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A SPECIAL CASE OF RATIONAL θS FOR TERMINATING θ-EXPANSIONS

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Abstract. There have been quite a few generalizations of the usual continued fraction expansions over the last few years. One very special generalization deals with θ-continued fraction expansions or simply θ-expansions introduced by Bhattacharya and Goswami [1]. Chakraborty and Rao [3] subsequently did elaborate studies on θ-expansions in their paper. They also obtained the unique invariant measure for the Markov process associated with the generalized Gauss transformation that generated θ-expansions for some special θs. In this work, we investigate an interesting question regarding the nature of θs for θ-expansion of $\frac{1}{\theta}$ terminating at stage two, particularly with θ rational.

1 Introduction

1.1 Usual Continued Fraction Expansions

An expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

where $a_0, a_1, a_2, \ldots$ are real or complex numbers is called a continued fraction expansion. However, in most of the interesting studies, $a_0, a_1, a_2, \ldots$ are assumed to be positive integers. If the resulting number is between zero and one, then a_0 is assumed to be zero. A continued fraction with these $a_0, a_1, a_2, \ldots$ is also very popularly written as

$$[a_0; a_1, a_2, \cdots]$$

In case a_0 is zero, one writes

$$[a_1, a_2, \cdots]$$

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The theory of usual continued fraction expansions has wide applications in analysis, probability theory and number theory. As of now, there had been studies on continued fraction expansions for quite a few centuries. A very important reference on continued fractions is by Khinchin [5] that was published in the form of a monograph. Later there had been quite a few generalized versions of continued fractions found in the works of Bissinger [2], Everett [4] and Renyi [6].

1.2 θ-Continued Fraction Expansions

In the new millennium, a very special kind of generalization can be found in the works of Bhattacharya and Goswami [1] who introduced the concept of θ-expansions for a number in $[0, \theta)$ with $0 < \theta < 1$.

An expression of the form

$$a_0\theta + \frac{1}{a_1\theta + \frac{1}{a_2\theta + \dots}}$$

where $a_0, a_1, a_2, \dots$ are real or complex numbers is called a θ-continued fraction expansion. For our study, we assume $a_0, a_1, a_2, \dots$ to be positive integers. Like usual continued fraction expansions, we can also write it as

$$[a_0\theta; a_1\theta, a_2\theta, \cdots]_\theta$$

where the suffix θ is to stress that it is a θ-expansion. But since this article only considers θ-expansions, we will suppress the suffix and will always write it as

$$[a_0\theta; a_1\theta, a_2\theta, \cdots]$$

If the resulting number is between 0 and θ, then a_0 is assumed to be zero and we write it as

$$[a_1\theta, a_2\theta, \cdots]$$

An elaborate study on necessary and sufficient conditions for existence and uniqueness of θ-expansions for a number in $[0, \theta)$ was done in Chakraborty and Rao [3]. These conditions are quite non-trivial when one compares them with the ones for the usual continued fraction expansion. We give a brief overview in the next section to familiarize the readers with θ-expansions.

1.3 Generalized Gauss Map and Invariant Measure

The usual continued fraction expansion is generated by the Gauss transformation and there is a unique invariant measure for the Gauss transformation. This transformation is given by

$$U(x) = \left(\frac{1}{x} - \left[\frac{1}{x}\right]\right) I_x(0,1)$$

where I is the indicator function. The corresponding invariant measure is

$$\frac{1}{log\, 2}\, \frac{1}{1+x}\, dx$$

Chakraborty and Rao [3] obtained a generalized version of the Gauss transformation that generates the θ-expansions and they call it generalized Gauss map:

$$T(x) = \left(\frac{1}{x} - \theta\left[\frac{1}{\theta x}\right]\right) I_x(0,\theta)$$

where I is again the indicator function. They showed that when θ-expansion of $\frac{1}{\theta}$ terminates at the first stage, that is, when θ^2 is the reciprocal of a positive integer, the invariant measure for the Gauss map generalizes to that for the generalized Gauss map [3]:

$$\frac{1}{log\,(1+\theta^2)}\, \frac{\theta}{1+\theta x}\, dx$$

But there is no clue as to how to obtain the invariant measure for other values of θ in the work of Chakraborty and Rao [3].

1.4 Motivation for our present problem

Our main goal was to obtain an invariant measure for the generalized Gauss transformation when the θ-expansion of $\frac{1}{\theta}$ does not terminate at stage one and we understood that even the θ-expansion of $\frac{1}{\theta}$ terminating at stage two is not easy to dela with.

To understand the complexity of the problem, we look at the following comparison. In case of usual continued fraction expansion, a number is rational if the expansion is terminating. But for θ-expansions, the situation is not so simple. It can be easily seen that θ is not necessarily rational, even when the θ-expansion of $\frac{1}{\theta}$ terminates at stage one. For example, if $\theta^2 = 0.2$, then $\frac{1}{\theta^2} = 5$, and hence $\frac{1}{\theta^2}$ is a positive integer, but θ is not rational. Therefore, it makes sense to study terminating θ- expansions with rational θ. In this article, we only study the case when θ-expansion of $\frac{1}{\theta}$ terminates at stage two, obtain necessary and sufficient condition when θ is rational in this case and use this condition to generate all possible rational θs. So, using this expression for the solution of θ, we will try to investigate the situations when θ is rational and the θ-expansion of $\frac{1}{\theta}$ terminates at stage two, i.e., $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ for two positive integers n_1 and n_2.

In the next section, we discuss some preliminaries with respect to θ-continued fraction expansions. In section 3, we discuss the necessary and sufficient conditions for θ to be rational when the expansion terminates at stage 2. In section 4, we discuss more properties of n_1 and n_2 for θ to be rational. In section 5, we discuss how to generate rationals of various types using the conditions derived earlier on n_1 and n_2 under a special situation. We consider three subcases. We conclude with some remarks in section 6.

2 Preliminaries in θ-expansions

In this section, we give a brief overview of θ expansions. As mentioned in the previous section, an expression of the form

$$a_0\theta + \cfrac{1}{a_1\theta + \frac{1}{a_2\theta + \dots}}$$

is called a θ-continued fraction expansion or simply θ-expansion where $a_0, a_1, a_2, ...$ are assumed to be positive integers. If the resulting number is between 0 and θ, then a_0 is assumed to be zero. Starting with a positive real number x, one can obtain its θ-continued fraction expansion in terms of $a_0, a_1, a_2, \cdots$ as follows:

2.1 Definitions of $a_0, a_1, a_2, \cdots$ in θ-expansion

Let $x > 0$. Let $a_0 = max\{n \geq 0 : n\theta \leq x\}$. If x already equals $a_0\theta$, we write $x = [a_0\theta]$, otherwise define r_1 by $x = a_0\theta + \frac{1}{r_1}$ where $0 < \frac{1}{r_1} < \theta$. Then $r_1 > \frac{1}{\theta} \geq \theta$ and let $a_1 = max\{n \geq 0 : n\theta \leq r_1\}$. If $r_1 = a_1\theta$, then we write $x = [a_0\theta; a_1\theta]$. Otherwise we define r_2 like r_1 and proceed accordingly. This way, the process terminates at some stage n or continues indefinitely. In the former case, we write the expansion as $[a_0\theta; a_1\theta, ..., a_n\theta]$ and is called a terminating continued fraction expansion. In the latter case, we write the expansion as $[a_0\theta; a_1\theta, ...]$ and is called a non-terminating continued fraction expansion. Then one can define the n-th convergent of x as follows

$$\frac{p_n}{q_n} = [a_0\theta; a_1\theta, ..., a_n\theta]$$

for $n \geq 1$ where $[a_0\theta; a_1\theta, ..., a_n\theta]$ is a θ-expansion terminating at stage n. Here a_0 is zero if the number x is less than θ and we write the n-th convergent as $[a_1\theta, ..., a_n\theta]$. Then one can show that the n-th convergent of x, $\frac{p_n}{q_n}$ converges to x as n tends to ∞.

2.2 When does a θ-expansion of the form $[a_1\theta, a_2\theta, ...]$ arise as the θ-expansion of a number x less than θ?

The conditions were given in Chakraborty and Rao [3] for both terminating and non-terminating expansions of $\frac{1}{\theta}$.

2.2.1 θ-expansion of $\frac{1}{\theta}$ terminating at stage m (finite)

For terminating expansion of $\frac{1}{\theta}$, say $\frac{1}{\theta} = [n_1\theta;\, n_2\theta, ..., n_m\theta]$, we have the following conditions on x:

i) Each $a_i \geq n_1$.

ii) In case for some $i \geq 1$ and $p < m$,

$$< a_{i+1}, ..., a_{i+p} >=< n_1, ..., n_p >$$

then we should have $a_{i+p+1} \leq n_{p+1}$ if $p+1$ is even and $a_{i+p+1} \geq n_{p+1}$ if $p+1$ is odd. Moreover, if m is even and $p+1$ equals m, then $a_{i+p+1} < n_{p+1}$.

iii) If θ-expansion of x terminates at a finite stage, say, k, then a_k must satisfy $a_k > n_1$ and further if for some even $p < m$,

$$< a_{k-p}, ..., a_{k-1} >=< n_1, ..., n_p >$$

then $a_k > n_{p+1}$.

2.2.2 θ-expansion of $\frac{1}{\theta}$ is not terminating

For non-terminating expansion of $\frac{1}{\theta}$, say, $\frac{1}{\theta} = [n_1\theta;\, n_2\theta, ...]$, following are the conditions on x:

i) Each $a_i \geq n_1$.

ii) In case for some $i \geq 1$ and $p \geq 1$,

$$< a_{i+1}, ..., a_{i+p} >=< n_1, ..., n_p >$$

then $a_{i+p+1} \leq n_{p+1}$ if $p+1$ is even and $a_{i+p+1} \geq n_{p+1}$ if $p+1$ is odd.

iii) In the terminating case, the last a_k must satisfy $a_k > n_1$ and further if for some even $p \geq 1$,

$$< a_{k-p}, ..., a_{k-1} >=< n_1, ..., n_p >$$

then $a_k > n_{p+1}$.

2.3 θ-expansion of $\frac{1}{\theta}$: conditions on $n_1, n_2, \cdots$

Chakraborty and Rao [3] stated the conditions for both terminating and non-terminating θ-expansions. Here we only mention the conditions for θ-expansion terminating at the mth stage:

i) $n_i > n_1$ for $i = 2$ or m whereas $n_i \geq n_1$ for $2 < i < m$.

ii) If for some i and p with $i+p < m$, $< n_{i+1}, ..., n_{i+p} >=< n_1, ..., n_p >$, then we should have $n_{i+p+1} \leq n_{p+1}$ if $p+1$ is even and $n_{i+p+1} \geq n_{p+1}$ if $p+1$ is odd.

For $m = 2$ or 3, they derived the conditions explicitly. For our purpose, we only need $m = 2$ and so, from now on, we will only consider rational θs with θ-expansion of $\frac{1}{\theta}$ terminating at stage two. For $m = 2$, Chakraborty and Rao [3] has the following result that we wil use in subsequent sections:

Theorem 1. *The θ-expansion of $\frac{1}{\theta}$ terminates at stage two, i.e., $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ for two positive integers n_1 and n_2 if and only if $n_2 > n_1 + 1$ and $\theta = \sqrt{\frac{n_2-1}{n_1 n_2}}$.*

3 Conditions for θ rational

As mentioned earlier, we will only investigate the situation for rational θs with the θ-expansion of $\frac{1}{\theta}$ terminating at stage two. So, from now on, we fix our notations as follows: $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ for two positive integers n_1,n_2 ;θ is rational, say, $\theta = \frac{p}{q}$ where p and q are relatively prime positive integers. The question we now ask is the following: what should be the necessary and sufficient conditions on n_1 and n_2 so that θ is rational?

Theorem 2. *Given $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$, the necessary and sufficient condition on n_1 and n_2 so that θ is rational is as follows: there exists a positive integer m and two more squared positive integers m_1 and m_2 so that*

$$n_1 = \frac{mm_2}{mm_1+1} \quad \textit{and} \quad n_2 = mm_1 + 1$$

Proof. First we assume

$$n_1 = \frac{mm_2}{mm_1+1} \quad \text{and} \quad n_2 = mm_1 + 1$$

where m is a positive integer and m_1 and m_2 are squared positive integers. We will show that θ is rational. By the assumption, we have, $n_2 - 1 = mm_1$ and $n_1 n_2 = mm_2$. Therefore, $\frac{n_2-1}{n_1n_2} = \frac{m_1}{m_2}$. Now, since m_1 and m_2 are squared integers, let $m_1 = p^2$

and $m_2 = q^2$ for two positive integers p and q. So, $\theta = \sqrt{\frac{n_2-1}{n_1 n_2}} = \sqrt{\frac{m_1}{m_2}} = \frac{p}{q}$. Since both p and q are positive integers, θ must be a rational.

Conversely, if θ is rational, say, $\theta = \frac{p}{q}$, where p and q are relatively prime positive integers, then we would like to show that

$$n_1 = \frac{mm_2}{mm_1+1} \quad \text{and} \quad n_2 = mm_1 + 1$$

where m is a positive integer and m_1 and m_2 are squared positive integers.

It is clear that

$$\theta = \sqrt{\frac{n_2-1}{n_1 n_2}} = \frac{p}{q}$$

which implies

$$\theta^2 = \frac{n_2-1}{n_1 n_2} = \frac{p^2}{q^2}$$

Since p and q are relatively prime, i.e., $(p,\, q) = 1$, so, $(p^2,\, q^2) = 1$. Now, if there exists a positive integer m such that $(n_2 - 1,\, n_1 n_2) = m$ for some positive integer m, then, $n_2 - 1 = mp^2$ and $n_1 n_2 = mq^2$. So, $n_2 = mp^2 + 1$ and $n_1 = \frac{mq^2}{mp^2+1}$. Call $m_1 = p^2$ and $m_2 = q^2$. Then, obviously, m_1 and m_2 are two squared positive integers and n_1, n_2 satisfy

$$n_1 = \frac{mm_2}{mm_1+1} \quad \text{and} \quad n_2 = mm_1 + 1$$

for some positive integer m. Thus the proof is complete. □

Remark 3. *When* $(n_2 - 1,\, n_1 n_2) = 1$, *we have,* $n_2 - 1 = p^2$ *and* $n_1 n_2 = q^2$. *Therefore,* $p^2 + 1$ *divides* q^2.

Example 4. *Let* $\theta = \frac{7}{10}$. *Then,* $\frac{1}{\theta} = 2\theta + \frac{1}{50\theta}$. *So,* $n_1 = 2$, $n_2 = 50$ *and obviously,* $(n_2 - 1,\, n_1 n_2) = 1$.

Remark 5. *When* $(n_2 - 1,\, n_1 n_2) = m > 1$, *we have,* $n_2 - 1 = mp^2$ *and* $n_1 n_2 = mq^2$. *So,* $n_2 = mp^2 + 1$ *divides* q^2 *because* m *and* $mp^2 + 1$ *can not have any common factor. Since* m *divides* $n_2 - 1$, *it can not divide* n_2 *and hence, it must divide* n_1. *So, if* $n_1 = mr$, *then* $n_2 r = q^2$.

Example 6. *Consider* $\theta = \frac{2}{3}$. *Then,* $\frac{1}{\theta} = 2\theta + \frac{1}{9\theta}$. *So,* $m = n_1 = 2$, $n_2 = 9$ *and* $r = 1$. *Obviously,* $(n_2 - 1,\, n_1 n_2) = 2 > 1$.

We shall discuss the two cases mentioned in the above two remarks elaborately in the next section to further investigate conditions on n_1 and n_2 for θ to be rational.

4 More Conditions on n_1 and n_2

In this section, we investigate the nature of n_1 and n_2 given that $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ and θ rational with $\theta = \frac{p}{q}$ where p and q are relatively prime positive integers. Then, by Theorem 2 from the previous section, there exists a positive integer m so that $(n_2 - 1, n_1 n_2) = m$. We consider two cases depending on $m = 1$ or $m > 1$.

Case 1: $\mathbf{m = 1}$

In this case, $n_2 - 1$ is coprime with $n_1 n_2$. We start with some trivial examples.

In the first example, we consider $p = 2$. Then, $n_2 = 5$. So, $n_1 n_2 = 5n_1 = q^2$. Now, q^2 being a squared number, 25 must divide q^2 thereby implying that n_1 is divisible by 5. But this is not possible as $n_1 < n_2$.

In the second example, we consider $p = 3$. Then, $n_2 = 10$. So, $n_1 n_2 = 10n_1 = q^2$. Therefore, 100 must divide q^2 and hence, 10 should divide n_1. Once again this is not possible as $n_1 < n_2$.

In the above mentioned examples, n_2 was chosen either a prime or a product of distinct primes and we failed to obtain a rational θ satisfying $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$. We are going to see that in such a case, we can not have an example of rational θ satisfying $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ with $(n_2 - 1, n_1 n_2) = 1$.

But if n_2 is neither a prime nor a product of distinct primes, then there is indeed some example giving rise to rational θ of our need. Let us go back to Example 6 in the previous section. In that example, we had $\theta = \frac{7}{10}$ so that $\frac{1}{\theta} = 2\theta + \frac{1}{50\theta}$. So, $n_2 = 50$. Now in the prime factor decomposition of 50, 5 appears twice. So, 50 is neither a prime nor a product of distinct primes. Now let us state our theorem.

Theorem 7. *Let θ be a rational and the θ-expansion of $\frac{1}{\theta}$ is given by $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ with n_1 and n_2 are positive integers satisfying $(n_2 - 1, n_1 n_2) = 1$. Then, n_2 is neither a prime nor a product of distinct primes.*

Proof. The proof is easy and goes by the method of contradiction. Suppose we assume that n_2 is a product of distinct primes $p_1, \cdots, p_k$ (case $k = 1$ takes care of prime n_2s). Then, from $n_1 n_2 = q^2$, we see that $p_1, \cdots, p_k$ must divide n_1. But this is not possible as this leads to $n_1 \geq n_2$ which is false. Hence, the proof is complete by the method of contradiction. $\square$

So, we conclude that $n_2 = p^2 + 1$ must be of the form $p_1^{k_1} \cdots p_l^{k_l}$ where $p_1, \cdots, p_l$ are prime numbers and at least one of $k_1, \cdots, k_l$ must be bigger than 1. Is it possible that $l = 1$? We don't know. But we can answer some questions. Suppose $n_2 = p^2 + 1 = p_1^{k_1}$. Then how should be p_1 and k_1?

Our next theorem provides us with the answer.

Theorem 8. *Let θ be a rational given by $\theta = \frac{p}{q}$ and the θ-expansion of $\frac{1}{\theta}$ is given by $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ with n_1 and n_2 are positive integers satisfying $(n_2 - 1, n_1 n_2) = 1$.*

Also, let $p^2+1=p_1^{k_1}$ where p_1 is prime and $k_1>1$. Then the following conditions are satisfied:

(i) k_1 must be odd, say, $k_1=2k_1^*+1$,

(ii) p_1 is different from 2,

(iii) p must be even, i.e., $p=2p^*$ for some positive integer p^*,

(iv) If $p_1=2p_1^*+1$, then p_1^* must be even, i.e., there exists a positive integer p_1^{**} such that $p_1^*=2p_1^{**}$.

Proof. (i) If k_1 is even, say, $k_1=2k_1^*$, then $p^2+1=p_1^{2k_1^*}$ which implies two squared numbers p^2 and $(p_1^{k_1^*})^2$ differ only by 1. This is not possible. So, k_1 must be odd, say, $k_1=2k_1^*+1$ for some positive integer k_1^*.

(ii) If p is even, it is clear that p_1 can not be 2. But if p is odd, say, $p=2p^*+1$ then $p_1=2$ so that $p^2+1=2^{2k_1^*+1}$. This implies $4p^{*^2}+4p^*+2=2^{2k_1^*+1}\Rightarrow 2p^{*2}+2p^*+1=2^{2k_1^*}$ which is not possible. So, p_1 can not be 2 and p has to be even so that p_1 is odd. Therefore, $p_1=2p_1^*+1$ for some positive integer p_1^*.

(iii) From (ii), p is already even. Hence, $p=2p^*$ for some positive integer p^*.

(iv) From (i), (ii) and (iii), we have,

$$p^2+1=4p^{*2}+1=(2p_1^*+1)^{2k_1^*+1}$$

$$=\sum_{j=0}^{2k_1^*+1}\binom{2k_1^*+1}{j}(2p_1^*)^j=\sum_{j=2}^{2k_1^*+1}\binom{2k_1^*+1}{j}(2p_1^*)^j+2p_1^*+1$$

Now each term in $\sum_{j=2}^{2k_1^*+1}\binom{2k_1^*+1}{j}(2p_1^*)^j$ is divisible by 4 and hence $2p_1^*$ is divisible by 4 which means p_1^* is even. So, $p_1=4p_1^*+1$. This completes the proof. □

We now investigate the various combinations of p, q, n_1, and n_2 that appear in the θ-expansion of $\frac{1}{\theta}$.

For example, if p is even, p^2 is even and p^2+1 is odd. So, n_2 (which is same as p^2+1) is odd. Now since p is even, q has to be odd and so, n_1 (which divides q^2) has to be odd too. For p even, we can think of the example $\theta=\frac{18}{65}$ where $n_2=325$ is odd and $n_1=13$ is also odd. One can observe that $(n_2-1,n_1n_2)=(324,4225)=1$.

On the other hand, if p is odd, p^2 is odd and p^2+1 is even. So, n_2 (which is same as p^2+1) is even and hence q^2 (which is divisible by n_2) is even implying q even. So, n_1 (which divides q^2) can be odd or even. For p odd, consider once again the example $\theta=\frac{7}{10}$ so that $n_2=50$ is even and $n_1=2$ is also even. In this case, $(n_2-1,n_1n_2)=(49,100)=1$.

Case 2: m > 1

In this case, we have, $n_2 = mp^2 + 1$ which should divide q^2. Going by the arguments in the previous case, we can state the following Theorem 9 which is analogous to Theorem 7.

Theorem 9. *Let θ be a rational and the θ-expansion of $\frac{1}{\theta}$ is given by $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ with n_1 and n_2 are positive integers satisfying $(n_2 - 1, n_1 n_2) = m > 1$. Then, $n_2 = mp^2 + 1$ is neither a prime nor a product of distinct primes.*

The proof of Theorem 9 is exactly similar to that of Theorem 7. Therefore, n_2 is of the form $p_1^{k_1} \cdots p_l^{k_l}$ where $p_1, \cdots, p_l$ are distinct primes and at least one of $k_1, \cdots, k_l$ is bigger than 1.

One can pose a similar question as to whether l could be 1 or not and try to formulate a theorem as Theorem 8. In other words, is it possible to get a prime p_1 and a positive integer k_1 so that $mp^2 + 1 = p_1^{k_1}$? The answer is yes.

A subsequent question is: Is k_1 necessarily odd (like Theorem 8)? The answer is no. Look at the following example:

Example 10. $m = 3$, $p = 4$, $p_1 = 7$, $k_1 = 2$. *Then,* $mp^2 + 1 = 49 = p_1^{k_1}$. *In this example, choose* $q = 7$. *Then,* $n_1 = 3$, $n_2 = 49$ *so that* $\theta = \frac{4}{7}$.

We may also ask: Is p_1 necessarily different from 2 (like Theorem 8)? The answer is no again:

Example 11. *Choose* $m = 7$, $p = 3$. *Then,* $mp^2 + 1 = 64$ *so that* $p_1 = 2$, $k_1 = 6$. *In this case, choose* $q = 8$ *so that* $n_1 = 7$, $n_2 = 64$ *so that* $\theta = \frac{3}{8}$.

Another question is: Is p necessarily even? The answer is no again because of the above example, i.e., Example 11 where $p = 3$.

Finally, if p_1 is odd, is it of the form $4p_1^* + 1$ for some positive integer p_1^* (like Theorem 8)? The answer is, once again, no; this is because of the Example 10 above. In this example, $p_1 = 7$ which is of the form $4p_1^* + 3$ for $p_1^* = 1$. So, an analogous theorem as Theorem 8 does not hold good when $m > 1$.

In the following section, we investigate various possibilities as to how to get different rationals θ so that $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ when $r = 1$. For $r > 1$, we do not have any concrete result and we only mention some examples in the concluding section.

5 Generating rational θs for $r = 1$

Throughout this section, we assume $r = 1$ where $q^2 = n_2 r$ (r is the quotient when n_2 divides q^2). Then we first assume the following:

($\star$) If p is even, say, $p = 2p^*$, then p^* equals 1 or p^* is a prime or a power of a prime. And if p is odd, then p is a prime or a power of a prime.

Then consider the following theorem:

Theorem 12. *Assume* $(\star)$. *Then*

(i) Assume p *to be even. Then obviously* q *is odd and for every positive integer* i, $q = 2p^{*2}i \pm 1$ *so that* $\theta = \dfrac{p}{2p^{*2}i \pm 1}$.

(ii) Assume p *to be odd and* q *to be even. Then for all odd positive integers* i, $q = p^2 i \pm 1$ *so that* $\theta = \dfrac{p}{p^2 i \pm 1}$.

(iii) Assume p *to be odd and* q *to be odd too. Then for every positive integer* i, $q = 2p^2 i \pm 1$ *and* $\theta = \dfrac{p}{2p^2 i \pm 1}$.

Proof. For (i), since p is even, $p = 2p^*$ and since q is odd, $q = 2j + 1$ for some positive integer j. Then q^2 has to be odd. Therefore, $n_2 = mp^2 + 1$ is odd. Since $r = 1$, we have, $q^2 = n_2 = mp^2 + 1$, so that

$$(2j+1)^2 = 4mp^{*2} + 1 \Rightarrow 4j^2 + 4j + 1 = 4mp^{*2} + 1 \Rightarrow mp^{*2} = j(j+1) \tag{5.1}$$

Since we assume p^* to be a prime or power of a prime, therefore, p^{*2} divides either j or $j+1$. So, j is either $p^{*2}i$ or $p^{*2}i - 1$ for any positive integer i and hence, m should be of the form $i(p^{*2}i \pm 1)$ and q should be of the form $2p^{*2}i \pm 1$. So, $n_2 = (2p^{*2}i \pm 1)^2$. Also, since $n_1 n_2 = mq^2 = mn_2$, we have, $n_1 = m = i(p^{*2}i \pm 1)$. So, for each positive integer i, $\theta = \dfrac{p}{2p^{*2}i \pm 1}$ is a possible value.

For (ii), let $q = 2j$ for some positive integer j. Then, $n_2 = mp^2 + 1$ is even and so m is odd. Now since $r = 1$, we have,

$$q^2 = n_2 = mp^2 + 1 \Rightarrow 4j^2 = mp^2 + 1 \Rightarrow mp^2 = (2j-1)(2j+1) \tag{5.2}$$

Now, since p is a prime or a power of a prime, p^2 divides either $2j - 1$ or $2j + 1$. Therefore, $p^2 i = 2j - 1$ or $p^2 i = 2j + 1$, i.e., $j = \dfrac{p^2 i \pm 1}{2}$ for odd integers i. This implies, $m = i(p^2 i \pm 2)$ for odd integers i. Since r is 1, n_1 is same as m and hence it is odd and $n_2 = p^2 i(p^2 i \pm 2) + 1 = (p^2 i \pm 1)^2$ for odd positive integers i. As a result, $q = p^2 i \pm 1$ and $\theta = \dfrac{p}{p^2 i \pm 1}$ for odd positive integers i.

For (iii) let $q = 2j + 1$ for some positive integer j. In this case, since $r = 1$, we have,

$$q^2 = n_2 = mp^2 + 1 \Rightarrow 4j^2 + 4j + 1 = mp^2 + 1 \Rightarrow mp^2 = 4j(j+1) \tag{5.3}$$

Since p is a prime or a power of a prime, p^2 divides either j or $j+1$. Hence, $j = p^2 i$ or $p^2 i - 1$ so that $m = 4i(p^2 i \pm 1)$, $q = 2p^2 i \pm 1$ and $\theta = \dfrac{p}{2p^2 i \pm 1}$ for every positive integer i. Since r is 1, so n_1 is same as m and hence it is even. $\square$

For part (i), consider the following examples:

Example 13. *If $p^* = 1$ (i.e. $p = 2$), then $m = j(j+1)$ for every positive integer j and hence, for every j, $\theta = \frac{2}{2j+1}$ has its θ-expansion terminating at stage 2. If $p^* = 2$ or $p^* = 3$, then for every i, $\theta = \frac{4}{8i \pm 1}$ or $\theta = \frac{6}{12i \pm 1}$ are possible values.*

Example 14. *In Example 13, $p^* = 1$ and $n_1 = m$ is even. In general, if p^* is odd and i is also odd, then m is even: consider $i = 1$, $p^* = 3$, $m = 10$, $\theta = \frac{6}{19}$. In this case, $n_1 = 10$ and $n_2 = 361$.*

Example 15. *On the other hand, if p^* is even and i is odd, then m is odd: consider $i = 1$, $p^* = 4$, $m = 15$, $\theta = \frac{8}{31}$. In this case, $n_1 = 15$ and $n_2 = 961$.*

For part (ii), consider the following example:

Example 16. *For p prime or power of a prime and q even, some examples are: $p = 7$, $m = i(49i \pm 2)$, $\theta = \frac{7}{49i \pm 1}$; $p = 9$, $m = i(81i \pm 2)$, $\theta = \frac{9}{81i \pm 1}$ etc. where the i-s are odd so that m is odd and q is even.*

For part (iii), consider the following examples:

Example 17. *For p prime and q odd, some examples are: $p = 3$, $m = 4i(9i \pm 1)$, $\theta = \frac{3}{18i \pm 1}$; $p = 5$, $m = 4i(25i \pm 1)$, $\theta = \frac{5}{50i \pm 1}$; $p = 7$, $m = 4i(49i \pm 1)$, $\theta = \frac{7}{98i \pm 1}$ etc. where the i-s are odd so that m is odd and q is also odd.*

Now we remove assumption ($\star$). Then if p is even, p^* is neither a prime nor a power of a prime. Let $p^* = p_1^{l_1} ... p_s^{l_s}$ be the prime factorization of p^*. Then, we don't know which p_i divides j and which p_i divides $j+1$. But from equation (5.1), it is clear that if for some i_1 and i_2, p_{i_1} divides j, $p_{i_1}^{l_{i_1}}$ also divides j and if p_{i_2} divides $j+1$, $p_{i_2}^{l_{i_2}}$ also divides $j+1$. On the other hand, if p is odd, p is neither a prime nor a power of a prime. Then let $p = p_1^{l_1} \cdots p_s^{l_s}$. So, if q is even, from (5.2), $mp^2 = (2j-1)(2j+1)$, it is clear that for a certain pair (i_1, i_2), if p_{i_1} divides $2j-1$, $p_{i_1}^{l_{i_1}}$ should also divide $2j-1$ and if p_{i_2} divides $2j+1$, $p_{i_2}^{l_{i_2}}$ should also divide $2j+1$. And if q is odd, from (5.3), $mp^2 = 4j(j+1)$ and it is clear that for a certain pair (i_1, i_2), if p_{i_1} divides j, then $p_{i_1}^{l_{i_1}}$ should also divide j and if p_{i_2} divides $j+1$, then $p_{i_2}^{l_{i_2}}$ should also divide $j+1$. With all these understanding, we can prove a more general theorem:

Theorem 18. *Let us remove assumption ($\star$). Then*

(i) If p is even and equals $2p^*$, p^* is neither a prime nor a power of a prime. Let $p^* = p_1^* p_2^*$ where $p_1^* = p_1^{l_1} \cdots p_s^{l_s}$ divides j and $p_2^* = p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}}$ divides $j+1$. Then, there exists a pair (i_1, i_2) so that j is of the form $(p_1^{l_1} \cdots p_s^{l_s})^2 i_1$ and $j+1$ is of the form $(p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2$ so that $(p_1^{l_1} \cdots p_s^{l_s})^2 i_1 + 1 = (p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2$. For such a pair (i_1, i_2), we have, $q = \sqrt{4p_1^{2l_1} \cdots p_s^{2l_s} p_{s+1}^{2l_{s+1}} \cdots p_{s+t}^{2l_{s+t}} i_1 i_2 + 1}$. So, θ is of the form

$\theta = \dfrac{p}{\sqrt{4\prod_{i=1}^{s+t} p_i^{2l_i}\, i_1 i_2 + 1}}$. *Also, if* (i_1, i_2) *is a pair, then* $(i_1 + \prod_{i=1}^{s+t} p_i^{2l_i},\ i_2 + \prod_{i=1}^{s+t} p_i^{2l_i})$ *is also a pair that gives rise to another* q *and another* θ.

(ii) p *is odd and* p *is neither a prime nor a power of a prime. Since* q *is even, let* $p = p_1^* p_2^*$ *where* $p_1^* = p_1^{l_1} \cdots p_s^{l_s}$ *divides* $2j-1$ *and* $p_2^* = p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}}$ *divides* $2j+1$. *Then, there exists a pair* (i_1, i_2) *so that* $2j-1$ *is of the form* $(p_1^{l_1} \cdots p_s^{l_s})^2 i_1$ *and* $2j+1$ *is of the form* $(p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2$ *so that* $(p_1^{l_1} \cdots p_s^{l_s})^2 i_1 + 2 = (p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2$. *For such a pair* (i_1, i_2), *we have,* $q = \sqrt{\prod_{i=1}^{s+t} p_i^{2l_i} i_1 i_2 + 1} = (p_1^{l_1} \cdots p_s^{l_s})^2 i_1 + 1 = (p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2 - 1$. *So,* θ *is of the form* $\theta = \dfrac{p}{\sqrt{\prod_{i=1}^{s+t} p_i^{2l_i}\, i_1 i_2 + 1}}$. *Also, if* (i_1, i_2) *is a pair, then* $(i_1 + \prod_{i=1}^{s+t} p_i^{2l_i},\ i_2 + \prod_{i=1}^{s+t} p_i^{2l_i})$ *is also a pair that gives rise to another* q *and another* θ.

(iii) p *is odd and* p *is neither a prime nor a power of a prime. Since* q *is odd, let* $p = p_1^* p_2^*$ *where* $p_1^* = p_1^{l_1} \cdots p_s^{l_s}$ *divides* j *and* $p_2^* = p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}}$ *divides* $j+1$. *Then, there exists a pair* (i_1, i_2) *so that* j *is of the form* $(p_1^{l_1} \cdots p_s^{l_s})^2 i_1$ *and* $j+1$ *is of the form* $(p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2$ *so that* $(p_1^{l_1} \cdots p_s^{l_s})^2 i_1 + 1 = (p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2$. *For such a pair* (i_1, i_2), *we have,* $q = \sqrt{4\prod_{i=1}^{s+t} p_i^{2l_i} i_1 i_2 + 1} = 2(p_1^{l_1} \cdots p_s^{l_s})^2 i_1 + 1 = 2(p_{s+1}^{l_{s+1}} \cdots p_{s+t}^{l_{s+t}})^2 i_2 - 1$. *So,* θ *is of the form* $\theta = \dfrac{p}{\sqrt{4\prod_{i=1}^{s+t} p_i^{2l_i}\, i_1 i_2 + 1}}$. *Also, if* (i_1, i_2) *is a pair, then* $(i_1 + \prod_{i=1}^{s+t} p_i^{2l_i},\ i_2 + \prod_{i=1}^{s+t} p_i^{2l_i})$ *is also a pair that gives rise to another* q *and another* θ.

We also mention a simpler version of theorem 6 for understanding the subsequent examples:

Theorem 19. *Let us remove assumption* $(\star)$.

(i) If p *is even, assume* $p^* = p_1 p_2$ *where* p_1 *and* p_2 *are distinct primes. If* $p_1 p_2$ *divides* j *or* $j+1$, *the conclusion is exactly like theorem 5(i). If* p_1 *divides* j *and* p_2 *divides* $j+1$, *then there exists a pair* (i_1, i_2) *so that* j *is of the form* $p_1^2 i_1$ *and* $j+1$ *is of the form* $p_2^2 i_2$ *so that* $p_1^2 i_1 + 1 = p_2^2 i_2$. *For such a pair* (i_1, i_2), *using (5.1), we have,* $m = i_1 i_2$ *so that* $q = \sqrt{mp^2+1} = \sqrt{4p_1^2 p_2^2 i_1 i_2 + 1}$. *So,* θ *is of the form* $\dfrac{p}{\sqrt{4p_1^2 p_2^2 i_1 i_2 + 1}}$. *If* $2j+1$ *is a possible value of* q, *then* $2(j + p_1^2 p_2^2) + 1$ *is also a possible value of* q.

(ii) If p *is odd, assume* $p = p_1 p_2$ *where* p_1 *and* p_2 *are distinct primes. Let* $q = 2j$ *be even. If* $p_1 p_2$ *divides* $2j-1$ *or* $p_1 p_2$ *divides* $2j+1$, *the conclusion is exactly like*

theorem 5(ii). If p_1 divides $2j-1$ and p_2 divides $2j+1$, then there exists a pair (i_1, i_2) so that $2j-1$ is of the form $p_1^2 i_1$ and $2j+1$ is of the form $p_2^2 i_2$ so that $p_1^2 i_1 + 2 = p_2^2 i_2$ where i_1 and i_2 are odd. For such a pair (i_1, i_2), from (5.2), we have, $m = i_1 i_2$ so that $q = \sqrt{mp^2+1} = \sqrt{p_1^2 p_2^2 i_1 i_2 + 1} = p_1^2 i_1 + 1 = p_2^2 i_2 - 1$. So, θ is of the form $\frac{p}{\sqrt{p_1^2 p_2^2 i_1 i_2 + 1}} = \frac{p}{p_1^2 i_1 + 1} = \frac{p}{p_2^2 i_2 - 1}$. If $2j$ is a possible value of q, then $2(j + p_1^2 p_2^2)$ is also a possible value of q.

(iii) If p is odd, assume $p = p_1 p_2$ where p_1 and p_2 are distinct primes. Let $q = 2j+1$ be odd. If $p_1 p_2$ divides j or $j+1$, the conclusion is exactly like theorem 5(iii). If p_1 divides j and p_2 divides $j+1$, then there exists a pair (i_1, i_2) so that j is of the form $p_1^2 i_1$ and $j+1$ is of the form $p_2^2 i_2$ so that $p_1^2 i_1 + 1 = p_2^2 i_2$ where i_1 and i_2 are odd. For such a pair (i_1, i_2), from (5.3), we have, $m = 4 i_1 i_2$ so that $q = \sqrt{mp^2+1} = \sqrt{4p_1^2 p_2^2 i_1 i_2 + 1} = 2p_1^2 i_1 + 1 = 2p_2^2 i_2 - 1$. So, θ is of the form $\frac{p}{\sqrt{4p_1^2 p_2^2 i_1 i_2 + 1}} = \frac{p}{2p_1^2 i_1 + 1} = \frac{p}{2p_2^2 i_2 - 1}$. If $2j+1$ is a possible value of q, then $2(j + p_1^2 p_2^2) + 1$ is also a possible value of q.

We consider the following examples to understand Theorem 19:

Example 20. *For part (i) of Theorem 19, suppose $p^* = 6$. Then, p^* has two distinct prime factors, namely, 2 and 3. Now, from (5.1), we already have $mp^{*2} = 36m = j(j+1)$. In case 36 divides j or 36 divides $j+1$, it is like Theorem 12 (i). So, we assume the following: either 4 divides j and 9 divides $j+1$ or 9 divides j and 4 divides $j+1$ or 36 divides j. If 4 divides j and 9 divides $j+1$, then there exist i_1 and i_2 so that $4i_1 + 1 = 9i_2$. One has to choose possible pairs of (i_1, i_2). The first possible pair is $(2, 1)$ so that $j = 8$, $j+1 = 9$ and $j(j+1)$ equals 72. Therefore, $m = i_1 i_2 = 2$, $q = 17$ and $\theta = \frac{12}{17}$. One can observe that, one can obtain subsequent pairs of $(j, j+1)$ by simply adding multiples of 36 to $(8, 9)$: $(44, 45)$, $(80, 81)$, $(116, 117)$, $(152, 153)$ etc. In these cases, corresponding θs and ms are given by $m = 180$, $\theta = \frac{12}{161}$; $m = 377$, $\theta = \frac{12}{233}$; $m = 646, \theta = \frac{12}{305}$ etc. Similarly, if 9 divides j and 4 divides $j+1$, we need pairs (i_1, i_2) so that $9i_1 + 1 = 4i_2$. The first possible pair of (i_1, i_2) is $(3, 7)$ which gives rise to the first possible pair of $(j, j+1)$ to be $(27, 28)$. In this case, $\theta = \frac{12}{55}$ and $m = 21$. The subsequent pairs of $(j, j+1)$ are obtained by adding 36 to both the coordinates, namely, $(63, 64)$, $(99, 100)$, $(135, 136)$ etc. The corresponding values of θ and m are as follows: $m = 112$, $\theta = \frac{12}{127}$; $m = 275$, $\theta = \frac{12}{199}$; $m = 510$, $\theta = \frac{12}{271}$, $m = 510$ etc. We can consider many such examples where p^* is just the product of two distinct primes, like, $p^* = 10, 14, 15$ etc. In these cases, once we get the first pairs of $(j, j+1)$, the subsequent ones are obtained by adding 100, 196, 225 etc. respectively to each of the co-ordinates of the previous pair. In general, if $p^* = p_1 p_2$ where p_1 and p_2 are two distinct primes, then after obtaining the first pair, the subsequent pairs are obtained by adding p^{*2} to each of the co-ordinates of the previous pair and* *so on.*

Example 21. *For part (ii) of Theorem 19, let $p = 15$. Then, from $mp^2 = (2j - 1)(2j + 1)$, either 225 divides $2j - 1$ or 225 divides $2j + 1$ or 9 divides $2j - 1$ and 25 divides $2j + 1$ or 25 divides $2j - 1$ and 9 divides $2j + 1$. The case with 225 dividing $2j - 1$ or $2j + 1$ is similar to the case when p is a prime or a power of a prime. So, we concentrate on the last two cases. Then, an example of such a pair $(2j - 1, 2j + 1)$ is $(25, 27)$ where 25 divides $2j - 1$ and 9 divides $2j + 1$ is: $m = 3$, $\theta = \frac{15}{26}$. Then it is clear that $n_2 = 676$, $n_1 = 3$. Subsequent pairs of $(2j - 1, 2j + 1)$ are obtained by adding multiples of 450 to both coordinates. So, the next pair is $(475, 477)$. On the other hand, if we desire to have a pair where 9 divides $2j - 1$ and 25 divides $2j + 1$, then such a pair for $(2j - 1, 2j + 1)$ is $(423, 425)$. In this case, $m = 799$, $\theta = \frac{15}{424}$ etc. The next subsequent pair of $(2j - 1, 2j + 1)$ by adding 450 to both coordinates is $(873, 875)$.*

Example 22. *For part (iii) of Theorem 19, let $p = 15$. Then, from $mp^2 = 4j(j+1)$, either 225 divides j or 225 divides $j + 1$ or 9 divides j and 25 divides $j + 1$ or 25 divides j and 9 divides $j + 1$. The case with 225 dividing j or $j + 1$ is similar to the case when p is a prime or a power of a prime. So, we concentrate on the last two cases. Then, an example of such a pair $(j, j + 1)$ is $(99, 100)$ where 9 divides j and 25 divides $j + 1$ is: $m = 176$, $\theta = \frac{15}{199}$. Then it is clear that $n_2 = 39601$, $n_1 = 176$. Another example of such a pair $(j, j + 1)$ is $(125, 126)$ where 25 divides j and 9 divides $j + 1$ is: $m = 280$, $\theta = \frac{15}{251}$. In this case, $n_2 = 63001$, $n_1 = 280$. Subsequent pairs are obtained by adding 450 to both coordiantes of $(j, j + 1)$ in both these cases.*

However, the situation is not so simple as Theorem 19 as Theorem 18 suggests. So, we can write down more examples to illustrate Theorem 18:

Example 23. *For part (i) of Theorem 18, let $p = 144$ so that $p^* = 72 = 2^3 \cdot 3^2 = 8 \cdot 9$. We need a $q = 2j + 1$ so that 81 divides j and 64 divides $j + 1$. Choose $j = 1215$. Then 81 divides 1215 and 64 divides 1216. In this case, the pair is $(i_1, i_2) = (15, 19)$. So, $m = 285$ and q equals $q = 2 \cdot 1215 + 1 = 2431 = \sqrt{4 \cdot 8^2 \cdot 9^2 \cdot 15 \cdot 19 + 1}$. Then θ equals $\theta = \frac{144}{2431}$. Next we need a $q = 2j + 1$ so that 64 divides j and 81 divides $j + 1$. Choose $j = 3968$. Then, 64 divides 3968 and 81 divides 3969. In this case, the pair is $(62, 49)$. So, $m = 3038$ and $q = 2 \cdot 3968 + 1 = 7937$. Therefore, θ equals $\theta = \frac{144}{7937}$.*

Example 24. *For part (ii) of Theorem 18, let $p = 45 = 3^2 \cdot 5 = 9 \cdot 5$. Then $9^2 = 81$ divides $2j - 1$ and $5^2 = 25$ divides $2j + 1$. Then in order to find a q, we have to find a pair (i_1, i_2) so that $2j - 1 = 81i_1$ and $2j + 1 = 25i_2$. Such a pair is $(i_1, i_2) = (33, 107)$ so that $2j - 1 = 2673$ and $2j + 1 = 2675$. For such a pair, we have, $q = 2j = 2674$ and $\theta = \frac{45}{2674}$. Similarly, we can find a q so that 25 divides $2j - 1$ and 81 divides $2j + 1$. We have the pair $(i_1, i_2) = (55, 17)$ so that $2j - 1 = 1375$ and $2j + 1 = 1377$. For this pair, $q = 2j = 1376$ and $\theta = \frac{45}{1376}$.*

Example 25. *For part (iii) of Theorem 18, consider once again* $p = 45 = 3^2 \cdot 5 = 9 \cdot 5$. *So, we first find a* q *so that* $9^2 = 81$ *divides* j *and* $5^2 = 25$ *divides* $j+1$. *Then, for the pair* $(i_1, i_2) = (4, 13)$, *we have,* $j = 324$ *and* $j+1 = 325$ *so that* $m = 208$, $q = 649$ *and* $\theta = \frac{45}{649}$. *Next, we find a* q *so that* 25 *divides* j *and* 81 *divides* $j+1$. *Then, the pair* $(68, 21)$ *is our key. It gives us* $j = 1700$ *and* $j+1 = 1701$ *so that* $m = 5712$, $q = 3401$ *and* $\theta = \frac{45}{3401}$.

6 Concluding Remarks

Remark 26. *It would be interesting to see whether there is an example for* $p^2 + 1 = p_1^{k_1}$ *in section 4. Of course, one has to have the conditions that* k_1 *is odd,* p *is even, and* p_1 *is of the form* $4p_1^{**} + 1$ *for some positive integer* p_1^{**}.

Remark 27. *In the previous section, we had the assumption* $r = 1$ *and we came up with formulas as to how to generate all possible such rational* θ*s. But when* $r > 1$, *the situation is not easily understandable. Suppose we start with some 'meaningful'* m *and* p. *Then, from* $(mp^2 + 1)r = q^2$, *one understands that* $r < mp^2 + 1$ *as otherwise, we get* $n_1 \geq n_2$ *which is not possible. Also, we observe that either both* $mp^2 + 1$ *and* r *are perfect squares, say,* $mp^2 + 1 = m_1^2$ *and* $r = m_2^2$ *with the obvious understanding that* $m_1 > m_2$ *so that* $q = m_1 m_2$ *and* $\theta = \dfrac{p}{m_1 m_2}$ *or* r *is a factor of* $mp^2 + 1$ *with* $mp^2 + 1$ *containing a perfect square and of the form* $mp^2 + 1 = m_3^2 m_4$ *where* $m_3 > 1$ *and* $m_3^2 > m_4$ *and* m_4 *- a prime or a product of distinct primes in which case* $r = m_4$ *so that* $q = m_3 m_4$ *and* $\theta = \dfrac{p}{m_3 m_4}$. *But the question that always remains is the following: how to choose* p *and* m? *Below, we make some observations in all three of our cases, namely, (i)* p *even,* q *odd, (ii)* p *odd,* q *even and (iii)* p *odd,* q *odd:*

(i) For $p = 2p^*$ *even, we have,* $q^2 = (mp^2 + 1)r = (4mp^{*2} + 1)r \Rightarrow 4j^2 + 4j + 1 = 4mp^{*2}r + r$. *Then,* $r - 1$ *should be divisible by* 4. *Let* $r = 4r^* + 1$ *for some positive integer* r^*. *Then, we have,* $q^2 = 4j^2 + 4j + 1 = 16mp^{*2}r^* + 4mp^{*2} + 4r^* + 1$. *This implies,* $j(j+1) = 4mp^{*2}r^* + mp^{*2} + r^*$ *etc.*

(ii) For p *odd and* q *even,* n_2 *is even and both* n_1 *and* m *are odd. Then, from* $(mp^2 + 1)r = q^2 = (2j)^2 = 4j^2$. *If* r *is odd,* $mp^2 + 1$ *is divisible by* 4. *It can be seen that under the given assumptions on* m *and* p *(both odd),* m *should be of the form* $4m^* - 1$ *and* p *should be of the form* $2p^* + 1$. *But if* r *is even, then, if* m *is even (odd),* n_2 *is odd (even) and* n_1 *is always even irrespective of* m *being even or odd. (a) First we consider* m *to be even. Then, in* $(mp^2 + 1)r = q^2 = 4j^2$, m *is of the form* $2m^*$ *and* $p = 2p^* + 1$. *(b) Then we consider* m *to be odd. So, if* $(mp^2 + 1)r = q^2 = 4j^2$, m *is of the form* $2m^* + 1$ *and* $p = 2p^* + 1$.

(iii) For p *odd and* q *odd,* r *is also odd. As a result,* n_2 *(a factor of* q^2*) is odd and so* m *is even (since* $n_2 = mp^2 + 1$*) and hence,* $n_1 = mr$ *is even. Then, from* $(mp^2 + 1)r = 4j^2 + 4j + 1$ *where* m *is of the form* $m = 2m^*$ *and* $p = 2p^* + 1$.

So, we do not have any concrete result and we mention only some examples to understand Remark 27:

Example 28. *For (i), we choose* $r^* = 1$ *so that* $r = 5$ *and* $j(j+1) = 5mp^{*2} + 1$. *One possible value of* $(j, j+1) = (42, 43)$. *In this case,* $p^* = 19$, $m = 1$ *and* $\theta = \frac{38}{85}$. *Choosing* $r^* = 2$, *we have,* $r = 9$ *and* $j(j+1) = 9mp^{*2} + 2$. *In this case, we can get a pair,* $(j, j+1) = (13, 14)$ *so that* $\theta = \frac{4}{27}$ *and* $m = 5$ *or say,* $(j, j+1) = (22, 23)$ *so that* $\theta = \frac{4}{45}$ *and* $m = 14$. *Another example with* m *odd is:* $m = 15$, $\theta = \frac{8}{93}$ *where* $q^2 = 961r$ *where* $r = 9 = 4.2 + 1$ *so that* $r^* = 2$. *So,* $n_1 = 135$ *and* $n_2 = 961$ *here.*

Example 29. *For (ii), first we consider even* q *and odd* r. *Take* $m = 3$, $p^* = 2$, $p = 5$ *so that* $n_2 = mp^2 + 1 = 76$ *and* $q^2 = n_2 r = 76r = 4j^2$ *which implies* $19r = j^2$. *If we choose* $r = 19$, *then* $n_1 = 3r = 57$, $q = 38$ *and* $\theta = \frac{5}{38}$. *Here* $r = 19$ *is odd and* $n_2 = 76$ *is even.*

Example 30. *For (ii), next we consider even* q, *even* r *and even* m. *Take* $m = 2$, $p^* = 3$, $p = 7$ *so that* $n_2 = mp^2 + 1 = 99$ *and* $q^2 = n_2 r = 99r = 4j^2$. *If we choose* $r = 44$, *it is even and* $n_1 = 2r = 88$, $q = 66$, $n_2 = 99$ *and* $\theta = \frac{7}{66}$.

Example 31. *For (ii), finally we consider even* q, *even* r *and odd* m. *Take* $m = 3$, $p^* = 7$, $p = 15$ *so that* $n_2 = mp^2 + 1 = 676$ *and* $q^2 = n_2 r = 676r_1 = 4j^2 \Rightarrow 169r = j^2$. *If we choose* $r = 4$ *is even,* $n_2 = 676$ *is even and* $n_1 = 12$ *is also even. So,* $q = 52$ *and* $\theta = \frac{15}{52}$.

Example 32. *For (iii), choose* $m = 2$, $p^* = 3$, $p = 7$ *so that* $n_2 = mp^2+1 = 99$ *and* $q^2 = (mp^2 + 1)r = 99r = (2j+1)^2$. *If we choose* $r = 11$ *is odd, then* $n_1 = 2r = 22$ *and* $n_2 = 99$ *is odd. So,* $q = 33$ *and* $\theta = \frac{7}{33}$.

Remark 33. *Even though the results are interesting from number theoretic point of view, the author believes that these results might help in further analysis related to continued fraction expansions and ergodic theory.*

Remark 34. *Although we studied only the situation when* θ*-expansion of* $\dfrac{1}{\theta}$ *terminates at stage two, it will be interesting to look into the cases with* $\frac{1}{\theta}$ *having terminating* θ*-expansion that terminates at stages higher than* 2. *Some examples of such non-trivial rational* θ*s with* $\theta = \frac{p}{q}$ *are as follows:*

(a) p *even,* q *odd: consider* $\theta = \frac{4}{11}$. *Then, one can show that*

$$\frac{1}{\theta} = [7\theta; 13\theta, 17\theta, 484\theta]$$

(b) p *odd,* q *even: consider* $\theta = \frac{5}{6}$. *Then, one can show that*

$$\frac{1}{\theta} = [\theta; 3\theta, 5\theta, 5\theta, 10\theta, 18\theta]$$

(c) p odd, q odd: consider $\theta = \frac{3}{7}$. *Then, one can show that*

$$\frac{1}{\theta} = [5\theta; 12\theta, 21\theta, 7\theta]$$

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