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A Laurent Phenomenon for the Cayley Plane

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Abstract. We describe a Laurent phenomenon for the Cayley plane, which is the homogeneous variety associated to the cominuscule representation of E_6. The corresponding Laurent phenomenon algebra has finite type and appears in a natural sequence of LPAs indexed by the E_n Dynkin diagrams for $n \leq 6$. We conjecture the existence of a further finite type LPA, associated to the Freudenthal variety of type E_7.

Key words: Laurent phenomenon; cluster structure; mirror symmetry; Cayley plane

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1 Introduction

1.1 Background

In this paper we consider the (complex) Cayley plane, which is the cominuscule homogeneous space

$$\mathbb{OP}^2 = E_6/P_6.$$

The Cayley plane is a 16-dimensional algebraic variety, with a projective 'octonic spinor embedding' $\mathbb{OP}^2 \subset \mathbb{P}^{26}$ of codimension 10. We let $\mathcal{A}_6 := \mathbb{C}\big[\mathbb{OP}^2\big]$ denote the homogeneous coordinate ring of the Cayley plane with respect to this embedding.

Despite the name, the Cayley plane was first discovered by Ruth Moufang in 1933 [9]. It was named after Cayley since it can be realised as the projective plane over the octonions.

Laurent phenomenon algebras. *Laurent phenomenon algebras* (LPAs) were introduced by Lam and Pylyavskyy [8] as a generalisation of cluster algebras. As the name suggests, they possess the same Laurent phenomenon property of a cluster algebra but with the flexibility of having much more general exchange polynomials (rather than the more restrictive binomial exchange relations of a cluster algebra).

Cluster algebra structures on homogeneous spaces. A cluster structure (or more generally, a LPA structure) on the coordinate ring of an algebraic variety $\mathcal{V}$ need not be uniquely determined, since it depends on a choice of anticanonical divisor $\mathcal{D} \subset \mathcal{V}$, on which a set of *frozen coefficients* vanish. The simplest examples of cluster structures are *finite type cluster structures*, in which the cluster algebra has only finitely many seeds, although these are rather uncommon since cluster algebras are typically not of finite type.

The existence of cluster algebra structures on the coordinate rings of homogeneous spaces have been an active area of study for some time, beginning with the case Grassmannians [10] and followed by the case of partial flag varieties by Geiß, Leclerc and Schröer [5]. Their construction provides cluster algebra structures for both the Cayley plane and the Freudenthal variety, albeit not ones of finite type.

Given the non-uniqueness of LPA structures in general, it is quite possible (as we show for the Cayley plane in this paper) that a homogeneous variety may admit a finite type LPA structure, even when it only has cluster algebra structures that are not of finite type.

Mirror symmetry for cluster varieties. Cluster varieties have played an important role in the study of mirror symmetry, since they have a well-known mirror construction (the duality between so-called $\mathcal{A}$- and $\mathcal{X}$-cluster varieties). In particular, homogeneous spaces have become a fruitful set of examples for studying various aspects of mirror symmetry, due to the existence of the aforementioned cluster structures. In particular, Spacek and Wang [12] recently studied mirrors for both the Cayley plane and the Freudenthal variety using the cluster algebra structures of [5].

Mirror symmetry for LPAs. An LPA is the coordinate ring of a maximal log Calabi–Yau variety and, as such, it is still expected to have a mirror (cf. [4, Conjecture 2.5]), although an explicit construction for the mirror of an LPA is not yet known. A general approach towards constructing a mirror follows the Gross–Siebert program, which involves building a scattering diagram and then counting broken lines in this scattering diagram to compute coefficients in the equations that define the mirror algebra. In general there may be infinitely many broken lines contributing to these counts, and the mirror can only be defined formally due to convergence issues. However, LPAs of finite type are of particular interest because the corresponding scattering diagram will have only finitely many walls and chambers, and hence only finitely many broken lines contributing to the coefficients of a given equation.

1.2 Summary of the paper

Main result. Our main result, Theorem 5.4, is the description of a LPA structure of finite type on the ring $\mathcal{A}_6$. This finite type LPA is of rank 5, and it has 264 seeds and 32 cluster variables.

The key to constructing this LPA structure lies in deriving an initial seed (see Proposition 5.3) which is compatible with the symmetry of the action of a Coxeter rotation on $\mathcal{A}_6$. Once we have discovered this seed, the proof of Theorem 5.4 follows by plugging our seed into the Sage package LPASeed (written by the first author [2]) and verifying the result.

From the output of our code we also verify that our finite type LPA has a positivity phenomenon in Corollary 5.5.

Contents. The material in this paper is divided into the following sections.

§2 A recap of Lam and Pylyavskyy's Laurent phenomenon algebras.

§3 The description of a sequence of homogeneous varieties $\mathcal{V}_n$ that we consider in this paper, corresponding to a sequence of type E_n Dynkin diagrams.

§4 A recap of the finite type LPA structure on the homogeneous coordinate ring of $\mathcal{V}_n$ for two simpler cases in our sequence. Namely,

> E_4 case: $\mathcal{V}_4 = \mathrm{Gr}(2,5)$ has a finite type cluster algebra structure of type A_2, and
> E_5 case: $\mathcal{V}_5 = \mathrm{OGr}(5,10)$ has a finite type LPA structure (considered in [4]).

§5 Our main E_6 case. By generalising the examples of Section 5, we construct a finite type LPA structure on the homogeneous coordinate ring of the Cayley plane $\mathcal{V}_6 = \mathbb{OP}^2$.

§6 A conjecture and some limited progress on the E_7 case.

§A Examples of the Sage code.

2 Laurent phenomenon algebras

2.1 The initial seed

In the most general setting we consider a coefficient ring A, which we take to be an integral domain, and $\mathcal{F}$ a degree n transcendental field extension of $\mathrm{Frac}(A)$.

Definition 2.1. A *seed* (of rank n) consists of a pair $(\boldsymbol{x}, \boldsymbol{F})$, where

(1) $\boldsymbol{x} = \{x_1, \dots, x_n\}$, the *cluster*, is a transcendence basis for $\mathcal{F}$. The variables $x_1, \dots, x_n$ are called *cluster variables*, and

(2) $\boldsymbol{F} = \{F_1, \dots, F_n\} \subset A[x_1, \dots, x_n]$ is a collection of n polynomials, called *exchange polynomials*, with the following properties:

(LP1) The exchange polynomial F_i is considered to correspond to the cluster variable x_i, and $F_i \in A[x_1, \dots, \widehat{x_i}, \dots, x_n]$ does not depend on x_i, and

(LP2) each F_i is irreducible and not divisible by any of the cluster variables x_j.

Definition 2.2. Given a seed $S = (\boldsymbol{x}, \boldsymbol{F})$, the *exchange Laurent polynomial* associated to F_i is the Laurent polynomial $\hat{F}_i$ defined according to the following rule: we set $\hat{F}_i = MF_i$ where $M = x_1^{a_1} \cdots \widehat{x_i} \cdots x_n^{a_n}$ for some $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in \mathbb{Z}_{\leq 0}$, and the power a_j of x_j in the denominator of $\hat{F}_i$ is $-m$, where m is the maximal power such that $F_i|_{x_j \leftarrow F_j/x_j}$ is divisible by F_j^m.

The exchange Laurent polynomials are uniquely determined from the exchange polynomials, and vice versa. Indeed, to obtain the exchange Laurent monomials, one sets the power of x_j in the Laurent monomial denominator M equal to the largest power of F_j that divides F_i upon the substitution $x_j \leftarrow F_j/x_j$. To get the exchange polynomials back, one multiplies $\hat{F}_i$ by the unique monomial M (up to a unit) such that $M\hat{F}_i$ is an irreducible polynomial. One important difference to the classical cluster algebra setup is that we use the exchange Laurent polynomials to determine new cluster variables, as opposed to just the exchange polynomials.

2.2 Mutation procedure

To mutate a seed S at an index i, one proceeds with the following (non-deterministic, cf. Remark 2.4) process:

(1) The cluster variables of the mutated seed $\mu_i(S) = (\mu_i(\boldsymbol{x}), \mu_i(\boldsymbol{F}))$ are the same, except we replace x_i according to the exchange relation prescribed by its exchange Laurent polynomial. That is, $\mu_i(\boldsymbol{x}) = \{x_1, \dots, x_i', \dots, x_n\}$, where $x_i x_i' = \hat{F}_i$. Note that we use the exchange *Laurent* polynomial in this relation.

(2) The exchange polynomials $\mu_i(\boldsymbol{F}) = \{F_1', \dots, F_n'\}$ are updated as follows.

The exchange polynomial F_i' corresponding to x_i' remains the same, that is $F_i' := F_i$.

The exchange polynomials F_j' for $j \neq i$ are defined through the following procedure. If F_j does not depend on x_i, then we set F_j' to be any polynomial satisfying $F_j' = u_j F_j$, where u_j is a unit in A. If F_j does depend on x_i, then we perform the following:

- *Substitution step.* We set

$$(F_j')^* = F_j|_{x_i \leftarrow (\hat{F}_i|_{x_j \leftarrow 0})/x_i'}.$$

The substitution is well defined since, by [8, Lemma 2.7], we see that x_i cannot appear in the denominator of $\hat{F}_j$.

- *Cancellation step.* We divide out by any common factors that $(F_j')^*$ shares with $\hat{F}_i|_{x_j \leftarrow 0}$. This then defines F_j' up to a monomial multiplier.
- *Normalisation step.* We multiply through by a monomial in $x_1, \ldots, x_i', \ldots, x_n$ to make F_j' satisfy (LP1) and (LP2) as an exchange polynomial in S_i. Such a monomial will be uniquely defined only up to a unit. Thus F_j' is only defined up to a unit multiplier u_j.

Example 2.3. Consider the initial seed with exchange variables $\{x_1, x_2\}$ and corresponding exchange polynomials $\{F_1, F_2\} = \{1 + x_2, 1 + x_1\}$. One checks that the exchange Laurent polynomials have trivial denominators, so that $\{F_1, F_2\} = \{\hat{F}_1, \hat{F}_2\}$. Mutating at x_1 replaces x_1 with

$$x_3 := \frac{1 + x_2}{x_1}$$

and leaves x_2 invariant. The exchange polynomial at x_2 is changed by this mutation. The substitution step yields

$$(F_2')^* = 1 + \frac{1}{x_3}.$$

There are no common factors to cancel out. Finally, multiplying by x_3 gives an exchange polynomial that satisfies (LP1) and (LP2), so we have $F_2' = 1 + x_3$. The mutation is therefore

$$(\{x_1, x_2\}, \{1 + x_2, 1 + x_1\}) \mapsto \left(\left\{ x_3 = \frac{1 + x_2}{x_1}, x_2 \right\}, \{1 + x_2, 1 + x_3\} \right).$$

Remark 2.4. The reason that the process is non-deterministic, is the choice of unit multipliers u_i for the mutated exchange polynomials F_i'. If u_i is the unit multiplier for F_i chosen in the mutation $\mu_i \colon S \to \mu_i(S)$, we assume that u_i^{-1} is chosen for the mutation $\mu_i \colon \mu_i(S) \to \mu_i(\mu_i(S))$. Thus mutations are involutive, meaning that for any index i, the seed $\mu_i(\mu_i(S))$ is equal to S.

In practice we assume that all unit multipliers are equal to 1, and identify two seeds whenever they are equivalent, as we now define.

Definition 2.5. Two seeds $S = (\boldsymbol{x}, \boldsymbol{F})$ and $S' = (\boldsymbol{x}', \boldsymbol{F}')$ of rank n are said to be *equivalent* if for each $1 \leq i \leq n$, there exists units $\mu_i, \tau_i \in \mathcal{F}$ such that $x_i = \mu_i x_i'$ and $F_i = \tau_i F_i'$.

2.3 Obtaining the LPA

In the literature, an LPA is defined as a pair $(\mathcal{A}, \{S_i\}_{i \in I})$, where $\mathcal{A}$ is a subring of the ambient field $\mathcal{F}$, and $\{S_i\}_{i \in I}$ is a distinguished collection of seeds with cluster variables in $\mathcal{F}$. The cluster variables generate $\mathcal{A}$ over A, and any two seeds in the collection are mutation-equivalent. For our purposes, it is convenient to present the following definition:

Definition 2.6. Given a seed $S = (\boldsymbol{x}, \boldsymbol{F})$, we define the LPA $\mathcal{A}(S)$ generated by S to be the A-algebra given by

$$\mathcal{A} = \bigcap_{S_i} A\left[x_{i1}^{\pm 1}, \ldots, x_{in}^{\pm 1}\right],$$

where the index S_i runs over all seeds obtainable through mutation of S, and $\{x_{i1}, \ldots, x_{in}\}$ is the cluster for S_i.

One can view this definition as the analogue of the *upper cluster algebra*. However, all of the examples we will consider have finitely many cluster variables, and in this case there is no difference between these two definitions for the LPA: both are equal to the ring generated by all of the cluster variables.

Example 2.7. Continuing with the mutated seed obtained in Example 2.3, one can compute the mutation at x_2 as

$$(\{x_3, x_2\}, \{1 + x_2, 1 + x_3\}) \mapsto (\{x_3, x_4\}, \{1 + x_4, 1 + x_3\}),$$

where

$$x_4 = \frac{1 + x_3}{x_2} = \frac{1 + \frac{1+x_2}{x_1}}{x_2} = \frac{1 + x_1 + x_2}{x_1 x_2}.$$

Similarly, one obtains by mutation at x_3 the seed with cluster variables $\{x_5, x_4\}$ with

$$x_5 = \frac{1 + x_4}{x_3} = \frac{x_1(1 + x_2)(1 + x_1)}{(1 + x_2)x_1 x_2} = \frac{1 + x_1}{x_2}$$

and one can compute that, continuing in this sequence, $x_6 = x_1$, and mutating at x_5 returns the initial seed. Thus the corresponding LPA $\mathcal{A}$ is generated by five cluster variables

$$x_1, \qquad x_2, \qquad x_3 = \frac{1 + x_2}{x_1}, \qquad x_4 = \frac{1 + x_1 + x_2}{x_1 x_2}, \qquad x_5 = \frac{1 + x_1}{x_2},$$

and is isomorphic to the ring

$$\mathcal{A} \cong \mathbb{C}[x_1, x_2, x_3, x_4, x_5]/I,$$

where $I = (x_5 x_2 = 1 + x_1, x_1 x_3 = 1 + x_2, \dots, x_4 x_1 = 1 + x_5)$ is the ideal of relations holding between $x_1, \dots, x_5$.

Geometrical interpretation. From a more geometrical point of view we consider a LPA $\mathcal{A}$ as the ring of regular functions on an affine algebraic variety $U = \operatorname{Spec} \mathcal{A}$. Each seed S corresponds to the inclusion of a torus chart

$$S = (\boldsymbol{x}, \boldsymbol{F}), \qquad \mathbb{T}_S := \operatorname{Spec} A\big[x_1^{\pm 1}, \dots, x_n^{\pm 1}\big] \hookrightarrow U,$$

by the Laurent phenomenon $\mathcal{A} \subseteq A\big[x_1^{\pm 1}, \dots, x_n^{\pm 1}\big]$. Each mutation then corresponds to a birational map, e.g.,

$$\mu_1\colon\ \mathbb{T}_S \dashrightarrow \mathbb{T}_{\mu_1 S}, \qquad \mu_1^*(x_1', x_2, \dots, x_n) = \big(x_1^{-1}\hat{F}_1, x_2, \dots, x_n\big),$$

and these torus charts are glued together by identifying points according to these mutations. The LPA (as we have defined it) is then the ring of regular functions on the union of all these seed tori, and $U = \operatorname{Spec} \mathcal{A}$ is the 'affinisation' of $\bigcup_S \mathbb{T}_S$. This geometrical point of view is described in the context of cluster algebras in [6, Section 3], and more generally in [4, Section 2.2] (see [4, Remark 2.7] in particular).

2.4 Finite type LPAs

As with the traditional case of cluster algebras, a typical LPA will have infinitely many seeds.

Definition 2.8. If it is only possible to obtain finitely many seeds via mutations of S, then we say that $\mathcal{A}(S)$ is *finite type*, or even that the seed S is finite type.

One can check that this definition of finite type matches the definition given in the literature (as the finiteness of seeds in the normalisation of $\big(\mathcal{A}, \{S_i\}_{i \in I}\big)$).

Cluster algebras of finite type have a particularly neat classification, and are in one-to-one correspondence with Dynkin diagrams of finite type. Lam and Pylyavskyy [8, Theorem 6.6] classify finite type LPAs of rank 2 and their classification is essentially equivalent to that of cluster algebras of rank 2. However there is no such classification in general, starting with case of LPAs of rank 3. Indeed there are many more finite type LPAs than there are cluster algebras, with Lam and Pylyavskyy showing that the number of finite type LPAs grows exponentially with respect to the rank.

Definition 2.9. Two seeds S, S' are said to be *similar* if there exists a seed S'' equivalent to S', such that S'' can be obtained from S by renaming the cluster variables and substituting this renaming into the exchange polynomials. If an LPA has finitely many similarity classes of seeds, we say it is of *finite mutation type*.

Finite type implies finite mutation type, but not conversely, as shown by Lam and Pylyavskyy in their two-layer brick wall example [8, Section 7.2]. One should contrast this situation to the cluster algebra setting in which exceptional quivers arise of infinite type, but finite mutation type.

Borrowing terminology from cluster algebras, we may construct the *exchange graph* of a given seed S. The vertices of this graph are given by all seeds obtainable by mutation of S, and any two vertices are connected by an edge if one may be obtained from the other (up to similarity) by a single mutation.

2.5 Description of Sage code

The first author has implemented the algorithms for computing mutations in Sage. Examples of using the Sage code are available in Appendix A, and a Sage cell for interactive use of the code in a web browser is also available [2].

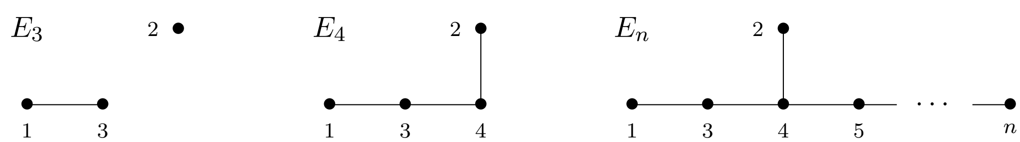
Remarks on the code. As in Remark 2.4, when we compute a mutation, in the normalisation step, we choose to keep the coefficient of the multiplying monomial equal to 1. Since mutations are defined up to units, this does not change the resulting structure of the exchange graph, neither does it affect the LPA that the cluster variables generate. It does however mean that our statement of the cluster variables is only well-defined up to a unit multiplier.

In our computations and the Sage code, we identify two seeds S, S' when they are equivalent.

3 A type E sequence

The main result of this paper concerns the existence of a finite type LPA, which we view as the $n = 6$ case in a sequence indexed by type E_n Dynkin diagrams.

We consider the Dynkin diagram E_n for $3 \leq n \leq 8$ (where $E_3 = A_1A_2$, $E_4 = A_4$ and $E_5 = D_5$) with the nodes labelled according to the convention adopted by Bourbaki:



We can associate a number of mathematical objects to this sequence:

(1) a smooth del Pezzo surface dP_n of degree $9 - n$ [3, Section 8], obtained by blowing up n general points in $\mathbb{P}^2$,

(2) the homogeneous space $\mathcal{V}_n := E_n/P_n$, where P_n is the parabolic subgroup associated to the nth node of the Dynkin diagram,

(3) the n-dimensional semiregular Gosset polytope Ξ_n with Coxeter symbol $(n-4)_{21}$ [1, Section 11.8].

3.1 Numerical invariants

We collect some numerical invariants associated to this sequence in Table 1.

According to the three different points of view, for $3 \leq n \leq 7$ these numbers can be interpreted in the following ways.

n	$\mathcal{V}_n$	$\dim \mathcal{V}_n$	Coxeter number h	$\gamma_{n,1}$	$\gamma_{n,2}$	$\gamma_{n,3}$	$\gamma_{n,2} + \gamma_{n,3}$
3	$\mathbb{P}^2 \times \mathbb{P}^1$	3	3	6	3	2	5
4	$\mathrm{Gr}(2, 5)$	6	5	10	5	5	10
5	$\mathrm{OGr}(5, 10)$	10	8	16	10	16	26
6	$\mathbb{OP}^2$	16	12	27	27	72	99
7	Freudenthal variety	27	18	56	126	576	702

Table 1. Numerical invariants associated to the E_n root systems for $3 \leq n \leq 7$.

(1) The del Pezzo surface dP_n contains $\gamma_{n,1}$ lines, $\gamma_{n,2}$ conic classes ξ (which correspond to conic fibrations $\xi\colon \mathrm{dP}_n \to \mathbb{P}^1$), and $\gamma_{n,3}$ cubic classes η (which correspond to contractions $\eta\colon \mathrm{dP}_n \to \mathbb{P}^2$).

(2) The homogeneous space $\mathcal{V}_n \subset \mathbb{P}^{\gamma_{n,1}-1}$ has an embedding into a projective space of dimension $\gamma_{n,1} - 1$ and is cut out by $\gamma_{n,2}$ quadratic equations. It is a Fano variety of the specified dimension and Fano index h (or in other words, $-K_{\mathcal{V}_n} = \mathcal{O}_{\mathcal{V}_n}(h)$ for the given embedding).

(3) The effective cone $\mathrm{Eff}(\mathrm{dP}_n)$ is the cone over $\Xi_n \subset \mathrm{NS}(\mathrm{dP}_n) \cong \mathbb{R}^{n+1}$, where the polytope Ξ_n is obtained as the convex hull of the classes of the lines in $\mathrm{NS}(\mathrm{dP}_n)$. It has $\gamma_{n,1}$ vertices and $\gamma_{n,2} + \gamma_{n,3}$ facets, of which $\gamma_{n,2}$ facets are $(n-1)$-dimensional orthoplexes and $\gamma_{n,3}$ facets are $(n-1)$-dimensional simplexes.

3.2 Coxeter projection of Ξ_n

In Figure 1, we draw the projection of the polytope Ξ_n onto the Coxeter plane for the cases $n = 4, 5, 6, 7$. The action of the Coxeter rotation of order h is plainly visible. In these four cases, the vertices of the polytopes are split into orbits of the following sizes:

$$(10) = 2 \times (5), \qquad (16) = 2 \times (8), \qquad (27) = 2 \times (12) + (3), \qquad (56) = 3 \times (18) + (2).$$

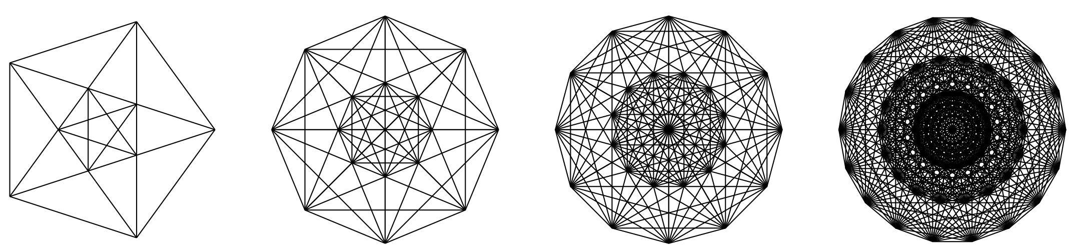


Figure 1. Coxeter projections of the polytopes Ξ_n for $n = 4, 5, 6, 7$.

3.3 A family of log Calabi–Yau varieties

Following interpretations (2) and (3) from Section 3.1, the $\gamma_{n,1}$ coordinates on the homogeneous space $\mathcal{V}_n \subset \mathbb{P}^{\gamma_{n,1}-1}$ can be placed in one-to-one correspondence with vertices of Ξ_n. Given that the Fano index of $\mathcal{V}_n$ is equal to the Coxeter number h, we can make a natural choice of anticanonical boundary divisor $\mathcal{D}_n \subset \mathcal{V}_n$ by taking $\mathcal{D}_n = \mathbb{V}(a_1) + \mathbb{V}(a_2) + \cdots + \mathbb{V}(a_h) \in |-K_{\mathcal{V}_n}|$, where $a_1, \ldots, a_h$ are the coordinates on $\mathcal{V}_n$ corresponding to the 'outside ring' of the Coxeter projection (see Figure 1).

Definition 3.1. We let $\mathcal{A}_n := \mathbb{C}[\mathcal{V}_n]$ be the homogeneous coordinate ring of $\mathcal{V}_n$, with respect to the given embedding, and consider the affine cone $C\mathcal{V}_n = \operatorname{Spec} \mathcal{A}_n$. We consider the fibration induced by the projection

$$\pi\colon\ C\mathcal{V}_n \to \mathbb{A}^h_{a_1,\dots,a_h}$$

and let $U_n := \pi^{-1}(\alpha_1, \dots, \alpha_h)$ denote a general fibre.

This log Calabi–Yau variety U_n is simply the affine variety obtained by substituting the value $\alpha_1, \dots, \alpha_h \in \mathbb{C}$ for each coordinate $a_1, \dots, a_h$ in $\mathcal{A}_n$ respectively. This projection π spreads out the components of $C\mathcal{D}_n$ over the coordinate hyperplanes of $\mathbb{A}^h$, giving a degenerating family of log Calabi–Yau varieties.

Remark 3.2. In the language of cluster algebras, the coordinates $a_1, \dots, a_h$ are *frozen variables* and we will interpret the remaining coordinates as *cluster variables* in a LPA. Moreover, a LPA structure for $\mathcal{A}_n$ over the base ring $A = \mathbb{C}[a_1, \dots, a_h]$ must have rank $r = \dim \mathcal{V}_n + 1 - h$, since the cluster of each seed will correspond to the inclusion of a torus chart

$$\operatorname{Spec} A\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big] = \mathbb{A}^h_{a_1,\dots,a_h} \times \big(\mathbb{C}^\times\big)^r_{x_1,\dots,x_r} \hookrightarrow C\mathcal{V}_n$$

that birationally cover $C\mathcal{V}_n$.

Remark 3.3. We do not extend our discussion to include the E_8 case, since the numerology of Table 1 breaks down for the homogeneous space $\mathcal{V}_8 = E_8/P_8$. In particular, the Fano index of $\mathcal{V}_8$ is 29 (as computed in [11]), rather than the Coxeter number $h = 30$, and thus we do not obtain an anticanonical divisor $\mathcal{D}_8 \subset \mathcal{V}_8$ in the same way.

4 The Laurent phenomenon for $\mathcal{V}_4$ and $\mathcal{V}_5$

We briefly summarise the finite type LPA structure on the homogeneous coordinate rings $\mathcal{A}_4 = \mathbb{C}[\mathcal{V}_4]$ and $\mathcal{A}_5 = \mathbb{C}[\mathcal{V}_5]$, corresponding to the E_4 and E_5 cases of our sequence.

4.1 The E_4 case

The LPA in this case is given by the famous example of the A_2 cluster algebra.

It is convenient to name the frozen variables $a_1, \dots, a_5$ and the non-frozen variables $x_1, \dots, x_5$ according to the labelling of Ξ_4 shown in Figure 2. Then the Grassmannian $\mathcal{V}_4 = \operatorname{Gr}(2, 5) \subset \mathbb{P}^9$ is cut out by five quadratic Plücker equations corresponding to the five octahedral faces of Ξ_4.

Structure of the equations. The five quadratic equations have a common structure in that they are comprised of three monomials, each one of which is a product of two opposite vertices in the corresponding octahedron. A coherent choice of signs for these equations is determined by the following *positivity rule*

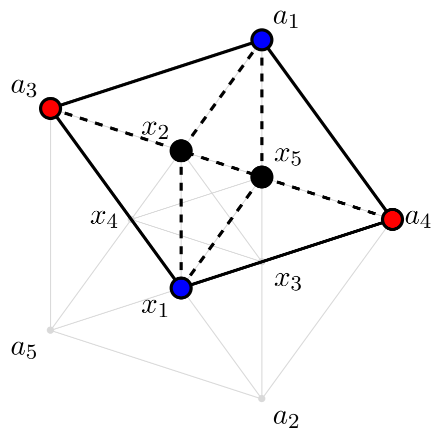
$$x_i x_{i+2} = \text{positive sum of the other monomials,} \tag{4.1}$$

where $x_i x_{i+2}$ is the monomial corresponding to the pair of 'internal' vertices of the projected octahedron, and the right hand side comprises of the monomials corresponding to all pairs of 'external' vertices.

Initial seed. We let $A = \mathbb{C}[a_1, \dots, a_5]$ be the ring generated by the frozen variables and, by Remark 3.2, a LPA structure on $\mathcal{A}_4$ will have rank 2.

As is well-known, each of the following Plücker coordinates

$$x_3 = \frac{a_2 x_2 + a_4 a_5}{x_1}, \qquad x_4 = \frac{a_5 a_1 x_1 + a_2 a_3 x_2 + a_3 a_4 a_5}{x_1 x_2}, \qquad x_5 = \frac{a_1 x_1 + a_3 a_4}{x_2}$$



$$x_5x_2 = a_1x_1 + a_3a_4$$
$$x_1x_3 = a_2x_2 + a_4a_5$$
$$x_2x_4 = a_3x_3 + a_5a_1$$
$$x_3x_5 = a_4x_4 + a_1a_2$$
$$x_4x_1 = a_5x_5 + a_2a_3$$

Figure 2. The equations of $\mathcal{V}_4$.

can be expressed as Laurent polynomials in $A\big[x_1^{\pm 1}, x_2^{\pm 1}\big]$. Moreover, an initial seed for the corresponding LPA structure on $\mathcal{A}_4$ is given by

$$S = \begin{cases} x_1, & a_2x_2 + a_4a_5, \\ x_2, & a_1x_1 + a_3a_4. \end{cases}$$

Mutating S at x_1 gives an almost identical seed (up to reordering) where the only difference is that all the indices of all the variables x_i and a_i have been shifted by $i \mapsto i+1 \mod 5$.

Exchange graph. The exchange graph of $\mathcal{A}_4$ is a pentagon, with vertices labelled by the five possible clusters $\{x_i, x_{i+1}\}$ for all $i \in \mathbb{Z}/5\mathbb{Z}$ and edges by the five possible mutations $\{x_{i-1}, x_i\} \to \{x_i, x_{i+1}\}$.

Positivity. The ring $\mathcal{A}_4$ also has a curious property known as the *positivity phenomenon*: every coefficient in the Laurent expansion of every cluster variable is positive, as well as every coefficient in the exchange polynomials of every seed.

4.2 The E_5 case

This is the LPA studied in [4]. In this case we label the variables $a_1, \ldots, a_8$ and $x_1, \ldots, x_8$ with $i \in \mathbb{Z}/8\mathbb{Z}$, as in Figure 3. The orthogonal Grassmannian $\mathcal{V}_5 = \mathrm{OGr}(5, 10) \subset \mathbb{P}^{15}$ is cut out by ten quadratic equations which correspond to the ten octahedral faces of Ξ_5. However this time the equations of $\mathcal{V}_5$ split into one orbit (a) of size eight and one orbit (b) of size two.

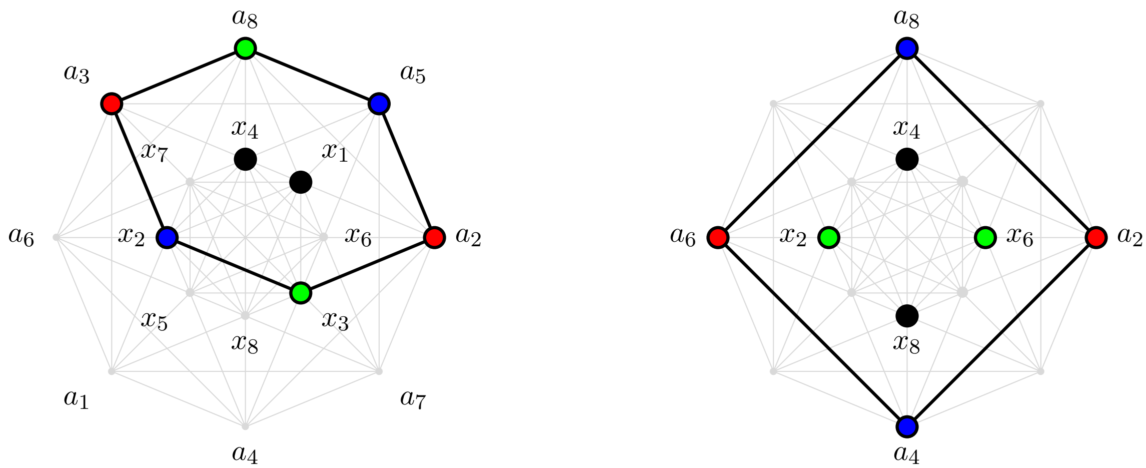
Structure of the equations. We can make a coherent choice of minus signs in the equations by asking that the eight (a) equations obey the analogous positivity rule to equation (4.1). Doing that uniquely determines the signs in the remaining two (b) equations.

Initial seed. By Remark 3.2, a LPA structure on $\mathcal{A}_5$ will have rank 3. Beginning with $\{x_1, x_2, x_3\}$ as a candidate for an initial cluster, we can check that each of the other x_i can be written as a Laurent polynomial in x_1, x_2, x_3. Thus we might hope that this initial cluster can be used to get an LPA structure on $\mathcal{V}_5$ which is analogous to the LPA structure on $\mathcal{V}_4$.

To promote this cluster into a seed, we have to specify what the exchange polynomial F_i corresponding to x_i should be for $i = 1, 2, 3$. The two equations $x_1x_4 = \cdots$ and $x_3x_8 = \cdots$ give easy and obvious candidates for the exchange polynomials F_1 and F_3:

$$F_1 = a_5x_2 + a_8x_3 + a_2a_3, \qquad F_3 = a_4x_1 + a_7x_2 + a_1a_2.$$

However, it is not immediately clear how to write down the exchange polynomial F_2.



(a) $x_1x_4 = a_5x_2 + a_8x_3 + a_2a_3$ (b) $x_2x_6 - a_2a_6 = x_4x_8 - a_4a_8$

Figure 3. The equations of $\mathcal{V}_5$.

Since we would like mutation in the LPA to be compatible with the Coxeter symmetry (as it was in the previous case), we can easily work out what F_2 should be by considering the mutation $\mu_1\colon \{x_1, x_2, x_3\} \mapsto \{x_2, x_3, x_4\}$, writing down the exchange polynomial $\mu_1 F_2 = a_6x_3 + a_1x_4 + a_3a_4$ that we expect to see for x_2 with respect to this seed, and then mutating back to get $F_2 = \mu_1^{-1}(\mu_1 F_2)$.

As seen in [4], this gives an initial seed

$$S = \begin{cases} x_1, & a_5x_2 + a_8x_3 + a_2a_3, \\ x_2, & a_6x_1x_3 + a_3a_4x_1 + a_8a_1x_3 + a_1a_2a_3, \\ x_3, & a_4x_1 + a_7x_2 + a_1a_2, \end{cases}$$

and, incredibly, the mutation of this LPA seed is compatible with the Dih_8-symmetry, in the sense that mutating S at x_1 returns an identical seed (up to reordering) with all indices shifted by $i \mapsto i+1 \mod 8$.

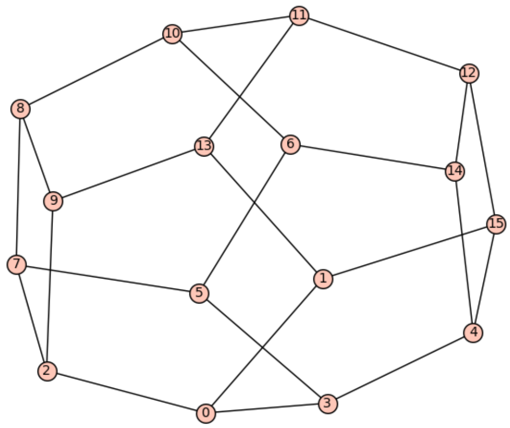
Moreover, mutating x_2 gives a quantity $q_1 = x_2^{-1}F_2$ which can be expressed as a quadratic $q_1 = x_1x_5 - a_1a_5 = x_3x_7 - a_3a_7$ in the other variables. If we also let $q_2 = x_2x_6 - a_2a_6 = x_4x_8 - a_4a_8$, then we get a finite type LPA structure with sixteen clusters:

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \dots, \{x_7, x_8, x_1\},$$
$$\{x_1, x_3, q_1\}, \{x_2, x_4, q_2\}, \{x_3, x_5, q_1\}, \dots, \{x_8, x_2, q_2\}.$$

Exchange graph. The exchange graph is the 1-skeleton of a 3-dimensional polytope with sixteen vertices, eight pentagonal faces (corresponding to $x_1, \dots, x_8$) and two square faces (corresponding to q_1, q_2). The exchange graph, shown in Figure 4, can be produced using the Sage code found in Appendix A.

Positivity. As with the previous case, $\mathcal{A}_5$ also has the positivity phenomenon. By an explicit calculation one can check that every Laurent expansion of a cluster variable, as well as every exchange polynomial in every seed, has positive coefficients.[1]

[1] The reader may be a little bit disturbed by the fact that the two equations of type (b) appear to have negative coefficients. However, after introducing the two new cluster variables q_1, q_2, they can be rewritten as exchange relations with positive coefficients, e.g., $x_1x_5 - a_1a_5 = x_3x_7 - a_3a_7 \implies x_1x_5 = q_1 + a_1a_5$.



$$f = \begin{pmatrix} 16 \\ 24 \\ 10 \\ 1 \end{pmatrix}$$

Figure 4. The exchange graph for the LPA in the $\mathcal{V}_5$ case with f-vector f.

5 The Laurent phenomenon for $\mathcal{V}_6$

To describe the projective embedding of the Cayley plane, we must first understand the equations of the projective embedding $\mathcal{V}_6 \subset \mathbb{P}^{26}$. We do this by thinking of the 27-dimensional representation of E_6 in terms of the 27 lines on a cubic surface.

5.1 The equations of the Cayley plane

We fix a birational model $\pi\colon \mathrm{dP}_6 \to \mathbb{P}^2$ for the smooth cubic surface $\mathrm{dP}_6 \subset \mathbb{P}^3$ obtained as the blowup of six points $p_1, \ldots, p_6 \in \mathbb{P}^2$, and we name the 27 lines in dP_6 according to the following conventions:

(1) let e_i be the line corresponding to the exceptional divisor over p_i,

(2) let ℓ_{ij} be the line corresponding to the line through p_i, p_j and

(3) let c_i be the line corresponding to the conic through the five points other than p_i.

The polytope Ξ_6. As mentioned in Section 3.1, the polytope $\Xi_6 \subset \mathrm{NS}(\mathrm{dP}_6)$ has 27 vertices, corresponding to the 27 lines of dP_6, and $99 = 27 + 72$ facets which are one of two types:

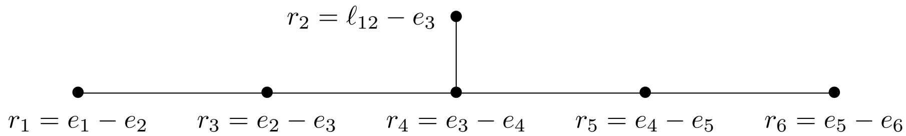
(1) There are 27 5-dimensional orthoplex facets, corresponding to the 27 extremal rays $.\xi \in \mathrm{Nef}(\mathrm{dP}_6)$ that define conic fibrations $\pi_\xi\colon \mathrm{dP}_6 \to \mathbb{P}^1$. Each face has ten vertices $\ell_1, \ldots, \ell_5$, $\ell'_1, \ldots, \ell'_5$ appearing in five 'opposite' pairs ℓ_i, ℓ'_i such that $\xi \sim \ell_i + \ell'_i$ for $i = 1, \ldots, 5$.

(2) There are 72 5-dimensional simplex facets, corresponding to the 72 extremal rays $\eta \in \mathrm{Nef}(\mathrm{dP}_6)$ that define contractions $\pi_\eta\colon \mathrm{dP}_6 \to \mathbb{P}^2$. Each face has six vertices $\ell_1, \ldots, \ell_6$; the six lines that are contracted by π_η.

Action of a Coxeter rotation. The 72 roots of the E_6 root system in $\mathrm{NS}(\mathrm{dP}_6)$ are given by all possible differences $\ell_i - \ell_j$, where ℓ_i, ℓ_j are a pair of non-intersecting lines in dP_6. Each root r_i specifies a reflection

$$\rho_i(r_j) = r_j + (r_i \cdot r_j) r_i,$$

and a Coxeter rotation is an element $\sigma = \rho_1\rho_2\rho_3\rho_4\rho_5\rho_6$ of order 12, obtained as a product of the reflections over a set of simple roots. For example, if σ is the Coxeter rotation obtained from the following choice of simple roots



then the action of σ on the set of the 27 lines has (ordered) orbits of length 12, 12, 3, as shown in the rows of Table 2. The 27 lines correspond to 27 spinor coordinates of $\mathcal{V}_6 \subset \mathbb{P}^{26}$, and we name these coordinates a_i, x_j, z_k for $i, j \in \mathbb{Z}/12\mathbb{Z}$ and $k \in \mathbb{Z}/3\mathbb{Z}$ according to which of these three orbits they belong to, as in Table 2. We have labelled the Coxeter projection of Ξ_6 with

i	1	2	3	4	5	6	7	8	9	10	11	12
a_i	e_3	ℓ_{34}	ℓ_{56}	e_6	c_2	e_4	c_6	ℓ_{16}	ℓ_{23}	c_3	e_5	c_1
x_i	ℓ_{26}	ℓ_{24}	ℓ_{46}	e_1	ℓ_{45}	e_2	ℓ_{35}	ℓ_{15}	ℓ_{13}	c_4	ℓ_{12}	c_5
z_i	ℓ_{14}	ℓ_{36}	ℓ_{25}									

Table 2. The frozen variables a_i and cluster variables x_i, z_i for $\mathcal{A}_6$.

these coordinate names, as shown in Figure 5 (i) (where the orbit $\{z_1, z_2, z_3\}$ of size three has been squished together in the centre).

The equations of the Cayley plane. The 27 octahedral faces of Ξ_6 also split up into two orbits of size 12 and one orbit of size three under the action of the rotation σ. These three types of octahedral face are shown in Figure 5 (ii), and a representative equation from each of these three Dih_{12}-orbits is given by

$$x_1x_6 = a_6x_3 + a_1z_2 + a_8x_4 + a_3a_{11}, \tag{a}$$
$$x_1x_5 = x_3z_3 - a_3x_2 - a_{10}x_4 + a_1a_{12}, \tag{b}$$
$$z_1z_2 = x_3x_9 + x_6x_{12} + a_2a_8 + a_5a_{11}. \tag{c}$$

The choice of $\pm$ sign in front of each monomial in each of these equations is uniquely determined by specifying that all of the equations in orbit (a) obey the analogous positivity rule to equation (4.1). Indeed, Macaulay2 agrees that these equations define an irreducible Gorenstein variety $\mathcal{V}_6 \subset \mathbb{P}^{26}$ which has codimension 10 and a Gorenstein resolution with Betti numbers $(1, 27, 78, 351, 650, 702, 650, 351, 78, 27, 1)$.

5.2 Finding an initial seed

We now describe how we found an initial seed for an LPA structure on $\mathcal{A}_6$. We do this in three steps:

(1) First find a candidate for an initial cluster (e.g., an appropriately sized subset of the spinor coordinates for which all other spinor coordinates can be expressed as Laurent polynomials).

(2) Work out corresponding exchange polynomials for this cluster.

(3) Check that the corresponding LPA is of finite type and contains all (non-frozen) spinor coordinates on $\mathcal{V}_6$ as cluster variables.

Rank of the LPA. We first note that we expect such a LPA structure to have rank 5, by Remark 3.2.

Finding an initial cluster. We describe three attempts we made in order to find an initial cluster.

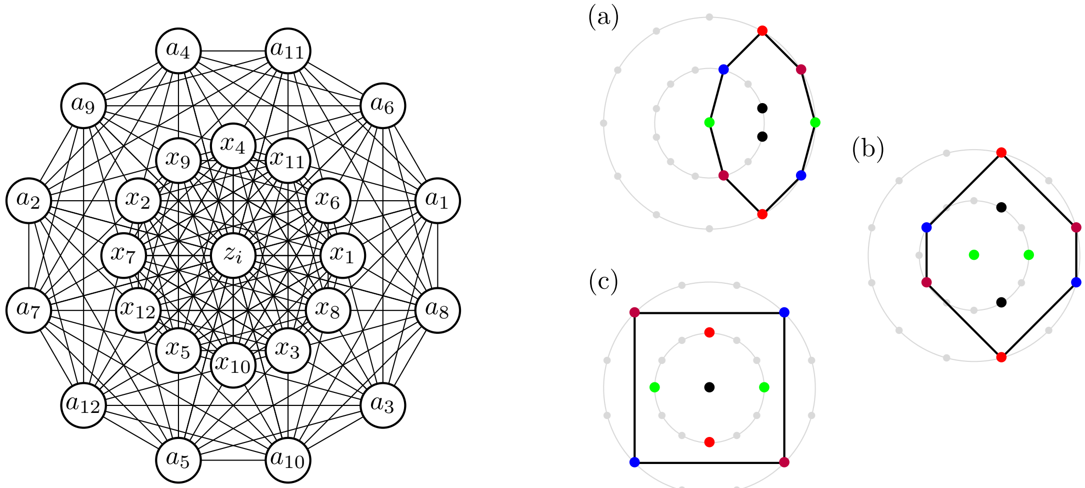


Figure 5. (i) A labelling of the vertices of Ξ_6, and (ii) the three types of octahedral face.

Attempt 1. As with the previous cases described in Section 4, our first thought was to take $\{x_1, x_2, x_3, x_4, x_5\}$ as an initial cluster. Unfortunately however, we have the equation

$$x_1x_5 = x_3z_3 - a_3x_2 - a_{10}x_4 + a_1a_{12}$$

that looks like it could make x_5 redundant as a cluster variable. It also doesn't help account for the variable z_3 appearing in the corresponding exchange polynomial.

Attempt 2. Next we replaced x_5 with z_3, hoping that $\{x_1, x_2, x_3, x_4, z_3\}$ would work as an initial cluster. Using computer algebra to eliminate variables from the ring $\mathcal{A}_6$ we discover that all of the other x_i and z_i variables can be written as rational functions in this cluster, but not, unfortunately, as *Laurent polynomials.*

Attempt 3. Finally, although the rational functions obtained in attempt 2 were not Laurent polynomials, a closer inspection reveals that they are *'almost'* Laurent polynomials. Indeed, the denominators are always monomials in the five terms

$$x_1, x_2, x_3, x_4, x_3z_3 - a_3x_2 - a_{10}x_4.$$

Therefore we introduce $y_3 := x_3z_3 - a_3x_2 - a_{10}x_4$ as a new cluster variable.

Definition 5.1. We let y_i be defined by the expression

$$y_i := x_iz_i - a_ix_{i-1} - a_{i+7}x_{i+1}, \qquad i \in \mathbb{Z}/12\mathbb{Z}.$$

We conclude that we have the following result.

Lemma 5.2. *We can expand all of the spinors variables x_i, z_i as Laurent polynomials in $\{x_1, x_2, x_3, x_4, y_3\}$, and thus we can use this as a candidate for an initial cluster. Moreover, the new variables $y_1, \dots, y_{12}$ are also all Laurent polynomials in our chosen initial cluster, and they are all distinct. By symmetry we find that both*

$$\{x_{i-1}, x_i, x_{i+1}, x_{i+2}, y_i\} \qquad \textit{and} \qquad \{x_{i-1}, x_i, x_{i+1}, x_{i+2}, y_{i+1}\}$$

are clusters for any value of i.

Finding the exchange polynomials. From equation (b) above we immediately have the relation $x_1x_5 = y_3 + a_1a_{12}$. Moreover, by manipulations with the equations of types (a), (b), (c), we can write the product y_2y_3 as a positive sum of monomials in terms of the frozen coefficients and x_1, x_2, x_3, x_4. Indeed we get

$$\begin{aligned} y_2y_3 &= (x_2z_2 - a_2x_1 - a_9x_3)(x_3z_3 - a_3x_2 - a_{10}x_4) \\ &= x_1x_4(a_5x_2 + a_7x_3 + a_2a_{10}) + a_{12}x_3(a_4x_1 + a_1a_9) + a_{12}x_2(a_6x_3 + a_8x_4 + a_3a_{11}). \end{aligned}$$

Thus we can use these as exchange relations in the following sequence of mutations, together with the expected Dih_{12} symmetry, to work out what all of the other exchange polynomials should be

$$\cdots \longleftrightarrow \left\{\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ & y_2 & & \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ & & y_3 & \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} x_2 & x_3 & x_4 & x_5 \\ & y_3 & & \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} x_2 & x_3 & x_4 & x_5 \\ & & y_4 & \end{matrix}\right\} \longleftrightarrow \cdots.$$

Doing this, we arrive at the following candidate for our the initial seed.

Proposition 5.3. *The following seed S is an initial seed for a LPA structure on $\mathcal{A}_6 = \mathbb{C}[\mathcal{V}_6]$, which is compatible with the* Dih_{12}*-symmetry:*

$$S = \begin{cases} x_1, & y_3 + a_{12}a_1, \\ x_2, & a_2x_1(y_3 + a_{10}x_4) + a_9x_3(y_3 + a_1a_{12}) + x_1x_3(a_7x_4 + a_4a_{12}), \\ x_3, & y_3 + a_3x_2 + a_{10}x_4, \\ x_4, & a_{11}(y_3 + a_3x_2) + x_3(a_4x_1 + a_6x_2 + a_1a_9), \\ y_3, & x_1x_4(a_5x_2 + a_7x_3 + a_2a_{10}) + a_{12}x_3(a_4x_1 + a_1a_9) \\ & \quad + a_{12}x_2(a_6x_3 + a_8x_4 + a_3a_{11}). \end{cases}$$

5.3 Summary of the LPA structure

Number of seeds and the exchange graph Once we are given the right initial seed it is easy to plug into our code and verify that it generates a LPA of finite type.

Theorem 5.4. *The LPA structure on $\mathcal{A}_6$, generated by the initial seed of Proposition 5.3, has finite type. In particular it has 264 seeds and 32 cluster variables.*

The cluster variables consist of the 15 spinor coordinates $x_1, \dots, x_{12}$, z_1, z_2, z_3 on $\mathcal{V}_6$, plus 17 additional cluster variables $y_1, \dots, y_{12}$, t_1, t_2, t_3, u_1, u_2 where

1. *$y_1, \dots, y_{12}$ are quadratics in the spinor variables introduced above,*
2. *t_1, t_2, t_3 are quartics in the spinor variables determined by the* Dih_{12}*-conjugates of the equation*

$$x_1x_4x_7x_{10} = t_1 + a_3a_6a_9a_{12},$$

3. *u_1, u_2 are cubics in the spinor variables determined by the* Dih_{12}*-conjugates of the equation*[2]

$$x_1x_5x_9 = u_1 + a_4a_5x_1 + a_8a_9x_5 + a_{12}a_1x_9 + a_1a_5a_9 + a_4a_8a_{12}.$$

Up to the Dih_{12}*-symmetry there are 15 different orbits of seeds; seven orbits have length 24, which we label A,…,G, and eight orbits have length 12, which we name H,…,O. They are related by mutation according to Table 3, and the* Dih_{12}*-quotient of the exchange graph is presented in Figure 6.*

A	seed	x_1	x_2	x_3	x_4	y_2
	mutation	y_4	z_2	y_{12}	x_{12}	y_3
	orbit	D	B	C	A	A
B	seed	x_1	x_2	x_4	y_3	z_3
	mutation	x_5	x_7	x_{11}	y_{12}	x_3
	orbit	B	E	F	F	A
C	seed	x_1	x_2	x_4	y_{12}	y_2
	mutation	u_2	x_{10}	x_{12}	x_3	z_3
	orbit	H	G	D	A	F
D	seed	x_1	x_2	x_3	y_1	y_3
	mutation	x_5	u_1	x_{11}	x_4	x_{12}
	orbit	C	I	C	A	A
E	seed	x_1	x_4	x_7	y_3	z_3
	mutation	x_5	y_9	x_2	t_1	y_5
	orbit	F	K	B	J	G
F	seed	x_1	x_2	x_4	y_{12}	z_3
	mutation	y_6	x_{10}	x_{11}	y_3	y_2
	orbit	L	E	B	B	C
G	seed	x_1	x_4	x_7	y_3	y_5
	mutation	x_5	u_1	x_3	z_2	z_3
	orbit	C	N	C	E	E
H	seed	x_1	x_3	y_3	y_5	u_1
	mutation	x_5	x_7	y_{11}	y_1	x_4
	orbit	I	N	M	I	C
I	seed	x_1	x_3	y_1	y_3	u_1
	mutation	x_5	x_{11}	y_5	y_{11}	x_2
	orbit	H	H	H	H	D
J	seed	x_1	x_4	x_7	z_2	t_1
	mutation	x_{10}	x_{10}	x_{10}	z_3	y_5
	orbit	J	J	J	J	E
K	seed	x_1	x_7	y_5	y_{11}	z_2
	mutation	x_9	x_3	x_{10}	x_4	u_2
	orbit	L	L	E	E	O
L	seed	x_1	x_3	y_5	y_{11}	z_2
	mutation	x_9	x_7	x_{12}	x_4	u_1
	orbit	K	K	F	F	M
M	seed	x_1	x_3	y_5	y_{11}	u_1
	mutation	x_9	x_7	y_1	y_3	z_2
	orbit	O	O	H	H	L
N	seed	x_1	x_7	y_3	y_5	u_1
	mutation	x_5	x_3	y_{11}	y_9	x_4
	orbit	H	H	O	O	G
O	seed	x_1	x_7	y_5	y_{11}	u_1
	mutation	x_9	x_3	y_9	y_3	z_2
	orbit	M	M	N	N	K

Table 3. Representatives for each of the 15 orbits of seeds A,. . .,O. The top row of each entry contains the five cluster variables in the seed. The second row records which cluster variable is obtained by mutating the seed at the variable directly above it, leading to a seed in the orbit given by the label on the third row.

We can also check various things about the structure of the exchange graph, such as the fact that every x (resp. y, z, t, u) variable belongs to 60 (resp. 32, 40, 8, 36) seeds.

Positivity. By inspecting the output of our computation, which consists of all of the seeds for $\mathcal{A}_6$ (including the Laurent expansion of all of the cluster variables), we have the following result.

Corollary 5.5. *The positivity phenomenon holds for $\mathcal{A}_6$. In other words, all of the coefficients in the Laurent expansions of the cluster variables and all the coefficients in exchange polynomials of each seed are positive.*

This is somewhat unexpected, since enforcing the positivity rule in equation (4.1) on the equations of type (a) necessarily creates minus signs in some of the other spinor equations defining the Cayley plane, e.g.,

$$x_1x_5 = x_3z_3 - a_3x_2 - a_{10}x_4 + a_1a_{12}.$$

However, in order to get a Laurent phenomenon we needed to introduce the new cluster variable $y_3 = x_3z_3 - a_3x_2 - a_{10}x_4$ and this allows us to rewrite this equation with positive coefficients as $x_1x_5 = y_3 + a_1a_{12}$.

Remark 5.6. Positivity was proved for cluster algebras by Gross, Hacking, Keel and Kontsevich [7] by associating a *consistent scattering diagram* to a cluster algebra. The proof follows by interpreting the coefficients in the Laurent expansion of each cluster monomial as a count of broken lines in the scattering diagram. The consistent scattering diagram for the LPA structure on $\mathcal{A}_5$ was constructed in [4], and it should be possible to construct one for $\mathcal{A}_6$ in a similar manner.

[2]It is clear from the equation that u_1 is invariant under the shift $i \mapsto i+4$ for $i \in \mathbb{Z}/12\mathbb{Z}$, but in fact, as a consequence of the other relations, it turns out that it is invariant under $i \mapsto i+2$ too.

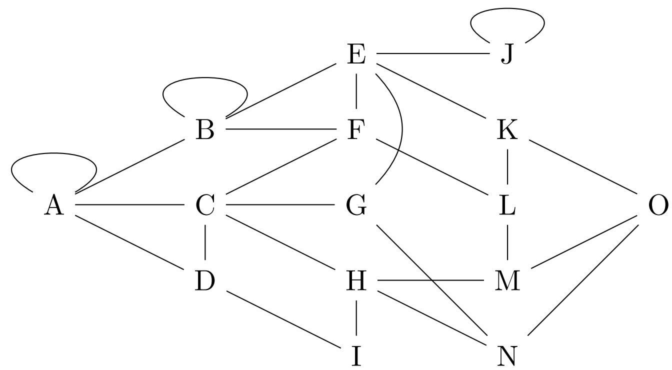


Figure 6. The Dih_{12}-quotient of the exchange graph of $\mathcal{A}_6$. Beginning with the initial seed S of Proposition 5.3, which is in the orbit A, the remaining orbits are named alphabetically, according to the order in which they were found during a breadth-first search of the exchange graph.

A final remark. We recall that the definition of mutation in an LPA uses the exchange Laurent polynomials $\widehat{F}_i$, rather than the exchange polynomials F_i. We always have $F_i = \widehat{F}_i$ in the case of cluster algebras, and it is tempting to think we might be able to dispense with the $\widehat{F}_i$ in the general case of an LPA. However, it is crucial to work with the $\widehat{F}_i$ in order for the LPA structure on $\mathcal{A}_6$ to have finite type. Moreover, this LPA provides an example for which every seed has at least one direction i in which $F_i \neq \widehat{F}_i$.

6 The Freudenthal variety

We conjecture the existence of a similar LPA structure on the homogeneous coordinate ring of the Freudenthal variety $\mathcal{V}_7$.

Conjecture 6.1. *There is a finite type LPA structure of rank* 10 *on* $\mathcal{A}_7 = \mathbb{C}[\mathcal{V}_7]$, *which has the positivity phenomenon.*

The LPA should have rank 10 by Remark 3.2. We get as far as writing down the equations for $\mathcal{V}_7$ (as we did for $\mathcal{V}_6$ in Section 5.1).

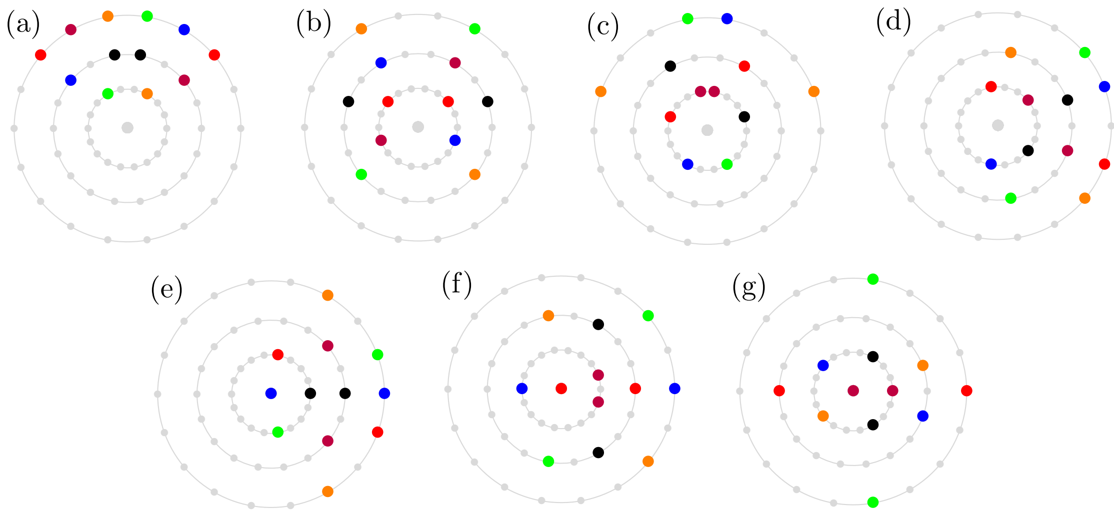
6.1 The equations of the Freudenthal variety

The Freudenthal variety has an embedding $\mathcal{V}_7 \subset \mathbb{P}^{55}$ where the 56 variables $a_1, \dots, a_{18}$, $x_1, \dots, x_{18}$, $y_1, \dots, y_{18}$, z_1, z_2 can be put into one-to-one correspondence with the 56 lines on a del Pezzo surface of degree 2. The Coxeter rotation has order 18 and the action on the variables is on the labels.

There are seven orbits of equations:

$$x_1x_2 = a_3x_{17} + a_2y_0 + a_1y_3 + a_0x_4 + a_{17}a_4, \tag{a}$$
$$y_{14}y_1 = x_{15}y_4 - a_0x_{10} - a_{15}x_5 + x_0y_{11} + x_{13}x_2, \tag{b}$$
$$y_2y_1 = a_2y_{12} + y_5x_0 + y_{16}x_3 + a_1y_9 + a_5a_{16}, \tag{c}$$
$$x_4y_1 = x_2y_5 + a_5x_{17} + a_4y_{16} - a_2y_8 - a_1x_7, \tag{d}$$
$$x_2x_6 = x_4y_4 \mp a_4z_2 - a_5y_0 - a_3y_8 + a_1a_7, \tag{e}$$
$$y_1y_3 = x_{17}x_5 \pm x_2z_1 + a_0x_7 + a_4x_{15} - a_2y_{11}, \tag{f}$$
$$y_4y_{16} = \pm y_1z_2 - y_8x_0 - y_{12}x_2 - a_1x_{10} + a_5a_{15}, \tag{g}$$

corresponding to the diagrams



and some additional quadratic equations

$$z_1z_2 = y_3y_{12} + y_7y_{16} + y_8y_{17} + x_1x_{10} + x_3x_{12} + x_5x_{14} + a_0a_9 + a_2a_{11} - a_3a_{12} + a_4a_{13} + a_6a_{15}$$

that are implied by the others. Here the choice of $\pm$ sign in the equations is due to the fact that the z_i variables do not appear in the equations of type (a), and thus the positivity rule of equation (4.1) does not determine the signs in front of the monomials containing exactly one z_i.

From our observations on the previous cases, it seems like it will be constructive to introduce new cluster variables which will allow us to rearrange the equations so that they satisfy the positivity phenomenon, such as

$$t_? := y_{14}y_1 - x_0y_{11} - x_{13}x_2 = x_{15}y_4 - a_0x_{10} - a_{15}x_5,$$
$$u_? := x_4y_1 - a_5x_{17} - a_4y_{16} = x_2y_5 - a_2y_8 - a_1x_7,$$

and so on.

However, at this point we get stuck. Trying to follow our previous approach of identifying an initial cluster, by using computer algebra to eliminate variables in a ring of codimension 28, proves to be a step too far.

A Sage code

The sequence of LPAs associated to the spaces in this paper can be computed using the Sage package LPASeed [2]. To define the initial seed for the E_4 case, one declares the variable names and corresponding polynomials as follows:

```
sage: var('x1, x2')
(x1, x2)
sage: var('a2, a4, a5, a1, a3')
(a2, a4, a5, a1, a3)
sage: S = LPASeed({x1: a2*x2 + a4*a5, x2: a1*x1 + a3*a4},
....: coefficients=[a2, a4, a5, a1, a3])
```

Then one can analyse the structure of the mutations using the built-in methods. For instance, to get the number of seeds in the exchange graph:

```
sage: len(S.mutation_class())
5
```

A list of all the cluster variables:

```
sage: S.variable_class()
[(x1*a5*a1 + x2*a2*a3 + a4*a5*a3)/(x1*x2),
 (x1*a1 + a4*a3)/x2,
 (x2*a2 + a4*a5)/x1,
 x2,
 x1]
```

We can work out larger examples rather quickly and even plot their exchange graphs. The LPA for the orthogonal $\mathrm{OGr}(5, 10)$ case corresponding to E_5 can be fully understood in a few lines:

```
sage: var('x1, x2, x3')
(x1, x2, x3)
sage: var('a1, a2, a3, a4, a5, a6, a7, a8')
(a1, a2, a3, a4, a5, a6, a7, a8)
sage: F1 = a5*x2 + a8*x3 + a2*a3
sage: F2 = a6*x1*x3 + a3*a4*x1 + a8*a1*x3 + a1*a2*a3
sage: F3 = a4*x1 + a7*x2 + a1*a2
sage: S = LPASeed({x1: F1, x2: F2, x3: F3},
....: coefficients=[a1,a2,a3,a4,a5,a6,a7,a8])
sage: len(S.mutation_class())
16
sage: len(S.variable_class())
10
sage: show(S.exchange_graph())
```

The final command shows the exchange graph featured in Figure 4 to the user.

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