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EXISTENCE AND DATA DEPENDENCE FOR MULTIVALUED WEAKLY REICH-CONTRACTIVE OPERATORS

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Abstract. In this paper we define the concept of weakly Reich-contractive operator and give a fixed point result for this type of operators. Then we study the data dependence for this new result.

1 Introduction

Let (X, d) be a metric space. A singlevalued operator T from X into itself is called contractive if there exists a real number $r \in [0, 1)$ such that $d(T(x), T(y)) \leq rd(x, y)$ for every $x, y \in X$. It is well know that if X is a complete metric space then a contractive operator from x into itself has a unique fixed point in X. In 1972 S. Reich was obtained some generalizations of this results for some classes of generalized contractive operators and in some recent papers [10]-[13] S. Reich et al. gave some applications of these results.

In 1996 the Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the concept of w-distance (see [4]) and discussed some properties of this new distance. Later, T. Suzuki and W. Takahashi starting by the definition above, gave some fixed points result for a new class of operators, weakly contractive operators (see [17]).

In 2001 T. Suzuki (see [15]) introduced the concept of τ-distance on a metric space which is a generalization of both w-distance and Tataru's distance. He gave some examples of τ-distance and improve the generalization of Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle and the Takahashi's nonconvex minimization theorem, see [15]. Also, some fixed point theorems for multivalued operators on a complete metric space endowed with a τ-distance were established in T. Suzuki [16].

The purpose of this paper is to give a fixed point theorem for a new class of operators, the so-called weakly Reich-contractive operators. Then we present a data

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dependence result for the fixed point set of these operators.

2 Preliminaries

Let (X,d) be a complete metric space. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;

$\mathcal{P}(X) = P(X)\bigcup\emptyset$

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_b(X)$ - *the set of all nonempty bounded subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of* X;

For $A, B \in P_b(X)$ we recall the following functionals.

$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, D(Z,Y) = \inf\{d(x,y) : x \in Z\, y \in Y\}$, $Z \subset X$ - *the gap functional.*

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A,B) := sup\{d(a,b)|x \in A, b \in B\}$ - *the diameter functional*;

$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \rho(A,B) := sup\{D(a,B)|a \in A\}$ - *the excess functional*;

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A,B) := max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}$ - *the Pompeiu-Hausdorff functional*;

$FixF := \{x \in X \mid x \in F(x)\}$ - *the set of the fixed points of* F;

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[4]) as follows:

Let (X,d) be a metric space. The functional $w : X \times X \to [0,\infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x,z) \leq w(x,y) + w(y,z)$, for any $x, y, z \in X$;
2. for any $x \in X : w(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z,x) \leq \delta$ and $w(z,y) \leq \delta$ implies $d(x,y) \leq \varepsilon$.

Some examples of w-distance can be find in [16].

For the proof of the main results we need the following crucial result for w-distance (see[17]).

Lemma 1. *Let* (X,d) *be a metric space and let* w *be a* w*-distance on* X*. Let* (x_n) *and* (y_n) *be two sequences in* X*, let* (α_n)*,* (β_n) *be sequences in* $[0,+\infty[$ *converging to zero and let* $x, y, z \in X$*. Then the following hold:*

1. *If* $w(x_n, y) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* $y = z$*.*
2. *If* $w(x_n, y_n) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* (y_n) *converges to* z*.*

3. *If* $w(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$*, then* (x_n) *is a Cauchy sequence.*

4. *If* $w(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$*, then* (x_n) *is a Cauchy sequence.*

The concept of τ-distance was introduced by T. Suzuki (see[1]) as follows.

Definition 2. *Let (X,d) be a metric space. Then* $\tau : X \times X \to [0, \infty)$ *is called* $\tau -$ *distance on X if there exists a function* $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ *and the following are satisfied :*

(τ_1) $\tau(x, z) \leq \tau(x, y) + \tau(y, z)$*, for any* $x, y, z \in X$*;*

(τ_2) $\eta(x, 0) = 0$ *and* $\eta(x, t) \geq t$ *for all* $x \in X$ *and* $t \in \mathbb{R}_+$*, and* η *is concave and continuous in its the second variable;*

(τ_3) $\lim_n x_n = x$ *and* $\lim_n sup\{\eta(z_n, \tau(z_n, x_m)) : m \geq n\} = 0$ *imply* $\tau(w, x) \leq \liminf_n(\tau(w, x_n))$ *for all* $w \in X$*;*

(τ_4) $\lim_n sup\{\tau(x_n, y_m)) : m \geq n\} = 0$ *and* $\lim_n \eta(x_n, t_n) = 0$ *imply* $\lim_n \eta(y_n, t_n) = 0$*;*

(τ_5) $\lim_n \eta(z_n, \tau(z_n, x_n)) = 0$ *and* $\lim_n \eta(z_n, \tau(z_n, y_n)) = 0$ *imply* $\lim_n d(x_n, y_n) = 0$.

Notice that one may replace (τ_2) by the following $(\tau_2)'$:

$(\tau_2)'$ $inf\{\eta(x, t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in the second variable.

Some examples of τ-distance are given in [15].

We recall the definition of τ-Cauchy sequence and some lemmas (see [16]), useful for the proofs of the fixed point results on metric spaces endowed with a τ-distance.

Definition 3. *Let* (X, d) *be a metric space and let* τ *be a* τ*-distance on* X*. Then a sequence* $\{x_n\}$ *in* X *is called* $\tau -$ *Cauchy if there exists a function* $\eta : X \times [0, \infty) \to [0, \infty)$ *satisfying* (τ_2)-(τ_5) *and a sequence* $\{z_n\}$ *in* X *such that* $\lim_n sup\{\eta(z_n, \tau(z_n, x_m)) : m \geq n\} = 0$.

A crucial results in order to obtain fixed point theorems by using τ-distance are the following lemmas.

Lemma 4. *Let* (X, d) *be a metric space and let* τ *be a* τ*-distance on* X*. If a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_n sup\{\tau(x_n, x_m) : m > n\} = 0$*, then* $\{x_n\}$ *is a* τ*-Cauchy sequence. Moreover, if a sequence* $\{y_n\}$ *in* X *satisfies* $\lim_n \tau(x_n, y_n) = 0$*, then* $\{y_n\}$ *is also a* τ*-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$.

Lemma 5. *Let* (X, d) *be a metric space and let* τ *be a* τ*-distance on* X*. If a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_n \tau(z, x_n) = 0$ *for* $z \in X$ *then* $\{x_n\}$ *is a* τ*-Cauchy sequence. Moreover, if a sequence* $\{y_n\}$ *in* X *also satisfies* $\lim_n \tau(z, y_n) = 0$*, then* $\lim_n d(x_n, y_n) = 0$*. In particular, for* $x, y, z \in X$*,* $\tau(z, x) = 0$ *and* $\tau(z, y) = 0$ *imply* $x = y$*.*

Lemma 6. *Let (X,d) be a metric space and let τ be a τ-distance on X. If $\{x_n\}$ is a τ-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $\lim_n sup\{\tau(x_n, y_m) : m > n\} = 0$, then $\{y_n\}$ is a τ-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

3 Existence of fixed points for multivalued weakly Reich-contractive operators

For the first result of this section, let us define the notion of multivalued weakly Reich-contractive operators.

Definition 7. *Let (X,d) be a metric space, $T : X \to P(X)$ is called multivalued weakly Reich-contractive operator if for every $a, b, c \in \mathbb{R}_+$ such that $a+b+c \in [0,1)$, there exists a w-distance on X such that for every $x, y \in X$ and $u \in T(x)$ there exists $v \in T(y)$ such that*

$$w(u,v) \leq aw(x,y) + bD_w(x,T(x)) + cD_w(y,T(y)),$$

where $D_w(x,T(x)) := \inf\{w(x,y) : y \in T(x)\}$.

Let (X,d) be a metric space, w be a w-distance on X $x_0 \in X$ and $r > 0$. Let us define:

$B_w(x_0;r) := \{x \in X | w(x_0,x) < r\}$ the open ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}^d(x_0;r)$- the closure in (X,d) of the set $B_w(x_0;r)$.

One of the main results is the following fixed point theorem for weakly Reich-contractive operators.

Theorem 8. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$, $\alpha := \frac{a+b}{1-c}$ for every $a, b, c \in \mathbb{R}_+$ with $a+b+c \in [0,1)$ and $T : \widetilde{B_w}(x_0;r) \to P_{cl}(X)$ a multivalued operator such that:*

1. *T is weakly Reich-contractive operator with respect to a w-distance;*
2. *For every $x, y \in X$, with $y \notin T(y)$ we have that*

$$\inf\{w(x,y) + D_w(x,T(x)) : \ x \in X\} > 0;$$

3. *$D_w(x_0,T(x_0)) < (1-\alpha)r$.*

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. Let $0 < s < r$ and $D_w(x_0, T(x_0)) < (1-\alpha)s < (1-\alpha)r$.

Then there exists $x_1 \in T(x_0)$ such that $w(x_0, x_1) < (1-\alpha)s \leq s$. Hence $x_1 \in B_w(x_0; s)$.

For $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$w(x_1, x_2) \leq aw(x_0, x_1) + bD_w(x_0, T(x_0)) + cD_w(x_1, T(x_1))$$

$$w(x_1, x_2) \leq aw(x_0, x_1) + bw(x_0, x_1) + cw(x_1, x_2))$$

$$w(x_1, x_2) \leq \frac{a+b}{1-c} w(x_0, x_1)$$

Then $w(x_1, x_2) \leq \alpha w(x_0, x_1) \leq \alpha(1-\alpha)s$.
Then $w(x_0, x_2) \leq w(x_0, x_1) + w(x_1, x_2) < (1-\alpha)s + \alpha(1-\alpha)s = (1-\alpha^2)s \leq s$. Hence $x_2 \in B_w(x_0; s)$.

For $x_1 \in B_w(x_0; s)$ and $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that

$$w(x_2, x_3) \leq aw(x_1, x_2) + bD_w(x_1, T(x_1)) + cD_w(x_2, T(x_2))$$

$$w(x_2, x_3) \leq aw(x_1, x_2) + bw(x_1, x_2) + cw(x_3, x_3))$$

$$w(x_2, x_3) \leq \frac{a+b}{1-c} w(x_1, x_2)$$

Then $w(x_2, x_3) \leq \alpha w(x_1, x_2) \leq \alpha^2 w(x_0, x_1) \leq \alpha^2(1-\alpha)s$.
Then $w(x_0, x_3) \leq w(x_0, x_2) + w(x_2, x_3) < (1-\alpha^2)s + \alpha^2(1-\alpha)s = (1-\alpha)(1+\alpha+\alpha^2)s = (1-\alpha^3)s < s$. Hence $x_3 \in B_w(x_0; s)$.

By induction we obtain in this way a sequence $(x_n)_{n\in\mathbb{N}} \in B_w(x_0; s)$ with the following properties:

(1) $x_n \in T(x_{n-1})$, for each $n \in \mathbb{N}$;

(2) $w(x_n, x_{n+1}) \leq \alpha^n(1-\alpha)s$, for each $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$ we have

$$w(x_n, x_m) \leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + ... + w(x_{m-1}, x_m) \leq$$

$$\leq \alpha^n(1-\alpha)s + \alpha^{n+1}(1-\alpha)s + ... + \alpha^{m-1}(1-\alpha)s \leq$$

$$\leq \frac{\alpha^n}{1-\alpha}(1-\alpha)s = \alpha^n s.$$

Using Lemma 1(3) we have that $(x_n)_{n\in\mathbb{N}} \in B_w(x_0; s)$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit $x^* \in \widetilde{B_w}^d(x_0; s)$.

Assume that $x^* \notin T(x^*)$. Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}} \in B_w(x_0; s)$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have

$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \alpha^n s$, for every $n \in \mathbb{N}$.

Therefore by hypothesis (2) and by using the above inequality, we obtain

$$\begin{aligned} 0 &< \inf\{w(x,x^*) + D_w(x,T(x)) : x \in X\} \\ &\leq \inf\{w(x_n,x^*) + w(x_n,x_{n+1}) : n \in \mathbb{N}\} \\ &\leq \inf\{\alpha^n s(2-\alpha)w(x_0,x_1) : n \in \mathbb{N}\} = 0. \end{aligned}$$

Which is a contradiction. Thus we conclude that $x^* \in T(x^*)$. □

A global result for previous theorem is the following fixed point result for multivalued weakly Reich-contractive operators.

Theorem 9. *Let (X,d) be a complete metric space, $T : X \to P_{cl}(X)$ a multivalued operator such that such that:*

1. *T is weakly Reich-contractive operatorwith respect to a w-distance;*
2. *For every $x, y \in X$, with $y \notin T(y)$ we have that*

$$\inf\{w(x,y) + D_w(x,T(x)) : \; x \in X\} > 0;$$

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Notice that some similar results can be found in [7].

Remark 10. *Similar results can be obtained for the case of τ-distance.*

4 Data dependence for multivalued weakly Reich-contractive operators

The main result of this section is the following data dependence theorem for the fixed point set of multivalued weakly Reich contractive operators.

Theorem 11. *Let (X,d) be a complete metric space, $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued weakly Reich-contractive operators with respect to a w-distance, with $\alpha \in [0,1)$ where $\alpha := \frac{a+b}{1-c}$, for every $a,b,c \in \mathbb{R}_+$ with $a+b+c \in [0,1)$ and satisfying for every $x,y \in X$, with $y \notin T_i(y)$, the following inequality $\inf\{w(x,y) + D_w(x,T_i(x)) : x \in X\} > 0$. Then the following are true:*

1. *$FixT_1 \neq \emptyset \neq FixT_2$;*
2. *We suppose that there exists $\eta > 0$ such that for every $u \in T_1(x)$ there exists $v \in T_2(x)$ such that $w(u,v) \leq \eta$, (respectively for every $v \in T_2(x)$ there exists $u \in T_1(x)$ such that $w(v,u) \leq \eta$).*

Then for every $u^* \in FixT_1$ there exists $v^* \in FixT_2$ such that
$w(u^*, v^*) \leq \frac{\eta}{1-\alpha}$*, where* $\alpha = \alpha_i$ *for* $i = \{1, 2\}$*;*
(respectively for every $v^* \in FixT_2$ there exists $u^* \in FixT_1$ such that
$w(v^*, u^*) \leq \frac{\eta}{1-\alpha}$*, where* $\alpha = \alpha_i$ *for* $i = \{1, 2\}$*)*

Proof. Let $u_0 \in FixT_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis (2) we have that there exists $u_1 \in T_2(u_0)$ such that $w(u_0, u_1) \leq \eta$.

Since T_1, T_2 are weakly Reich-contractive with $\alpha_i \in [0, 1)$, where $\alpha := \frac{a+b}{1-c}$, for every $a, b, c \in \mathbb{R}_+$ with $a + b + c \in [0, 1)$ and $i = \{1, 2\}$ we have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that

$$w(x_1, x_2) \leq aw(u_0, u_1) + bD_w(u_0, T_2(u_0)) + cD_w(u_1, T_2(u_1))$$

$$w(u_1, u_2) \leq aw(u_0, u_1) + bw(u_0, u_1) + cw(u_1, u_2))$$

$$w(u_1, u_2) \leq \frac{a+b}{1-c} w(u_0, u_1)$$

Then $w(u_1, u_2) \leq \alpha w(u_0, u_1)$.

For $u_1 \in X$ and $u_2 \in T_2(u_1)$ there exists $u_3 \in T_2(u_2)$ such that

$$w(u_2, u_3) \leq aw(u_1, u_2) + bD_w(u_1, T_2(u_1)) + cD_w(u_2, T_2(u_2))$$

$$w(u_2, u_3) \leq aw(u_1, u_2) + bw(u_1, u_2) + cw(u_3, u_3))$$

$$w(u_2, u_3) \leq \frac{a+b}{1-c} w(u_1, u_2)$$

Then $w(u_2, u_3) \leq \alpha w(u_1, u_2) \leq \alpha^2 w(u_0, u_1)$.

By induction we obtain a sequence $(u_n)_{n\in\mathbb{N}} \in X$ such that

(1) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(2) $w(u_n, u_{n+1}) \leq \alpha^n w(u_0, u_1)$

For $n, p \in \mathbb{N}$ we have the inequality

$w(u_n, u_{n+p}) \leq w(u_n, u_{n+1}) + w(u_{n+1}, u_{n+2}) + \cdots + w(u_{n+p-1}, u_{n+p}) \leq$
$< \alpha^n w(u_0, u_1) + \alpha^{n+1} w(u_0, u_1) + \cdots + \alpha^{n+p-1} w(u_0, u_1) \leq$
$\leq \frac{\alpha^n}{1-\alpha} w(u_0, u_1)$

By the Lemma 1(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.

Assume that $v^* \notin T_2(v^*)$. Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}} \in X$ converge to v^* and $w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous we have

$$w(u_n, v^*) \leq \lim_{p\to\infty} \inf w(u_n, u_{n+p}) \leq \frac{\alpha^n}{1-\alpha} w(u_0, u_1) \tag{4.1}$$

By hypothesis we have the following inequality:

$$\begin{aligned} 0 &< \inf\{w(u,v^*) + D_w(u,T_2(u)) : x \in X\} \\ &\leq \inf\{w(u_n,v^*) + w(u_n,u_{n+1}) : n \in \mathbb{N}\} \\ &\leq \inf\{\tfrac{\alpha^n}{1-\alpha} w(u_0,u_1) + \alpha^n w(u_0,u_1) : n \in \mathbb{N}\} = 0. \end{aligned}$$

Which is a contradiction. Thus we conclude that $v^* \in T(v^*)$.

Then, by $w(u_n,v^*) \leq \frac{\alpha^n}{1-\alpha} w(u_0,u_1)$, with $n \in \mathbb{N}$, for $n = 0$ we obtain

$$w(u_0,v^*) \leq \frac{1}{1-\alpha} w(u_0,u_1) \leq \frac{\eta}{1-\alpha}$$

which complete the proof. □

References

[1] A. Granas, J. Dugundji, *Fixed Point Theory*, Berlin, Springer-Verlag, 2003.

[2] Y. Feng, S. Liu, *Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings,* J. Math. Anal. Appl. **317**(2006), 103-112. Zbl 1094.47049.

[3] L. Guran, *Existence and data dependence for multivalued weakly contractive operators*, Studia Univ. Babeş-Bolyai, Mathematica, **54**(2009), no. 3, 67-76.

[4] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica **44**(1996) 381-391. MR1416281(97j:49011). Zbl 0897.54029.

[5] A. Latif, A.W. Albar , *Fixed point results for multivalued maps*, Int. J. Contemp. Math. Sci. **2**, 21-24(2007), 1129-1136. MR2373908. Zbl 1165.54018.

[6] A. Latif, A. A. N. Abdou, *Fixed point results for generalized contractive multimaps in metric spaces*, Fixed Point Theory and Applications Volume **2009**(2009), Article ID 432130, 16 pages doi:10.1155/2009/432130. MR2544343. Zbl 1179.54065.

[7] A. Latif, Saleh Abdullah Al-Mezel, *Fixed point results related to Reich'S problem*, Fixed Point Theory and Applications, Vol.4(2009), Issue 2, 147-159.

[8] T. Lazăr, A. Petruşel, N. Shahzad, *Fixed points for non-self operators and domanin invariance theorems*, Nonlinear Analysis: Theory, Methods and Applications, **70**(2009), 117-125.

[9] S. Reich, *Fixed points of contractive functions*, Boll. Unione Mat. Ital., **5**(1972), 26-42.

[10] S. Reich, A. J. Zaslavski, *Convergence of iterates of nonexpansive set-valued mappings*, Set Valued Mappings with Applications in Nonlinear Analysis, **4**(2002) of Mathematical Analysis and Applications, 411–420, Taylor and Francis, London, UK.

[11] S. Reich, A. J. Zaslavski, *Generic existence of fixed points for set-valued mappings*, Set-Valued Analysis, **10**(2002), no. 4, 287–296.

[12] S. Reich, A. J. Zaslavski, *Two results on fixed points of set-valued nonexpansive mappings*, Revue Roumaine de Mathématiques Pures et Appliquées, **51**(2006), no. 1, 89–94.

[13] S. Reich, A. J. Zaslavski, *Existence and Approximation of Fixed Points for Set-Valued Mappings*, Fixed Point Theory and Applications, Volume **2010**(2010), Article ID 351531, 10 pages doi:10.1155/2010/351531. Zbl 1189.54037

[14] I. A. Rus, *Generalized Contractions and Applications*, Presa Clujeană Universitară, Cluj-Napoca, 2001.

[15] T. Suzuki, *Generalized Distance and Existence Theorems in Complete Metric Spaces*, J.Math.Anal.Appl., **253**(2001), 440-458. MR1808147(2002f:49038). Zbl 0983.54034.

[16] T. Suzuki, *Several Fixed Point Theorems Concerning τ-distance*, Fixed Point Theory and Application, **3**(2004), 195-209. MR2096951. Zbl 1076.54532.

[17] T. Suzuki, W. Takahashi, *Fixed points theorems and characterizations of metric completeness*, Topological Methods in Nonlinear Analysis, Journal of Juliusz Schauder Center, **8**(1996), 371-382. MR1483635(99c:54064). Zbl 0902.47050.

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