Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 99 – 111

FRACTIONAL ORDER DIFFERENTIAL INCLUSIONS ON THE HALF-LINE

Mouffak Benchohra and Naima Hamidi

Abstract. We are concerned with the existence of bounded solutions of a boundary value problem on an unbounded domain for fractional order differential inclusions involving the Caputo fractional derivative. Our results are based on the fixed point theorem of Bohnnenblust-Karlin combined with the diagonalization method.

1 Introduction

This paper deals with the existence of bounded solutions for boundary value problems (BVP for short) for fractional order differential inclusions of the form

$$^{c}D^{\alpha}y(t) \in F(t, y(t)), \quad t \in J := [0, \infty), \tag{1.1}$$

$$y(0) = y_0, \quad y \text{ is bounded on } J, \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2]$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact, convex values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $y_0 \in \mathbb{R}$.

Fractional Differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, control, etc. (see [15, 17, 19, 18, 24, 27, 28]).

Recently, there has been a significant development in the study of ordinary and partial differential equations and inclusions involving fractional derivatives, see the monographs of Kilbas *et al.* [21], Lakshmikantham *et al.* [22], Miller and Ross [25], Podlubny [27], Samko *et al.* [29] and the papers by Agarwal *et al.* [1], Belarbi *et al.* [7, 8], Benchohra *et al.* [9, 10, 11, 12], Chang and Nieto [14], Diethelm *et al.* [15], and Ouahab [26].

Agarwal *et al.* [2] have considered a class of boundary value problems involving Riemann-Liouville fractional derivative on the half line. They used the diagonalization

2010 Mathematics Subject Classification: 26A33, 26A42, 34A60, 34B15.
Keywords: Boundary value problem; fractional order differential inclusions; fixed point; infinite intervals; diagonalization process.

process combined with the nonlinear alternative of Leray- Schauder type. This paper continues this study by considering a boundary value problem with the Caputo fractional derivative. We use the classical fixed point theorem of Bohnnenblust-Karlin [13] combined with the diagonalization process widely used for integer order differential equations; see for instance [3, 4]. Our results extend to the multivalued case those considered recently by Arara *et al.* [5].

2 Preliminary facts

We now introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.
Let $T > 0$ and $J := [0, T]$. $C(J, \mathbb{R})$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the usual norm

$$\|y\| = \sup\{|y(t)| : 0 \le t \le T\}.$$

$L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

$AC^1(J, \mathbb{R})$ denote the space of differentiable functions whose first derivative y' is absolutely continuous.

2.1 Fractional derivatives

Definition 1. *([21, 27]). Given an interval $[a, b]$ of $\mathbb{R}$. The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2. *([21]). For a given function h on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by*

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{m-\alpha-1} h^{(m)}(s)ds,$$

where $m = [\alpha] + 1$.

More details on fractional derivatives and their properties can be found in [21, 27]

Lemma 3. *(Lemma 2.22 [21]). Let $\alpha > 0$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{m-1}t^{m-1}, c_i \in \mathbb{R}, \quad i = 0, 1, 2, ..., m-1, \quad m = [\alpha] + 1.$$

Lemma 4. *(Lemma 2.22 [21]). Let $\alpha > 0$, then*

$$I^\alpha\ {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{m-1}t^{m-1}, \tag{2.1}$$

for arbitrary $c_i \in \mathbb{R}$, $\quad i = 0, 1, 2, ..., m-1, \quad m = [\alpha] + 1$.

2.2 Set-valued maps

Let X and Y be Banach spaces. A set-valued map $G : X \to \mathcal{P}(Y)$ is said to be compact if $G(X) = \bigcup\{G(y); y \in X\}$ is compact. G has convex (closed, compact) values if $G(y)$ is convex(closed, compact) for every $y \in X$. G is bounded on bounded subsets of X if $G(B)$ is bounded in Y for every bounded subset B of X. A set-valued map G is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set O containing Gz_0, there exists a neighborhood V of z_0 such that $G(V) \subset O$. G is usc on X if it is usc at every point of X if G is nonempty and compact-valued then G is usc if and only if G has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $\mathcal{P}_{b,cl,c}(X)$. A set-valued map $G : J \to \mathcal{P}_{cl}(X)$ is measurable if for each $y \in X$, the function $t \mapsto dist(y, G(t))$ is measurable on J. If $X \subset Y$, G has a fixed point if there exists $y \in X$ such that $y \in Gy$. Also, $\|G(y)\|_{\mathcal{P}} = sup\{|x|; x \in G(y)\}$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Frankowska [6], Deimling [16] and Hu and Papageorgiou [20].

Definition 5. *A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if*

(i) $t \longmapsto F(t, y)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t, x)\} \leq \varphi_q(t) \text{ for all } |x| \leq q \text{ and for a.e. } t \in J.$$

The multivalued map F is said of Carathéodory if it satisfies (i) and (ii).

For each $y \in C(J,\mathbb{R})$, define the set of selections of F by

$$S^1_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \ a.e.\ t \in J\}.$$

Definition 6. *By a solution of BVP (1.1)-(1.2) we mean a function $y \in AC^1(J,\mathbb{R})$ such that*

$$^cD^\alpha y(t) = g(t), \quad t \in J, \quad 1 < \alpha \le 2, \tag{2.2}$$

$$y(0) = y_0, \quad y \text{ bounded on } J, \tag{2.3}$$

where $g \in S^1_{F,y}$.

Remark 7. *Note that for an L^1-Carathéodory multifunction $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ the set $S^1_{F,y}$ is not empty (see [23]).*

Lemma 8. *(Bohnenblust-Karlin)([13]). Let X be a Banach space and $K \in P_{cl,c}(X)$ and suppose that the operator $G : K \to P_{cl,c}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

3 Main result

We first address a boundary value problem on a bounded domain. Let $n \in \mathbb{N}$, and consider the boundary value problem

$$^cD^\alpha y(t) \in F(t,y(t)), \quad t \in J_n := [0,n], \quad 1 < \alpha \le 2, \tag{3.1}$$

$$y(0) = y_0, \quad y'(n) = 0. \tag{3.2}$$

Let $h : J_n \to \mathbb{R}$ be continuous, and consider the linear fractional order differential equation

$$^cD^\alpha y(t) = h(t), \quad t \in J_n, \quad 1 < \alpha \le 2. \tag{3.3}$$

We shall refer to (3.3)-(3.2) as (LP).

By a solution to (LP) we mean a function $y \in AC^1(J_n,\mathbb{R})$ that satisfies equation (3.3) on J_n and condition (3.2).

We need the following auxiliary result:

Lemma 9. *A function y is a solution of the fractional integral equation*

$$y(t) = y_0 + \int_0^n G_n(t,s)h(s)ds, \tag{3.4}$$

if and only if y is a solution of (LP), where $G(t,s)$ is the Green's function defined by

$$G_n(t,s) = \begin{cases} \frac{-t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le n, \\ \frac{-t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s < n. \end{cases} \tag{3.5}$$

Proof. Let $y \in C(J_n,\mathbb{R})$ be a solution to (LP). Using Lemma 4, we have that

$$y(t) = I^\alpha h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0 - c_1 t,$$

for arbitrary constants c_0 and c_1. We have by derivation

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - c_1.$$

Applying the boundary conditions (3.2), we find that

$$c_0 = -y_0,$$
$$c_1 = \int_0^n \frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds.$$

Reciprocally, let $y \in C(J_n,\mathbb{R})$ satisfying (3.4), then

$$y(t) = y_0 + \int_0^t \left[\frac{-t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right] h(s)ds + \int_t^n \frac{-t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds.$$

Then $y(0) = y_0$ and

$$y'(t) = \int_0^n \frac{-(n-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds.$$

Thus, $y'(n) = 0$ and

$${}^cD^{\alpha-1}y(t) =^c D^\alpha y(t) =^c D^{\alpha-1} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds =^c D^{\alpha-1} I^{\alpha-1} h(t).$$

□

Remark 10. *For each $n > 0$, the function $t \in J_n \mapsto \int_0^n |G_n(t,s)|ds$ is continuous on $[0,n]$, and hence is bounded. Let*

$$\tilde{G}_n = \sup\left\{\int_0^n |G_n(t,s)|ds,\ t \in J_n\right\}.$$

Definition 11. *A function $y \in AC^1(J_n,\mathbb{R})$ is said to be a solution of (3.1)–(3.2) if there exists a function $v \in L^1(J_n,\mathbb{R})$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J_n$, such that the differential equation ${}^cD^\alpha y(t) = v(t)$ on J_n and*

$$y(0) = y_0, \quad y'(n) = 0$$

are satisfied.

Theorem 12. *Assume the following hypotheses hold:*

($\mathcal{H}_1$) $F : J_n \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory with compact convex values,*

($\mathcal{H}_2$) *there exist* $p \in C(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$\|F(t, u)\|_{\mathcal{P}} \leq p(t)\psi(|u|) \text{ for } t \in J_n \text{ and each } u \in \mathbb{R};$$

($\mathcal{H}_3$) *There exists a constant* $r > 0$ *such that*

$$r \geq |y_0| + p_n^* \psi(r)\tilde{G}_n,$$

where

$$p_n^* = \sup\{p(s), s \in J_n\}.$$

Then BVP (3.1)–(3.2) has at least one solution on J_n *with* $|y(t)| \leq r$ *for each* $t \in J_n$.

Proof. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$D^{\alpha} y(t) \in F(t, y(t)), \ \ t \in J_n, \quad 1 < \alpha \leq 2, \tag{3.6}$$

$$y(0) = y_0, \ \ y'(n) = 0. \tag{3.7}$$

We begin by showing that (3.6)-(3.7) has a solution $y_n \in C(J_n, \mathbb{R})$ with

$$|y_n(t)| \leq r \text{ for each } t \in J_n.$$

Consider the operator $N : C(J_n, \mathbb{R}) \longrightarrow 2^{C(J_n, \mathbb{R})}$ defined by

$$(Ny) = \left\{ h \in C(J, \mathbb{R}) : h(t) = y_0 + \int_0^n G_n(t, s)v(s)ds \right\}$$

where $v \in S^1_{F,y}$, and $G_n(t, s)$ is the Green's function given by (3.5). Clearly, from Lemma 8, the fixed points of N are solutions to (3.6)–(3.7). We shall show that N satisfies the assumptions of Bohnenblust-Karlins fixed point theorem. The proof will be given in several steps.

Let

$$K = \{y \in C(J_n, \mathbb{R}), \|y\|_n \leq r\},$$

where r is the constant given by ($\mathcal{H}_3$). It is clear that K is a closed, convex subset of $C(J_n, \mathbb{R})$.

Step1: *$N(y)$ is convex for each $y \in K$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S^1_{F,y}$ such that for each $t \in J_n$ we have

$$h_i(t) = y_0 + \int_0^n G_n(t,s)v_i(s)ds, \quad i = 1, 2.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^n G_n(t,s)(dv_1(s) + (1-d)v_2(s)ds.$$

Since $S^1_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *$N(K)$ is bounded.*

This is clear since $N(K) \subset K$ and K is bounded.

Step 3: *$N(K)$ is equicontinuous.*

Let $\xi_1, \xi_2 \in J,\ \ \xi_1 < \xi_2,\ \ y \in K$ and $h \in N(y)$, then

$$\begin{array}{rcl} |h(\xi_2) - h(\xi_1)| & \leq & \int_0^n |G(\xi_2, s) - G(\xi_1, s)|v(s)|ds \\ & \leq & p_n^* \psi(r) \int_0^n |G_n(\xi_2, s) - G_n(\xi_1, s)|ds. \end{array}$$

As $\xi_1 \to \xi_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : K \longrightarrow \mathcal{P}(K)$ is compact.

Step 4: *N has a closed graph.*

Let $y_n \to y_*,\ h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S^1_{F,y_n}$ such that, for each $t \in J_n$,

$$h_n(t) = y_0 + \int_0^n G_n(t,s)v_n(s)ds.$$

We must show that there exists $v_* \in S^1_{F,y_*}$ such that, for each $t \in J_n$,

$$h_*(t) = y_0 + \int_0^n G_n(t,s)v^*(s)ds.$$

We consider the continuous linear operator

$$\Gamma : L^1(J_n, \mathbb{R}) \to C(J_n, \mathbb{R}),$$

defined by

$$(\Gamma v)(t) = \int_0^n G_n(t,s)v(s)ds.$$

Since $h_n(t) - y_0 \in \Gamma(S^1_{F,y_n})$, $|(h_n(t) - y_0) - (h_*(t) - y_0)| \to 0$ as $n \to \infty$ and $\Gamma \circ S^1_F$ has a closed graph, then

$$h_* - y_0 \in \Gamma(S^1_{F,y}).$$

So

$$h_* \in N(y_*).$$

Therefore, we deduce from Bohnenblust-Karlin fixed point theorem that N has a fixed point y_n in K which is a solution of BVP (3.6)–(3.7) with

$$|y_n(t)| \leq r \text{ for each } t \in J_n.$$

Diagonalization process

We now use the following diagonalization process. For $k \in \mathbb{N}$, let

$$u_k(t) = \begin{cases} y_k(t), & t \in [0, n_k], \\ y_k(n_k) & t \in [n_k, \infty). \end{cases} \tag{3.8}$$

Here $\{n_k\}_k \in \mathbb{N}^*$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots < n_k < \ldots \uparrow \infty.$$

Let $S = \{u_k\}_{k=1}^{\infty}$. Notice that

$$|u_k(t)| \leq r \text{ for } \ t \in [0, n_1],\ k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$ and $t \in [0, n_1]$ we have

$$u_{n_k}(t) = y_0 + \int_0^{n_1} G_{n_1}(t,s)v_{n_k}(s)ds,$$

where $v_{n_k} \in S^1_{F,u_{n_k}}$ and thus, for $k \in \mathbb{N}$ and $t, x \in [0, n_1]$ we have

$$u_{n_k}(t) - u_{n_k}(x) = \int_0^{n_1} [G_{n_1}(t,s) - G_{n_1}(x,s)]v_{n_k}(s)ds$$

and by $(\mathcal{H}_2)$, we have

$$|u_{n_k}(t) - u_{n_k}(x)| \leq p_1^* \psi(r) \int_0^{n_1} |G_{n_1}(t,s) - G_{n_1}(x,s)|ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence N_1^* of $\mathbb{N}$ and a function $z_1 \in C([0,n_1],\mathbb{R})$ with $u_{n_k} \to z_1$ in $C([0,n_1],\mathbb{R})$ as $k \to \infty$ through N_1^*. Let $N_1 = N_1^* \backslash \{1\}$. Notice that

$$|u_{n_k}(t)| \leq r \text{ for } t \in [0,n_2],\ k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$ and $t,x \in [0,n_2]$ we have

$$|u_{n_k}(t) - u_{n_k}(x)| \leq p_2^* \psi(r) \int_0^{n_2} |G_{n_2}(t,s) - G_{n_2}(x,s)| ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence N_2^* of N_1 and a function $z_2 \in C([0,n_2],\mathbb{R})$ with $u_{n_k} \to z_2$ in $C([0,n_2],\mathbb{R})$ as $k \to \infty$ through N_2^*. Note that $z_1 = z_2$ on $[0,n_1]$ since $N_2^* \subseteq N_1$. Let $N_2 = N_2^* \backslash \{2\}$. Proceed inductively to obtain for $m \in \{3,4,...\}$ a subsequence N_m^* of N_{m-1} and a function $z_m \in C([0,n_m],\mathbb{R})$ with $u_{n_k} \to z_m$ in $C([0,n_m],\mathbb{R})$ as $k \to \infty$ through N_m^*. Let $N_m = N_m^* \backslash \{m\}$.
Define a function y as follows. Fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $s \leq n_m$. Then define $y(t) = z_m(t)$. Then $y \in C([0,\infty),\mathbb{R})$, $y(0) = y_0$ and $|y(t)| \leq r$ for $t \in [0,\infty)$.
Again fix $t \in [0,\infty)$ and let $m \in \mathbb{N}$ with $s \leq n_m$. Then for $n \in N_m$ we have

$$u_{n_k}(t) = y_0 + \int_0^{n_m} G_{n_m}(t,s) v_{n_k}(s) ds,$$

Let $n_k \to \infty$ through N_m to obtain

$$z_m(t) = y_0 + \int_0^{n_m} G_m(x,s) v_m(s) ds,$$

i.e

$$y(t) = y_0 + \int_0^{n_m} G_{n_m}(t,s) v(s) ds,$$

where $v_m \in S^1_{F,z_m}$.
We can use this method for each $x \in [0,n_m]$, and for each $m \in \mathbb{N}$. Thus

$$D^\alpha y(t) \in F(t,y(t)), \text{ for } t \in [0,n_m]$$

for each $m \in \mathbb{N}$ and $\alpha \in (1,2]$. □

4 An example

Consider the boundary value problem

$$^cD^\alpha y(t) \in F(t,y(t)), \text{ for } t \in J = [0,\infty), \quad 1 < \alpha \leq 2, \tag{4.1}$$

$$y(0) = 1, \; y \text{ is bounded on } \; [0, \infty), \tag{4.2}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative. Set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

where $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ are measurable in t. We assume that for each $t \in J$, $f_1(t, \cdot)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in J$, $f_2(t, \cdot)$ is upper semi-continuous (i.e the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there exists $p \in C(J, \mathbb{R}^+)$ and $\delta \in (0, 1)$ such that

$$\max(|f_1(t, y)|, \; |f_2(t, y)|) \leq p(t)|y|^\delta, \quad t \in J, \text{ and all } y \in \mathbb{R}.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [16]). Also conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied with

$$\psi(u) = u^\delta, \text{ for each } u \in [0, \infty).$$

From (3.5) we have for $s \leq t$

$$\int_0^t G_n(t, s)ds = \frac{t}{\Gamma(\alpha - 1)(\alpha - 1)}[(n - t)^{(\alpha-1)} - n^{(\alpha-1)}] + \frac{t^\alpha}{\alpha\Gamma(\alpha)}$$

and for $t \leq s$

$$\int_t^n G_n(t, s)ds = \frac{-t}{(\alpha - 1)\Gamma(\alpha - 1)}(n - t)^{\alpha-1}.$$

Also since

$$\lim_{c\to\infty} \frac{c}{1 + p_n^*\psi(c)\tilde{G}_n} = \lim_{c\to\infty} \frac{c}{\psi(c)} = \lim_{c\to\infty} \frac{c}{c^\delta} = \infty,$$

then there exists $r > 0$ such that

$$\frac{r}{1 + p_n^*\psi(r)\tilde{G}_n} \geq 1.$$

Hence $(\mathcal{H}_3)$ is satisfied. Then by Theorem 12, BVP (4.1)-(4.2) has a bounded solution on $[0, \infty)$.

Acknowledgement. The authors are grateful to the referee of his/her remarks.

References

[1] R.P Agarwal, M. Benchohra and S. Hamani, *A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math. **109** (3) (2010), 973-1033.

[2] R.P. Agarwal, M. Benchohra, S. Hamani and S. Pinelas, *Boundary value problem for differential equations involving Riemann-Liouville fractional derivative on the half line*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. (to appear).

[3] R.P Agarwal and D. O' Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 2001. MR1845855(2002g:34058). Zbl 0988.34002.

[4] R.P Agarwal and D. O' Regan, *Boundary value problems of nonsingular type on the semi-infinite interval*, Tohoku. Math. J. **51** (1999), 391-397. MR1707763(2000f:34033). Zbl 0942.34026.

[5] A. Arara, M. Benchohra, N. Hamidi and J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal. TMA **72** (2010), 580-586. MR2579326(2001m:22005). Zbl 1179.26015.

[6] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990. MR1048347(91d:49001). Zbl 0713.49021.

[7] A. Belarbi, M. Benchohra, S. Hamani and S.K. Ntouyas, *Perturbed functional differential equations with fractional order*, Commun. Appl. Anal. **11** (3-4) (2007), 429-440. MR2368194(2008j:34099). Zbl 1148.34042.

[8] A. Belarbi, M. Benchohra and A. Ouahab, *Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces*, Appl. Anal. **85** (2006), 1459-1470. MR2282996(2008b:34158). Zbl 1175.34080.

[9] M. Benchohra, J. R. Graef and S. Hamani, *Existence results for boundary value problems with non-linear fractional differential equations*, Appl. Anal. **87** (7) (2008), 851-863. MR2458962(2009g:34033). Zbl 05373332.

[10] M. Benchohra and S. Hamani, *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative*, Topol. Methods Nonlinear Anal. **32** (1) (2008), 115-130. MR2466806(2009h:34012). Zbl 1180.26002.

[11] M. Benchohra, S. Hamani and S.K. Ntouyas, *Boundary value problems for differential equations with fractional order*, Surv. Math. Appl. **3** (2008), 1-12. MR2390179(2009b:34188). Zbl 1157.26301.

[12] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl. **338** (2) (2008), 1340-1350. MR2386501(2008m:34182). Zbl 05242627.

[13] H.F. Bohnenblust and S. Karlin, *On a theorem of ville. Contribution to the theory of games.*155-160, Annals of Mathematics Studies, no. 24. Priceton University Press, Princeton. N. G. 1950. MR0041415(12,844c). Zbl 0041.25701.

[14] Y.-K. Chang and J.J. Nieto, *Some new existence results for fractional differential inclusions with boundary conditions*, Math. Comput. Model. **49** (2009), 605-609. MR2483665(2009m:34020). Zbl 1165.34313.

[15] K. Diethelm and A.D. Freed, *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity*, in "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), pp 217-224, Springer-Verlag, Heidelberg, 1999.

[16] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992. MR1189795(94b:34026). Zbl 0760.34002.

[17] W. G. Glockle and T. F. Nonnenmacher, *A fractional calculus approach of self-similar protein dynamics*, *Biophys. J.* **68** (1995), 46-53.

[18] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000. MR1890104(2002j:00009). Zbl 0998.26002.

[19] N. Heymans and I. Podlubny, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives.* Rheologica Acta **45** (5) (2006), 765—772.

[20] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory **I***, Kluwer, Dordrecht, 1997. MR1485775(98k:47001). Zbl 0887.47001.

[21] A.A. Kilbas, Hari M. Srivastava and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. MR2218073(2007a:34002). Zbl 1092.45003.

[22] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009. Zbl 05674847.

[23] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781-786. MR0196178(33#4370). Zbl 0151.10703.

[24] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. **103** (1995), 7180-7186.

[25] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993. MR1219954(94e:26013). Zbl 0789.26002.

[26] A. Ouahab, *Some results for fractional boundary value problem of differential inclusions*, Nonlinear Anal. **69** (11) (2008), 3877-3896. MR2463341(2009h:34014). Zbl 1169.34006.

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999. MR1658022(99m:26009). Zbl 0924.34008.

[28] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal. **5** (2002), 367-386. MR1967839(2004k:26011a). Zbl 1042.26003.

[29] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993. MR1347689(96d:26012). Zbl 0818.26003.

Mouffak Benchohra
Laboratory of Mathematics,
University of Sidi Bel Abbès,
P.O Box 89, Sidi Bel-Abbès, 22000,
Algeria.
e-mail: benchohra@univ-sba.dz

Naima Hamidi
Laboratory of Mathematics,
University of Sidi Bel Abbès,
P.O Box 89, Sidi Bel-Abbès, 22000,
Algeria.
e-mail: hamidi.naima@yahoo.fr