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AVERAGING FOR FUZZY DIFFERENTIAL EQUATIONS

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Abstract. We prove and discuss averaging results for fuzzy differential equations. Our results generalize previous ones.

1 Introduction and preliminaries

In recent years, the fuzzy set theory introduced by Zadeh [20] has emerged as a powerful tool for modeling of uncertainty and for processing of vague or subjective information in mathematical models, whose main directions of development have been diversified and applied in many varied real problems. For such mathematical modeling, using fuzzy differential equations are necessary. For significant results from the theory of fuzzy differential equations and their applications, among many works we refer the interested reader, for instance, to the books [12, 16] and the papers [1, 2, 4, 13, 17, 18, 19] and the references therein.

In the present paper, we establish averaging results for fuzzy differential equations. As in the previous works of the first author on the justification of the method of averaging (see [7, 8, 9, 10, 11]), the conditions we assume on the right-hand sides of the fuzzy differential equations under which our averaging results are stated are more general than those considered in the existing literature as in [5, 6, 15].

Let $\mathbb{R}^d$ denotes the d-dimensional space with the euclidian norm $|\cdot|$. $\mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ stands for the class of all nonempty compact and convex subsets of $\mathbb{R}^d$. In $\mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ the so-called Hausdorff metric is defined by

$$\rho(A,B) := \max\left(\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right), \quad A,B\in \mathrm{Conv}(\mathbb{R}^{\mathrm{d}}).$$

The metric space $(\mathrm{Conv}(\mathbb{R}^{\mathrm{d}}),\rho)$ is complete.

Denote $\mathbb{E}^d$ the space of functions $u:\mathbb{R}^d\to[0,1]$ that satisfy the following conditions:

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i) u is normal, i.e., there exists $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, i.e., for any $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$, one has

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};$$

iii) u is upper semicontinuous, i.e., for any $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that $u(x) < u(x_0) + \varepsilon$ for all $x \in \mathbb{R}^d$ that satisfy the condition $|x - x_0| < \delta$;

iv) the closure of the set $\{x \in \mathbb{R}^d : u(x) > 0\}$ is compact.

The zero element in $\mathbb{E}^d$ is defined by $\hat{0}(y) = 1$ if $y = 0$ and $\hat{0}(y) = 0$ if $y \neq 0$.

For $\alpha \in (0, 1]$, the α-section $[u]^\alpha$ of a mapping $u \in \mathbb{E}^d$ is defined as the set $\{x \in \mathbb{R}^d : u(x) \geq \alpha\}$. The zero section of a mapping $u \in \mathbb{E}^d$ is defined as the closure of the set $\{x \in \mathbb{R}^d : u(x) > 0\}$.

From i) – iv), it follows that $[u]^\alpha \in \mathrm{Conv}(\mathbb{R}^\mathrm{d})$ for all $\alpha \in [0, 1]$.

For addition and scalar multiplication, we have, for $\alpha \in [0, 1]$,

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda u]^\alpha = \lambda [u]^\alpha, \qquad u, v \in \mathbb{E}^d, \ \lambda \in \mathbb{R}.$$

In the space $\mathbb{E}^d$, define $D : \mathbb{E}^d \times \mathbb{E}^d \to \mathbb{R}_+$ by setting

$$D(u, v) = \sup_{\alpha \in [0,1]} \rho([u]^\alpha, [v]^\alpha), \qquad u, v \in \mathbb{E}^d$$

where ρ is the Hausdorff metric. D is a metric in $\mathbb{E}^d$ such that:

i) $(\mathbb{E}^d, D)$ is complete;

ii) $D(x + z, y + z) = D(x, y)$ for all $x, y, z \in \mathbb{E}^d$;

iii) $D(kx, ky) = |k| D(x, y)$ for all $x, y \in \mathbb{E}^d$ and $k \in \mathbb{R}$.

The following definitions and theorems are given in [3, 14]. Let I be an interval in $\mathbb{R}$.

Definition 1. *A function $f : I \to \mathbb{E}^d$ is called strongly measurable on I if, for all $\alpha \in [0, 1]$, the set-valued function $f_\alpha : I \to \mathrm{Conv}(\mathbb{R}^\mathrm{d})$ defined by: $f_\alpha(t) = [f(t)]^\alpha$, is Lebesgue-measurable.*

Definition 2. *A function $f : I \to \mathbb{E}^d$ is called integrally bounded on I if there exists a Lebesgue-integrable function $k(t)$ such that $|x| \leq k(t)$ for all $x \in f_0(t)$.*

Definition 3. *The integral of a function $f : I \to \mathbb{E}^d$ over I, denoted by $\int_I f(t)dt$, is defined as an element $G \in \mathbb{E}^d$ such that, for all $\alpha \in (0, 1]$,*

$$\begin{aligned} [G]^\alpha &= \int_I f_\alpha(t)dt \\ &= \left\{ \int_I \phi(t)dt \mid \phi : I \to \mathbb{R}^d \text{ is a measurable selection for } f_\alpha \right\}. \end{aligned}$$

Theorem 4. *If a function $f : I \to \mathbb{E}^d$ is strongly measurable and integrally bounded, then f is integrable on I.*

Theorem 5. *If $f, g : I \to \mathbb{E}^d$ are integrable on I and $\lambda \in \mathbb{R}$, then the following assertions are true:*

i) $\displaystyle\int_I [f(t) + g(t)]dt = \int_I f(t)dt + \int_I g(t)dt;$

ii) $\displaystyle\int_I \lambda f(t)dt = \lambda \int_I f(t)dt;$

iii) $\displaystyle D\left(\int_I f(t)dt, \int_I g(t)dt\right) \leq \int_I D(f(t), g(t))dt.$

Definition 6. *A function $f : I \to \mathbb{E}^d$ is called continuous at a point $t_0 \in I$ if, for any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $D(f(t), f(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta$, $t \in I$.*

A function $f : I \to \mathbb{E}^d$ is called continuous on I if it is continuous at every point $t_0 \in I$.

Definition 7. *A function $f : I \times \mathbb{E}^d \to \mathbb{E}^d$ is called continuous at a point $(t_0, x_0) \in I \times \mathbb{E}^d$ if, for any $\varepsilon > 0$ there exists $\delta = \delta(t_0, x_0, \varepsilon) > 0$ such that $D(f(t, x), f(t_0, x_0)) < \varepsilon$ whenever $|t - t_0| < \delta$ and $D(x, x_0) < \delta$, $t \in I$ and $x \in \mathbb{E}^d$.*

A function $f : I \times \mathbb{E}^d \to \mathbb{E}^d$ is called continuous on $I \times \mathbb{E}^d$ if it is continuous at every point $(t_0, x_0) \in I \times \mathbb{E}^d$.

Definition 8. *A function $f : I \times \mathbb{E}^d \to \mathbb{E}^d$ is called continuous in $x \in \mathbb{E}^d$ uniformly with respect to $t \in I$ if, for any $x_0 \in \mathbb{E}^d$ and $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that $D(f(t, x), f(t, x_0) < \varepsilon$ for all $t \in I$ whenever $D(x, x_0) < \delta$, $x \in \mathbb{E}^d$.*

Definition 9. *A function $f : I \to \mathbb{E}^d$ is called differentiable at a point $t_0 \in I$ if, for any $\alpha \in [0, 1]$, the set-valued function $f_\alpha : I \to \mathrm{Conv}(\mathbb{R}^d)$ is Hukuhara differentiable at the point t_0, its derivative is equal to $D_H f_\alpha(t_0)$, and the family of sets $\{D_H f_\alpha(t_0) : \alpha \in [0, 1]\}$ defines a function $f'(t_0) \in \mathbb{E}^d$ (which is called a fuzzy derivative of $f(t)$ at the point t_0).*

A function $f : I \to \mathbb{E}^d$ is called differentiable on I if it is differentiable at every point $t_0 \in I$.

Theorem 10. *Suppose that a function $f : I \to \mathbb{E}^d$ is differentiable and its fuzzy derivative $f' : I \to \mathbb{E}^d$ is integrable on I. Then, for any $a, t \in I$, we have*

$$f(t) = f(a) + \int_a^t f'(\tau)d\tau.$$

2 Averaging results

Consider the following initial value problem associated to a fuzzy differential equation with a small parameter

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right), \qquad x(0) = x_0, \tag{2.1}$$

where $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$, $\mathbb{U}$ an open subset of $\mathbb{E}^d$, $x_0 \in \mathbb{U}$ and $\varepsilon > 0$ is a small parameter.

Definition 11. *A function $x : I \to \mathbb{U}$, where $I = [0, \omega)$, $0 < \omega \leq \infty$, is called a solution of problem (2.1) if it is continuous and, for all $t \in I$, satisfies the integral equation*

$$x(t) = x_0 + \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau)\right) d\tau.$$

We associate (2.1) with the averaged initial value problem

$$\dot{y} = f^o(y), \qquad y(0) = x_0, \tag{2.2}$$

where the function $f^o : \mathbb{U} \to \mathbb{E}^d$ is such that, for any $x \in \mathbb{U}$

$$\lim_{T\to\infty} D\left(\frac{1}{T}\int_0^T f(\tau, x)d\tau, f^o(x)\right) = 0. \tag{2.3}$$

The main result of this paper establishes nearness of solutions of problems (2.1) and (2.2) on finite time intervals, and reads as follows.

Theorem 12. *Suppose that the following conditions are satisfied:*

(H1) the function $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$ in (2.1) is continuous;

(H2) the continuity of f in $x \in \mathbb{U}$ is uniform with respect to $t \in \mathbb{R}_+$;

(H3) there exist a locally integrable function $m : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $M > 0$ such that

$$D(f(t, x), \hat{0}) \leq m(t), \quad \forall t \in \mathbb{R}_+, \forall x \in \mathbb{U}$$

with

$$\int_{t_1}^{t_2} m(t)dt \leq M(t_2 - t_1), \ \forall t_1, t_2 \in \mathbb{R}_+;$$

(H4) for all $x \in \mathbb{U}$, the limit (2.3) exists;

(H5) for the function $f^o : \mathbb{U} \to \mathbb{E}^d$ in (2.2), there exists a constant $\lambda > 0$ such that for continuous functions $u, v : \mathbb{R}_+ \to \mathbb{U}$ and $t \in \mathbb{R}_+$,

$$D\left(\int_0^t f^o(u(\tau))d\tau, \int_0^t f^o(v(\tau))d\tau\right) \leq \lambda \int_0^t D(u(\tau), v(\tau))d\tau. \tag{2.4}$$

Let $x_0 \in \mathbb{U}$. Let x_ε be a solution of (2.1) and let $I = [0, \omega_\varepsilon)$, $0 < \omega_\varepsilon \leq \infty$, be its maximal positive interval of definition. Let y be the solution of (2.2) and let $J = [0, \omega_0)$, $0 < \omega_0 \leq \infty$, be its maximal positive interval of definition. Then, for any $L > 0$, $L \in I \cap J$, and $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, we have $D(x_\varepsilon(t), y(t)) < \delta$ for all $t \in [0, L]$.

Notice that in condition (H5) we do not require that the function f^o is Lipschitz continuous. On the other hand, the averaging results stated in [5, 6, 15] are proved under conditions that are stronger compared to the ones above. In particular, the authors assume that the function f satisfies the Lipschitz condition with respect to the second variable.

We discuss now the result of Theorem 12 when the function f is periodic or more generally almost periodic in t. In those cases some of the conditions in Theorem 12 can be removed. Indeed, if f is periodic in t, from continuity and periodicity properties one can easily deduce condition (H2). Periodicity also implies condition (H4) in an obvious way. The average of f is then given, for any $x \in \mathbb{U}$, by

$$D\left(\frac{1}{P}\int_0^P f(\tau, x)d\tau, f^o(x)\right) = 0, \tag{2.5}$$

where P is the period. If f is almost periodic in t, for all $x \in \mathbb{U}$, the limit

$$\lim_{T\to\infty} D\left(\frac{1}{T}\int_s^{s+T} f(\tau, x)d\tau, f^o(x)\right) = 0 \tag{2.6}$$

exists uniformly with respect to $s \in \mathbb{R}$. So, condition (H4) is satisfied when $s = 0$. In a number of cases encountered in applications the function f is a finite sum of periodic functions in t. As in the periodic case above, condition (H2) is then satisfied. Hence we have the following result.

Corollary 13 (Periodic and Almost periodic cases)**.** *The conclusion of Theorem 12 holds when the function f satisfies conditions (H1), (H3), (H5) and is periodic or a sum of periodic functions in the first variable. It holds also when f satisfies conditions (H1), (H2), (H3), (H5) and is almost periodic in the first variable.*

2.1 Technical lemmas

In what follows we will prove some results we need for the proof of Theorem 12.

Lemma 14. *Let $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$ be a function. Suppose that f satisfies conditions (H2), (H3) and (H4) in Theorem 12. Then the function $f^o : \mathbb{U} \to \mathbb{R}^d$ in (2.3) is continuous and satisfies*

$$D(f^o(x), \hat{0}) \leq M, \quad \forall x \in \mathbb{U}$$

where the constant M is as in condition (H4).

Proof. Continuity of f^o. Let $x_0 \in \mathbb{U}$. By condition (H2), for any $\xi > 0$ there exists $\delta > 0$ such that, for all $x \in \mathbb{U}$, $D(x, x_0) \leq \delta$ implies that

$$D(f(\tau, x), f(\tau, x_0)) \leq \xi, \ \forall \tau \in \mathbb{R}_+. \tag{2.7}$$

Now, by condition (H4), we can easily deduce that, for any $\eta > 0$ there exists $T_0 = T_0(x_0, x, \eta) > 0$ such that, for all $T \geq T_0$ we have

$$\begin{aligned} D(f^o(x), f^o(x_0)) &\leq D\left(f^o(x), \frac{1}{T}\int_0^T f(\tau, x)d\tau\right) \\ &\quad + D\left(\frac{1}{T}\int_0^T f(\tau, x)d\tau, \frac{1}{T}\int_0^T f(\tau, x_0))d\tau\right) \\ &\quad + D\left(f^o(x_0), \frac{1}{T}\int_0^T f(\tau, x_0)d\tau\right) \\ &\leq 2\eta + \frac{1}{T}\int_0^T D\left(f(\tau, x), f(\tau, x_0)\right)d\tau \leq 2\eta + \xi. \end{aligned}$$

Since the value of η is arbitrary, in the limit we obtain that $D(f^o(x), f^o(x_0)) \leq \xi$, which finishes to prove the continuity of f^o at the point x_0.

Boundedness of f^o by M. Let $x \in \mathbb{U}$. By condition (H3), we deduce that, for any $\eta > 0$ there exists $T_0 = T_0(x, \eta) > 0$ such that, for all $T \geq T_0$ we have

$$\begin{aligned} D(f^o(x), \hat{0}) &\leq D\left(f^o(x), \frac{1}{T}\int_0^T f(\tau, x)d\tau\right) + D\left(\frac{1}{T}\int_0^T f(\tau, x)d\tau, \hat{0}\right) \\ &\leq \eta + \frac{1}{T}\int_0^T D\left(f(\tau, x), \hat{0}\right)d\tau \leq \eta + M. \end{aligned}$$

Since the value of η is arbitrary, in the limit we obtain the desired result. □

Lemma 15. *Let $f : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{E}^d$ be a function. Suppose that f satisfies condition (H4) in Theorem 12. Then, for all $x \in \mathbb{U}$, $L > 0$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,L]} D\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0.$$

Proof. Let $x \in \mathbb{U}$, $L > 0$ and $\alpha > 0$. Let $t \in [0, L]$.

Case 1: $t = 0$. From condition (H4), it follows immediately that

$$\lim_{\varepsilon \to 0} D\left(\frac{\varepsilon}{\alpha}\int_0^{\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0.$$

Case 2: $t \in (0, L]$. We write

$$\begin{aligned}
&\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau \\
&\quad= \frac{1}{\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau - \frac{1}{\alpha/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau \\
&\quad= \frac{1}{t/\varepsilon+\alpha/\varepsilon}\left(\frac{t}{\alpha}+1\right)\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau \\
&\qquad\qquad -\frac{t}{\alpha}\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau \\
&\quad= \frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau \\
&\qquad +\frac{t}{\alpha}\left[\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau - \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau\right].
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\sup_{t\in(0,L]} D\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) \\
&\quad\le \sup_{t\in(0,L]} D\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) \\
&\qquad +\frac{L}{\alpha}\left[\sup_{t\in(0,L]} D\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right)\right. \\
&\qquad\qquad \left.+\sup_{t\in(0,L]} D\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau, f^o(x)\right)\right].
\end{aligned} \tag{2.8}$$

From condition (H4), we can easily deduce that

$$\lim_{\varepsilon\to 0}\sup_{t\in(0,L]} D\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0$$

and

$$\lim_{\varepsilon\to 0}\sup_{t\in(0,L]} D\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, x)d\tau, f^o(x)\right) = 0.$$

Therefore the right-hand side of (2.8) tends to zero as $\varepsilon \to 0$ and the result is proved. $\square$

The next corollary follows directly from Lemma 15.

Corollary 16. *Suppose that the function f in (2.1) satisfies conditions (H1), (H3) and (H4) in Theorem 12. Let $x_0 \in \mathbb{U}$. Let x_ε be a solution of (2.1) and let $I = [0, \omega_\varepsilon)$, $0 < \omega_\varepsilon \leq \infty$, be its maximal positive interval of definition. Then, for all $L > 0$, $L \in I$, and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,L]} D\left(\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x_\varepsilon(t)) d\tau, f^o(x_\varepsilon(t))\right) = 0. \tag{2.9}$$

Lemma 17. *Suppose that the function f in (2.1) satisfies conditions (H1)-(H4) in Theorem 12. Let $x_0 \in \mathbb{U}$. Let x_ε be a solution of (2.1) and let $I = [0, \omega_\varepsilon)$, $0 < \omega_\varepsilon \leq \infty$, be its maximal positive interval of definition. Then, for all $L > 0$, $L \in I$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,L]} D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t f^o(x_\varepsilon(\tau)) d\tau\right) = 0.$$

Proof. Let $L > 0$, $L \in I$, and $t_0 = 0 < t_1 < \cdots < t_n < \cdots < t_p = L$, $p \in \mathbb{N}$, a partition of $[0, L]$ with $\alpha = \alpha(\varepsilon) := t_{n+1} - t_n$, $n = 1, \ldots, p$ and $\lim_{\varepsilon \to 0} \alpha = 0$. Let $t \in [t_m, t_{m+1}]$ for any $m \in \{0, \cdots, p-1\}$. Then

$$\begin{aligned} D&\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t f^o(x_\varepsilon(\tau)) d\tau\right) \\ &\leq \sum_{n=0}^{m-1} D\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_{t_n}^{t_{n+1}} f^o(x_\varepsilon(\tau)) d\tau\right) \\ &\quad + D\left(\int_{t_m}^{t} f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_{t_m}^{t} f^o(x_\varepsilon(\tau)) d\tau\right). \end{aligned} \tag{2.10}$$

By condition (H3) and Lemma 14 we have

$$\begin{aligned} D&\left(\int_{t_m}^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_{t_m}^t f^o(x_\varepsilon(\tau)) d\tau\right) \\ &\leq D\left(\int_{t_m}^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \hat{0}\right) + D\left(\int_{t_m}^t f^o(x_\varepsilon(\tau)) d\tau, \hat{0}\right) \\ &\leq \int_{t_m}^t D\left(f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right), \hat{0}\right) d\tau + \int_{t_m}^t D\left(f^o(x_\varepsilon(\tau)), \hat{0}\right) d\tau \leq 2M\alpha. \end{aligned}$$

Now, for each $n = 0, \ldots, m-1$ and $\tau \in [t_n, t_{n+1}]$, by condition (H3) we can easily deduce that $D(x_\varepsilon(\tau), x_\varepsilon(t_n)) \leq M\alpha$ so that by conditions (H2) and the continuity of f^o (Lemma 15), it follows, respectively, that

$$D\left(f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right), f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(t_n)\right)\right) \leq \gamma_n = \gamma_n(\varepsilon)$$

and

$$D(f^o(x_\varepsilon(\tau)), f^o(x_\varepsilon(t_n))) \le \delta_n = \delta_n(\varepsilon),$$

with $\lim\limits_{\varepsilon\to 0} \gamma_n = \lim\limits_{\varepsilon\to 0} \delta_n = 0$.

Hence, from (2.10), it follows that

$$\begin{aligned} D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t f^o(x_\varepsilon(\tau))d\tau\right) \\ \le \sum_{n=0}^{m-1} D\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} f^o(x_\varepsilon(t_n))d\tau\right) \\ + \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} (\gamma_n + \delta_n)d\tau + 2M\alpha. \end{aligned} \tag{2.11}$$

For each $n = 0, \ldots, m-1$, we have

$$\begin{aligned} \beta_n &:= D\left(\int_{t_n}^{t_{n+1}} f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} f^o(x_\varepsilon(t_n))d\tau\right) \\ &= \alpha D\left(\frac{\varepsilon}{\alpha}\int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha/\varepsilon} f\left(\tau, x_\varepsilon(t_n)\right) d\tau, f^o(x_\varepsilon(t_n))\right) \\ &\le \alpha \sup_{t\in[0,L]} D\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_\varepsilon(t))d\tau, f^o(x_\varepsilon(t))\right) := \ \alpha\varrho \qquad (\varrho = \varrho(\varepsilon)). \end{aligned}$$

Then

$$\sum_{n=0}^{m-1} \beta_n \le \varrho \sum_{n=0}^{m-1} \alpha = \varrho \sum_{n=0}^{m-1} (t_{n+1} - t_n) = \varrho t \le \varrho L,$$

where, by Corollary 16, $\lim\limits_{\varepsilon\to 0} \varrho = 0$.

On the other hand, we have

$$\sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} (\gamma_n + \delta_n)d\tau \le \eta \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} d\tau = \eta t \le \eta L,$$

where $\eta = \eta(\varepsilon) = \max\{\gamma_n + \delta_n : n = 0, \ldots, m-1\}$ and $\lim\limits_{\varepsilon\to 0} \eta = 0$.

Finally, from (2.11) we obtain

$$\sup_{t\in[0,L]} D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t f^o(x_\varepsilon(\tau))d\tau\right) \le (\varrho + \eta)L + 2M\alpha. \tag{2.12}$$

As the right-hand side of (2.12) tends to zero as $\varepsilon \to 0$, the lemma is proved. $\square$

2.2 Proof of Theorem 12

We are now able to prove our main result (there is not much work left). We assume that the assumptions in Theorem 12 are fulfilled.

For $t \in [0, L] \subset I \cap J$, using condition (H5), we obtain

$$\begin{aligned} D(y(t), x_\varepsilon(t)) &= D\left(\int_0^t f^o(y(\tau))d\tau, \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau\right) \\ &\leq D\left(\int_0^t f^o(y(\tau))d\tau, \int_0^t f^o(x_\varepsilon(\tau))d\tau\right) \\ &\quad + D\left(\int_0^t f^o(x_\varepsilon(\tau))d\tau, \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau\right) \\ &\leq \lambda \int_0^t D\left(y(\tau), x_\varepsilon(\tau)\right) d\tau + \sigma \end{aligned} \tag{2.13}$$

where

$$\sigma = \sigma(\varepsilon) := \sup_{t\in[0,L]} D\left(\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t f^o(x_\varepsilon(\tau))d\tau\right).$$

By Lemma 17, we have $\lim_{\varepsilon\to 0} \sigma = 0$.

Now, by Gronwall Lemma, from (2.13) we deduce that

$$D(y(t), x_\varepsilon(t)) \leq \sigma \frac{|e^{\lambda L} - 1|}{\lambda},$$

which implies that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,L]} D(x_\varepsilon(t), y(t)) = 0.$$

The proof is complete. □

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