of M. This makes $(E_k, g|_{E_k}; \partial_k^-, \partial M)$ a Riemannian band. Finally, let $\rho_k := (\tau_k/a_k)\rho$, and define

$$\mu_k := \hat{\mu}_k \circ \rho_k.$$

By this arrangement, $\mu_k|_{\partial_k^-} = \infty$.

5.2 μ_k-bubbles in E_k

For each fixed k, consider $(E_k, g|_{E_k}; \partial_k^-, \partial M)$. Note that $H_{\partial M} \ge 3$; by construction, μ_k satisfies the barrier condition (see Definition A.1). By Fact A.2, a smooth μ_k-bubble Ω_k exists. Define $\Sigma_k := \partial\Omega_k \setminus \partial_k^-$, which is smooth, closed, and separates ∂_k^- from ∂M.

The following lemma shows that all Σ_k must meet a fixed compact subset of M.

Lemma 5.1. *Let $\mathcal{K} := \{x \in M : \mathrm{dist}_g(x, \partial M) \le 10\}$. Then $\Sigma_k \cap \mathcal{K} \neq \varnothing$.*

Proof. Suppose on the contrary that $\Sigma_k \cap \mathcal{K} = \varnothing$. This implies that $\eta \circ \rho_k = 0$ on Σ_k. Moreover, by assumption, $R_g \ge -12$, and by construction, $|\mathrm{d}\rho_k|_g < 1$. Thus, we have (see (A.1))

$$R_+^{\mu_k} > -12 + \frac{4}{3}[3\coth(2(\rho_k + \tau_k))]^2 - 12[\sinh(2(\rho_k+\tau_k)]^{-2} = 0 \qquad \text{on } \Sigma_k.$$

This, along with Fact A.5, implies that Σ_k admits a PSC metric; since Σ_k is separating, we get a contradiction, by Lemma A.11. ∎

5.3 Convergence of Σ_k

By using [37, Theorem 3.6], one can show that the second fundamental form II_{Σ_k} is uniformly bounded within any compact subset of M. Thus, by Lemma 5.1, Σ_k subconverges to a smooth hypersurface Σ in M (for convenience, denote the subsequence by the same symbol Σ_k). Within compact subsets of M, the convergence is uniform and has multiplicity one; moreover, Σ bounds a ‘minimizing 3-bubble’ for which minimality is interpreted with respect to compactly supported perturbations (cf. [25, Lemma 4.10]). Depending on whether Σ is compact, we consider the two cases below.

Case 1: Σ is compact. By minimality, we have (see Fact A.3)

$$H_\Sigma = 3 \qquad \text{and} \qquad L_\Sigma = -\Delta_\Sigma + \frac{1}{2}\big(R_\Sigma - R_+^3\big) \ge 0. \tag{5.1}$$

Since $R_+^3 = R_g + 12 \ge 0$, (5.1) implies that $-\Delta_\Sigma + \frac{1}{2}R_\Sigma \ge 0$; thus, there exists a smooth function $u > 0$ defined on Σ and a constant $\lambda \ge 0$ such that

$$\left(-\Delta_\Sigma + \frac{1}{2}R_\Sigma\right)u = \lambda u. \tag{5.2}$$

Define $\tilde{g}_\Sigma = u g_\Sigma$ where g_Σ is the metric on Σ induced by g. We have

$$R_{\tilde{g}_\Sigma} = u^{-1}\left(R_\Sigma + \frac{3}{2}\left|\frac{\nabla u}{u}\right|^2 - 2\frac{\Delta u}{u}\right) = u^{-1}\left(2\lambda + \frac{3}{2}\left|\frac{\nabla u}{u}\right|^2\right) \geq 0. \tag{5.3}$$

Since each Σ_k is separating, so is Σ. By Lemma A.11, Σ admits no PSC metric; then by (5.3) and the trichotomy theorem of Kazdan and Warner, $R_{\tilde{g}_\Sigma} = 0$. Thus, λ must vanish, and u must be a constant; (5.2) in turn implies that $R_\Sigma = 0$. Then by Bourguignon's theorem (see [27, Lemma 5.2]), g_Σ is Ricci-flat, which must be flat since $\dim \Sigma = 3$.

Now we prove that a neighborhood of Σ splits. When $\Sigma \cap \partial M = \varnothing$, since Σ is the boundary of minimizing 3-bubble, [2, Theorem 2.3] implies that there exists an open neighborhood of Σ that is isometric to a warped product $\big((-\epsilon,\epsilon)\times\Sigma, \mathrm{d}t^2 + \mathrm{e}^{2t} g_\Sigma\big)$, where t is the coordinate on $(-\epsilon,\epsilon)$ and Σ corresponds to $t=0$. When $\Sigma \cap \partial M \neq \varnothing$, we must have $\Sigma = \partial M$, by the maximum principle. In this case, the proof of [2, Theorem 2.3] still applies and gives an open neighborhood of Σ that is isometric to a warped product $\big((-\epsilon,0]\times\Sigma, \mathrm{d}t^2 + \mathrm{e}^{2t} g_\Sigma\big)$.

Thus, a neighborhood of Σ is foliated by the t-level sets. Note that moving along the foliation leaves the energy functional invariant; thus, each t-slice also bounds a minimizing 3-bubble, to which the same analysis above applies.

This implies that a maximal neighborhood $\mathcal{U}$ of Σ on which the metric splits as

$$\big(I\times\Sigma, \mathrm{d}t^2 + \mathrm{e}^{2t} g_\Sigma\big)$$

must be both open and closed in M. By connectedness, $\mathcal{U} = M$, and I must be of the form $(-\infty, c]$. This achieves the desired splitting.

Case 2: Σ is non-compact. By finding a contradiction, we prove that this case does not occur. The argument largely follows the proof of [38, Theorem 1.1], so we only sketch the steps.

Let

$$(M_k, g_k) = \big(\Sigma_k \times \mathbb{S}^1, g_{\Sigma_k} + u_k^2 dt^2\big),$$

where u_k is the first eigenfunction of L_{Σ_k}; that is, $L_{\Sigma_k} u_k = \lambda_k u_k$ with $\lambda_k \geq 0$. Since $\dim \Sigma_k = 3$, [9, Corollary 1.10] implies that M_k admits no PSC metric.

Now

$$R_{g_k} = R_{g_{\Sigma_k}} - 2\frac{\Delta_{g_{\Sigma_k}} u_k}{u_k} = R_+^{\mu_k} + 2\lambda_k. \tag{5.4}$$

By construction, $R_+^{\mu_k} \geq 0$ outside $\mathcal{K}$, and there exist $\delta_k > 0$, satisfying $\lim \delta_k = 0$, such that $R_+^{\mu_k} \geq -\delta_k$ on M. Since R_{g_k} cannot be positive and $\lambda_k \geq 0$, by (5.4), we must have $\lim \lambda_k = 0$.

Next, choose $q_k \in \Sigma_k \cap \mathcal{K}$ so that $\lim q_k = q \in \Sigma$, and let $p_k = (q_k, t_0) \in \Sigma_k \times \mathbb{S}^1$ and $p = (q, t_0) \in \widetilde{M} = \Sigma \times \mathbb{S}^1$. Normalize u_k such that $u_k(q_k) = 1$. By the Harnack inequality, u_k converges smoothly to a positive function u on Σ with $u(q) = 1$. Thus, (M_k, g_k) converges in the pointed smooth topology to $\big(\widetilde{M}, \widetilde{g}\big)$, where $\widetilde{g} = g_\Sigma + u^2 \mathrm{d}t^2$.

Now one can follow the proof[8] of [38, Proposition 3.2] to show that $\mathrm{Ric}_{\widetilde{g}} = 0$, and then follow the proof[9] of [38, Theorem 1.1] to show that u is constant, which implies $\mathrm{Ric}_{g_\Sigma} = 0$.

In summary, (Σ, g_Σ) is complete, non-compact, Ricci-flat, and with finite area; this contradicts [32, p. 25, Theorem 4.1]. ∎

[8] The proof of [38, Proposition 3.2] only relies on $\widetilde{M}$ admitting no PSC metric and the properties of $R_+^{\mu_k}$ mentioned above.

[9] In particular, the boundedness of area(Σ) follows from $\mathcal{A}_{\Omega_0}^{\mu_k}(\Omega_k) \leq \mathcal{A}_{\Omega_0}^{\mu_k}(E_k)$ and $\mu_k > 0$.

Remark 5.2. The PSC obstruction, provided by [9, Corollary 1.10], for manifolds of the form $\Sigma\times\mathbb{S}^1$ only works when $\dim\Sigma\neq 4$. On the other hand, if Σ $(2\leq\dim\Sigma\leq 6)$ is closed, orientable, and if it admits a map of nonzero degree to some $\Sigma'\in\mathcal{C}_{\deg}$, then by a similar argument as [11, Theorem 1.1], one can show that $\Sigma\times\mathbb{S}^1$ admits no PSC metric.

Proof of Theorem 1.10. The inequality $\inf_{\partial M} H\leq n-1$ follows directly from Proposition 4.1.

To prove the second part of the theorem, first obtain a covering $(\hat{M},\hat{g})$ of (M,g) as in the proof of Proposition 4.1, and then apply (essentially) the same proof of Theorem 1.7 to $(\hat{M},\hat{g})$; to assist the reader, we list a few points that may need attention.

- $\partial\hat{M}$ may not be connected, but Riemannian bands can still be constructed in a similar manner as in the proof of Proposition 4.1. To avoid clash of symbols, denote $S:=\partial M$ and let $\hat{S}$ be a fixed lifting of S in $\hat{M}$. Thus $\partial_+=\hat{S}$ and $\partial_\star\subset\partial_-$; $\mu_k>0$ can be defined such that $\mu_k|_{\partial_\star}=\infty$ and $\mu_k|_{\hat{S}}=(n-1)-1/k$; on $\partial_-\setminus\partial_\star$ (if nonempty) we have $H\geq n-1$; one can check that the barrier condition is satisfied, and the Σ_ks exist; restricting $j\colon\hat{M}\to S$ to Σ_k yields a map $\Sigma_k\to\hat{S}$ of nonzero degree.
- An adapted version of Lemma 5.1 holds; in the proof, invoke Lemma A.8 instead of Lemma A.11. It follows that Σ_k converges to some Σ.
- When Σ is compact, the corresponding part in Section 5.3 applies, apart from dimensional adjustments and the fact that Ricci-flatness may no longer imply flatness.
- When Σ is non-compact, we need to argue, without relying on [9, Corollary 1.10], that $M_k=\Sigma_k\times\mathbb{S}^1$ admits no PSC metric, and this is already addressed by Remark 5.2.

The consequence is that $(\hat{M},\hat{g})$ is of the form

$$\big((-\infty,0]\times\Sigma,\mathrm{d}t^2+\mathrm{e}^{2t}g_\Sigma\big),$$

where g_Σ is Ricci-flat. In particular, the covering $\hat{M}\to M$ is 1-fold and hence an isometry. Since $\Sigma=\partial M$ is assumed to be aspherical, g_Σ must be flat, which can be seen by applying the Cheeger–Gromoll splitting theorem to the universal cover; for details, see the beginning paragraph of [12, Section 6]. ■

A μ-bubbles

This section collects some 'definitions' and 'facts' concerning the μ-bubble technique, about which we make no claim to originality. For detailed expositions and proofs, the reader may consult [9, 12, 37, 39] and [22, Section 5]. This section also includes three supplementary 'lemmas'.

A common setting for μ-bubbles is a *Riemannian band*, namely a compact, connected Riemannian manifold (M^n,g) whose (nonempty) boundary is expressed as a disjoint union $\partial M=\partial_-\sqcup\partial_+$, where each of $\partial_\pm$ is a smooth, closed and possibly disconnected $(n-1)$-manifold.

Given a Riemannian band $(M^n,g;\partial_-,\partial_+)$ and a function $\mu\in C^\infty(\mathring{M})$, consider the following variational problem: Let Ω_0 be a smooth open neighborhood of ∂_-; among all Caccioppoli sets $\Omega\subset M$ that satisfy $\partial_-\subset\Omega$ and $\Omega\Delta\Omega_0\Subset\mathring{M}$, seek a *minimizer* of the functional

$$\mathcal{A}^\mu_{\Omega_0}(\Omega)=\mathcal{H}^{n-1}(\partial\Omega)-\mathcal{H}^{n-1}(\partial\Omega_0)-\int_M(\chi_\Omega-\chi_{\Omega_0})\mu\,\mathrm{d}\mathcal{H}^n,$$

where $\mathcal{H}^k$ is the induced k-dimensional Hausdorff measure, and $\chi_\Omega,\chi_{\Omega_0}$ are characteristic functions. Such a minimizer is called a *μ-bubble*.

Existence and regularity of μ-bubbles are well-established when μ satisfies the following 'barrier condition'.

Definition A.1. Let $(M^n, g; \partial_-, \partial_+)$ be a Riemannian band. A function $\mu \in C^\infty(\mathring{M})$ is said to satisfy the *barrier condition* if, for each connected component $S \subset \partial_+$ (resp., $S \subset \partial_-$),

- either μ smoothly extends to S and satisfies $H_S > \mu|_S$ (resp., $H_S > -\mu|_S$), where H_S is the mean curvature of S with respect to the outward normal;
- or $\mu \to -\infty$ (resp., $\mu \to +\infty$) towards S.

Fact A.2. *For $3 \leq n \leq 7$, if $\mu \in C^\infty(\mathring{M})$ satisfies the barrier condition, then there exists a smooth μ-bubble Ω. In particular, $\partial\Omega \setminus \partial_-$ is homologous to ∂_+ and is separating (see Definition A.6 below).*

Also well-known are the following variational properties. To fix notation, let Σ denote the hypersurface $\partial\Omega \setminus \partial_-$ with outward unit normal ν; let R_Σ and Δ_Σ be, respectively, the scalar curvature and the Laplacian along Σ (with the induced metric); let H_Σ and II be, respectively, the mean curvature and the second fundamental form of Σ, computed with respect to ν; define the operators

$$
\begin{aligned}
J_\Sigma &= -\Delta_\Sigma + \frac{1}{2}\big(R_\Sigma - R_g - \mu^2 - |\mathrm{II}|^2\big) - \nu(\mu) \\
L_\Sigma &= -\Delta_\Sigma + \frac{1}{2}\big(R_\Sigma - R_+^\mu\big),
\end{aligned}
$$

where

$$
R_+^\mu = R_g + \frac{n}{n-1}\mu^2 - 2|\mathrm{d}\mu|_g. \tag{A.1}
$$

Fact A.3. *Suppose that Ω is a smooth μ-bubble. We have*

(*a*) $H_\Sigma = \mu|_\Sigma$;

(*b*) $L_\Sigma \geq J_\Sigma \geq 0$.

The semi-positivity of L_Σ has several applications, and we shall list a few. To start with, let $u > 0$ be an eigenfunction associated to the first eigenvalue $\lambda \geq 0$ of L_Σ. Consider the warped-product metric $\hat{h} := g_\Sigma + u^2\mathrm{d}\theta^2$ defined on $\hat{\Sigma} := \Sigma \times \mathbb{S}^1$, where $\theta \in \mathbb{S}^1$.

Fact A.4. *Suppose that Ω is a smooth μ-bubble. The scalar curvature of $(\hat{\Sigma}, \hat{h})$ is*

$$
R_{\hat{h}} = R_\Sigma - 2u^{-1}\Delta_\Sigma u = R_+^\mu + 2\lambda.
$$

In particular, if $R_+^\mu > 0$ on Σ, then $\Sigma \times \mathbb{S}^1$ admits a PSC metric.

Alternatively, one can compare L_Σ with the conformal Laplacian on Σ and obtain the following.

Fact A.5. *For $n \geq 3$, suppose that Ω is a smooth μ-bubble on which $R_+^\mu > 0$. Then Σ admits a PSC metric.*

With additional topological assumptions on M, Fact A.5 can be used to prove width estimates for (M, g). To be precise, we start by recalling the following notion (cf. [9, Property A]).

Definition A.6. Given a (topological) band $(M^n; \partial_-, \partial_+)$, we say that a (closed) hypersurface $\mathcal{S}$ in M is *separating*, if all paths connecting ∂_- and ∂_+ must intersect $\mathcal{S}$. A band is said to satisfy the **NSep**$^+$*property* if no separating hypersurface admits a PSC metric.

Remark A.7. If $\mathcal{S} \subset M^n$ is a separating hypersurface, then there exists a minimal list of connected components S_i $(i = 1, \ldots, k)$ of $\mathcal{S}$ such that their union $\mathcal{S}'$ remains separating. For details, see [9, Lemma 2.2]. Using intersection theory, one can show that $[\mathcal{S}'] \neq 0 \in H_{n-1}(M; \mathbb{Z})$. Moreover, with suitable orientation, $\mathcal{S}'$ is homologous to ∂_+ in M.

Lemma A.8. *Let $\big(M^n, g; \partial_-, \partial_+\big)$ be a Riemannian band, and let $\iota\colon \partial_+ \hookrightarrow M$ be the inclusion map. Suppose that $\partial_+ \in \mathcal{C}_{\deg}$ (see Definition 1.9) and that there exists a continuous map $f\colon M \to \partial_+$ such that $f \circ \iota$ is homotopic to id_{∂_+}. Then (M, g) satisfies the* **NSep**$^+$*property.*

Proof. Suppose that $\mathcal{S}$ is a separating hypersurface in M, and let $\mathcal{S}'$ be as in Remark A.7; in particular, $\mathcal{S}'$ is homologous to ∂_+ in M. Now since $f \circ \iota$ is homotopic to id_{∂_+}, it is easy to see that the restriction $f|_{\mathcal{S}'}\colon \mathcal{S}' \to \partial M$ has degree 1. Since $\partial_+ \in \mathcal{C}_{\deg}$, $\mathcal{S}'$ admits no PSC metric. ■

The next lemma is a variant of Gromov's band-width estimate [22, Section 5.3].

Lemma A.9. *For $3 \le n \le 7$, let $\big(M^n, g; \partial_-, \partial_+\big)$ be a Riemannian band that satisfies the* **NSep**$^+$*property, and let $\partial_\star \subset \partial_-$ be a compact subset without boundary. Suppose that*

(a) $R_g \ge -n(n-1)$*;*

(b) $H_{\partial_- \setminus \partial_\star} \ge -(n-1)$*;*

(c) $H_{\partial_+} \ge (n-1)(1+\delta)$ *for some constant $\delta > 0$.*

Then there exists a constant $T_\delta > 0$, depending only on δ, such that

$$\mathrm{dist}_g(\partial_\star, \partial_+) \le T_\delta.$$

Proof. Set $\epsilon = \delta/3$, and define $C_\delta, T_\delta > 0$ by

$$\coth(C_\delta/2) = \frac{1+\delta/2}{1+\epsilon} \qquad \text{and} \qquad T_\delta = \frac{C_\delta}{n(1+\epsilon)}.$$

For the sake of deriving a contradiction, suppose that $\mathrm{dist}_g(\partial_\star, \partial_+) > T_\delta$. By the proof of [39, Lemma 4.1], there exists a smooth, proper function $\rho\colon M \to [-T_\delta, 0]$ such that

$$\rho^{-1}(-T_\delta) = \partial_\star, \qquad \rho^{-1}(0) = \partial_+, \qquad \text{and} \qquad |\mathrm{d}\rho|_g < 1. \tag{A.2}$$

Now consider the function

$$h(t) = (n-1)(1+\epsilon)\coth\left(\frac{n(1+\epsilon)t + C_\delta}{2}\right), \qquad t \in (-T_\delta, 0].$$

By construction, h is decreasing, strictly greater than $n-1$, and satisfies

$$h(0) < H_{\partial_+}, \qquad \lim_{t \to -T_\delta} h(t) = \infty, \qquad \frac{n}{n-1}h(t)^2 + 2h'(t) \equiv n(n-1)(1+\epsilon)^2. \tag{A.3}$$

Combining (A.2), (A.3), and the assumptions (a), (b), (c), one can easily check that the function $\mu := h \circ \rho$, defined on $M \setminus \partial_\star$, satisfies both the barrier condition and the inequality $R_+^\mu > 0$. By Facts A.2 and A.5, there exists a separating hypersurface Σ in $(M; \partial_-, \partial_+)$ that admits a PSC metric. This contradicts the **NSep**$^+$hypothesis. ■

Remark A.10. We mention two variants of Lemma A.9, both of which can be obtained by slightly modifying the proof above. (A) For $3 \le n \le 7$, no Riemannian band can simultaneously satisfy the **NSep**$^+$property and the conditions $R_g \ge -n(n-1)$, $H_{\partial_-} \ge -(n-1)$ and $H_{\partial_+} > n-1$. (B) For $3 \le n \le 7$, let (M^n, g) be a complete, non-compact Riemannian manifold with compact boundary ∂M. Suppose that M satisfies the **NSep**$^+$property (see below); then (M, g) cannot satisfy the conditions $R_g \ge -n(n-1)$ and $H_{\partial M} > n-1$ simultaneously.

The concept of separating hypersurfaces can also be defined for complete, non-compact Riemannian manifolds (M, g) with compact boundary—just require that $\mathcal{S}$ intersects with all paths connecting ∂M and infinity. The $\mathbf{NSep}^{+}$property can be extended to such manifolds.

Lemma A.11. *Let $\left(M^4, g\right)$ be a complete, non-compact Riemannian 4-manifold with compact (nonempty) boundary ∂M. Suppose that the homotopy groups $\pi_2(M) = \pi_3(M) = 0$. Then (M, g) satisfies the $\mathbf{NSep}^{+}$property.*

Proof. Suppose that $\mathcal{S} \subset M$ is a (closed) separating hypersurface that admits a PSC metric, and let $\mathcal{S}' \subset \mathcal{S}$ be as indicated in Remark A.7. In particular, $\mathcal{S}'$ admits a PSC metric, and $[\mathcal{S}'] \neq 0 \in H_3(M, \mathbb{Z})$. Since $\pi_2(M)$ is trivial, the topological classification of closed 3-manifolds admitting a PSC metric implies that $\mathcal{S}'$ is homologous to a spherical class in $H_3(M, \mathbb{Z})$ (see [35, p. 112]). Since $\pi_3(M)$ is also trivial, this violates Lemma B.1 below. ∎

B Topological lemmas

Lemma B.1. *Let M be a non-compact 4-manifold satisfying $\pi_3(M) = 0$. Then $H_3(M, \mathbb{Z})$ contains no nontrivial spherical class (i.e., classes of the form $\left[\mathbb{S}^3/\Gamma\right]$).*

Proof. Let $[\beta]$ denote the fundamental class of $\mathbb{S}^3/\Gamma$ where Γ is a discrete subgroup of $O(4)$. Let $i\colon \mathbb{S}^3/\Gamma \to M$ be a continuous map. The goal is to prove that $i_*[\beta] = 0 \in H_3(M, \mathbb{Z})$. Now let $[\alpha]$ be the fundamental class of $\mathbb{S}^3$. The composition $\mathbb{S}^3 \xrightarrow{\pi} \mathbb{S}^3/\Gamma \xrightarrow{i} M$ induces a map at the level of $H_3(\cdot, \mathbb{Z})$, such that $[\alpha] \xrightarrow{\pi_*} d[\beta] \xrightarrow{i_*} di_*[\beta]$ where d is the degree of π. Since $\pi_3(M) = 0$, Hurewicz homomorphism implies that

$$di_*[\beta] = (i \circ \pi)_*[\alpha] = h([i \circ \pi]) = 0 \in H_3(M, \mathbb{Z}),$$

where $h\colon \pi_3(M) \to H_3(M, \mathbb{Z})$ is the Hurewicz map. Thus, in order to show that $i_*[\beta] = 0$, it suffices to show that $H_3(M, \mathbb{Z})$ is torsion free, and this follows from M being non-compact (see [7, Corollary 7.12]). ∎

Lemma B.2. *For $1 \leq k \leq n-2$, let $M_1 = \left(\mathbb{R} \times \mathbb{T}^{n-k-1} - \mathbf{B}\right) \times \mathbb{T}^k$, where $\mathbf{B}$ is an embedded $(n-k)$-ball in $\mathbb{R} \times \mathbb{T}^{n-k-1}$. Let M_2 be a smooth, possibly non-compact, manifold with boundary ∂M_2. Suppose that $\Phi\colon \partial M_1 \to \partial M_2$ is a diffeomorphism, and let $M := M_1 \sqcup_\Phi M_2$ be the manifold obtained by identifying $\partial M_1, \partial M_2$ via Φ. Let $t \in \mathbb{R}$ be such that $\{t\} \times \mathbb{T}^{n-k-1}$ is disjoint from $\mathbf{B}$. Then the hypersurface $\Sigma = \{t\} \times \mathbb{T}^{n-1}$ is incompressible in M if and only if the $\mathbb{T}^k$-factor[10] of ∂M_2 is incompressible in M.*

Proof. In M, the $\mathbb{T}^k$-factor of Σ is homotopic to that of ∂M_1 and hence to that of ∂M_2. Thus, $(\Rightarrow)$ is clear.

For $(\Leftarrow)$, we prove its contrapositive. Suppose that $L \subset \Sigma$ is a non-contractible loop that is contractible in M. Write

$$[L] = (m_i\alpha_i, n_j\beta_j) \in \pi_1\left(\mathbb{T}^{n-k-1}\right) \times \pi_1\left(\mathbb{T}^k\right) \cong \pi_1(\Sigma),$$

where α_i generates the fundamental group of the i-th $\mathbb{S}^1$-factor in $\mathbb{T}^{n-k-1}$ and $m_i \in \mathbb{Z}$, similarly for β_j and n_j. Let us write $\hat{\alpha}_i$, $\hat{\beta}_j$ for the corresponding elements in the homology class $H_1(\Sigma; \mathbb{Z})$.

It will be convenient to view the $\mathbb{R}$-factor in $\mathbb{R} \times \mathbb{T}^{n-k-1}$ as $\mathbb{S}^1$ minus a point, and to view M as a subset of $\hat{M} := \hat{M}_1 \sqcup_\Phi M_2$, where $\hat{M}_1 := \left(\mathbb{S}^1 \times \mathbb{T}^{n-k-1} - \mathbf{B}\right) \times \mathbb{T}^k$.

[10]Note that ∂M_2 has the product structure $\mathbb{S}^{n-k-1} \times \mathbb{T}^k$ induced by Φ.

Let $\iota\colon \Sigma \hookrightarrow \hat{M}$ be the inclusion map. For $1 \le i \le n-k-1$, let θ_i be the coordinate on the i-th $\mathbb{S}^1$-factor of $\mathbb{T}^{n-k-1}$. By construction, there exists $t_i \in \mathbb{S}^1$ such that $\theta_i = t_i$ defines a hypersurface S_i in $\hat{M}$ that is 'dual' to $\iota_*\hat{\alpha}_i$, in the sense that the intersection products

$$[S_i] \cdot \iota_*\hat{\alpha}_i = 1 \qquad \text{and} \qquad [S_i] \cdot \iota_*\hat{\alpha}_{i'} = [S_i] \cdot \iota_*\hat{\beta}_j = 0, \qquad i' \neq i.$$

Since L is contractible in $M \subset \hat{M}$, we have

$$\sum_i m_i \iota_*\hat{\alpha}_i + \sum_j n_j \iota_*\hat{\beta}_j = 0 \in H_1(\hat{M};\mathbb{Z});$$

by taking intersection products with $[S_i]$, we see that $m_i = 0$ for all $i = 1,\dots,n-k-1$, so L is homotopic to a loop in the $\mathbb{T}^k$-factor of Σ. Thus, the $\mathbb{T}^k$-factor of Σ is not incompressible in M. By homotopy, the same is true for the $\mathbb{T}^k$-factor of ∂M_2. This completes the proof. ■

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