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THE n-DIMENSIONAL CONTINUOUS WAVELET TRANSFORMATION ON GELFAND AND SHILOV TYPE SPACES

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Abstract. In this paper the wavelet transformation on Gelfand and Shilov spaces of type $W_M(\square^n)$, $W^{\Omega}(\Delta^n)$ and $W_M^{\Omega}(\Delta^n)$ is studied. It is shown that $W_\psi\phi : W_M(\square^n) \to W_M(\square^n \times \square_+^n)$, $W_\psi\phi : W^{\Omega}(\Delta^n) \to W^{\Omega}(\Delta^n \times \square_+^n)$ and $W_\psi\phi : W_M^{\Omega}(\Delta^n) \to W_M^{\Omega}(\Delta^n \times \square_+^n)$ is linear and continuous where $\square^n$ and Δ^n are n-dimensional real numbers and complex numbers. A boundedness result in a generalized Sobolev space is derived.

1 Introduction

The spaces of type $W_M(\square^n)$, $W^{\Omega}(\Delta^n)$ and $W_M^{\Omega}(\Delta^n)$ were investigated by Gelfand and Shilov ([3]) and Friedman ([2]). It was shown that the Fourier transformation $\mathcal{F} : W_M(\square^n) \to W^{\Omega}(\Delta^n)$, $\mathcal{F} : W^{\Omega}(\Delta^n) \to W_M(\square^n)$ and $\mathcal{F} : W_M^{\Omega}(\Delta^n) \to W_{M_1}^{\Omega_1}(\Delta^n)$ are linear and continuous. The main objective of this paper is to study the continuous and wavelet transform of these W-type spaces. A complex-valued continuous function ψ with property

$$\int_{\square^n} \psi(t)\, dt = 0$$

is called a *wavelet*.

The n-dimensional continuous wavelet transformation of function ϕ with respect to wavelet ψ is defined by

$$\widetilde{\phi}(\sigma, a) = (W_\psi\phi)(\sigma, a) = (2\pi)^{-n/2} \int_{\square^n} \phi(t)\, \psi\left(\frac{t-\sigma}{a}\right) \frac{dt}{a^{n/2}} \tag{1.1}$$

provided the integral exist for $a = (a_1, a_2, a_3, \cdots, a_n) \in \square_+^n$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n) \in \square^n$. If $\phi \in L^2(\square^n)$ and $\psi \in L^2(\square^n)$ then by the Parseval formula of the Fourier

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transformation, the wavelet transformation (1.1) can be defined in the Fourier space by the following formula:

$$\widetilde{\phi}(\sigma,a)=(W_\psi\phi)(\sigma,a)=(2\pi)^{-n/2}\int_{\square^n}e^{i\langle x,\sigma\rangle}\overline{\widehat{\psi}(ax)}\,\widehat{\phi}(x)\,dx. \tag{1.2}$$

The reconstruction formula for (1.1) is given by

$$\begin{aligned}\phi(x) &= \left(\left(W_\psi^{-1}\phi\right)(\sigma,a)\right)(x)\\ &= C_\psi^{-1}(2\pi)^{-n/2}\left(\int_{\square_+^n}\left(\int_{\square^n}\widetilde{\phi}(\sigma,a)\,\psi\left(\frac{x-\sigma}{a}\right)\frac{da}{a^{2n}}\right)d\sigma\right)\end{aligned} \tag{1.3}$$

where $C_\psi=\int_{\square^n}\frac{\left|\widehat{\psi}(x)\right|^2}{x}dx>0$.

We use [2] and [3] for the definitions of W-spaces of type $W_M(\square^n)$, $W^\Omega(\Delta^n)$ and $W_M^\Omega(\Delta^n)$. Let M_j and Ω_j be the convex functions such that

$$M_j(x_j) = \int_0^{x_j}\mu(\xi_j)\,d\xi_j \quad (x_j\geq 0), \tag{1.4}$$

$$\Omega_j(y_j) = \int_0^{y_j}\omega(\eta_j)\,d\eta_j \quad (y_j\geq 0) \tag{1.5}$$

for $j=1,\,2,\,3,\,\cdots,\,n$.

We set

$$\mu(\xi) = (\mu_1(\xi_1)),\cdots,(\mu_n(\xi_n)), \qquad \omega(\eta)=(\omega_1(\eta_1)),\cdots,(\omega_n(\eta_n))$$

and

$$M_j(-x_j) = M_j(x_j), \qquad M_j(x_j)+M_j\left(x_j'\right)\leq M_j\left(x_j+x_j'\right) \tag{1.6}$$

$$\Omega_j(-y_j) = \Omega_j(y_j), \qquad \Omega_j(y_j)+\Omega_j\left(y_j'\right)\leq\Omega_j\left(y_j+y_j'\right). \tag{1.7}$$

The space $W_M(\square^n)$ consists of all C^∞-complex valued function $\phi(x)$ on $\square^n$ which satisfy the inequalities

$$\left|D_x^k\phi(x)\right| \leq C_k\exp\left[-M(\alpha x)\right], \tag{1.8}$$

where $\exp\left[-M(\alpha x)\right]=\exp\left[-M_1(\alpha_1x_1)\cdots-M_j(\alpha_jx_j)\cdots-M_n(\alpha_nx_n)\right]$ and C_k, $\alpha>0$ are constants which may depend on the function ϕ.

A function $\phi(z)\in W^\Omega(\Delta^n)$ if and only if for $\beta>0$ there exists a constant $C_k>0$ such that

$$\left|z^k\phi(z)\right| \leq C_k\exp\left[\Omega(\beta y)\right], \qquad z=x+iy \tag{1.9}$$

and

$$\exp\left[\Omega\left(\beta y\right)\right] = \exp\left[\Omega\left[\left(\beta_1 y_1\right) + \cdots + \Omega\left(\beta_j y_j\right) + \cdots + \Omega\left(\beta_n y_n\right)\right]\right],$$

where the constants C_k and β depend on the function ϕ.

The space $W_M^{\Omega}\left(\Delta^n\right)$ consists of all entire analytic functions $\phi\left(z\right)$ there exist constants $\alpha > 0$, $\beta > 0$ and $C > 0$ such that

$$\left|\phi\left(z\right)\right| \leq C\left[\exp\left[-M\left(\alpha x\right)\right] + \Omega\left[\left(\beta y\right)\right]\right], \qquad z = x + iy \tag{1.10}$$

where $\exp\left[-M\left(\alpha x\right)\right]$ and $\exp\left[\Omega\left[\left(\beta y\right)\right]\right]$ have usual meaning like (1.8) and (1.9) and constants C, α and β depend on the function ϕ.

Now we define the duality of functions $M\left(x\right)$ and $\Omega\left(y\right)$ following the Young's concept. Let $M\left(x\right)$ and $\Omega\left(y\right)$ be defined by (1.6) and (1.7) respectively by the function $\mu\left(\xi\right)$ and $\omega\left(\eta\right)$ which occur in these equations are mutually inverse, that is $\mu\left(\omega\left(\eta\right)\right) = \eta$ and $\omega\left(\mu\left(\xi\right)\right) = \xi$, then the corresponding functions $M\left(x\right)$ and $\Omega\left(y\right)$ are said to be dual in sense of Young. In this case, the Young inequality

$$x_j\, y_j \leq M_j\left(x_j\right) + \Omega_j\left(y_j\right) \tag{1.11}$$

holds for any $x_j \geq 0$, $y_j \geq 0$, where equality holds if and only if $y_j = \mu_j\left(x_j\right)$ and x_j varies in the interval $x_j^0 < x_j < \infty$, y_j varies in $y_j^0 < y_j < \infty$. The Fourier duality relation is given by $\mathcal{F}\left[W_M\left(\square^n\right)\right] = W^{\Omega}\left(\Delta^n\right)$, $\mathcal{F}\left[W^{\Omega}\left(\Delta^n\right)\right] = W_M\left(\square^n\right)$ and $\mathcal{F}\left[W_M^{\Omega}\left(\Delta^n\right)\right] = W_{M_1}^{\Omega_1}\left(\Delta^n\right)$ from [2] and [3].

The present article is divided into three sections. Section 1 is an introduction with notations, auxiliary results of the wavelet transformation and W-type spaces. Section 2 gives the characterization of W-spaces by using the concepts of the wavelet transformation. In the last section we derive some results of the wavelet transformation on L^p-Sobolev spaces.

2 Characterization of W-type spaces

In this section, we study the characterization of W-type space by using wavelet transformation

Lemma 1. *Let* $\widehat{\psi}\left(z\right) \in W^{\Omega}\left(\Delta^n\right)$ *and* $\widehat{\phi} \in W^{\Omega}\left(\Delta^n\right)$. *Then* $\widehat{\psi}\,\widehat{\phi} \in W^{\Omega}\left(\Delta^n\right)$.

Proof. Since $a \in \square^n$ and $z \in \Delta^n$. Therefore $\widehat{\psi}\left(az\right) \in W^{\Omega}\left(\Delta^n\right)$ and $\widehat{\phi} \in W^{\Omega}\left(\Delta^n\right)$.

This indicates that

$$\begin{aligned}\left|z^{k+m}\widehat{\psi}(az)\,\widehat{\phi}(z)\right| &= \left|z^{k}\widehat{\psi}(az)\,z^{m}\widehat{\phi}(z)\right| \\ &= \left|z^{k}\widehat{\psi}(az)\right|\left|z^{m}\widehat{\phi}(z)\right| \\ &\leq C_k \exp\left[\Omega\left[(\alpha a y)\right]\right] C_m \exp\left[\Omega\left[(ay)\right]\right] \\ &\leq C_k C_m \exp\left[\Omega\left[\alpha\,(a+1)\,y\right]\right] \\ &\leq C_{k,m} \exp\left[\Omega\left[\alpha\,(a+1)\,y\right]\right].\end{aligned}$$

□

Theorem 2. *Let the $M(x)$ and $\Omega(y)$ be the functions, which are dual in sense of Young and $\psi \in W_M(\square^n)$ and $\phi \in W_M(\square^n)$. Then the wavelet transformation $W_\psi\phi : W_M(\square^n) \to W_M\left(\square^n \times \square^n_+\right)$ is continuous and linear.*

Proof. The wavelet transformation $W_\psi\phi$ with respect to wavelet ψ is given by

$$(W_\psi\phi)(\sigma, a) = (2\pi)^{-n/2} \int_{\square^n} e^{i\langle\sigma,x\rangle}\widehat{\psi}(ax)\,\widehat{\phi}(x)\,dx.$$

It follows from [2, p. 121.] that

$$\begin{aligned}(W_\psi\phi)(\sigma, a) &= (2\pi)^{-n/2} \int_{\square^n} e^{i\langle\sigma,z\rangle}\widehat{\psi}(az)\,\widehat{\phi}(z)\,dx \\ &= (2\pi)^{-n/2} \int_{\square^n} e^{i\langle\sigma,z\rangle}\widehat{\psi}(az)\,\widehat{\phi}(z)\,dx.\end{aligned}$$

For $\Phi(z,a) = \widehat{\psi}(az)\,\widehat{\phi}(z)$. Therefore, $(W_\psi\phi)(\sigma,a) = (2\pi)^{-n/2}\int_{\square^n} e^{i\langle\sigma,z\rangle}\Phi(z,a)\,dx$, where $z = (z_1, z_2, z_3 \cdots z_n)$, $a = (a_1, a_2, a_3 \cdots a_n)$ and $z = x + iy$.
Differentiating the above function with respect to σ and a gives

$$D^k_\sigma D^m_a (W_\psi\phi)(\sigma, a) = (2\pi)^{-n/2} \int_{\square^n} (iz)^k\, e^{i\langle\sigma,z\rangle} D^m_a \Phi(z, a)\,dx.$$

Then, applying Lemma 1, we get

$$\begin{aligned}
&\left|D_\sigma^k D_a^m \left(W_\psi \phi\right)(\sigma,a)\right| \\
\leq\ & (2\pi)^{-n/2}\left(\int_{\square^n} e^{-\sigma y}\left|z^k D_a^m \Phi(z,a)\right| dx\right) && (2.1)\\
\leq\ & (2\pi)^{-n/2}\left(\int_{\square^n} e^{-\sigma y}\frac{|z|^{k+2}+|z|^k}{(x^2+1)^n}\left|D_a^m \Phi(z,a)\right| dx\right) \\
\leq\ & (2\pi)^{-n/2}\left[\exp(-\sigma y)\int_{\square^n}(C_{k+2}+C_k)\exp\left[\Omega\left[(a+1)\alpha y\right]\right]\frac{dx}{(x^2+1)^n}\right] \\
\leq\ & (2\pi)^{-n/2}\exp\left[-\sigma y+\Omega\left[(a+1)\alpha y\right]\right](C_{k+2}+C_k)C_n \\
\leq\ & (2\pi)^{-n/2}\exp\left[-\sigma y+\Omega\left[(a+1)\alpha y\right]\right](C_{k+2,n}+C_{k,n}) \\
\leq\ & (2\pi)^{-n/2}D_{k,n}\exp\left[-\sigma y+\Omega\left[(a+1)\alpha y\right]\right]. && (2.2)
\end{aligned}$$

Since y is found an arbitrary number then using [2, p. 22], we choose the sign of y in such a way that the following equality holds

$$|\sigma|\,|y| = M\left[\frac{\sigma}{(a+1)\alpha}\right]+\Omega\left[(a+1)\alpha y\right].$$

The exponent in the expression (2.2) becomes

$$-\sigma y+\Omega\left[(a+1)\alpha y\right] = -M\left[\frac{\sigma}{(a+1)\alpha}\right].$$

Then we get

$$\left|D_\sigma^k D_a^m \left(W_\psi \phi\right)(\sigma,a)\right| \leq E_{k,n}\exp\left[-M\left[\frac{\sigma}{(a+1)\alpha}\right]\right].$$

This implies that

$$\left(W_\psi \phi\right)(\sigma,a)\in W_M\left(\square^n\times\square_+^n\right).$$

□

Lemma 3. *Let* $\psi(ax)\in W^{\Omega}(\Delta^n)$. *Then* $D_x^\beta\widehat{\psi}(ax)\in W_M(\square^n)$.

Proof. We obtain from [3]

$$\widehat{\psi}(ax) = (2\pi)^{-n/2}\int_{\square^n} e^{i\langle ax,s\rangle}\psi(s)\,d\sigma,$$

where $s=\sigma+i\tau$, $\sigma=(\sigma_1,\sigma_2,\sigma_3,\cdots,\sigma_n)$ and $\tau=(\tau_1,\tau_2,\tau_3,\cdots,\tau_n)$.

Then

$$\begin{aligned} D_x^\beta \widehat{\psi}(ax) &= (2\pi)^{-n/2} \int_{\square^n} (ias)^\beta e^{i\langle ax,s\rangle} \psi(s)\, d\sigma \\ &= (2\pi)^{-n/2} \int_{\square^n} (ias)^\beta e^{i\langle ax,\sigma+i\tau\rangle} \psi(s)\, d\sigma \\ &= (2\pi)^{-n/2} \int_{\square^n} (ias)^\beta e^{i\langle ax,\sigma\rangle} \psi(s)\, d\sigma. \end{aligned}$$

Hence

$$\begin{aligned} &\left| D_x^\beta \widehat{\psi}(ax) \right| \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} |a|^\beta |s|^\beta e^{-ax\tau} |\psi(s)|\, d\sigma \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} \left(1+|a|^\beta\right) \left| s^\beta \psi(s) \right| \exp(-ax\tau)\, d\sigma \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} \left(1+|a|^\beta\right) \left| \left(1+s^2\right) s^\beta \psi(s) \right| \exp(-ax\tau) \frac{d\sigma}{(1+\sigma^2)^{n/2}}. \end{aligned}$$

From (1.7) and (1.9) we obtain

$$\left| D_x^\beta \widehat{\psi}(ax) \right| \leq C_n \left(1+|a|^\beta\right) (C_\beta + C_{\beta+2}) \exp\left[\Omega(\beta\tau) - ax\tau\right].$$

Using (1.11) gives

$$\left| D_x^\beta \widehat{\psi}(ax) \right| \leq (2\pi)^{-n/2} C_n \left(1+|a|^\beta\right) (C_\beta + C_{\beta+2}) \exp\left[-M\left(\frac{ax}{\beta}\right)\right].$$

Then we obtain

$$\left| \exp\left[M\left(\frac{ax}{\beta}\right)\right] D_x^\beta \widehat{\psi}(ax) \right| \leq C_{n,\alpha,\beta} \left(1+|a|^\beta\right).$$

□

Lemma 4. *Let* $\widehat{\phi}(x) \in W_M(\square^n)$ *and* $\widehat{\psi}(ax) \in W_M(\square^n)$*. Then*

$$|D_x^m \Phi(x,a)| \leq (1+|a|^m)\, D_m \exp\left[-M\left(\frac{a-1}{\alpha}\right)x\right], \qquad m>0 \quad \textit{and} \quad \alpha>0,$$

where $\Phi(x,a) = \widehat{\phi}(x)\,\widehat{\psi}(ax)$.

Proof. We have

$$\Phi(x,a)=\widehat{\phi}(x)\,\widehat{\psi}(ax)\,.$$

Then

$$\begin{aligned}\left|D_x^m\Phi(x,a)\right| &= \left|D_x^m\left[\widehat{\phi}(x)\,\widehat{\psi}(ax)\right]\right|\\ &= \left|\sum_{\delta\le m}\binom{m}{\delta}D_x^{m-\delta}\widehat{\phi}(x)\,D_x^{\delta}\widehat{\psi}(ax)\right|\\ &= \sum_{\delta\le m}\binom{m}{\delta}\left|D_x^{m-\delta}\widehat{\phi}(x)\right|\left|D_x^{\delta}\widehat{\psi}(ax)\right|.\end{aligned}$$

From Lemma 3, the above expression yields

$$\begin{aligned}&\left|D_x^m\Phi(x,a)\right|\\ \le\ & \sum_{\delta\le m}\binom{m}{\delta}C_{m-\delta}\exp\left[-M\left(\frac{x}{\alpha}\right)\right]C_{\delta}\left(1+|a|\right)^{\delta}\exp\left[-M\left(\frac{ax}{\alpha}\right)\right]\\ \le\ & \sum_{\delta\le m}\binom{m}{\delta}C_{m-\delta}C_{\delta}\left(1+|a|\right)^{\delta}\exp\left[M\left(\frac{x}{\alpha}\right)-M\left(\frac{ax}{\alpha}\right)\right].\end{aligned}$$

Using (1.6), we obtain

$$\begin{aligned}&\left|D_x^m\Phi(x,a)\right|\\ \le\ & (1+|a|)^m\left(\sum_{\delta\le m}\binom{m}{\delta}C_{m-\delta}C_{\delta}\exp\left[-M\left(\frac{a-1}{\alpha}\right)x\right]\right)\\ \le\ & (1+|a|)^m\exp\left[-M\left(\frac{a-1}{\alpha}\right)x\right]\left(\sum_{\delta\le m}\binom{m}{\delta}C_{m-\delta}\,C_{\delta}\right)\\ \le\ & (1+|a|)^m\,D_m\exp\left[-M\left(\frac{a-1}{\alpha}\right)x\right].\end{aligned}$$

□

Theorem 5. *Let $M(x)$ and $\Omega(y)$ be the functions which are same as in Theorem 2.2 and $\phi\in W^{\Omega}(\Delta^n)$, $\psi\in W^{\Omega}(\Delta^n)$. Then the wavelet transformation $W_{\psi}\phi$ is a continuous linear mapping from $W^{\Omega}(\Delta^n)$ into $W^{\Omega}\left(\Delta^n\times\square_+^n\right)$.*

Proof. The wavelet transformation of a function ϕ with respect to wavelet ψ is given by

$$\left(W_{\psi}\phi\right)(\sigma,a)=(2\pi)^{-n/2}\int_{\square^n}e^{i\langle x,\sigma\rangle}\widehat{\phi}(x)\,\widehat{\psi}(ax)\,dx.$$

From [2, p. 20-21], we find that

$$(W_\psi \phi)(s,a) = (2\pi)^{-n/2} \int_{\square^n} e^{i\langle x,s\rangle} \Phi(x,a)\, dx,$$

where $\Phi(x,a) = \widehat{\phi}(x)\, \widehat{\psi}(ax)$.
Hence

$$(is)^m (W_\psi \phi)(s,a) = (-1)^{|m|} (2\pi)^{-n/2} \int_{\square^n} e^{i\langle x,s\rangle} D_x^m \Phi(x,a)\, dx.$$

Therefore,

$$\begin{aligned} & |(is)^m (W_\psi \phi)(s,a)| \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} \exp(-|x\tau|)\, |D_x^m \Phi(x,a)|\, dx \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} \exp(-|x|\,|\tau|)\, |D_x^m \Phi(x,a)|\, dx. \end{aligned}$$

Using Lemma 4, we get

$$\begin{aligned} & |(is)^m (W_\psi \phi)(s,a)| \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} D_m (1+|a|)^m \exp\left[-M\left(\frac{a-1}{\alpha}\right)x\right] \exp(|x|\,|\tau|)\, dx \\ \leq\ & (2\pi)^{-n/2} D_m (1+|a|)^m \left(\int_{\square^n} \exp\left[|x|\,|\tau| - M\left[\left(\frac{a-1}{\alpha}\right)x\right]\right] dx\right). \end{aligned}$$

Using Young inequality (1.11), the above integral can be evaluated as follows

$$\begin{aligned} |x|\,|\tau| - M\left[\left(\frac{a-1}{\alpha}\right)x\right] &\leq M\left[\left(\frac{a-2}{\alpha}\right)x\right] + \Omega\left[\left(\frac{\alpha}{a-2}\right)\tau\right] - M\left[\left(\frac{a-1}{\alpha}\right)x\right] \\ &\leq -M\left(\frac{x}{\alpha}\right) + \Omega\left[\left(\frac{\alpha}{a-2}\right)\tau\right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & |(is)^m (W_\psi \phi)(s,a)| \\ \leq\ & (2\pi)^{-n/2} D_m (1+|a|)^m \exp\left[\Omega\left[\frac{\alpha}{(a-2)}\right]\tau\right] \int_{\square^n} \exp\left[-M\left(\frac{x}{\alpha}\right)\right] dx \\ \leq\ & D_{m,n,\alpha} (1+|a|)^m \exp\left[\Omega\left[\frac{\alpha}{(a-2)}\right]\tau\right]. \end{aligned}$$

Hence,

$$\left|(is)^m (1+a)^{-m} (W_\psi \phi)(s,a)\right| \leq D_{m,n,\alpha} \exp\left[\Omega\left\{(a-2)^{-1}\alpha\right\}\tau\right].$$

This implies that

$$W_\psi \phi \in W^\Omega\left(\Delta^n \times \square^n_+\right).$$

□

Lemma 6. *Let* $\widehat{\phi} \in W_{M_1}^{\Omega_1}(\Delta^n)$ *and* $\widehat{\psi}(az) \in W_{M_1}^{\Omega_1}(\Delta^n)$. *Then*

$$|\Phi(z,a)| \leq D \exp\left[-M_1\{(a-1)\alpha x\} + \Omega_1[(a+1)\beta y]\right].$$

Proof. Now,

$$\begin{aligned} |\Phi(z,a)| &= \left|\widehat{\phi}(z)\,\widehat{\psi}(az)\right| \\ &= \left|\widehat{\phi}(z)\right|\left|\widehat{\psi}(az)\right| \\ &\leq C\exp\left[-M_1(\alpha x) + \Omega_1(\beta y)\right] C' \exp\left[-M_1(a\alpha x) + \Omega_1(a\beta y)\right]. \end{aligned}$$

From (1.6) and (1.7) we have

$$\begin{aligned} |\Phi(z,a)| &\leq CC' \exp\left[\{-M_1(a\alpha x) + M_1(\alpha x)\} + \Omega_1[(\beta y) + \Omega_1(a\beta y)]\right] \\ &\leq D\exp\left[-M_1\{(a-1)\alpha x\} + \Omega_1\{(a+1)\beta y\}\right]. \end{aligned}$$

Therefore,

$$|\Phi(z,a)| \leq D\exp\left[-M_1\{(a-1)\alpha x\} + \Omega_1\{(a+1)\beta y\}\right].$$

□

Theorem 7. *Let* $\Omega_1(x)$ *and* $M_1(x)$ *be the functions which are dual in the sense of Young to the functions* $M(x)$ *and* $\Omega(x)$ *and* $\phi \in W_M^{\Omega}(\Delta^n)$, $\psi \in W_M^{\Omega}(\Delta^n)$. *Then the wavelet transformation* $W_\psi \phi$ *is a continuous linear mapping from* $W_M^{\Omega}(\Delta^n)$ *into* $W_M^{\Omega}\left(\Delta^n \times \square_+^n\right)$.

Proof. From (1.2) the wavelet transformation is given by

$$\begin{aligned} (W_\psi \phi)(\sigma,a) &= (2\pi)^{-n/2}\int_{\square^n} e^{i\langle x,\sigma\rangle}\,\widehat{\phi}(x)\,\widehat{\psi}(ax)\,dx \\ &= (2\pi)^{-n/2}\int_{\square^n} e^{i\langle x,\sigma\rangle}\,\Phi(x,a)\,dx, \end{aligned}$$

where $\Phi(x,a) = \widehat{\phi}(x)\,\widehat{\psi}(ax)$.
It follows from [2, p. 23] that

$$\begin{aligned} &(W_\psi \phi)(s,a) \\ = \ & (2\pi)^{-n/2}\int_{\square^n} e^{i\langle s,z\rangle}\,\Phi(z,a)\,dz, \\ = \ & (2\pi)^{-n/2}\left(\int_{\square^n} \exp\left[i(\sigma + i\tau)(x+iy)\right]\Phi(z,a)\,dx\right) \\ = \ & (2\pi)^{-n/2}\left(\int_{\square^n} \exp\left[i(-\sigma y - \tau x) + i(\sigma x - \tau y)\right]\Phi(z,a)\,dx\right). \end{aligned}$$

So that

$$\begin{aligned} & |(W_\psi \phi)(s,a)| \\ \leq\ & (2\pi)^{-n/2} \int_{\square^n} \exp\left[(-\tau x - \sigma y)\right] |\Phi(z,a)|\, dx \\ \leq\ & (2\pi)^{-n/2} D \int_{\square^n} \exp\left[(\tau x - \sigma y)\right] \exp\left[-M_1 (a-1)\,\alpha x\right] + \Omega_1 \left[(a+1)\,\beta y\right] dx \\ =\ & (2\pi)^{-n/2} D \exp\left[-\sigma y + \Omega_1 \left\{(a+1)\,\beta y\right\}\right] \int_{\square^n} \exp\left[\tau x - M_1 \left\{(a-1)\,\alpha x\right\}\right] dx. \end{aligned}$$

Now using the arguments similar to those of Theorem 2.2 and Theorem 2.5, we find that

$$\begin{aligned} |(W_\psi \phi)(s,a)| & \leq D \exp\left[-M\left\{\frac{\sigma}{\beta(1+a)}\right\} + \Omega\left\{\frac{\tau}{(a+2)}\right\}\right] \int_{\square^n} \exp\left[-M(\alpha x)\right] dx \\ & \leq D' \exp\left[-M\left\{\frac{\sigma}{\beta(1+a)}\right\} + \Omega\left\{\frac{\tau}{(a+2)}\right\}\right]. \end{aligned}$$

Hence

$$(W_\psi \phi)(s,a) \in W_M^\Omega\left(\Delta^n \times \square_+^n\right).$$

□

3 Wavelet transformation on $L^p(\square^n)$-Sobolev spaces

In this section we study the wavelet transformation of a function $\phi \in W_M^\Omega(\Delta^n)$ with respect to wavelet $\psi \in W_M^\Omega(\Delta^n)$ on L^p-type Sobolev space.

Definition 8. *For $m \in \square$ and $1 \leq p < \infty$, the space $G^{m,p}(\Delta^n)$ is defined to the set of all $\phi \in \left(W_M^\Omega(\Delta^n)\right)'$ such that*

$$\|\phi\|_{G^{m,p}} = \left(\int_{\square^n} \left(1+|s|^2\right)^{m/2} \left|\widehat{\phi}(s)\right|^p d\sigma\right)^{\frac{1}{p}},$$

where $\widehat{\phi} = \mathcal{F}\phi$ and $s = \sigma + i\tau$.

Definition 9. *We define the space of all measurable function ϕ on $\square_+^n \times \Delta^n$ such that*

$$\left\|\widetilde{\Phi}(s,a)\right\|_{H_p^{p',m}} = \left[\int_{\square_+^n} \left(\int_{\square^n} |\Phi(s,a)|\, d\sigma\right)^{p'/p} a^{-m-1} da\right],$$

where $s = \sigma + i\tau$.

Lemma 10. *Let* $\phi \in W_M^{\Omega}(\Delta^n)$ *and* $\psi \in W_M^{\Omega}(\Delta^n)$. *Then*

$$\left[\int_{\square_+^n} a^{m-1}\left\{\int_{\square^n}\left|\widehat{\psi}(as)\,\widehat{\phi}(s)\right|^{p'}\right\}da\right] = C_{\psi}^{m,p'}\left[\int_{\square^n}\left(1+|s|^2\right)^{m/2}\left|\widehat{\phi}(s)\right|^{p'}d\sigma\right],$$

where $m \in \square^n$, $a \in \square_+^n$ *and* $s = \sigma + i\tau$,

$$C_{\psi}^{m,p'} = \sup_s\left[\int_{\square_+^n} a^{-m-1}\left(1+|s|^2\right)^{-m/2}\left|\widehat{\psi}(as)\right|^{p'}da\right].$$

Proof. Now

$$\begin{aligned}
&\int_{\square_+^n} a^{-m-1}\left\{\int_{\square^n}\left|\widehat{\psi}(as)\,\widehat{\phi}(s)\right|^{p'}d\sigma\right\}da \\
&= \int_{\square_+^n} a^{-m-1}\left\{\int_{\square^n}\left(1+|s|^2\right)^{-m/2}\left|\widehat{\psi}(as)\right|^{p'}\left(1+|s|^2\right)^{m/2}\left|\widehat{\phi}(s)\right|^{p'}d\sigma\right\}da \\
&= \int_{\square^n}\left\{\int_{\square_+^n} a^{-m-1}\left(1+|s|^2\right)^{-m/2}\left|\widehat{\psi}(as)\right|^{p'}da\right\}\left(1+|s|^2\right)^{m/2}\left|\widehat{\phi}(s)\right|^{p'}d\sigma \\
&= \sup_s\left[\int_{\square_+^n}\left(1+|s|^2\right)^{-m/2}a^{-m-1}\left|\widehat{\psi}(as)\right|^{p'}da\right]\left[\int_{\square^n}\left(1+|s|^2\right)^{m/2}\left|\widehat{\phi}(s)\right|^{p'}d\sigma\right] \\
&= C_{\psi}^{m,p'}\left[\int_{\square^n}\left(1+|s|^2\right)^{m/2}\left|\widehat{\phi}(s)\right|^{p'}d\sigma\right].
\end{aligned}$$

□

Theorem 11. *If* $m \in \square^n$, $\frac{1}{p}+\frac{1}{p'}=1$ *and* $1 \le p < 2$, *then, we have the following relation:*

$$\left\|\widetilde{\Phi}(s,a)\right\|_{H_p^{p',m}} = \left(C_{\psi}^{m,p}\right)^{1/p'}\|\phi\|_{G^{m,p}}.$$

Proof. From Theorem 5 the wavelet transformation of a function $\psi \in W_M^{\Omega}(\Delta^n)$ can be written as

$$(W_{\psi}\phi)(s,a) = \left(\int_{\square^n} e^{isz}\widehat{\phi}(s)\,\widehat{\psi}(az)\,dx\right), \tag{3.1}$$

where $s = \sigma + i\tau$ and $z = x + iy$.

Therefore,

$$(W_{\psi}\phi)(s,a) = \mathcal{F}\left[\widehat{\phi}(z)\,\widehat{\psi}(az)\right](s),$$

so that

$$\mathcal{F}^{-1}\left[W_{\psi}(s,a)\right](z) = \widehat{\phi}(z)\,\widehat{\psi}(az).$$

Using the Hansdorff–Young inequality for $1 \le p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned}\left[\int_{\square^n} \left|\overline{\widehat{\psi}(az)}\,\widehat{\phi}(z)\right|^{p'} dx\right]^{1/p'} &= \left[\int_{\square^n} \left|\mathcal{F}^{-1}\left[W_\psi \phi(s,a)\right](z)\right|^{p'} dx\right]^{1/p'} \\ &\le C_p \left[\int_{\square^n} \left|(W_\psi \phi)(s,a)\right|^p d\sigma\right]^{1/p},\end{aligned}$$

where $C_p > 0$ is a constant. Then

$$\begin{aligned}&\int_{\square^n_+} a^{-m-1} \left[\int_{\square^n} \left|\widehat{\psi}(az)\,\widehat{\phi}(z)\right|^{p'} dx\right] da \\ \le\;& (C_p)^{p'} \left[\int_{\square^n_+} \left\{\int_{\square^n} \left|(W_\psi \phi)(s,a)\right|^p d\sigma\right\}^{p'/p} a^{-m-1} da\right].\end{aligned}$$

Using Lemma 3, we get

$$\|\phi\|_{G^{p',m}} \le \left(\frac{C_p^{p'}}{C_\psi^{m,p'}}\right)^{1/p'} \left\|\widetilde{\Phi}(s,a)\right\|_{H_p^{p',m}}.$$

Again we can write from (3.1) so that

$$(W_\psi \phi)(s,a) = \mathcal{F}\left[\widehat{\phi}(z)\,\widehat{\psi}(az)\right](s).$$

Then

$$\begin{aligned}\left[\int_{\square^n} \left|(W_\psi \phi)(s,a)\right|^{p'} d\sigma\right]^{\frac{1}{p'}} &= \left[\int_{\square^n} \left|F\left[\widehat{\phi}(z)\,\widehat{\psi}(az)\right](s)\right|^{p'} d\sigma\right]^{\frac{1}{p'}} \\ &\le \left[\int_{\square^n} \left|\widehat{\phi}(z)\,\widehat{\psi}(az)\right|^p dx\right]^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\left[\int_{\square^n} \left|(W_\psi \phi)(s,a)\right|^{p'} d\sigma\right]^{\frac{p}{p'}} \le D_{p'} \int_{\square^n} \left|\widehat{\phi}(z)\,\widehat{\psi}(az)\right|^p dx.$$

so that

$$\begin{aligned}&\left[\int_{\square^n_+} a^{m-1} \left\{\int_{\square^n} \left|(W_\psi \phi)(s,a)\right|^{p'} d\sigma\right\}^{p/p'} da\right] \\ \le\;& D_{p'} \left[\int_{\square^n_+} a^{m-1} \left\{\int_{\square^n} \left|\widehat{\phi}(z)\,\widehat{\psi}(az)\right|^p dx\right\} da\right].\end{aligned}$$

Using Lemma 10 yields

$$\begin{aligned} & \left[\int_{\square_+^n} a^{m-1}\left\{\int_{\square^n} \left|\left(W_\psi\phi\right)(s,a)\right|^{p'} d\sigma\right\}^{p/p'} da\right] \\ \leq \;& D_{p'}C_\psi^{m,p}\left[\int_{\square^n}\left(1+|s|^2\right)^{m/2}\left|\widehat{\phi}(s)\right|^p d\sigma\right] \\ = \;& D_{p'}C_\psi^{m,p}\left\|\phi\right\|_{G^{p,m}}. \end{aligned}$$

Then

$$\left\|\widetilde{\Phi}(s,a)\right\|_{H_{p'}^{p,m}} \leq E_\psi^{m,p,p'}\left\|\phi\right\|_{G^{p,m}}.$$

□

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