Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 12 (2017), 81 – 88

A REPRESENTATION FOR A WEIGHTED L^2 SPACE

Martha Guzmán-Partida and Carlos Robles-Corbala

Abstract. Using elementary tools of complex analysis and Hilbert space theory, we present a realization of a weighted L^2 space on the unit disc. In the way, we show some additional properties.

1 Introduction

A way to study the behavior of functions defined on the euclidean space is to decompose their domain into pieces of a certain nature. There are many examples in mathematics literature that illustrate this fact. Just to mention a case, consider the euclidean space $\mathbb{R}^n$ decomposed into the following disjoint pieces

$$\mathbb{R}^n = \bigcup_{k=0}^{\infty} C_k,$$

where

$$C_0 = \{x : |x| \leq 1\},$$
$$C_k = \left\{x : 2^{k-1} < |x| \leq 2^k\right\},$$

for $k = 1, 2, \ldots$. This kind of dyadic decompositions is employed in harmonic analysis, when we deal with maximal functions or singular integral operators. Also, such decompositions can be used to define spaces that include the classical ones studied in analysis. For example, one might consider the following generalizations of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$ (see [1], [3]):

$$A^p = \left\{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{A^p} < \infty\right\},$$

$$B^p = \left\{g \in L^p_{loc}(\mathbb{R}^n) : \|g\|_{B^p} < \infty\right\},$$

2010 Mathematics Subject Classification: 46C15, 46E22.
Keywords: Reproducing kernel, orthonormal basis, Hilbert space.

where

$$\|f\|_{A^p} = \sum_{k=0}^{\infty} 2^{kn/p'} \|f\chi_{C_k}\|_{L^p}, \tag{1.1}$$

$$\|g\|_{B^p} = \sup_{k\geq 0} 2^{-kn/p} \|f\chi_{C_k}\|_{L^p},$$

and $1/p + 1/p' = 1$. It is well known that A^p and B^p are Banach spaces and, for $1 < p < q < \infty$ we have the inclusions $A^q \subset A^p$ and $B^q \subset B^p$. Moreover, $L^\infty \subset B^p$ and $A^p \subset L^1$ for each p. Although we do not go into detail in this matter, it is worth to mention that the pair of spaces $\left(A^p, B^{p'}\right)$ play in a certain way, the role of the dual pair $\left(L^1, L^\infty\right)$. Moreover, the spaces A^p and B^p defined above, can be generalized introducing more parameters (see [5]) and they are usually called Herz-type spaces.

In this paper, we shall pay our attention to certain generalization of decomposition (1.1), but now applied to families of holomorphic functions in the unit disc. We will highlight some properties of these spaces and also, we will see how, such kind of decompositions lead us in a natural way to another representation of a weighted L^2 space on the disc. All this will be done in the following sections.

2 Preliminaries

Let D be the unit disc on $\mathbb{R}^2$, that is, $D = \{z \in \mathbb{C} : |z| < 1\}$. For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. The Herz-type space on the unit disc is defined as

$$\mathcal{K}_q^{p,\alpha}(D) = \left\{ f \in L^p_{loc}(D) : \|f\|_{\mathcal{K}_q^{p,\alpha}} < \infty \right\},$$

where

$$\|f\|_{\mathcal{K}_q^{p,\alpha}} = \left\| 2^{-m\alpha} \|f_m\|_{L^p} \right\|_{l^q(\mathbb{N}\cup\{0\})}$$

$$= \left[\sum_{m=0}^{\infty} 2^{-mq\alpha} \|f_m\|_{L^p}^q \right]^{1/q},$$

and $f_m = f\chi_{A_m}$, $A_m = \{z \in D : r_m \leq |z| < r_{m+1}\}$, $r_m = 1 - 2^{-m}$, $m = 0, 1, \ldots$.

Notice that, when $f \in L^2(D)$ and $\alpha \leq 0$ we have

$$\|f\|_{L^2}^2 = \sum_{m=0}^{\infty} \|f_m\|_{L^2}^2 \leq \sum_{m=0}^{\infty} 2^{-2m\alpha} \|f_m\|_{L^2}^2 = \|f\|_{\mathcal{K}_2^{2,\alpha}}^2.$$

In fact, for $\alpha \leq 0 \leq \beta$ we have the continuous inclusions

$$\mathcal{K}_2^{2,\alpha}(D) \hookrightarrow L^2(D) \hookrightarrow \mathcal{K}_2^{2,\beta}(D) \tag{2.1}$$

since

$$\sum_{m=0}^{\infty} 2^{-2m\beta} \|f_m\|_{L^2}^2 \leq \|f\|_{L^2}^2 .$$

Now, let us define an inner product. For $f,\, g \in \mathcal{K}_2^{2,\alpha}(D)$

$$\langle f, g\rangle := \sum_{m=0}^{\infty} 2^{-2m\alpha} \int_{A_m} f\,(z)\, \overline{g\,(z)} dz, \tag{2.2}$$

where dz denotes the ordinary Lebesgue measure. This inner product is well defined since by Hölder inequality

$$\begin{aligned} \left| \sum_{m=0}^{\infty} 2^{-2m\alpha} \int_{A_m} f\,(z)\, \overline{g\,(z)} dz \right| &\leq \sum_{m=0}^{\infty} \left(2^{-m\alpha} \|f_m\|_{L^2}\right) \left(2^{-m\alpha} \|g_m\|_{L^2}\right) \\ &\leq \|f\|_{\mathcal{K}_2^{2,\alpha}} \|g\|_{\mathcal{K}_2^{2,\alpha}} < \infty. \end{aligned}$$

Clearly, for $\alpha = 0$, we recover the usual inner product on $L^2\,(D)$.

Denote by $H\,(D)$ the family of holomorphic functions on D and let $H^2\,(D)$ be the classical Hardy-Hilbert space

$$H^2\,(D) = \left\{ f \in H\,(D) : \|f\|_{H^2} < \infty \right\},$$

where

$$\|f\|_{H^2} = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 dt \right)^{1/2} .$$

Then, for $f \in H^2\,(D)$ we can see that

$$\begin{aligned} \sum_{m=0}^{\infty} 2^{-2m\alpha} \int_{A_m} |f\,(z)|^2\, dz &= \sum_{m=0}^{\infty} 2^{-2m\alpha} \int_{r_m}^{r_{m+1}} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 dtrdr \\ &\leq \pi \|f\|_{H^2}^2 \sum_{m=0}^{\infty} 2^{-m(1+2\alpha)} < \infty \end{aligned}$$

if $\alpha > -1/2$.

Thus, for $\alpha > -1/2$ we also have the continuous inclusion

$$H^2\,(D) \hookrightarrow \mathcal{K}_2^{2,\alpha}(D) \cap H\,(D)\,. \tag{2.3}$$

So far, thanks to (2.1) and (2.3), we have identified some known spaces inside these “new” spaces. We will discuss more about their structure in the following section.

3 A particular Hilbert space

Let us observe that for $\alpha \leq 0$ and $f \in \mathcal{K}_2^{2,\alpha}(D) \cap H(D)$, we can obtain for any $z_0 \in D$, after applying the mean value property for holomorphic functions (see [2], p. 6)

$$\begin{aligned}
|f(z_0)|^2 &\leq C_{z_0} \|f\|_{L^2}^2 \\
&\leq C_{z_0} \sum_{m=0}^{\infty} 2^{-2m\alpha} \|f_m\|_{L^2}^2 \\
&= C_{z_0} \|f\|_{\mathcal{K}_2^{2,\alpha}}^2 , \qquad (3.1)
\end{aligned}$$

where C_{z_0} is a constant only depending on z_0. The estimate (3.1) shows that the evaluation map is continuous. If we denote by $L_a^2(D)$ the Bergman space for the unit disc, that is, $L^2(D) \cap H(D)$, we have proved

Proposition 1. *For $\alpha \leq 0$, $\mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ is a reproducing kernel Hilbert space continuously included in the Bergman space $L_a^2(D)$.*

Our next goal is to determine an orthonormal basis for $\mathcal{K}_2^{2,\alpha}(D) \cap H(D)$, $\alpha \leq 0$. It is natural to consider the sequence of monomials $\{z^n\}_{n\geq 0}$, $z \in D$. Let us call $\varphi_n(z) = z^n$, $n \geq 0$. Thus, proceeding formally

$$\begin{aligned}
\|\varphi_n\|_{\mathcal{K}_2^{2,\alpha}}^2 &= \sum_{k=0}^{\infty} 2^{-2k\alpha} \int_{r_k}^{r_{k+1}} \int_0^{2\pi} r^{2n+1} dt dr \\
&= \frac{\pi}{n+1} \sum_{k=0}^{\infty} 2^{-2k\alpha} \left[r_{k+1}^{2(n+1)} - r_k^{2(n+1)} \right]. \qquad (3.2)
\end{aligned}$$

We can write

$$\begin{aligned}
r_{k+1}^{2(n+1)} - r_k^{2(n+1)} &= \left(r_{k+1}^{n+1} - r_k^{n+1}\right)\left(r_{k+1}^{n+1} + r_k^{n+1}\right) \\
&\leq 2\left(r_{k+1}^{n+1} - r_k^{n+1}\right) \\
&= 2\left(r_{k+1} - r_k\right)\left(r_{k+1}^n + r_{k+1}^{n-1} r_k + ... + r_{k+1} r_k^{n-1} + r_k^n\right) \\
&\leq 2(n+1)(r_{k+1} - r_k) \\
&= \frac{n+1}{2^k}.
\end{aligned}$$

In this way, for each $m \in \mathbb{N}$

$$\frac{\pi}{n+1} \sum_{k=0}^{m} 2^{-2k\alpha} \left[r_{k+1}^{2(n+1)} - r_k^{2(n+1)} \right] \leq \pi \sum_{k=0}^{m} 2^{-k(2\alpha+1)},$$

hence, the series in (3.2) converges if $\alpha > -1/2$, which implies that the sequence of monomials $\{z^n\}_{n\geq 0}$ belongs to $\mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ for $-1/2 < \alpha \leq 0$.

Next, notice that if $f \in \mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ with Taylor series $f(z) = \sum\limits_{j=0}^{\infty} a_j z^j$, and satisfies $\langle f, z^n \rangle = 0$ for each $n \geq 0$, then by uniform convergence

$$0 = a_n \frac{\pi}{n+1} \sum_{k=0}^{\infty} 2^{-2k\alpha} \left[r_{k+1}^{2(n+1)} - r_k^{2(n+1)} \right]$$

which implies that $a_n = 0$ for every $n \geq 0$. This shows that $\{z^n\}_{n\geq 0}$ is an orthogonal basis of $\mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ for $-1/2 < \alpha \leq 0$. If we denote by

$$\theta_n = \left(\frac{\pi}{n+1} \sum_{k=0}^{\infty} 2^{-2k\alpha} \left[r_{k+1}^{2(n+1)} - r_k^{2(n+1)} \right] \right)^{-1/2},$$

we have proved

Proposition 2. *For $-1/2 < \alpha \leq 0$, $\mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ is a reproducing kernel Hilbert space of holomorphic functions on D with orthonormal basis $\{\Psi_n\}_{n\geq 0}$, where $\Psi_n(z) = \theta_n z^n$.*

As it is well known (see [4],[6]), the reproducing kernel $K_w(z)$, $z, w \in D$, for $\mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ can be computed as

$$\begin{aligned} K_w(z) &= \sum_{n=0}^{\infty} \Psi_n(z) \overline{\Psi_n(w)} \\ &= \sum_{n=0}^{\infty} \frac{(n+1)}{b_{\alpha,n}} (z\overline{w})^n, \end{aligned}$$

where

$$b_{\alpha,n} = \pi \sum_{k=0}^{\infty} 2^{-2k\alpha} \left[r_{k+1}^{2(n+1)} - r_k^{2(n+1)} \right].$$

We notice that for $\alpha = 0$, $\sum\limits_{k=0}^{\infty} \left[r_{k+1}^{2(n+1)} - r_k^{2(n+1)} \right] = 1$ and so, in this case,

$$K_w(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\overline{w})^n = \frac{1}{\pi} \frac{1}{(1 - z\overline{w})^2}.$$

which is the reproducing kernel for the Bergman space $L_a^2(D)$.

Also, the previous computations allow us to give a description of $f \in \mathcal{K}_2^{2,\alpha}(D) \cap H(D)$ by means of its Taylor coefficients at 0.

Indeed, $f(z)=\sum_{j=0}^{\infty} a_j z^j \in \mathcal{K}_2^{2,\alpha}(D)\cap H(D)$ if and only if

$$\|f\|_{\mathcal{K}_2^{2,\alpha}}^2=\sum_{j=0}^{\infty}\left[\frac{|a_j|^2}{j+1}\left(\pi\sum_{k=0}^{\infty}\left(r_{k+1}^{2(j+1)}-r_k^{2(j+1)}\right)2^{-2k\alpha}\right)\right]<\infty. \tag{3.3}$$

Again, for $\alpha=0$, we recover the classical condition on Taylor coefficients for functions in the Bergman space $L_a^2(D)$, i.e., $f\in L_a^2(D)$ if and only if

$$\sum_{j=0}^{\infty}\frac{|a_j|^2}{j+1}<\infty.$$

Now, taking into account that

$$A_m=\left\{z\in D: 2^{-(m+1)}<1-|z|\leq 2^{-m}\right\},$$

we can easily see that for $\alpha\leq 0$, $f\in\mathcal{K}_2^{2,\alpha}(D)$ if and only if

$$\sum_{m=0}^{\infty}\int_{A_m}(1-|z|)^{2\alpha}|f(z)|^2\,dz=\int_D|f(z)|^2(1-|z|)^{2\alpha}\,dz<\infty.$$

Moreover, for $-1/2<\alpha\leq 0$ and $f(z)=\sum_{j=0}^{\infty}a_jz^j\in\mathcal{K}_2^{2,\alpha}(D)\cap H(D)$ we have

$$\begin{aligned}\|f\|_{\mathcal{K}_2^{2,\alpha}}^2&=\|f\|_{L^2\left(D,(1-|z|)^{2\alpha}dz\right)}^2\\&=2\pi\sum_{j=0}^{\infty}|a_j|^2\int_0^1(1-r)^{2\alpha}r^{2j+1}dr\\&=(2\pi)\sum_{j=0}^{\infty}\left[|a_j|^2\frac{\Gamma(2j+2)\Gamma(2\alpha+1)}{\Gamma(2j+2\alpha+3)}\right],\end{aligned}\tag{3.4}$$

where Γ is the gamma function.

If we compare (3.3) with (3.4) we obtain as additional information that

$$\frac{1}{j+1}\sum_{k=0}^{\infty}\left(r_{k+1}^{2(j+1)}-r_k^{2(j+1)}\right)2^{-2k\alpha}=2\frac{\Gamma(2j+2)\Gamma(2\alpha+1)}{\Gamma(2j+2\alpha+3)}. \tag{3.5}$$

Hence, we can give an explicit formula for the orthonormal basis $\{\Psi_n\}_{n\geq 0}$ obtained in Proposition 2, namely

$$\Psi_n(z)=\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\Gamma(2n+2\alpha+3)}}{\sqrt{\Gamma(2n+2)\Gamma(2\alpha+1)}}z^n.$$

We can summarize the work previously done in the following theorem.

Theorem 3. *For* $-1/2 < \alpha \leq 0$

$$\mathcal{K}_2^{2,\alpha}(D) \cap H(D) = L^2\left(D, (1-|z|)^{2\alpha}\, dz\right) \cap H(D)\,. \tag{3.6}$$

Finally, as the reader should have noticed in the above development, a crucial fact to obtain such characterizations was the analiticity of our functions. The space obtained in the right hand side of (3.6) is called Bergman space on the unit disc respect to the weight $(1-|z|)^{2\alpha}$, and plays an important role in complex analysis and operator theory.

References

[1] Y. Chen and K. Lau, Some new classes of Hardy spaces, *J. Funct. Anal.* **84** (1989), 255-278. MR1001460.

[2] J. Conway, *A Course in Functional Analysis,* Springer, 1990. MR1070713. Zbl 0706.46003.

[3] J. García-Cuerva, Hardy spaces and Beurling algebras, *J. London Math. Soc.* (2) **39** (1989), 499-513. MR1002462. Zbl 0681.42014.

[4] C. Goffman and G. Pedrick, *Functional Analysis,* Chelsea, 1983. MR0184064. Zbl 0502.46001.

[5] E. Hernández, G. Weiss and D. Yang, The φ-transform and wavelet characterizations of Herz-type spaces, *Collect. Math.* **47** (1996), 285-320. MR1437657. Zbl 0878.42011.

[6] K. Zhu, *Operator Theory in Function Spaces,* American Mathematical Society, 2007. MR2311536. Zbl 1123.47001.

Martha Guzmán-Partida
Departamento de Matemáticas,
Universidad de Sonora,
Hermosillo, Sonora, 83000, México.
e-mail: martha@mat.uson.mx

Carlos Robles-Corbala
Departamento de Matemáticas,
Universidad de Sonora,
Hermosillo, Sonora, 83000, México.
crobles@mat.uson.mx

License