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ON A RECURRENCE RELATION OF GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. The principal aim of this paper is to investigate a recurrence relation and an integral representation of generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,q}(z)$. In the end several special cases have also been discussed.

1 Introduction and Preliminaries

In 1903, the Swedish mathematician Gosta Mittag-Leffler [2] introduced the function $E_\alpha(z)$, defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} (\alpha \in \mathrm{C},\, \Re(\alpha) > 0) \tag{1.1}$$

where $\Gamma(z)$ is the familiar Gamma function. The Mittag–Leffler function (1.1) reduces immediately to the exponential function $e^z = E_1(z)$ when $\alpha = 1$. For $0 < \alpha < 1$ it interpolates between the pure exponential e^z and a geometric function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $(|z| < 1)$. Its importance has been realized during the last two decades due to its involvement in the problems of applied sciences such as physics, chemistry, biology and engineering. Mittag–Leffler function occurs naturally in the solution of fractional order differential or integral equations.

In 1905, a generalization of $E_\alpha(z)$ was studied by Wiman [6] who defined the function $E_{\alpha,\beta}(z)$ as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}; (\alpha, \beta \in \mathrm{C},\, \Re(\alpha) > 0,\, \Re(\beta) > 0). \tag{1.2}$$

The function $E_{\alpha,\beta}(z)$ is now known as Wiman function.

In 1971, Prabhakar [3] introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ defined by

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$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}(\alpha,\beta,\gamma\in C;\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(\gamma)>0),\quad (1.3)$$

where $(\lambda)_n$ is the Pochammer symbol (see, e.g. , [4]) defined ($\lambda \in C$) by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0;\ \lambda\neq 0)\\ \lambda(\lambda+1).......(\lambda+n-1) & (n\in N;\ \lambda\in C)\end{cases}$$

N being the set of positive integers. In the sequel to this study, Shukla and Prajapati [5] investigated the function $E_{\alpha,\beta}^{\gamma,q}(z)$ defined by

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad (1.4)$$

$$(\alpha,\beta,\gamma\in C,\Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(\gamma)>0,\ q\in(0,1)\cup N).$$

It is noted (See, e.g., [4, Lemma 6 p.22]) that
$(\gamma)_{qn}=q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_n\ (q\in N,\ n\in N_0:=N\cup\{0\}).$

The function $E_{\alpha,\beta}^{\gamma,q}(z)$ converges absolutely for all $z\in C$ if $q<\ \Re(\alpha)+1$ (an entire function of order $\Re(\alpha)^{-1}$ and for $|z|<\ 1$ if $q=\ \Re(\alpha)+1$. It is easily seen that (1.4) is an obvious generalization of (1.1), (1.2) , (1.3) and the exponential function e^z as follows:

$$E_{1,1}^{1,1}(z)=e^z,\quad E_{\alpha,1}^{1,1}(z)=E_{\alpha}(z),\quad E_{\alpha,\beta}^{1,1}(z)=E_{\alpha,\beta}(z),\quad E_{\alpha,\beta}^{\gamma,1}(z)=E_{\alpha,\beta}^{\gamma}(z).$$

2 Recurrence Relation

We begin by stating our main theorem

Theorem 1. *For* $\Re(\alpha+p)>0,\ \Re(\beta+s)>0,\ \Re(\gamma)>0,\ q\in(0,1)\cup N$ *we get*

$$\begin{aligned}&E_{\alpha+p,\beta+s+1}^{\gamma,q}(z)-E_{\alpha+p,\beta+s+2}^{\gamma,q}(z)\\ &=(\beta+s)(\beta+s+2)\,E_{\alpha+p,\beta+s+3}^{\gamma,q}(z)+(\alpha+p)^2z^2\,\ddot{E}_{\alpha+p,\beta+s+3}^{\gamma,q}(z)\\ &\quad+(\alpha+p)\ \{\alpha+p+2(\beta+s+1)\}\ z\ \dot{E}_{\alpha+p,\beta+s+3}^{\gamma,q}(z),\end{aligned}\quad (2.1)$$

where $\dot{E}_{\alpha,\beta}^{\gamma,q}(z)=\frac{d}{dz}E_{\alpha,\beta}^{\gamma,q}(z)$ *and* $\ddot{E}_{\alpha,\beta}^{\gamma,q}(z)=\frac{d^2}{dz^2}E_{\alpha,\beta}^{\gamma,q}(z)$.

It is easy to obtain the following corollary by letting $\alpha + p = k$ and $\beta + s = m$.

Corollary 2. *We have, for* k, $m \in \mathrm{N}$

$$\begin{array}{l} E^{\gamma,q}_{k,m+1}(z) = E^{\gamma,q}_{k,m+2}(z) + m(m+2)\, E^{\gamma,q}_{k,m+3}(z) \\ +k^2 z^2\, \ddot{E}^{\gamma,q}_{k,m+3}(z) + k\,(k+2m+2)\, \dot{E}^{\gamma,q}_{k,m+3}(z). \end{array} \tag{2.2}$$

Proof of the Theorem 1. By applying the fundamental relation of the Gamma function $\Gamma(z+1) = z\Gamma(z)$ to (1.4), we can write

$$E^{\gamma,q}_{\alpha+p,\beta+s+1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\{(\alpha+p)\,n+\beta+s\}\,\Gamma\left((\alpha+p)\,n+\beta+s\right)} \frac{z^n}{n\,!} \tag{2.3}$$

and

$$E^{\gamma,q}_{\alpha+p,\beta+s+2}(z)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\{(\alpha+p)\,n+\beta+s+1\}\,\{(\alpha+p)\,n+\beta+s\}\,\Gamma\left((\alpha+p)\,n+\beta+s\right)} \frac{z^n}{n\,!}. \tag{2.4}$$

Equation (2.4) can be written as follows:

$$E^{\gamma,q}_{\alpha+p,\beta+s+2}(z)$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{(\alpha+p)\,n+\beta+s} - \frac{1}{(\alpha+p)\,n+\beta+s+1} \right\} \frac{(\gamma)_{qn}}{\Gamma\left((\alpha+p)\,n+\beta+s\right)} \frac{z^n}{n\,!} \tag{2.5}$$

$$= E^{\gamma,q}_{\alpha+p,\beta+s+1}(z) - \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{((\alpha+p)\,n+\beta+s+1)\,\Gamma\left((\alpha+p)\,n+\beta+s\right)} \frac{z^n}{n\,!}.$$

We, for convenience, denote the last summation in (2.5) by S:

$$S = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{((\alpha+p)\,n+\beta+s+1)\,\Gamma\left((\alpha+p)\,n+\beta+s\right)} \frac{z^n}{n\,!} \tag{2.6}$$

$$= E^{\gamma,q}_{\alpha+p,\beta+s+1}(z) - E^{\gamma,q}_{\alpha+p,\beta+s+2}(z).$$

Applying a simple identity $\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1}$ $((u = (\alpha+p)\,n+\beta+s+1))$ to (2.6), we obtain

$$S = \sum_{n=0}^{\infty} \frac{\{(\alpha+p)\,n+\beta+s\}\,(\gamma)_{qn}}{\Gamma\left((\alpha+p)\,n+\beta+s+3\right)} \frac{z^n}{n\,!} + \sum_{n=0}^{\infty} \frac{\{(\alpha+p)\,n+\beta+s\}\,\{(\alpha+p)\,n+\beta+s+1\}\,(\gamma)_{qn}}{\Gamma\left((\alpha+p)\,n+\beta+s+3\right)} \frac{z^n}{n\,!}$$

$$= (\alpha+p)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma((\alpha+p)n+\beta+s+3)}\frac{z^n}{(n-1)!} + (\beta+s)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma((\alpha+p)n+\beta+s+3)}\frac{z^n}{n!}$$
$$+ (\alpha+p)^2\sum_{n=0}^{\infty}\frac{n\,(\gamma)_{qn}}{\Gamma((\alpha+p)n+\beta+s+3)}\frac{z^n}{(n-1)!} + b\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma((\alpha+p)n+\beta+s+3)}\frac{z^n}{(n-1)!}$$
$$+ c\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma((\alpha+p)n+\beta+s+3)}\frac{z^n}{n!} \tag{2.7}$$

where $b=(\alpha+p)(2\beta+2s+1)$ and $c=(\beta+s)(\beta+s+1)$.
We now express each summation in the right hand side of (2.7) as follows:

$$\frac{d^2}{dz^2}\{z^2 E^{\gamma,q}_{\alpha+p,\beta+s+3}(z)\} = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)\;(\gamma)_{qn}\;z^n}{\Gamma((\alpha+p)n+\beta+s+3)\;\;n!}. \tag{2.8}$$

We find from (2.8) that

$$\sum_{n=1}^{\infty}\frac{n\;(\gamma)_{qn}}{\Gamma\;((\alpha+p)n+\beta+s+3)}\;\frac{z^n}{(n-1)!} = z^2\,\ddot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z) + 4z\,\dot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z)$$
$$-\,3\sum_{n=1}^{\infty}\frac{(\gamma)_{qn}}{\Gamma\,((\alpha+p)n+\beta+s+3)}\;\frac{z^n}{(n-1)!}. \tag{2.9}$$

Considering

$$\frac{d}{dz}\{z\,E^{\gamma,q}_{\alpha+p,\beta+s+3}(z)\} = \sum_{n=0}^{\infty}\frac{(n+1)\;(\gamma)_{qn}}{\Gamma\,((\alpha+p)n+\beta+s+3)}\;\frac{z^n}{n!},$$

similarly we have

$$\sum_{n=1}^{\infty}\frac{(\gamma)_{qn}}{\Gamma\,((\alpha+p)n+\beta+s+3)}\;\frac{z^n}{(n-1)!} = z\,\dot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z). \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\sum_{n=1}^{\infty}\frac{n\,(\gamma)_{qn}}{\Gamma\,((\alpha+p)n+\beta+s+3)}\;\frac{z^n}{(n-1)!} = z\,\dot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z) + z^2\,\ddot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z). \tag{2.11}$$

Applying (2.10) and (2.11) to (2.7), we get

$$S = (\alpha+p)^2 z^2\,\ddot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z)$$
$$+\{(\alpha+p)^2+(\alpha+p)+b\}\,z\,\dot{E}^{\gamma,q}_{\alpha+p,\beta+s+3}(z) + (\beta+s+c)\,E^{\gamma,q}_{\alpha+p,\beta+s+3}(z)\,.$$

Now setting the last identity into (2.6) completes the proof of Theorem 1. □

3 Integral Representation

Theorem 3. *We get*

$$\int_0^1 t^{\beta+s} E^{\gamma,q}_{\alpha+p,\beta+s}(t^{\alpha+p})\,dt = E^{\gamma,q}_{\alpha+p,\beta+s+1}(1) - E^{\gamma,q}_{\alpha+p,\beta+s+2}(1). \quad (3.1)$$

$(\Re(\alpha+p) > 0,\ \Re(\beta+s) > 0,\ \Re(\gamma) > 0,\ q \in (0,1) \cup \mathrm{N})$.

Setting $\alpha + p = k \in \mathrm{N}$ and$\beta + s = m \in \mathrm{N}$ in (3.1) yields

Corollary 4.

$$\int_0^1 t^m E^{\gamma,q}_{k,m}(t^k)\,dt = E^{\gamma,q}_{k,m+1}(1) - E^{\gamma,q}_{k,m+2}(1), \quad (3.2)$$

$(k,\ m \in \mathrm{N})$

Proof of the Theorem 3. Putting $z = 1$ in (2.6) gives,

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\{(\alpha+p)\,n+\beta+s+1)\}\ \Gamma\ ((\alpha+p)\,n+\beta+s)\ \ n\,!}$$
$$= E^{\gamma,q}_{\alpha+p,\beta+s+1}(1) - E^{\gamma,q}_{\alpha+p,\beta+s+2}(1). \quad (3.3)$$

It is easy to find that

$$\int_0^z t^{\beta+s} E^{\gamma,q}_{\alpha+p,\beta+s}(t^{\alpha+p})\,dt$$
$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\ z^{(\alpha+p)n+\beta+s+1}}{\{(\alpha+p)\,n+\beta+s+1\}\ \Gamma\ ((\alpha+p)\,n+\beta+s)\ \ n\,!}. \quad (3.4)$$

Comparing (3.3) with the identity obtaining by setting $z = 1$ in (3.4) is seen to yields (3.1) in Theorem 3. □

4 Special Cases

1. Setting $p = 0,\ \gamma = q = 1$ and $\beta + s = m \in \mathrm{N}$ in (2.1) reduces to a known recurrence relation of $E_{\alpha,\beta}(z)$ (see Gupta and Debnath [1]):

$$\begin{aligned} &E_{\alpha,m+1}(z) = \alpha^2 z^2\, \ddot{E}_{\alpha,m+3}(z) + \alpha\,(\alpha+2m+2)\,z\,\dot{E}_{\alpha,m+3}(z) \\ &+m(m+2)\,E_{\alpha,m+3}(z)\ + E_{\alpha,m+2}(z) \end{aligned} \quad (4.1)$$

where $\dot{E}_{\alpha,\beta}(z) = \frac{d}{dz}[E_{\alpha,\beta}(z)]$ and$\ddot{E}_{\alpha,\beta}(z) = \frac{d^2}{dz^2}[E_{\alpha,\beta}(z)]$.

2. Setting ($k = m = q = \gamma = 1$)and ($k = m = q = 1$ *and* $\gamma = 2$) in (3.2) respectively, yields

$$\int_0^1 t\, e^t\, dt = E_{1,2}^{1,1}(1) - E_{1,3}^{1,1}(1) = E_{1,2}(1) - E_{1,3}(1)$$

and

$$\int_0^1 t\, E_{1,1}^{2,1}(t)\, dt = E_{1,2}^{2,1}(1) - E_{1,3}^{2,1}(1).$$

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