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POSITIVE DEFINITE SOLUTION OF TWO KINDS OF NONLINEAR MATRIX EQUATIONS

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Abstract. Based on the elegant properties of the Thompson metric, we prove that the following two kinds of nonlinear matrix equations $X = \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i$ and $X = \sum_{i=1}^{m} (A_i^* X A_i)^{\delta_i}$, $(0 < |\delta_i| < 1)$ always have a unique positive definite solution. Iterative methods are proposed to compute the unique positive definite solution. We show that the iterative methods are more effective as $\delta = max\{|\delta_i|, \ i = 1, 2, \cdots, m\}$ decreases. Perturbation bounds for the unique positive definite solution are derived in the end.

1 Introduction

We consider the following nonlinear matrix equation

$$X = \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i, \quad 0 < |\delta_i| < 1, \tag{1.1}$$

and

$$X = \sum_{i=1}^{m} (A_i^* X A_i)^{\delta_i}, \quad 0 < |\delta_i| < 1, \tag{1.2}$$

where $A_1, A_2, \cdots, A_m$ are $n \times n$ nonsingular complex matrix, and m is a positive integer. Here, A_i^* denotes the conjugate transpose of the matrix A_i. We mainly discuss the positive definite solution of Eqs. (1.1) and (1.2).

In the last few years there has been a constantly increasing interest in developing the theory and numerical approaches for positive definite solutions to the nonlinear matrix equation of the form (1.1) [1]-[11], [15]-[17], [13, 19]. Shi-Liu-Umoh-Gibson [17] used Brouwer's fixed point theorem to study the existence of solutions of the Eqs. (1.1) and (1.2). Multi-step stationary iterative methods were proposed to solve them

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in that paper. Duan-Liao-Tang [3] and Huang-Huang-Tsai[10] gave some theorems on the existence and uniqueness of positive definite solution for the similar form of Eqs. (1.1) and (1.2) by using fixed point theorems and projective metric technique, respectively.

In this paper, we firstly use Thompson metric method to prove that Eqs. (1.1) and (1.2) always have a unique positive definite solution. Iterative methods are proposed to compute the unique positive definite solution. We also show that the iterative methods are more effective as $\delta = max\{|\delta_i|, \quad i = 1, 2, \cdots, m\}$ decreases. Based on the perturbation theorem of contraction map [16], two perturbation bounds for the unique positive definite solution of Eqs. (1.1) and (1.2) are derived in the end.

Throughout this paper, we use $P(n)$ to denote the set of all $n \times n$ positive definite matrix. We use $\lambda_{max}(A)$ to denote the maximal eigenvalue of an Hermitian matrix A. The symbol $\|\cdot\|$ stands for the spectral norm. The notation $A \geq B$ indicates that $A - B$ is positive semidefinite. The symbol $M(\Omega, \alpha)$ denotes the set of all strict contraction maps on Ω with the contraction constant α, that is to say, for arbitrary $f \in M(\Omega, \alpha)$, then there exists $0 \leq \alpha < 1$ such that

$$\delta(f(x), f(y)) \leq \alpha\delta(x, y), \quad \forall x, y \in \Omega,$$

where $\delta(\cdot, \cdot)$ is a metric on Ω.

2 Positive definite solution of Eqs. (1.1) and (1.2)

In this section, we first review the Thompson metric on the open convex cone $P(n)$. And then we give some theorems on the existence of positive definite solution of Eqs. (1.1) and (1.2) by using the elegant properties of the Thompson metric. Iterative methods are constructed to solve Eqs. (1.1) and (1.2). We also show that the iterative methods are more effective as δ decreases.

The Thompson metric on $P(n)$ is defined by

$$d(A, B) = max\{logM(A/B), \quad logM(B/A)\},$$

where $M(A/B) = inf\{\lambda > 0 : A \leq \lambda B\} = \lambda_{max}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$. From Nussbaum [14] we obtain that $P(n)$ is a complete metric space with respect to the Thompson metric and $d(A, B) = \|log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|$. Now we shortly introduce the elegant properties of the Thompson metric [18]. It is invariant under the matrix inversion and congruence transformations

$$d(A, B) = d(A^{-1}, B^{-1}) = d(M^*AM, M^*BM) \tag{2.1}$$

for any $n \times n$ nonsingular matrix M. The other useful result is the nonpositive curvature property of the Thompson metric

$$d(X^r, Y^r) \leq rd(X, Y), \quad r \in [0, 1]. \tag{2.2}$$

According to (2.1) and (2.2), we have

$$d(M^*X^rM, M^*Y^rM) \leq |r|d(X,Y), \quad r \in [-1,1]. \tag{2.3}$$

We begin with two lemmas.

Lemma 1. *[13, Lemma 2.1] For any $A, B, C, D \in P(n)$,*

$$d(A+B, C+D) \leq max\{d(A,C), d(B,D)\}.$$

Especially,

$$d(A+B, A+C) \leq d(B,C).$$

Lemma 2. *[12, Theorem 1.1.6] Let $\phi \in M(\Omega, \alpha)$. Then the map ϕ has a unique fixed point $x^*(\phi)$ on Ω. Moreover, for every x_0 we have*

$$x^* = \lim_{m\to\infty} x_m,$$

where the sequence $\{x_m\}$ are determined by the iteration $\forall x_0 \in \Omega,\ \ x_{m+1} = \phi(x_m), m = 0,1,2,\cdots$, and the error estimation is given by

$$\delta(x_m, x^*) \leq \frac{\alpha^m}{1-\alpha}\delta(x_1, x_0).$$

Theorem 3. *Eq.(1.1) always has a unique positive definite solution $\widehat{X}$. Moreover, we have*

$$\widehat{X} = \lim_{k\to\infty} X_k,$$

where the matrix sequence $\{X_k\}$ are determined by the iteration

$$\forall X_0 \in P(n), \quad X_{k+1} = \sum_{i=1}^{m} A_i^* X_k^{\delta_i} A_i, \quad k = 0,1,2,\cdots, \tag{2.4}$$

and the error estimation is given by

$$d(X_k, \widehat{X}) \leq \frac{\delta^k}{1-\delta} d(X_1, X_0), \quad k = 0,1,2,\cdots. \tag{2.5}$$

Proof. Let

$$G(X) = \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i, \quad X \in P(n). \tag{2.6}$$

Observe that the solution of Eq.(1.1) is a fixed point of G. Now we will prove that G is a contraction map. It is easy to verify that

$$G : P(n) \to P(n).$$

For arbitrary $X, Y \in P(n)$, by Lemma 1 and (2.3), we have,

$$\begin{aligned}
d(G(X),G(Y)) &= d(\sum_{i=1}^{m} A_i^* X^{\delta_i} A_i, \sum_{i=1}^{m} A_i^* Y^{\delta_i} A_i) \\
&\leq max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), d(\sum_{i=2}^{m} A_i^* X^{\delta_i} A_i, \sum_{i=2}^{m} A_i^* Y^{\delta_i} A_i)\} \\
&\leq max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), max\{d(A_2^* X^{\delta_2} A_2, A_2^* Y^{\delta_2} A_2), \\
&\quad d(\sum_{i=3}^{m} A_i^* X^{\delta_i} A_i, \sum_{i=3}^{m} A_i^* Y^{\delta_i} A_i)\}\} \\
&= max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), d(A_2^* X^{\delta_2} A_2, A_2^* Y^{\delta_2} A_2), \\
&\quad d(\sum_{i=3}^{m} A_i^* X^{\delta_i} A_i, \sum_{i=3}^{m} A_i^* Y^{\delta_i} A_i)\} \\
&\leq \cdots\cdots \\
&\leq max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), d(A_2^* X^{\delta_2} A_2, A_2^* Y^{\delta_2} A_2), \\
&\quad \cdots, d(A_m^* X^{\delta_m} A_m, A_m^* Y^{\delta_m} A_m)\} \\
&\leq max\{|\delta_1| d(X,Y), |\delta_2| d(X,Y), \cdots, |\delta_m| d(X,Y)\} \\
&= \delta d(X,Y),
\end{aligned}$$

where $\delta = max\{|\delta_i|, \quad i = 1, 2, \cdots, m\}$.

Since $0 < \delta < 1$, we know that the map G is a strict contraction for the Thompson metric $d(\cdot,\cdot)$ on P(n) with the contraction constant δ, that is to say, $G \in M(P(n), \delta)$. By Lemma 2, the map G has a unique fixed point $\widehat{X}$ on $P(n)$, which implies that Eq.(1.1) has a unique solution $\widehat{X}$ on $P(n)$, and for every $X_0 \in P(n)$, the iterative sequence $\{X_k\}$ generated by (2.4) convergence to $\widehat{X}$, and the error estimation is given by (2.5). □ □

Remark 4. *Similar result in Theorem 3 can be also found in Lim [13].*

Theorem 5. *Eq. (1.2) always has a unique positive definite solution $\overline{X}$. Moreover, we have*

$$\overline{X} = \lim_{k\to\infty} X_k,$$

where the matrix sequence $\{X_k\}$ are determined by the iteration

$$\forall X_0 \in P(n), \quad X_{k+1} = \sum_{i=1}^{m} (A_i^* X_k A_i)^{\delta_i}, \quad k = 0, 1, 2, \cdots, \tag{2.7}$$

and the error estimation is given by

$$d(X_k, \overline{X}) \leq \frac{\delta^k}{1-\delta} d(X_1, X_0), \quad k = 0, 1, 2, \cdots. \tag{2.8}$$

Proof. Let

$$F(X) = \sum_{i=1}^{m} (A_i^* X A_i)^{\delta_i}, \quad X \in P(n). \tag{2.9}$$

Observe that the solution of Eq.(1.2) is a fixed point of F. Now we will prove that F is a contraction map. It is easy to verify that

$$F : P(n) \to P(n).$$

For arbitrary $X, Y \in P(n)$, by Lemma 1 and (2.3), we have,

$$\begin{aligned}
d(F(X), F(Y)) &= d(\sum_{i=1}^{m}(A_i^* X A_i)^{\delta_i}, \sum_{i=1}^{m}(A_i^* Y A_i)^{\delta_i}) \\
&\leq max\{d((A_1^* X A_1)^{\delta_1}, (A_1^* Y A_1)^{\delta_1}), d(\sum_{i=2}^{m}(A_i^* X A_i)^{\delta_i}, \sum_{i=2}^{m}(A_i^* Y A_i)^{\delta_i})\} \\
&\leq max\{d((A_1^* X A_1)^{\delta_1}, (A_1^* Y A_1)^{\delta_1}), max\{d((A_2^* X A_2)^{\delta_2}, (A_2^* Y A_2)^{\delta_2}), \\
&\quad d(\sum_{i=3}^{m}(A_i^* X A_i)^{\delta_i}, \sum_{i=3}^{m}(A_i^* Y A_i)^{\delta_i}\}\} \\
&= max\{d((A_1^* X A_1)^{\delta_1}, (A_1^* Y A_1)^{\delta_1}), d((A_2^* X A_2)^{\delta_2}, (A_2^* Y A_2)^{\delta_2}), \\
&\quad d(\sum_{i=3}^{m}(A_i^* X A_i)^{\delta_i}, \sum_{i=3}^{m}(A_i^* Y A_i)^{\delta_i})\} \\
&\leq \cdots\cdots \\
&\leq max\{d((A_1^* X A_1)^{\delta_1}, (A_1^* Y A_1)^{\delta_1}), d((A_2^* X A_2)^{\delta_2}, (A_2^* Y A_2)^{\delta_2}), \\
&\quad \cdots, d((A_m^* X A_m)^{\delta_m}, (A_m^* Y A_m)^{\delta_m})\} \\
&\leq max\{|\delta_1| d(X, Y), |\delta_2| d(X, Y), \cdots, |\delta_m| d(X, Y)\} \\
&= \delta d(X, Y).
\end{aligned}$$

Since $0 < \delta < 1$, we know that the map F is a strict contraction for the Thompson metric $d(\cdot, \cdot)$ on P(n) with the contraction constant δ, that is to say, $G \in M(P(n), \delta)$. By Lemma 2, the map F has a unique fixed point $\overline{X}$ on $P(n)$, which implies that Eq. (2.1) has a unique solution $\overline{X}$ on $P(n)$, and for every $X_0 \in P(n)$, the iterative sequence $\{X_k\}$ generated by (2.7) convergence to $\overline{X}$, and the error estimation is given by (2.8). ☐ ☐

Remark 6. *From (2.5) and (2.8) it is easy to see that the convergence rate of iterative method (2.4) and (2.7) are more rapid as δ decreases.*

3 Perturbation bounds of Positive definite solution of Eqs. (1.1) and (1.2)

In this section, we discuss the perturbation bound of the unique positive definite solution of Eqs. (1.1) and (1.2) by using the perturbation theorem of contraction map in Ran-Reurings-Rodman [16]. Consider the perturbed equation

$$\widetilde{X} = \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i, \quad 0 < |\delta_i| < 1, \tag{3.1}$$

and

$$\widetilde{X} = \sum_{i=1}^{m} (\widetilde{A}_i^* \widetilde{X} \widetilde{A}_i)^{\delta_i}, \quad 0 < |\delta_i| < 1, \tag{3.2}$$

where $\widetilde{A}_i$ are small perturbations of A_i.

Lemma 7. *[16, Theorem 2.2] Let the map $\phi \in M(\Omega, \alpha)$. Then for every $\varepsilon > 0$ and for all maps $\psi \in M(\Omega, \alpha)$ satisfying*

$$\sup_{X \in \Omega} \delta(\psi(X), \phi(X)) < min\{\frac{1-\alpha}{3}\varepsilon, 1\},$$

we have the inequality

$$\delta(x^*(\psi), x^*(\phi)) < \varepsilon,$$

where the symbols $x^*(\psi)$ and $x^*(\phi)$ denote the unique fixed point of ψ and ϕ on Ω, respectively.

Theorem 8. *Let*

$$a_i = d(\widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, A_i^* X^{\delta_i} A_i) = \| log((\widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i)^{-\frac{1}{2}} (A_i^* X^{\delta_i} A_i)(\widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i)^{-\frac{1}{2}})\|, \quad i = \overline{1, m}.$$

For every $\varepsilon > 0$, if

$$\sup_{X \in P(n)} \; max\{a_1, a_2, \cdots, a_m\} \leq min\{\frac{1-\delta}{3}\varepsilon, 1\}, \tag{3.3}$$

then we have

$$d(\widetilde{X}_1, \widehat{X}) < \varepsilon,$$

where $\widehat{X}$ and $\widetilde{X}_1$ are the unique positive definite solution of Eq.(1.1) and its perturbed equation (3.1), respectively.

Proof. From the proof of Theorem 3 it follows that G is a strict contraction map on $P(n)$ with the contraction constant δ, that is to say,

$$G \in M(P(n), \delta).$$

Now we consider the perturbed equation (3.1). Let

$$\widetilde{G}(X) = \sum_{i=1}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \quad X \in P(n).$$

Using the similar method in Theorem 3, it is easy to verify that the map $\widetilde{G}$ is a strict contraction with the contraction constant δ, i.e. $\widetilde{G} \in M(P(n), \delta)$ and this show that the perturbed equation (3.1) has a unique positive definite solution $\widetilde{X}_1$, i.e.

$$\widetilde{X}_1 = \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}_1^{\delta_i} \widetilde{A}_i.$$

We will give an upper bound for $d(\widetilde{X}_1, \widehat{X})$ by making use of the contraction map's perturbation Theorem 8.

For arbitrary $X \in P(n)$, according to (2.3) and Lemma 1, we have

$$\begin{aligned}
d(\widetilde{G}(X), G(X)) &= d(\sum_{i=1}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i) \\
&\leq max\{d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), d(\sum_{i=2}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=2}^{m} A_i^* X^{\delta_i} A_i)\} \\
&\leq max\{d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), max\{d(\widetilde{A}_2^* X^{\delta_2} \widetilde{A}_2, A_2^* X^{\delta_2} A_2), \\
&\quad d(\sum_{i=3}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=3}^{m} A_i^* X^{\delta_i} A_i)\}\} \\
&= max\{d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), d(\widetilde{A}_2^* X^{\delta_2} \widetilde{A}_2, A_2^* X^{\delta_2} A_2), \\
&\quad d(\sum_{i=3}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=3}^{m} A_i^* X^{\delta_i} A_i)\} \\
&\leq \cdots\cdots \\
&\leq max\{d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), d(\widetilde{A}_2^* X^{\delta_2} \widetilde{A}_2, A_2^* X^{\delta_2} A_2), \\
&\quad \cdots, d(\widetilde{A}_m^* X^{\delta_m} \widetilde{A}_m, A_m^* X^{\delta_m} A_m)\} \\
&= max\{a_1, a_2, \cdots, a_m\}.
\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we have

$$\sup_{X \in P(n)} d(\widetilde{G}(X), G(X)) \leq \sup_{X \in P(n)} max\{a_1, a_2, \cdots, a_m\} \leq min\{\frac{1-\delta}{3}\varepsilon, 1\},$$

and from Lemma 7 it follows that

$$d(\widetilde{X}_1, \widehat{X}) < \varepsilon. \quad \square$$

$\square$

Theorem 9. *Let*

$$a_i^{'} = d((\widetilde{A}_i^* X \widetilde{A}_i)^{\delta_i}, (A_i^* X A_i)^{\delta_i}) = \|log((\widetilde{A}_i^* X \widetilde{A}_i)^{-\frac{1}{2}\delta_i} (A_i^* X A_i)^{\delta_i} (\widetilde{A}_i^* X \widetilde{A}_i)^{-\frac{1}{2}\delta_i})\|, \quad i = \overline{1, m}.$$

For every $\varepsilon > 0$, if

$$\sup_{X \in P(n)} max\{a_1^{'}, a_2^{'}, \cdots, a_m^{'}\} \leq min\{\frac{1-\delta}{3}\varepsilon, 1\}, \tag{3.5}$$

then we have

$$d(\widetilde{X}_2, \overline{X}) < \varepsilon,$$

where $\overline{X}$ and $\widetilde{X}_2$ are the unique positive definite solution of Eq. (1.2) and its perturbed equation (3.2), respectively.

Proof. From the proof of Theorem 5 it follows that F is a strict contraction map on $P(n)$ with the contraction constant δ, that is to say,

$$F \in M(P(n), \delta).$$

Now we consider the perturbed equation (*3.2*). Let

$$\widetilde{F}(X) = \sum_{i=1}^{m} (\widetilde{A}_i^* X \widetilde{A}_i)^{\delta_i}, \quad X \in P(n).$$

Using the similar method in Theorem 5, it is easy to verify that the map $\widetilde{F}$ is a strict contraction with the contraction constant δ, i.e. $\widetilde{F} \in M(P(n), \delta)$ and this show that the perturbed equation (*3.2*) has a unique positive definite solution $\widetilde{X}_2$, i.e.

$$\widetilde{X}_2 = \sum_{i=1}^{m} (\widetilde{A}_i^* \widetilde{X}_2 \widetilde{A}_i)^{\delta_i}.$$

We will give an upper bound for $d(\widetilde{X}_2, \overline{X})$ by making use of the contraction map's perturbation Theorem 8. For arbitrary $X \in P(n)$, according to (2.3) and Lemma 1, we have

$$\begin{array}{rcl} d(\widetilde{F}(X), F(X)) & = & d(\sum\limits_{i=1}^{m} (\widetilde{A}_i^* X \widetilde{A}_i)^{\delta_i}, \sum\limits_{i=1}^{m} (A_i^* X A_i)^{\delta_i}) \\ & \leq & max\{d((\widetilde{A}_1^* X \widetilde{A}_1)^{\delta_1}, (A_1^* X A_1)^{\delta_1}), d(\sum\limits_{i=2}^{m} (\widetilde{A}_i^* X \widetilde{A}_i)^{\delta_i}, \sum\limits_{i=2}^{m} (A_i^* X A_i)^{\delta_i})\} \\ & \leq & max\{d((\widetilde{A}_1^* X \widetilde{A}_1)^{\delta_1}, (A_1^* X A_1)^{\delta_1}), max\{d((\widetilde{A}_2^* X \widetilde{A}_2)^{\delta_2}, (A_2^* X A_2)^{\delta_2}), \\ & & d(\sum\limits_{i=3}^{m} (\widetilde{A}_i^* X \widetilde{A}_i)^{\delta_i}, \sum\limits_{i=3}^{m} (A_i^* X A_i)^{\delta_i})\}\} \\ & = & max\{d((\widetilde{A}_1^* X \widetilde{A}_1)^{\delta_1}, (A_1^* X A_1)^{\delta_1}), d((\widetilde{A}_2^* X \widetilde{A}_2)^{\delta_2}, (A_2^* X A_2)^{\delta_2}), \\ & & d(\sum\limits_{i=3}^{m} (\widetilde{A}_i^* X \widetilde{A}_i)^{\delta_i}, \sum\limits_{i=3}^{m} (A_i^* X A_i)^{\delta_i})\} \\ & \leq & \cdots\cdots \\ & \leq & max\{d((\widetilde{A}_1^* X \widetilde{A}_1)^{\delta_1}, (A_1^* X A_1)^{\delta_1}), d((\widetilde{A}_2^* X \widetilde{A}_2)^{\delta_2}, (A_2^* X A_2)^{\delta_2}), \\ & & \cdots, d((\widetilde{A}_m^* X \widetilde{A}_m)^{\delta_m}, (A_m^* X A_m)^{\delta_m})\} \\ & = & max\{a_1', a_2', \cdots, a_m'\}. \end{array} \tag{3.6}$$

By (*3.5*) and (*3.6*), we have

$$\sup_{X \in P(n)} d(\widetilde{F}(X), F(X)) \leq \sup_{X \in P(n)} max\{a_1', a_2', \cdots, a_m'\} \leq min\{\frac{1-\delta}{3}\varepsilon, 1\},$$

and from Lemma 7 it follows that

$$d(\widetilde{X}_2, \overline{X}) < \varepsilon. \quad \square$$

□

4 Numerical Experiments

In this Section, we use the iterative methods (2.4) and (2.7) to compute the unique positive definite solution of Eqs. (1.1) and (1.2), respectively. The positive definite solutions are computed for different matrices A_i, $i = 1, 2, \cdots, m$ and different values of δ_i $i = 1, 2, \cdots, m$. All programs are written in MATLAB version 7.1.

Example 10. *Consider the matrix equation*

$$X = A_1^* X^{\frac{1}{2}} A_1 + A_2^* X^{-\frac{1}{3}} A_2, \tag{4.1}$$

where

$$A_1 = \begin{pmatrix} 0.3060 & 0.6894 & 0.6093 \\ 0.2514 & 0.4285 & 0.7642 \\ 0.0222 & 0.0987 & 0.8519 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.9529 & 0.6450 & 0.4801 \\ 0.4410 & 0.1993 & 0.9823 \\ 0.9712 & 0.0052 & 0.9200 \end{pmatrix}.$$

We use the iterative method (2.4) to solve Eq.(4.1). Let

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

After 21 *iterations of iterative method (2.4), we get the unique positive definite solution*

$$X \approx X_{21} = \begin{pmatrix} 1.4739 & 1.0322 & 1.7366 \\ 1.0322 & 1.4710 & 2.1083 \\ 1.7366 & 2.1083 & 5.4395 \end{pmatrix},$$

and its residual error $R(X_{21}) = \|X - A_1^* X_{21}^{\frac{1}{2}} A_1 - A_2^* X_{21}^{-\frac{1}{3}} A_2\|_2 = 3.91 \times 10^{-15}$.

Example 11. *Consider the matrix equation*

$$X = (A_1^* X A_1)^{-\frac{1}{3}} + (A_2^* X A_2)^{\frac{1}{2}}, \tag{4.2}$$

where

$$A_1 = \begin{pmatrix} 0.4710 & 0.0020 & 0.0400 \\ 0.0200 & 0.4720 & -0.0200 \\ -0.0400 & -0.0010 & 0.4700 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.2000 & 0.2000 & 0.1000 \\ 0.3000 & 0.1500 & 0.1500 \\ 0.1000 & 0.1000 & 0.2500 \end{pmatrix}.$$

We use the iterative method (2.7) to solve Eq.(4.2). Let

$$X_0 = \begin{pmatrix} 0.2000 & 0.1000 & 0 \\ 0.1000 & 0.2000 & 0.1000 \\ 0 & 0.1000 & 0.2000 \end{pmatrix}.$$

After 40 *iterations of the iterative method (2.7), we get the unique positive definite solution*

$$X \approx X_{40} = \begin{pmatrix} 5.8128 & 0.7622 & 0.4909 \\ 0.7622 & 4.0749 & 0.4465 \\ 0.4909 & 0.4465 & 7.8583 \end{pmatrix},$$

and its residual error $R(X_{40}) = \|X - (A_1^* X A_1)^{-\frac{1}{3}} - (A_2^* X A_2)^{\frac{1}{2}}\|_2 = 4.03 \times 10^{-15}$.

The above examples show that the iterative methods (2.4) and (2.7) is feasible and effective to compute the unique positive definite solution of Eqs. (1.1) and (1.2), respectively.

5 Conclusion

In this paper, we consider the nonlinear matrix equations (1.1) and (1.2). We firstly use Thompson metric method to obtain that Eqs. (1.1) and (1.2) always have a unique positive definite solution. Iterative methods are proposed to compute the unique positive definite solution of Eqs. (1.1) and (1.2). We also show that the iterative methods are more effective as δ decreases. Based on the perturbation theorem of contraction map [16], we derived two perturbation bounds for the unique positive definite solution of Eqs. (1.1) and (1.2) in the end.

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