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IDENTIFIABILITY OF THE MULTIVARIATE NORMAL BY THE MAXIMUM AND THE MINIMUM

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Abstract. In this paper, we have discussed theoretical problems in statistics on identification of parameters of a non-singular multi-variate normal when only either the distribution of the maximum or the distribution of the minimum is known.

1 Introduction

Let $X_1, X_2, \ldots, X_m$, where $X_i = (X_{i_1}, X_{i_2}, \ldots, X_{i_n})$ be m independent random n-vectors each with a n-variate non-singular density in some class F. Let $Y_1, Y_2, \ldots, Y_p$ be another such independent family of random n-vectors with each Y_i having its n-variate non-singular density in F. Suppose that

$$X_0 = (X_{0_1}, \ldots, X_{0_n}),\ where\ X_{0_j} = max\{X_{i_j} : 1 \leq i \leq m\},$$

and

$$Y_0 = (Y_{0_1}, \ldots, Y_{0_n}),\ where\ Y_{0_j} = max\{Y_{i_j} : 1 \leq i \leq p\},$$

have the same n-dimensional distribution function. The problem is to determine if m must equal p, and the distributions of $\{X_1, X_2, \ldots X_m\}$ are simply a rearrangement of those of $\{Y_1, Y_2, \ldots, Y_p\}$.

This problem comes up naturally in the context of a supply-demand problem in econometrics, and as far as we know, was considered first in [2], where it was solved (in the affirmative) for the class of univariate normal distributions, and also, for the class of bivariate normal distributions with positive correlations. In [18], it was solved in the affirmative for the class of bivariate normal distributions with positive or negative correlations. However, if such distributions with zero correlations are allowed, then unique factorization of product of such distributions no longer holds, and this can be verified by considering simply four univariate normal distributions

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$F_1(x), F_2(x), F_3(y), F_4(y)$ and observing that for all x, y, we have the equality:

$$[F_1(x)F_3(y)][F_2(x)F_4(y)] = [F_1(x)F_4(y)][F_2(x)F_3(y)].$$

In [19], it was shown in the general n-variate normal case, that when the n-variate normal distributions of each X_i, $1 \le i \le m$, and each Y_j, $1 \le j \le p$, have positive partial correlations (that is, the off-diagonal entries of the covariance matrix are all negative), then when X_o and Y_o, as defined earlier, have the same distribution function, we must have $m = p$, and the distributions of the X_i, $1 \le i \le m$, must be a permutation of those of the Y_j, $1 \le j \le m$. In [17], this problem was solved in the affirmative for the general n-variate case when the covariance matrices of all the n-variate normal distributions are of the form: $\Sigma_{ij} = \rho\sigma_i\sigma_j$ for $i \ne j$. As far as we know, the general problem discussed above is still open.

The maximum problem above occurs in m-component systems where the components are connected in parallel. The system lifetime is then given by

$$max\{X_1, X_2, \ldots, X_m\},$$

and this is observable. There are instances where this maximum is observable, but the individual X_is are not.

It may also be noted that if the distribution function of Y is $F(x)$, and if X_1 and X_2 are two independent random variables each with distribution $\sqrt{F(x)}$, then $max\{X_1, X_2\}$ and Y have the same distribution function.

In [8], a corresponding minimum problem was discussed in the context of a probability model describing the death of an individual from one of several competing causes. Let $X_1, X_2, \ldots, X_n$ be independent random variables with continuous distribution functions, $Z = min\{X_1, X_2, \ldots, X_i\}$ and $I = k$ iff $Z = X_k$. If the X_i have a common distribution function, then it is uniquely determined by the distribution function of Z. In [8], it was shown that when the distribution of the X_i are not all the same, then the joint distribution function of the identified minimum (that is, that of (I, Z)) uniquely determines each $F_i(x)$, the distribution function of X_i, $i = 1, 2, \ldots, n$.

Since $max\{X_1, X_2, \ldots, X_n\} = -min\{-X_1, -X_2, \ldots, -X_n\}$, the distribution of the identified maximum also uniquely determines the distribution of the X_i. Notice that if we consider $n = 2$ and the case where the independent random variables X_1 and X_2 are both exponential such that their density functions are given by

$$\begin{aligned} f_1(x) =& \lambda e^{-\lambda x},\ x > 0; \\ =& 0,\ otherwise, \end{aligned}$$

and

$$\begin{aligned} f_2(x) =& \mu e^{-\mu x},\ x > 0; \\ =& 0,\ otherwise. \end{aligned}$$

where λ and μ are both positive, then even though the joint distribution of their identified minimum (that is, that of $(I, min\{X_1, X_2\})$ uniquely determines the parameters λ and μ, the $min\{X_1, X_2\}$, by itself, does not identify uniquely the parameters λ and μ. Thus, one natural question comes up: when does the minimum of a n-variate random vector uniquely determine the parameters of the distribution of the random vector? In what follows, in the rest of this section, we discuss this problem.

As far as we know, the problem on identification of parameters of a random vector $(X_1, X_2, \dots, X_n)$ by the distribution of the minimum (namely, $min\{X_i : 1 \le i \le n\}$) is still unsolved even in the case of a n-variate normal vector. In the bivariate normal case, the problem was solved in the affirmative (in the natural sense) in [5] and [16] independently. The problem was considered also in the tri-variate normal case in [5] in the context of an identified minimum, and solved partially. The general minimum problem in the n-variate normal case, in the case of a common correlation, was solved in [9], and in the case of the tri-variate normal with negative correlations was solved in [12].

In the next section, we present asymptotic orders of certain tail probabilities for a multi-variate normal random vector that are useful in the context of the problems mentioned above. The purpose of this note here is to present some essential results which are useful in solving the identified minimum problem in the general tri-variate normal case.

2 Tail probabilities of a multivariate normal

Let $(X_1, X_2, \dots, X_n)$ be a n-variate normal random vector with a symmetric positive definite covariance matrix Σ such that the vector $1\Sigma^{-1} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $1 = (1, 1, \dots, 1)$, is positive (that is, each α_i is positive). Then it was proven in [9] that as $t \to \infty$, the tail probability

$$P(X_1 > t, X_2 > t, \dots, X_n > t)$$

is of the same (asymptotic) orders as that of

$$C\, exp(-\frac{1}{2}t^2[1\Sigma^{-1}1^T]),$$

where

$$\frac{1}{c} = (2\pi)^{\frac{n}{2}} \sqrt{|det\Sigma|}(\alpha_1\alpha_2 \dots \alpha_n)t^n.$$

Here we consider two functions $f(t)$ and $g(t)$ having the same order as $t \to \infty$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$. Thus, we can state, when $n = 2$, the following lemma.

Lemma 1. *Let (X_1, X_2) be a bivariate normal with zero means and variances σ_1^2, σ_2^2 (where $\sigma_1^2 \geq \sigma_2^2$, and correlation ρ, $|\rho| < 1$, such that $\rho < \frac{\sigma_2}{\sigma_1}$. Then, we have: as $t \to \infty$,*

$$P(X_1 > t, X_2 > t) \sim C\, exp\left(-\frac{1}{2}t^2[\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2(1-\rho^2)}]\right),$$

where

$$\frac{1}{C} = 2\pi(1-\rho^2)^{-\frac{3}{2}}t^2(\sigma_2 - \rho\sigma_1)(\sigma_1 - \rho\sigma_2).$$

Let us now consider the case $\rho > \frac{\sigma_2}{\sigma_1}$ for the general bivariate normal (with zero means, for simplicity) considered in Lemma 1 In this case, we no longer have $1\Sigma^{-1} > 0$, and thus, we need to use another idea. We can write

$$\begin{aligned}
&P(X_1 > t, X_2 > t)\\
&= \int_t^\infty P(X_1 > t | X_2 = x) f_{X_2}(x) dx\\
&= \int_t^\infty f_{X_2}(x)dx \int_t^\infty \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\, exp\left[-\frac{1}{2}\left(\frac{y - \left(\frac{\rho\sigma_1 x}{\sigma_2}\right)}{\sigma_1\sqrt{1-\rho^2}}\right)^2\right] dy\\
&= \int_t^\infty f_{X_2}(x)dx \int_{g(x,t)}^\infty \frac{1}{\sqrt{2\pi}}\, exp\left(-\frac{1}{2}z^2\right) dz,
\end{aligned}$$

where

$$g(x,t) = \frac{t - \frac{\rho\sigma_1 x}{\sigma_2}}{\sigma_1\sqrt{1-\rho^2}}.$$

Since $\rho > \frac{\sigma_2}{\sigma_1}$, for t sufficiently large (for a pre-assigned positive δ), we can write:

$$P(X_1 > t, X_2 > t) > (1-\delta)\int_t^\infty f_{X_2}(x)dx.$$

It follows easily that as $t \to \infty$, when $\rho > \frac{\sigma_2}{\sigma_1}$, we have:

$$P(X_1 > t, X_2 > t) \sim \frac{\sigma_2}{\sqrt{2\pi}\, t} exp\left[-\frac{1}{2}\frac{t^2}{\sigma_2^2}\right] \tag{2.1}$$

When $\rho = \frac{\sigma_2}{\sigma_1} (< 1)$ for the bivariate normal considered above, we have, similarly, for any $\epsilon > 0$ and all sufficiently large t,

$$\frac{1}{2}\frac{(1-\epsilon)\sigma_2}{\sqrt{2\pi}\, t}\, exp\left[-\frac{1}{2}\frac{t^2}{\sigma_2^2}\right] \leq P(X_1 > t, X_2 > t) \leq \frac{(1+\epsilon)\sigma_2}{\sqrt{2\pi}\, t}\, exp\left[-\frac{1}{2}\frac{t^2}{\sigma_2^2}\right] \tag{2.2}$$

Lemma 2. *Let (X_1, X_2) be a bivariate normal with zero means, variances each 1, and correlation ρ, $|\rho| < 1$. Let $\alpha > 0$, $\beta > 0$ and $\rho < \frac{\alpha}{\beta}$. Then we have:*

$$P(X_1 \geq \alpha t, X_2 \geq \beta t) \sim \frac{1}{C} exp\left[-\frac{1}{2}t^2\left(\frac{\alpha^2+\beta^2-2\rho\alpha\beta}{1-\rho^2}\right)\right], \text{ as } t \to \infty,$$

which is $o(exp\left[-\frac{1}{2}\beta^2t^2\right])$ and also $o(exp\left[-\frac{1}{2}\alpha^2t^2\right])$, where

$$C = 2\pi t^2(\alpha-\rho\beta)(\beta-\rho\alpha)(1-\rho^2)^{-\frac{3}{2}}\alpha^{-2}\beta^{-2}.$$

Lemma 3. *Let (X_1, X_2) be as in Lemma 2.2 let$\alpha > 0, \beta > 0, \alpha \leq \beta$ and $\rho > \frac{\alpha}{\beta}$. Then we have*

$$P(X_1 > \alpha t, X_2 > \beta t) \sim \frac{1}{\sqrt{2\pi}\,\beta t}\exp\left[-\frac{1}{2}\beta^2t^2\right] \text{ as } t \to \infty.$$

Both Lemmas 2 and 3 follow from Lemma 1 and equation (2.1) above. In Lemma 3, if we take $\rho = \frac{\alpha}{\beta}$, then given $\epsilon > 0$ and for all all sufficiently large t,

$$\frac{1-\epsilon}{2}\frac{1}{\sqrt{2\pi}\,\beta t} exp\left[-\frac{1}{2}\beta^2t^2\right] \leq P(X_1 > \alpha t, X_2 > \beta t) \leq \frac{1+\epsilon}{\sqrt{2\pi}\,\beta t} exp\left[-\frac{1}{2}\beta^2t^2\right]. \tag{2.3}$$

Lemma 4. *Let (X_1, X_2) be as in Lemma 2. Let $\alpha > 0, \beta > 0$. Let $\rho < min\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\}$. Then as $t \to -\infty$,*

$$1 - P(X_1 > \alpha t, X_2 > \beta t) \sim \frac{1}{\sqrt{2\pi}\,\mu|t|} exp\left[-\frac{1}{2}\mu^2t^2\right], \ \mu = min\{\alpha, \beta\}.$$

Proof. Notice that

$$\begin{aligned} & 1 - P(X_1 > \alpha t, X_2 > \beta t) \\ =& P(X_1 \leq \alpha t, X_2 \leq \beta t) + P(X_1 \leq \alpha t, X_2 \leq \beta t) + P(X_1 \leq \alpha t, X_2 > \beta t) \end{aligned}$$

The first term on the right hand side is, by Lemma 2, $o\left(\frac{1}{\sqrt{2\pi}\,\mu|t|} exp\left[-\frac{1}{2}\mu^2t^2\right]\right)$ as $t \to -\infty$. The second term there can be written as

$$\begin{aligned} & P(X_2 \leq \beta t) - P(-X_1 > \alpha(-t), -X_2 > \beta(-t)) \\ \sim & P(X_2 \leq \beta t) \text{ as } t \to -\infty, \end{aligned}$$

and similarly, the third term is $\sim P(X_1 \leq \alpha t)$ as $t \to -\infty$. The lemma is now clear. □

Lemma 5. *Let* (X_1, X_2) *be as in Lemma 2. Let* $\alpha < 0, \beta > 0, -\alpha > \beta$. *Then*

$$P(X_1 > \alpha t, X_2 > \beta t) \sim \frac{1}{\sqrt{2\pi}\,\beta t} exp\left[-\frac{1}{2}\beta^2 t^2\right] \ \ as\ t \to \infty$$

Proof. Let us write:

$$\begin{aligned} &P(X_1 > \alpha t, X_2 > \beta t) \\ =&P(X_2 > \beta t) - P(-X_1 > (-\alpha)t), X_2 > \beta t) \end{aligned}$$

Let $t \to \infty$. Since the correlation of $(-X_1, X_2)$ is $-\rho$, it follows from Lemma 2 that when $-\rho < -\frac{\beta}{\alpha}$,

$$P(-X_1 > (-\alpha)t, X_2 > \beta t) = o\left(exp\left[-\frac{1}{2}\beta^2 t^2\right]\right).$$

Also, it follows from Lemma 3 and inequalities in (2.3) that when $-\rho \geq \frac{\beta}{\alpha}$, as $t \to \infty$,

$$\begin{aligned} &P(-X_1 > (-\alpha)t, X_2 > \beta t) \\ \sim&C(t)\frac{1}{\sqrt{2\pi}|\alpha t|} exp\left[-\frac{1}{2}\alpha^2 t^2\right] \\ =&o\left(exp\left[-\frac{1}{2}\beta^2 t^2\right]\right). \end{aligned}$$

where $\frac{1}{2} \leq C(t) \leq 1$. The lemma now follows easily. □

Lemma 6. *Let* (X_1, X_2) *be as in Lemma 2. Let* $\alpha < 0$. *Then as* $t \to \infty$,

$$P\left(X_1 > \alpha t, X_2 > (-\alpha)t\right) \sim \frac{1}{\sqrt{2\pi}\,|\alpha| t} exp\left[-\frac{1}{2}\alpha^2 t^2\right]$$

Proof. It is enough to observe that

$$\begin{aligned} &P(X_1 > \alpha t, X_2 > (-\alpha)t) \\ =&P(X_2 > (-\alpha)t) - P(-X_1 > (-\alpha)t, X_2 > (-\alpha)t) \end{aligned}$$

and then Lemma 2 applies. □

3 The pdf of the identified minimum

Let (X_1, X_2, X_3) be a tri-variate normal random vector with zero means and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$, and a non-singular covariance matrix Σ, where $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, $i \neq j$, $\Sigma_{ii} = \sigma_i^2$.

We assume, with no loss of generality, that $\sigma_1^2, \sigma_2^2, \sigma_3^2$. Let $Y = min\{X_1, X_2, X_3\}$. We define the random variable I by $I = i$ iff $Y = X_i$, $i = 1, 2, 3$. Let $F(y, i)$ be the joint distribution of (Y, I) such that

$$\begin{aligned} F(y, 1) &= P(Y \leq y, I = 1), \\ F(y, 2) &= P(Y \leq y, I = 2), \\ F(y, 3) &= P(Y \leq y, I = 3). \end{aligned}$$

Then we have:

$$\begin{aligned} &P(Y \leq y, I = 1) \\ =&P(X_1 \leq y, X_1 \leq X_2, X_1 \leq X_3) \\ =&\int_{-\infty}^{y} P(X_2 \leq x_1, X_3 \leq x_1 | X_1 = x_1) f_{X_1}(x_1) dx_1 \end{aligned}$$

Now by differentiating with respect to y, we obtain:

$$\begin{aligned} &f_1(y) \\ =&\frac{d}{dy} F(y, 1) \\ =&f_{X_1}(y) P(X_2 \geq y, X_3 \geq y | X_1 = y) \end{aligned}$$

Notice that the conditional density of (X_2, X_3), given $X_1 = y$, is a bivariate normal with means $\rho_{12}(\frac{\sigma_2}{\sigma_1})y, \rho_{13}(\frac{\sigma_3}{\sigma_1})y$, and variances $\sigma_2^2(1 - {\rho_{12}}^2), \sigma_3^2(1 - {\rho_{13}}^2)$. Thus, we can write:

$$f_1(y) = \frac{1}{\sigma_1} \varphi\left(\frac{y}{\sigma_1}\right) P\left(W_{21} \geq \frac{1 - \rho_{12}\left(\frac{\sigma_2}{\sigma_1}\right)}{\sigma_2\sqrt{1 - {\rho_{12}}^2}}\, y, W_{31} \geq \frac{1 - \rho_{13}\left(\frac{\sigma_3}{\sigma_1}\right)}{\sigma_3\sqrt{1 - {\rho_{13}}^2}}\, y\right) \tag{3.1}$$

where (W_{21}, W_{31}) is a bivariate normal with zero means, variances each one, and correlation $\rho_{23.1} = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - {\rho_{12}}^2}\sqrt{1 - {\rho_{13}}^2}}$, and φ is the standard normal density. Similarly, we also get the expressions for $f_2(y)$ and $f_3(y)$, where $f_2(y) = \frac{d}{dy}[F(y, 2)]$ and $f_3(y) = \frac{d}{dy}[F(y, 3)]$ given in (3.2) and (3.3) below. We have

$$f_2(y) = \frac{1}{\sigma_2} \varphi\left(\frac{y}{\sigma_2}\right) P\left(W_{12} \geq \frac{1 - \rho_{12}\left(\frac{\sigma_1}{\sigma_2}\right)}{\sigma_1\sqrt{1 - {\rho_{12}}^2}}\, y, W_{32} \geq \frac{1 - \rho_{23}\left(\frac{\sigma_3}{\sigma_2}\right)}{\sigma_3\sqrt{1 - {\rho_{23}}^2}}\, y\right) \tag{3.2}$$

$$f_3(y) = \frac{1}{\sigma_3}\varphi\left(\frac{y}{\sigma_3}\right) P\left(W_{13} \geq \frac{1-\rho_{13}\left(\frac{\sigma_1}{\sigma_3}\right)}{\sigma_1\sqrt{1-\rho_{13}{}^2}}\, y, W_{23} \geq \frac{1-\rho_{23}\left(\frac{\sigma_2}{\sigma_3}\right)}{\sigma_2\sqrt{1-\rho_{23}{}^2}}\, y\right) \tag{3.3}$$

where (W_{12}, W_{32}) and (W_{13}, W_{23}) are both bivariate normals with zero means and variances all ones, and correlations given, respectively, by $\rho_{13.2} = \frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{1-\rho_{12}{}^2}\sqrt{1-\rho_{23}{}^2}}$ and $\rho_{12.3} = \frac{\rho_{12}-\rho_{13}\rho_{23}}{\sqrt{1-\rho_{13}{}^2}\sqrt{1-\rho_{23}{}^2}}$. Now the problem of identified minimum is the following: we will assume that the functions $f_1(y), f_2(y)$ and $f_3(y)$ are given, and we need to prove that there can be only a unique set of parameters $\sigma_1^2, \sigma_2^2, \sigma_3^2, \rho_{12}, \rho_{23}, \rho_{13}$ that can lead to the same three given functions f_1, f_2 and f_3.

The proof, besides other arguments, uses mainly the lemmas given in section 2. The proof is rather involved and will appear elsewhere in full.

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