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ON THE APPROXIMATE SOLUTION OF SOME FREDHOLM INTEGRAL EQUATIONS BY NEWTON'S METHOD

J. M. GUTIÉRREZ, M. A. HERNÁNDEZ AND M. A. SALANOVA

Abstract. The aim of this paper is to apply Newton's method to solve a kind of nonlinear integral equations of Fredholm type. The study follows two directions: firstly we give a theoretical result on existence and uniqueness of solution. Secondly we illustrate with an example the technique for constructing the functional sequence that approaches the solution.

A.M.S. (MOS) Subject Classification Codes. 45B05,47H15, 65J15
Key Words and Phrases. Fredholm integral equations, Newton's method.

1. Introduction

In this paper we give an existence and uniqueness of solution result for a nonlinear integral equation of Fredholm type:

$$(1) \qquad \phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)^p \, dt, \quad x \in [a,b], \quad p \geq 2,$$

where λ is a real number, the kernel $K(x,t)$ is a continuous function in $[a,b] \times [a,b]$ and $f(x)$ is a given continuous function defined in $[a,b]$.

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There exist various results about Fredholm integral equations of second kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x, t, \phi(t))\, dt, \quad x \in [a, b]$$

when the kernel $K(x, t, \phi(t))$ is linear in ϕ or it is of Lipschitz type in the third component. These two points have been considered, for instance, in [7] or [3] respectively. However the above equation (1) does not satisfy either of these two conditions.

In [3] we can also find a particular case of (1), for $f(x) = 0$ and $K(x, t)$ a degenerate kernel. In this paper we study the general case. The technique will consist in writing equation (1) in the form:

$$F(\phi) = 0, \tag{2}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator defined by

$$F(\phi)(x) = \phi(x) - f(x) - \lambda \int_a^b K(x, t)\phi(t)^p\, dt, \qquad p \geq 2,$$

and $X = Y = C([a, b])$ is the space of continuous functions on the interval $[a, b]$, equipped with the max-norm

$$\|\phi\| = \max_{x \in [0,1]} |\phi(x)|, \quad \phi \in X.$$

In addition, $\Omega = X$ if $p \in \mathbb{N}, \quad p \geq 2$, and when it will be necessary, $\Omega = C_+([a, b]) = \{\phi \in C([a, b]);\ \phi(t) > 0,\ t \in [a, b]\}$ for $p \in \mathbb{R}$, with $p > 2$.

The aim of this paper is to apply Newton's method to equation (2) in order to obtain a result on the existence and unicity of solution for such equation. This idea has been considered previously in different situations [1], [2], [4], [6].

At it is well known, Newton's iteration is defined by

$$\phi_{n+1} = \phi_n - \Gamma_n F(\phi_n), \quad n \geq 0, \tag{3}$$

where Γ_n is the inverse of the linear operator F'_{ϕ_n}. Notice that for each $\phi \in \Omega$, the first derivative F'_ϕ is a linear operator defined from X to Y by the following formula:

$$F'_\phi[\psi](x) = \psi(x) - \lambda p \int_a^b K(x, t)\phi(t)^{p-1}\psi(t)\, dt, \quad x \in [a, b], \quad \psi \in X. \tag{4}$$

In the second section we establish two main theorems, one about the existence of solution for (2) and other about the unicity of solution for the same equation. In the third section we illustrate these theoretical results with an example. For this particular case, we construct some iterates of Newton's sequence.

2. The main result

Let us denote $N = \max_{x\in[a,b]} \int_a^b |K(x,t)|\,dt$. Let ϕ_0 be a function in Ω such that $\Gamma_0 = [F'_{\phi_0}]^{-1}$ exists and $\|\Gamma_0 F(\phi_0)\| \le \eta$. We consider the following auxiliary scalar function

$$(5)\quad f(t) = 2(\eta - t) + M(\|\phi_0\| + t)^{p-2}\left[(p-1)\eta t - 2(\eta - t)(\|\phi_0\| + t)\right],$$

where, $M = |\lambda| pN$. Let us note that if $p \in \mathbb{N}$, with $p \ge 2$, $f(t)$ is a polynomial of degree $p-2$. Firstly, we establish the following two technical lemmas:

Lemma 2.1. *Let us assume that the equation $f(t) = 0$ has at least a positive real solution and let us denote by R the smaller one. Then we have the following relations:*

i) $\eta < R$.
ii) $a = M(\|\phi_0\| + R)^{p-1} < 1$.
iii) *If we denote* $b = \dfrac{(p-1)\eta}{2(\|\phi_0\| + R)}$ *and* $h(t) = \dfrac{1}{1-t}$, *then,* $abh(a) < 1$.
iv) $R = \dfrac{\eta}{1 - abh(a)}$.

Proof: First, notice that *iv)* follows from the relation $f(R) = 0$. So, as $R > 0$, we deduce that $abh(a) < 1$, and *iii)* holds. Moreover, $1 > 1 - abh(a) > 0$, then $1 < \dfrac{1}{1-abh(a)}$, so $\eta < R$, and *i)* also holds.

To prove *ii)*, we consider the relation $f(R) = 0$ that can be written in the form:

$$2(\eta - R)\left[1 - M(\|\phi_0\| + R)^{p-1}\right] = -M\eta(p-1)R(\|\phi_0\| + R)^{p-2} < 0.$$

As $\eta - R < 0$, $1 - M(\|\phi_0\| + R)^{p-1} = 1 - a > 0$, and therefore $a < 1$.

Let us denote $B(\phi_0, R) = \{\phi \in X;\ \|\phi - \phi_0\| < R\}$ and $\overline{B(\phi_0, R)} = \{\phi \in X;\ \|\phi - \phi_0\| \le R\}$.

Lemma 2.2. *If $B(\phi_0, R) \subseteq \Omega$, the following conditions hold*

i) *For all $\phi \in B(\phi_0, R)$ there exists $[F'_\phi]^{-1}$ and $\|[F'_\phi]^{-1}\| \le h(a)$.*

ii) *If* $\phi_n, \phi_{n-1} \in B(\phi_0, R)$*, then*

$$\|F(\phi_n)\| \leq \frac{(p-1)a}{2(\|\phi_0\| + R)}\|\phi_n - \phi_{n-1}\|^2.$$

Proof: To prove *i)* we apply the Banach lemma on invertible operators [5]. Taking into account

$$(I - F'_\phi)\psi(x) = \lambda p \int_a^b K(x,t)\phi(t)^{p-1}\psi(t)\, dt,$$

then

$$\|I - F'_\phi\| \leq |\lambda| pN \|\phi\|^{p-1} \leq M(\|\phi_0\| + R)^{p-1} = a < 1,$$

therefore, there exists $[F'_\phi]^{-1}$ and $\|[F'_\phi]^{-1}\| \leq \dfrac{1}{1-a} = h(a)$.

To prove *ii)*, using Taylor's formula, we have

$$F(\phi_n)(x) = \int_0^1 [F'_{\phi_{n-1}+s(\phi_n-\phi_{n-1})} - F'_{\phi_{n-1}}](\phi_n - \phi_{n-1})(x)\, ds$$

$$= -\lambda p \int_0^1 \int_a^b K(x,t)\left[\rho_n(s,t)^{p-1} - \phi_{n-1}(t)^{p-1}\right](\phi_n(t) - \phi_{n-1}(t))\, dt\, ds,$$

$$-\lambda p \int_0^1 \int_a^b K(x,t)\left[\sum_{j=0}^{p-2} \rho_n(s,t)^{p-2-j}\phi_{n-1}(t)^j\right][\phi_n(t) - \phi_{n-1}(t)]^2 s\, dt\, ds,$$

where $\rho_n(s,t) = \phi_{n-1}(t) + s(\phi_n - \phi_{n-1})$ and we have considered the equality

$$x^{p-1} - y^{p-1} = \left(\sum_{j=0}^{p-2} x^{p-2-j}y^j\right)(x-y), \quad x, y \in \mathbb{R}.$$

As $\phi_{n-1}, \phi_n \in B(\phi_0, R)$, for each $s \in [0,1]$, $\rho_n(s,\cdot) \in B(\phi_0, R)$, then $\|\rho_n(s,\cdot)\| \leq \|\phi_0\| + R$. Consequently

$$\|F(\phi_n)\| \leq \frac{|\lambda| pN}{2}\left(\sum_{j=0}^{p-2} (\|\phi_0\| + R))^{p-2-j}\|\phi_{n-1}\|^j\right)\|\phi_n - \phi_{n-1}\|^2$$

$$\leq |\lambda| \frac{p(p-1)N}{2}[\|\phi_0\| + R]^{p-2}\|\phi_n - \phi_{n-1}\|^2 = \frac{(p-1)a}{2(\|\phi_0\| + R)}\|\phi_n - \phi_{n-1}\|^2,$$

and the proof is complete.

Next, we give the following results on existence and uniqueness of solutions for the equation (2). Besides, we obtain that the sequence given by Newton's method has R-order two.

Theorem 2.3. *Let us assume that equation $f(t) = 0$, with f defined in (5) has at least a positive solution and let R be the smaller one. If $B(\phi_0, R) \subseteq \Omega$, then there exists at least a solution ϕ^* of (2) in $\overline{B(\phi_0, R)}$. In addition, the Newton's sequence (3) converges to ϕ^* with at least R-order two.*

Proof: Firstly, as $\|\phi_1 - \phi_0\| \le \eta < R$, we have $\phi_1 \in B(\phi_0, R)$. Then, Γ_1 exists and $\|\Gamma_1\| \le h(a)$. In addition,

$$\|F(\phi_1)\| \le \frac{(p-1)a}{2(\|\phi_0\| + R)}\|\phi_1 - \phi_0\|^2 = ab\eta$$

and therefore

$$\|\phi_2 - \phi_1\| \le abh(a)\eta.$$

Then, applying *iv)* from Lemma 2.1,

$$\|\phi_2 - \phi_0\| \le \|\phi_2 - \phi_1\| + \|\phi_1 - \phi_0\| \le (1 - (abh(a))^2)R < R,$$

and we have that $x_2 \in B(\phi_0, R)$. By induction is easy to prove that

$$\|\phi_n - \phi_{n-1}\| \le (abh(a))^{2^{n-1}-1}\|\phi_1 - \phi_0\|. \tag{6}$$

In addition, taking into account Bernoulli's inequality, we also have:

$$\|\phi_n - \phi_0\| \le \left(\sum_{j=0}^{n-1}(abh(a))^{2^j-1}\right)\|\phi_1 - \phi_0\| < \left(\sum_{j=0}^{\infty}(abh(a))^{2^j-1}\right)\eta$$

$$< \left(\sum_{j=0}^{\infty}(abh(a))^{j}\right)\eta = R$$

Consequently, $\phi_n \in B(\phi_0, R)$ for all $n \ge 0$.

Next, we prove that $\{\phi_n\}$ is a Cauchy sequence. From (6), Lemma 2.1 and Bernouilli's inequality, we deduce

$$\|\phi_{n+m}-\phi_n\| \le \|\phi_{n+m}-\phi_{n+m-1}\|+\|\phi_{n+m-1}-\phi_{n+m-2}\|+\cdots+\|\phi_n-\phi_{n-1}\|$$

$$\le \left[(abh(a))^{2^{n+m-1}-1} + (abh(a))^{2^{n+m-2}-1} + \cdots + (abh(a))^{2^{n}-1}\right]\|\phi_1-\phi_0\|$$

$$\le (abh(a))^{2^n-1}\left[(abh(a))^{2^n(2^{m-1}-1)} + (abh(a))^{2^n(2^{m-2}-1)} + \cdots + (abh(a))^{2^n} + 1\right]\eta$$

$$< (abh(a))^{2^n-1}\left[(abh(a))^{2^n(m-1)} + (abh(a))^{2^n(m-2)} + \cdots + (abh(a))^{2^n} + 1\right]\eta$$

$$= (abh(a))^{2^n-1}\frac{1-(abh(a))^{2^n m}}{1-(abh(a))^{2^n}}\eta.$$

But this last quantity goes to zero when $n \to \infty$. Let $\phi^* = \lim_{n\to\infty} \phi_n$, then, by letting $m \to \infty$, we have

$$\|\phi^* - \phi_n\| \leq (abh(a))^{2^n-1}\frac{\eta}{1-(abh(a))^{2^n}} = \frac{\eta}{(1-(abh(a))^{2^n})(abh(a))}(abh(a))^{2^n}$$

$$\leq \frac{\eta}{(1-(abh(a)))(abh(a))}(abh(a))^{2^n} = C\gamma^{2^n}$$

with $C > 0$ and $\gamma = abh(a) < 1$. This inequality guarantees that $\{\phi_n\}$ has at least R-order of convergence two [8].

Finally, for $n = 0$, we obtain

$$\|\phi^* - \phi_0\| < \frac{\eta}{1-abh(a)} = R$$

then, $\phi^* \in B(\phi_0, R)$. Moreover, as

$$\|F(\phi_n)\| \leq \frac{1}{2}M(p-1)(\|\phi_0\| + R)^{p-2}\|\phi_n - \phi_{n-1}\|^2,$$

when $n \to \infty$ we obtain $F(\phi^*) = 0$, and ϕ^* is a solution of $F(x) = 0$.

Now we give a uniqueness result:

Theorem 2.4. *Let $\|\Gamma_0\| \leq \beta$, then the solution of (2) is unique in $B(\phi_0, \overline{R}) \bigcap \Omega$, with $\overline{R}$ is the bigger positive solution of the equation*

$$\frac{M\beta(p-1)}{2}(2\|\phi_0\| + R + x)^{p-2}(R + x) = 1. \tag{7}$$

Proof: To show the uniqueness, we suppose that $\gamma^* \in B(\phi_0, \overline{R}) \bigcap \Omega$ is another solution of (2). Then

$$0 = \Gamma_0 F(\gamma^*) - \Gamma_0 F(\phi^*) = \int_0^1 \Gamma_0 F'_{\phi^*+s(\gamma^*-\phi^*)}\, ds(\gamma^* - \phi^*).$$

We are going to prove that A^{-1} exists, where A is a linear operator defined by

$$A = \int_0^1 \Gamma_0 F'_{\phi^*+s(\gamma^*-\phi^*)}\, ds,$$

then $\gamma^* = \phi^*$. For this, notice that for each $\psi \in X$ and $x \in [a, b]$, we have

$$(A - I)(\psi)(x) = \int_0^1 \Gamma_0[F'_{\phi^*+s(\gamma^*-\phi^*)} - F'_{\phi_0}]\psi(x)\, ds,$$

$$= -\lambda p \int_0^1 \Gamma_0 \int_a^b K(x,t)\left[\rho^*(s,t)^{p-1} - \phi_0(t)^{p-1}\right]\psi(t)\, dt\, ds$$

$$= -\lambda p \int_0^1 \Gamma_0 \int_a^b K(x,t) \left[\sum_{j=0}^{p-2} \rho^*(s,t)^{p-2-j} \phi_0(t)^j \right] (\rho^*(s,t) - \phi_0(t)) \psi(t)\, dt\, ds,$$

where $\rho^*(s,t) = \phi^*(t) + s(\gamma^*(t) - \phi^*(t))$.

Taking into account that

$$|\rho^*(s,t) - \phi_0(t)| \leq \|\phi^* - \phi_0 + s(\gamma^* - \phi^*)\| \leq (1-s)\|\phi^* - \phi_0\| + s\|\gamma^* - \phi_0\| < (1-s)R + s\overline{R},$$

we obtain

$$\|(A-I)\psi\| \leq |\lambda| pN \|\Gamma_0\| \left[\int_0^1 \left(\sum_{j=0}^{p-2} \|\rho^*(s,\cdot)\|^{p-2-j} \|\phi_0\|^j \right) ((1-s)R + s\overline{R})\, ds \right] \|\psi\|.$$

Therefore, as

$$\|\rho^*(s,\cdot)\| \leq (1-s)\|\phi^*\| + s\|\gamma^*\| \leq (1-s)(\|\phi_0\| + R) + s(\|\phi_0\| + \overline{R}) \leq 2\|\phi_0\| + R + \overline{R},$$

we have, from (7),

$$\|A - I\| \leq \frac{\|\Gamma_0\| M}{2}(R + \overline{R}) \left[\sum_{j=0}^{p-2} \left(\frac{\|\phi_0\|}{2\|\phi_0\| + R + \overline{R}} \right)^j \right] (2\|\phi_0\| + R + \overline{R})^{p-2}$$

$$< \frac{M\beta}{2}(R + \overline{R})(p-1)(2\|\phi_0\| + R + \overline{R})^{p-2} = 1.$$

So, the operator $\int_0^1 F'(\phi^* + t(\gamma^* - \phi^*))\, dt$ has an inverse and consequently, $\gamma^* = \phi^*$. Then, the proof is complete.

3. An example

To illustrate the above theoretical results, we consider the following example

$$(8) \qquad \phi(x) = \sin(\pi x) + \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi t) \phi(t)^3\, dt, \quad x \in [0,1].$$

Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, with the max-norm and let $F : X \to X$ be the operator given by

(9)

$$F(\phi)(x) = \phi(x) - \sin(\pi x) - \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi t) \phi((t)^3\, dt, \quad x \in [0,1].$$

By differentiating (9) we have:

$$(10) \qquad F'_\phi[u](x) = u(x) - \frac{3}{5} \cos(\pi x) \int_0^1 \sin(\pi t) \phi(t)^2 u(t)\, dt.$$

With the notation of section 2,

$$\lambda = \frac{1}{5}, \quad N = \max_{x\in[0,1]} \int_0^1 |\sin(\pi t)|\,dt = 1 \quad \text{and} \quad M = |\lambda| pN = \frac{3}{5}.$$

We take as starting-point $\phi_0(x) = \sin(\pi x)$, then we obtain from (10)

$$F'_{\phi_0}[u](x) = u(x) - \frac{3}{5}\cos(\pi x)\int_0^1 \sin^3(\pi t)u(t)\,dt$$

If $F'_\phi[u](x) = \omega(x)$, then $[F'_\phi]^{-1}[\omega](x) = u(x)$ and $u(x) = \omega(x) + \frac{3}{5}\cos(\pi x)J_u$, where

$$J_u = \int_0^1 \sin(\pi t)\phi(t)^2 u(t)\,dt.$$

Therefore the inverse of F'_{ϕ_0} is given by

$$[F'_{\phi_0}]^{-1}[\omega](x) = \omega(x) + \frac{3}{5}\,\frac{\int_0^1 \sin^3(\pi t)w(t)\,dt}{1 - \frac{3}{5}\int_0^1 \cos(\pi t)\sin^3(\pi t)\,dt}\cos(\pi x).$$

Then

$$\|\Gamma_0\| \le \|I + \frac{4}{5\pi}\cos(\pi x)\| \le 1.25468\cdots = \beta,$$

and $\|F(\phi_0)\| \le \frac{3}{40} = 0.075$. Consequently $\|\Gamma_0 F(\phi_0)\| \le 0.094098\cdots = \eta$.

The equation $f(t) = 0$, with f given by (5) is now

$$1.2t^3 + 2.4t^2 - 0.912918t + 0.0752789 = 0.$$

This equation has two positive solutions. The smaller one is $R = 0.129115\ldots$. Then, by Theorem 2.3, we know there exists a solution of (8) in $\overline{B(\phi_0, R)}$. To obtain the uniqueness domain we consider the equation (7) whose positive solution is the uniqueness ratio. In this case, the solution is unique in $B(\phi_0, 0.396793\ldots)$.

Finally, we are going to deal with the computational aspects to solve (8) applying Newton's method (3). To calculate the iterations $\phi_{n+1}(x) = \phi_n(x) - [F'_{\phi_n}]^{-1}[F(\phi_n)](x)$ with the function $\phi_0(x)$ as starting-point, we proceed in the following way:

(1) First we compute the integrals

$$A_n = \int_0^1 \sin(\pi t)\phi_n(t)^3\,dt; \quad B_n = \int_0^1 \sin(\pi t)^2\phi_n(t)^2\,dt;$$

$$C_n = \int_0^1 \cos(\pi t)\sin(\pi t)\phi_n(t)^2\,dt.$$

(2) Next we define

$$\phi_{n+1}(x) = \sin(\pi x) + \frac{1}{5}\frac{-2A_n + 3B_n}{1 - \frac{3}{5}C_n}\cos(\pi x).$$

So we obtain the following approximations

$$\begin{aligned}
\phi_0(x) &= \sin \pi x, \\
\phi_1(x) &= \sin \pi x + 0.075 \cos \pi x, \\
\phi_2(x) &= \sin \pi x + 0.07542667509481667 \cos \pi x, \\
\phi_3(x) &= \sin \pi x + 0.07542668890493719 \cos \pi x, \\
\phi_4(x) &= \sin \pi x + 0.07542668890493714 \cos \pi x, \\
\phi_5(x) &= \sin \pi x + 0.07542668890493713 \cos \pi x,
\end{aligned}$$

As we can see, in this case Newton's method converges to the solution

$$\phi^*(x) = \sin \pi x + \frac{20 - \sqrt{391}}{3} \cos \pi x.$$

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ORLICZ-SOBOLEV SPACES WITH ZERO BOUNDARY VALUES ON METRIC SPACES

NOUREDDINE AÏSSAOUI

Abstract. In this paper we study two approaches for the definition of the first order Orlicz-Sobolev spaces with zero boundary values on arbitrary metric spaces. The first generalization, denoted by $M^{1,0}_{\Phi}(E)$, where E is a subset of the metric space X, is defined by the mean of the notion of the trace and is a Banach space when the N-function satisfies the Δ_2 condition. We give also some properties of these spaces. The second, following another definition of Orlicz-Sobolev spaces on metric spaces, leads us to three definitions that coincide for a large class of metric spaces and N-functions. These spaces are Banach spaces for any N-function.

A.M.S. (MOS) Subject Classification Codes. 46E35, 31B15, 28A80.
Key Words and Phrases. Orlicz spaces, Orlicz-Sobolev spaces, modulus of a family of paths, capacities.

1. Introduction

This paper treats definitions and study of the first order Orlicz-Sobolev spaces with zero boundary values on metric spaces. Since we have introduce two definitions of Orlicz-Sobolev spaces on metric spaces, we are leading to examine two approaches.

The first approach follows the one given in the paper [7] relative to Sobolev spaces. This generalization, denoted by $M^{1,0}_{\Phi}(E)$, where E is a subset of the metric space X, is defined as Orlicz-Sobolev functions on X, whose trace on $X \setminus E$ vanishes.

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This is a Banach space when the N-function satisfies the Δ_2 condition. For the definition of the trace of Orlicz-Sobolev functions we need the notion of Φ-capacity on metric spaces developed in [2]. We show that sets of Φ-capacity zero are removable in the Orlicz-Sobolev spaces with zero boundary values. We give some results closely related to questions of approximation of Orlicz-Sobolev functions with zero boundary values by compactly supported functions. The approximation is not valid on general sets. As in Sobolev case, we study the approximation on open sets. Hence we give sufficient conditions, based on Hardy type inequalities, for an Orlicz-Sobolev function to be approximated by Lipschitz functions vanishing outside an open set.

The second approach follows the one given in the paper [13] relative to Sobolev spaces; see also [12]. We need the rudiments developed in [3]. Hence we consider the set of Lipschitz functions on X vanishing on $X \setminus E$, and close that set under an appropriate norm. Another definition is to consider the space of Orlicz-Sobolev functions on X vanishing Φ-q.e. in $X \setminus E$. A third space is obtained by considering the closure of the set of compactly supported Lipschitz functions with support in E. These spaces are Banach for any N-function and are, in general, different. For a large class of metric spaces and a broad family of N-functions, we show that these spaces coincide.

2. Preliminaries

An *$\mathcal{N}$-function* is a continuous convex and even function Φ defined on $\mathbb{R}$, verifying $\Phi(t) > 0$ for $t > 0$, $\lim_{t\to 0} t^{-1}\Phi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\Phi(t) = +\infty$.

We have the representation $\Phi(t) = \int\limits_0^{|t|} \varphi(x)d\mathfrak{L}(x)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, $\lim_{t\to 0^+} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$. Here $\mathfrak{L}$ stands for the Lebesgue measure. We put in the sequel, as usually, $dx = d\mathfrak{L}(x)$.

The $\mathcal{N}$-function Φ^* *conjugate* to Φ is defined by $\Phi^*(t) = \int\limits_0^{|t|} \varphi^*(x)dx$, where φ^* is given by $\varphi^*(s) = \sup\{t : \varphi(t) \le s\}$.

Let (X, Γ, μ) be a measure space and Φ an $\mathcal{N}$-function. The *Orlicz class* $\mathcal{L}_{\Phi,\mu}(X)$ is defined by

$$\mathcal{L}_{\Phi,\mu}(X) = \left\{f : X \to \mathbb{R} \text{ measurable} : \int_X \Phi(f(x))d\mu(x) < \infty\right\}.$$

We define the *Orlicz space* $\mathbf{L}_{\Phi,\mu}(X)$ by

$$\mathbf{L}_{\Phi,\mu}(X) = \left\{ f : X \to \mathbb{R} \text{ measurable} : \int_X \Phi(\alpha f(x)) d\mu(x) < \infty \text{ for some } \alpha > 0 \right\}.$$

The Orlicz space $\mathbf{L}_{\Phi,\mu}(X)$ is a Banach space with the following norm, called the *Luxemburg norm,*

$$|||f|||_{\Phi,\mu,X} = \inf \left\{ r > 0 : \int_X \Phi\left(\frac{f(x)}{r}\right) d\mu(x) \leq 1 \right\}.$$

If there is no confusion, we set $|||f|||_\Phi = |||f|||_{\Phi,\mu,X}$.

The Hölder inequality extends to Orlicz spaces as follows: if $f \in \mathbf{L}_{\Phi,\mu}(X)$ and $g \in \mathbf{L}_{\Phi^*,\mu}(X)$, then $fg \in \mathbf{L}^1$ and

$$\int_X |fg|\, d\mu \leq 2 |||f|||_{\Phi,\mu,X}.\ |||g|||_{\Phi^*,\mu,X}.$$

Let Φ be an $\mathcal{N}$-function. We say that Φ *verifies the* Δ_2 *condition* if there is a constant $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t \geq 0$.

The Δ_2 condition for Φ can be formulated in the following equivalent way: for every $C > 0$ there exists $C' > 0$ such that $\Phi(Ct) \leq C'\Phi(t)$ for all $t \geq 0$.

We have always $\mathcal{L}_{\Phi,\mu}(X) \subset \mathbf{L}_{\Phi,\mu}(X)$. The equality $\mathcal{L}_{\Phi,\mu}(X) = \mathbf{L}_{\Phi,\mu}(X)$ occurs if Φ verifies the Δ_2 condition.

We know that $\mathbf{L}_{\Phi,\mu}(X)$ is reflexive if Φ and Φ^* verify the Δ_2 condition.

Note that if Φ verifies the Δ_2 condition, then $\int \Phi(f_i(x)) d\mu \to 0$ as $i \to \infty$ if and only if $|||f_i|||_{\Phi,\mu,X} \to 0$ as $i \to \infty$.

Recall that an $\mathcal{N}$-function Φ satisfies the Δ' condition if there is a positive constant C such that for all $x, y \geq 0$, $\Phi(xy) \leq C\Phi(x)\Phi(y)$. See [9] and [12]. If an $\mathcal{N}$-function Φ satisfies the Δ' condition, then it satisfies also the Δ_2 condition.

Let Ω be an open set in $\mathbb{R}^N$, $\mathbf{C}^\infty(\Omega)$ be the space of functions which, together with all their partial derivatives of any order, are continuous on Ω, and $\mathbf{C}_0^\infty(\mathbb{R}^N) = \mathbf{C}_0^\infty$ stands for all functions in $\mathbf{C}^\infty(\mathbb{R}^N)$ which have compact support in $\mathbb{R}^N$. The space $\mathbf{C}^k(\Omega)$ stands for the space of functions having all derivatives of order $\leq k$ continuous on Ω, and $\mathbf{C}(\Omega)$ is the space of continuous functions on Ω.

The (weak) partial derivative of f of order $|\beta|$ is denoted by

$$D^\beta f = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} . \partial x_2^{\beta_2} ... \partial x_N^{\beta_N}} f.$$

Let Φ be an $\mathcal{N}$-function and $m \in \mathbb{N}$. We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ has a distributional (weak partial) derivative of order m, denoted $D^\beta f$, $|\beta| = m$, if

$$\int f D^\beta \theta dx = (-1)^{|\beta|} \int (D^\beta f) \theta dx, \ \forall \theta \in \mathbf{C}_0^\infty.$$

Let Ω be an open set in $\mathbb{R}^N$ and denote $\mathbf{L}_{\Phi,\mathfrak{L}}(\Omega)$ by $\mathbf{L}_\Phi(\Omega)$. The *Orlicz-Sobolev space* $W^m\mathbf{L}_\Phi(\Omega)$ is the space of real functions f, such that f and its distributional derivatives up to the order m, are in $\mathbf{L}_\Phi(\Omega)$.

The space $W^m\mathbf{L}_\Phi(\Omega)$ is a Banach space equipped with the norm

$$|||f|||_{m,\Phi,\Omega} = \sum_{0\leq|\beta|\leq m} |||D^\beta f|||_\Phi,\ f \in W^m\mathbf{L}_\Phi(\Omega),$$

where $|||D^\beta f|||_\Phi = |||D^\beta f|||_{\Phi,\mathfrak{L},\Omega}$.

Recall that if Φ verifies the Δ_2 condition, then $\mathbf{C}^\infty(\Omega) \cap W^m\mathbf{L}_\Phi(\Omega)$ is dense in $W^m\mathbf{L}_\Phi(\Omega)$, and $\mathbf{C}_0^\infty(\mathbb{R}^N)$ is dense in $W^m\mathbf{L}_\Phi(\mathbb{R}^N)$.

For more details on the theory of Orlicz spaces, see [1, 8, 9, 10, 11].

In this paper, the letter C will denote various constants which may differ from one formula to the next one even within a single string of estimates.

3. Orlicz-Sobolev space with zero boundary values $M_\Phi^{1,0}(E)$

3.1. **The Orlicz-Sobolev space $M_\Phi^1(X)$.** We begin by recalling the definition of the space $M_\Phi^1(X)$.

Let $u : X \to [-\infty, +\infty]$ be a μ-measurable function defined on X. We denote by $D(u)$ the set of all μ-measurable functions $g : X \to [0, +\infty]$ such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \tag{3.1}$$

for every $x, y \in X \setminus F$, $x \neq y$, with $\mu(F) = 0$. The set F is called the exceptional set for g.

Note that the right hand side of (3.1) is always defined for $x \neq y$. For the points $x, y \in X$, $x \neq y$ such that the left hand side of (3.1) is undefined we may assume that the left hand side is $+\infty$.

Let Φ be an $\mathcal{N}$-function. The Dirichlet-Orlicz space $\mathbf{L}_\Phi^1(X)$ is the space of all μ-measurable functions u such that $D(u) \cap \mathbf{L}_\Phi(X) \neq \emptyset$. This space is equipped with the seminorm

$$|||u|||_{\mathbf{L}_\Phi^1(X)} = \inf\{|||g|||_\Phi : g \in D(u) \cap \mathbf{L}_\Phi(X)\}. \tag{3.2}$$

The Orlicz-Sobolev space $M_\Phi^1(X)$ is defined by $M_\Phi^1(X) = \mathbf{L}_\Phi(X) \cap \mathbf{L}_\Phi^1(X)$ equipped with the norm

$$|||u|||_{M_\Phi^1(X)} = |||u|||_\Phi + |||u|||_{\mathbf{L}_\Phi^1(X)}. \tag{3.3}$$

We define a *capacity* as an increasing positive set function C given on a σ-additive class of sets Γ, which contains compact sets and such that $C(\emptyset) = 0$ and $C(\bigcup_{i\geq 1} X_i) \leq \sum_{i\geq 1} C(X_i)$ for $X_i \in \Gamma$, $i = 1, 2, \ldots$.

C is called outer capacity if for every $X \in \Gamma$,

$$C(X) = \inf \{C(O) : O \text{ open}, X \subset O\}.$$

Let C be a capacity. If a statement holds except on a set E where $C(E) = 0$, then we say that the statement holds C-quasieverywhere (abbreviated C-q.e.). A function $u : X \to [-\infty, \infty]$ is C-quasicontinuous in X if for every $\varepsilon > 0$ there is a set E such that $C(E) < \varepsilon$ and the restriction of u to $X \setminus E$ is continuous. When C is an outer capacity, we may assume that E is open.

Recall the following definition in [2]

Definition 1. *Let Φ be an $\mathcal{N}$-function. For a set $E \subset X$, define $C_\Phi(E)$ by*

$$C_\Phi(E) = \inf\{|||u|||_{M^1_\Phi(X)} : u \in B(E)\},$$

where $B(E) = \{u \in M^1_\Phi(X) : u \geq 1$ on a neighborhood of $E\}$.

If $B(E) = \emptyset$, we set $C_{\Phi,\mu}(E) = \infty$.

Functions belonging to $B(E)$ are called admissible functions for E.

In the definition of $C_\Phi(E)$, we can restrict ourselves to those admissible functions u such that $0 \leq u \leq 1$. On the other hand, C_Φ is an outer capacity.

Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition, then by [2 Theorem 3.10] the set

$$Lip^1_\Phi(X) = \{u \in M^1_\Phi(X) : u \text{ is Lipschitz in } X\}$$

is a dense subspace of $M^1_\Phi(X)$. Recall the following result in [2, Theorem 4.10]

Theorem 1. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and $u \in M^1_\Phi(X)$. Then there is a function $v \in M^1_\Phi(X)$ such that $u = v$ μ-a.e. and v is C_Φ-quasicontinuous in X.*

The function v is called a C_Φ-quasicontinuous representative of u.

Recall also the following theorem, see [6]

Theorem 2. *Let C be an outer capacity on X and μ be a nonnegative, monotone set function on X such that the following compatibility condition is satisfied: If G is open and $\mu(E) = 0$, then*

$$C(G) = C(G \setminus E).$$

Let f and g be C-quasicontinuous on X such that

$$\mu(\{x : f(x) \neq g(x)\}) = 0.$$

Then $f = g$ C-quasi everywhere on X.

It is easily verified that the capacity C_Φ satisfies the compatibility condition. Thus from Theorem 2, we get the following corollary.

Corollary 1. *Let Φ be an $\mathcal{N}$-function. If u and v are C_Φ-quasicontinuous on an open set O and if $u = v$ μ-a.e. in O, then $u = v$ C_Φ-q.e. in O.*

Corollary 1 make it possible to define the trace of an Orlicz-Sobolev function to an arbitrary set.

Definition 2. *Let Φ be an $\mathcal{N}$-function, $u \in M^1_\Phi(X)$ and E be such that $C_\Phi(E) > 0$. The trace of u to E is the restriction to E of any C_Φ-quasicontinuous representative of u.*

Remark 1. *Let Φ be an $\mathcal{N}$-function. If u and v are C_Φ-quasicontinuous and $u \leq v$ μ-a.e. in an open set O, then $\max(u - v, 0) = 0$ μ-a.e. in O and $\max(u - v, 0)$ is C_Φ-quasicontinuous. Hence by Corollary 1, $\max(u - v, 0) = 0$ C_Φ-q.e. in O, and consequently $u \leq v$ C_Φ-q.e. in O.*

Now we give a characterization of the capacity C_Φ in terms of quasicontinuous functions. We begin by a definition

Definition 3. *Let Φ be an $\mathcal{N}$-function. For a set $E \subset X$, define $D_\Phi(E)$ by*

$$D_\Phi(E) = \inf\{|||u|||_{M^1_\Phi(X)} : u \in \mathcal{B}(E)\},$$

where

$$\mathcal{B}(E) = \{u \in M^1_\Phi(X) : u \text{ is } C_\Phi\text{-quasicontinuous and } u \geq 1 \ C_\Phi\text{-q.e. in } E\}.$$

If $\mathcal{B}(E) = \emptyset$, we set $D_\Phi(E) = \infty$.

Theorem 3. *Let Φ be an $\mathcal{N}$-function and E a subset in X. Then*

$$C_\Phi(E) = D_\Phi(E).$$

Proof. Let $u \in M^1_\Phi(X)$ be such that $u \geq 1$ on an open neighborhood O of E. Then, by Remark 1, the C_Φ-quasicontinuous representative v of u satisfies $v \geq 1$ C_Φ-q.e. on O, and hence $v \geq 1$ C_Φ-q.e. on E. Thus $D_\Phi(E) \leq C_\Phi(E)$.

For the reverse inequality, let $v \in \mathcal{B}(E)$. By truncation we may assume that $0 \leq v \leq 1$. Let ε be such that $0 < \varepsilon < 1$ and choose an open set V such that $C_\Phi(V) < \varepsilon$ with $v = 1$ on $E \setminus V$ and $v\big|_{X\setminus V}$ is continuous. We can find, by topology, an open set $U \subset X$ such that $\{x \in X : v(x) > 1 - \varepsilon\} \setminus V = U \setminus V$. We have $E \setminus V \subset U \setminus V$. We choose $u \in B(V)$ such that $|||u|||_{M^1_\Phi(X)} < \varepsilon$ and that $0 \leq u \leq 1$. We define $w = \frac{v}{1-\varepsilon} + u$. Then $w \geq 1$ μ-a.e. in $(U \setminus V) \cup V = U \cup V$, which is an open neighbourhood of E. Hence $w \in B(E)$. This implies that

$$\begin{aligned} C_\Phi(E) &\leq |||w|||_{M^1_\Phi(X)} \leq \frac{1}{1-\varepsilon} |||v|||_{M^1_\Phi(X)} + |||u|||_{M^1_\Phi(X)} \\ &\leq \frac{1}{1-\varepsilon} |||v|||_{M^1_\Phi(X)} + \varepsilon. \end{aligned}$$

We get the desired inequality since ε and v are arbitrary. The proof is complete. ∎

We give a sharpening of [2, Theorem 4.8].

Theorem 4. *Let Φ be an $\mathcal{N}$-function and $(u_i)_i$ be a sequence of C_Φ-quasicontinuous functions in $M^1_\Phi(X)$ such that $(u_i)_i$ converges in $M^1_\Phi(X)$ to a C_Φ-quasicontinuous function u. Then there is a subsequence of $(u_i)_i$ which converges to u C_Φ-q.e. in X.*

Proof. There is a subsequence of $(u_i)_i$, which we denote again by $(u_i)_i$, such that

$$\sum_{i=1}^{\infty} 2^i |||u_i - u|||_{M^1_\Phi(X)} < \infty. \tag{3.4}$$

We set $E_i = \{x \in X : |u_i(x) - u(x)| > 2^{-i}\}$ for $i = 1, 2, ...$, and $F_j = \bigcup_{i=j}^{\infty} E_i$. Then $2^i |u_i - u| \in \mathcal{B}(E_i)$ and by Theorem 3 we obtain $C_\Phi(E_i) \leq 2^i |||u_i - u|||_{M^1_\Phi(X)}$. By subadditivity we get

$$C_\Phi(F_j) \leq \sum_{i=j}^{\infty} C_\Phi(E_i) \leq \sum_{i=j}^{\infty} 2^i |||u_i - u|||_{M^1_\Phi(X)}.$$

Hence

$$C_\Phi(\bigcap_{j=1}^{\infty} F_j) \leq \lim_{j\to\infty} C_\Phi(F_j) = 0.$$

Thus $u_i \to u$ pointwise in $X \setminus \bigcap_{j=1}^{\infty} F_j$ and the proof is complete. ∎

3.2. The Orlicz-Sobolev space with zero boundary values $M^{1,0}_\Phi(E)$.

Definition 4. *Let Φ be an $\mathcal{N}$-function and E a subspace of X. We say that u belongs to the Orlicz-Sobolev space with zero boundary values, and denote $u \in M^{1,0}_\Phi(E)$, if there is a C_Φ-quasicontinuous function $\widetilde{u} \in M^1_\Phi(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$.*

In other words, u belongs to $M^{1,0}_\Phi(E)$ if there is $\widetilde{u} \in M^1_\Phi(X)$ as above such that the trace of $\widetilde{u}$ vanishes C_Φ-q.e. in $X \setminus E$.

The space $M_\Phi^{1,0}(E)$ *is equipped with the norm*

$$|||u|||_{M_\Phi^{1,0}(E)} = |||\widetilde{u}|||_{M_\Phi^1(X)}\,.$$

Recall that $C_\Phi(E) = 0$ implies that $\mu(E) = 0$ for every $E \subset X$; see [2]. It follows that the norm does not depend on the choice of the quasicontinuous representative.

Theorem 5. *Let* Φ *be an* $\mathcal{N}$*-function satisfying the* Δ_2 *condition and* E *a subspace of* X*. Then* $M_\Phi^{1,0}(E)$ *is a Banach space.*

Proof. Let $(u_i)_i$ be a Cauchy sequence in $M_\Phi^{1,0}(E)$. Then for every u_i, there is a C_Φ-quasicontinuous function $\widetilde{u_i} \in M_\Phi^1(X)$ such that $\widetilde{u_i} = u_i$ μ-a.e. in E and $\widetilde{u_i} = 0$ C_Φ-q.e. in $X \setminus E$. By [2, Theorem 3.6] $M_\Phi^1(X)$ is complete. Hence there is $u \in M_\Phi^1(X)$ such that $\widetilde{u_i} \to u$ in $M_\Phi^1(X)$ as $i \to \infty$. Let $\widetilde{u}$ be a C_Φ-quasicontinuous representative of u given by Theorem 1. By Theorem 4 there is a subsequence $(\widetilde{u_i})_i$ such that $\widetilde{u_i} \to \widetilde{u}$ C_Φ-q.e. in X as $i \to \infty$. This implies that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$ and hence $u \in M_\Phi^{1,0}(E)$. The proof is complete. ∎

Moreover the space $M_\Phi^{1,0}(E)$ has the following lattice properties. The proof is easily verified.

Lemma 1. *Let* Φ *be an* $\mathcal{N}$*-function and let* E *be a subset in* X*. If* $u, v \in M_\Phi^{1,0}(E)$*, then the following claims are true.*

1) *If* $\alpha \geq 0$*, then* $\min(u,\alpha) \in M_\Phi^{1,0}(E)$ *and* $|||\min(u,\alpha)|||_{M_\Phi^{1,0}(E)} \leq |||u|||_{M_\Phi^{1,0}(E)}$.
2) *If* $\alpha \leq 0$*, then* $\max(u,\alpha) \in M_\Phi^{1,0}(E)$ *and* $|||\max(u,\alpha)|||_{M_\Phi^{1,0}(E)} \leq |||u|||_{M_\Phi^{1,0}(E)}$.
3) $|u| \in M_\Phi^{1,0}(E)$ *and* $||||u||||_{M_\Phi^{1,0}(E)} \leq |||u|||_{M_\Phi^{1,0}(E)}$.
4) $\min(u,v) \in M_\Phi^{1,0}(E)$ *and* $\max(u,v) \in M_\Phi^{1,0}(E)$.

Theorem 6. *Let* Φ *be an* $\mathcal{N}$*-function satisfying the* Δ_2 *condition and* E *a* μ*-measurable subset in* X*. If* $u \in M_\Phi^{1,0}(E)$ *and* $v \in M_\Phi^1(X)$ *are such that* $|v| \leq u$ μ*-a.e. in* E*, then* $v \in M_\Phi^{1,0}(E)$.

Proof. Let w be the zero extension of v to $X \setminus E$ and let $\widetilde{u} \in M_\Phi^1(X)$ be a C_Φ-quasicontinuous function such that $\widetilde{u} = u$ μ-a.e. in E and that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. Let $g_1 \in D(\widetilde{u}) \cap \mathbf{L}_\Phi(X)$ and $g_2 \in D(v) \cap \mathbf{L}_\Phi(X)$. Define the function g_3 by

$$g_3(x) = \begin{cases} \max(g_1(x), g_2(x)), & x \in E \\ g_1(x), & x \in X \setminus E. \end{cases}$$

Then it is easy to verify that $g_3 \in D(w) \cap \mathbf{L}_\Phi(X)$. Hence $w \in M^1_\Phi(X)$. Let $\widetilde{w} \in M^1_\Phi(X)$ be a C_Φ-quasicontinuous function such that $\widetilde{w} = w$ μ-a.e. in X given by Theorem 1. Then $|\widetilde{w}| \le \widetilde{u}$ μ-a.e. in X. By Remark 1 we get $|\widetilde{w}| \le \widetilde{u}$ C_Φ-q.e. in X and consequently $\widetilde{w} = 0$ C_Φ-q.e. in $X \setminus E$. This shows that $v \in M^{1,0}_\Phi(E)$. The proof is complete. ∎

The following lemma is easy to verify.

Lemma 2. *Let Φ be an $\mathcal{N}$-function and let E be a subset in X. If $u \in M^{1,0}_\Phi(E)$ and $v \in M^1_\Phi(X)$ are bounded functions, then $uv \in M^{1,0}_\Phi(E)$.*

We show in the next theorem that the sets of capacity zero are removable in the Orlicz-Sobolev spaces with zero boundary values.

Theorem 7. *Let Φ be an $\mathcal{N}$-function and let E be a subset in X. Let $F \subset E$ be such that $C_\Phi(F) = 0$. Then $M^{1,0}_\Phi(E) = M^{1,0}_\Phi(E \setminus F)$.*

Proof. It is evident that $M^{1,0}_\Phi(E \setminus F) \subset M^{1,0}_\Phi(E)$. For the reverse inclusion, let $u \in M^{1,0}_\Phi(E)$, then there is a C_Φ-quasicontinuous function $\widetilde{u} \in M^1_\Phi(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. Since $C_\Phi(F) = 0$, we get that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus (E \setminus F)$. This implies that $u_{|E\setminus F} \in M^{1,0}_\Phi(E \setminus F)$. Moreover we have $\left|\left|\left|u_{|E\setminus F}\right|\right|\right|_{M^{1,0}_\Phi(E\setminus F)} = |||u|||_{M^{1,0}_\Phi(E)}$. The proof is complete. ∎

As in the Sobolev case, we have the following remark.

Remark 2. 1) *If $C_\Phi(\partial F) = 0$, then $M^{1,0}_\Phi(int\ E) = M^{1,0}_\Phi(\overline{E})$.*

2) *We have the equivalence: $M^{1,0}_\Phi(X \setminus F) = M^{1,0}_\Phi(X) = M^1_\Phi(X)$ if and only if $C_\Phi(F) = 0$.*

The converse of Theorem 7 is not true in general. In fact it suffices to take $\Phi(t) = \frac{1}{p}t^p$ $(p > 1)$ and consider the example in [7].

Nevertheless the converse of Theorem 7 holds for open sets.

Theorem 8. *Let Φ be an $\mathcal{N}$-function and suppose that μ is finite in bounded sets and that O is an open set. Then $M^{1,0}_\Phi(O) = M^{1,0}_\Phi(O \setminus F)$ if and only if $C_\Phi(F \cap O) = 0$.*

Proof. We must show only the necessity. We can assume that $F \subset O$. Let $x_0 \in O$ and for $i \in \mathbb{N}^*$, pose $O_i = B(x_0, i) \cap \{x \in O : \operatorname{dist}(x, X \setminus O) > 1/i\}$. We define for $i \in \mathbb{N}^*$, $u_i : X \to \mathbb{R}$ by $u_i(x) = \max(0, 1 - \operatorname{dist}(x, F \cap O_i))$. Then $u_i \in M^1_\Phi(X)$, u_i is continuous, $u_i = 1$ in $F \cap O_i$ and $0 \le u_i \le 1$. For $i \in \mathbb{N}^*$, define $v_i : O_i \to \mathbb{R}$ by $v_i(x) = \operatorname{dist}(x, X \setminus O_i)$. Then $v_i \in M^{1,0}_\Phi(O_i) \subset M^{1,0}_\Phi(O)$. By Lemma 2 we have, for every $i \in \mathbb{N}^*$, $u_i v_i \in M^{1,0}_\Phi(O) = M^{1,0}_\Phi(O \setminus F)$. If w is a C_Φ-quasicontinuous function such that $w = u_i v_i$ μ-a.e. in $O \setminus F$, then $w = u_i v_i$ μ-a.e. in O since

$\mu(F) = 0$. By Corollary 1 we get $w = u_i v_i$ C_Φ-q.e. in O. In particular $w = u_i v_i > 0$ C_Φ-q.e. in $F \cap O_i$. Since $u_i v_i \in M_\Phi^{1,0}(O \setminus F)$ we may define $w = 0$ C_Φ-q.e. in $X \setminus (O \setminus F)$. Hence $w = 0$ C_Φ-q.e. in $F \cap O_i$. This is possible only if $C_\Phi(F \cap O_i) = 0$ for every $i \in \mathbb{N}^*$. Hence $C_\Phi(F) \leq \sum_{i=1}^{\infty} C_\Phi(F \cap O_i) = 0$. The proof is complete. ∎

3.3. **Some relations between $H_\Phi^{1,0}(E)$ and $M_\Phi^{1,0}(E)$.** We would describe the Orlicz-Sobolev space with zero boundary values on $E \subset X$ as the completion of the set $Lip_\Phi^{1,0}(E)$ defined by

$$Lip_\Phi^{1,0}(E) = \{u \in M_\Phi^1(X) : u \text{ is Lipschitz in } X \text{ and } u = 0 \text{ in } X \setminus E\}$$

in the norm defined by (3.3). Since $M_\Phi^1(X)$ is complete, this completion is the closure of $Lip_\Phi^{1,0}(E)$ in $M_\Phi^1(X)$. We denote this completion by $H_\Phi^{1,0}(E)$.

Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and E a subspace of X. By [2, Theorem 3.10] we have $H_\Phi^{1,0}(X) = M_\Phi^{1,0}(X)$. Since $Lip_\Phi^{1,0}(E) \subset M_\Phi^{1,0}(E)$ and $M_\Phi^{1,0}(E)$ is complete, then $H_\Phi^{1,0}(E) \subset M_\Phi^{1,0}(E)$. When $\Phi(t) = \frac{1}{p}t^p$ $(p > 1)$, simple examples show that the equality is not true in general; see [7]. Hence for the study of the equality, we restrict ourselves to open sets as in the Sobolev case. We begin by a sufficient condition.

Theorem 9. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition, O an open subspace of X and suppose that $u \in M_\Phi^1(O)$. Let v be the function defined on O by $v(x) = \dfrac{u(x)}{dist(x, X \setminus O)}$. If $v \in \mathbf{L}_\Phi(O)$, then $u \in H_\Phi^{1,0}(O)$.*

Proof. Let $g \in D(u) \cap \mathbf{L}_\Phi(O)$ and define the function $\overline{g}$ by

$$\begin{aligned} \overline{g}(x) &= \max(g(x), v(x)) \text{ if } x \in O \\ \overline{g}(x) &= 0 \text{ if } x \in X \setminus O. \end{aligned}$$

Then $\overline{g} \in \mathbf{L}_\Phi(X)$. Define the function $\overline{u}$ as the zero extension of u to $X \setminus O$. For μ-a.e. $x, y \in O$ or $x, y \in X \setminus O$, we have

$$|\overline{u}(x) - \overline{u}(y)| \leq d(x,y)(\overline{g}(x) + \overline{g}(y)).$$

For μ-a.e. $x \in O$ and $y \in X \setminus O$, we get

$$|\overline{u}(x) - \overline{u}(y)| = |u(x)| \leq d(x,y)\frac{|u(x)|}{\mathrm{dist}(x, X \setminus O)} \leq d(x,y)(\overline{g}(x) + \overline{g}(y)).$$

Thus $\overline{g} \in D(\overline{u}) \cap \mathbf{L}_\Phi(X)$ which implies that $\overline{u} \in M_\Phi^1(O)$. Hence

$$|\overline{u}(x) - \overline{u}(y)| \leq d(x,y)(\overline{g}(x) + \overline{g}(y)) \tag{3.5}$$

for every $x, y \in X \setminus F$ with $\mu(F) = 0$.

For $i \in \mathbb{N}^*$, set

$$F_i = \{x \in O \setminus F : |\overline{u}(x)| \leq i,\ \overline{g}(x) \leq i\} \cup X \setminus O. \tag{3.6}$$

From (3.5) we see that $\overline{u}_{|F_i}$ is $2i$-Lipschitz and by the McShane extension

$$\overline{u}_i(x) = \inf \{\overline{u}(y) + 2id(x, y) : y \in F_i\}$$

we extend it to a $2i$-Lipschitz function on X. We truncate $\overline{u}_i$ at the level i and set $u_i(x) = \min(\max(\overline{u}_i(x), -i), i)$. Then u_i is such that u_i is $2i$-Lipschitz function in X, $|u_i| \leq i$ in X and $u_i = \overline{u}$ in F_i and, in particular, $u_i = 0$ in $X \setminus O$. We show that $u_i \in M^1_\Phi(X)$. Define the function g_i by

$$\begin{aligned} g_i(x) &= \overline{g}(x), \text{ if } x \in F_i, \\ g_i(x) &= 2i, \text{ if } \ x \in X \setminus F_i. \end{aligned}$$

We begin by showing that

$$|u_i(x) - u_i(y)| \leq d(x, y)(g_i(x) + g_i(y)), \tag{3.7}$$

for $x, y \in X \setminus F$. If $x, y \in F_i$, then (3.7) is evident. For $y \in X \setminus F_i$, we have

$$\begin{aligned} |u_i(x) - u_i(y)| &\leq 2id(x, y) \leq d(x, y)(g_i(x) + g_i(y)), \text{ if } x \in X \setminus F_i, \\ |u_i(x) - u_i(y)| &\leq 2id(x, y) \leq d(x, y)(\overline{g}(x) + 2i), \text{ if } x \in X \setminus F_i. \end{aligned}$$

This implies that (3.7) is true and thus $g_i \in D(u_i)$. Now we have

$$\begin{aligned} |||g_i|||_\Phi &\leq |||g_i|||_{\Phi,F_i} + 2i|||1|||_{\Phi,X\setminus F_i} \\ &\leq |||\overline{g}|||_{\Phi,F_i} + \frac{2i}{\Phi^{-1}(\frac{1}{\mu(X\setminus F_i)})} < \infty, \end{aligned}$$

and

$$\begin{aligned} |||u_i|||_\Phi &\leq |||\overline{u}|||_{\Phi,F_i} + 2i|||1|||_{\Phi,X\setminus F_i} \\ &\leq |||\overline{u}|||_{\Phi,F_i} + \frac{2i}{\Phi^{-1}(\frac{1}{\mu(X\setminus F_i)})} < \infty. \end{aligned}$$

Hence $u_i \in M^1_\Phi(X)$. It follows that $u_i \in Lip^{1,0}_\Phi(O)$.

It remains to prove that $u_i \to \overline{u}$ in $M^1_\Phi(X)$. By (3.6) we have

$$\mu(X \setminus F_i) \leq \mu(\{x \in X : |\overline{u}(x)| > i\}) + \mu(\{x \in X :\ \overline{g}(x) > i\}).$$

Since $\overline{u} \in \mathbf{L}_\Phi(X)$ and Φ satisfies the Δ_2 condition, we get

$$\int_{\{x\in X:|\overline{u}(x)|>i\}} \Phi(\overline{u}(x))d\mu(x) \geq \Phi(i)\mu \{x \in X : |\overline{u}(x)| > i\},$$

which implies that $\Phi(i)\mu \{x \in X : |\overline{u}(x)| > i\} \to 0$ as $i \to \infty$.

By the same argument we deduce that $\Phi(i)\mu\left\{x \in X : \overline{g}(x) > i\right\} \to 0$ as $i \to \infty$.

Thus

$$\Phi(i)\mu(X \setminus F_i) \to 0 \text{ as } i \to \infty. \tag{3.8}$$

Using the convexity of Φ and the fact that Φ satisfies the Δ_2 condition, we get

$$\begin{aligned}\int_X \Phi(\overline{u} - u_i)d\mu &\leq \int_{X\setminus F_i} \Phi(|\overline{u}| + |u_i|)d\mu \\ &\leq \tfrac{C}{2}\Big[\int_{X\setminus F_i} \Phi \circ |\overline{u}|\, d\mu + \Phi(i)\mu(X \setminus F_i)\Big] \to 0 \text{ as } i \to \infty.\end{aligned}$$

On the other hand, for each $i \in \mathbb{N}^*$ we define the function h_i by

$$\begin{aligned} h_i(x) &= \overline{g}(x) + 3i, \text{ if } x \in X \setminus F_i, \\ h_i(x) &= 0, \text{ if } x \in F_i. \end{aligned}$$

We claim that $h_i \in D(\overline{u} - u_i) \cap \mathbf{L}_\Phi(X)$. In fact, the only nontrivial case is $x \in F_i$ and $y \in X \setminus F_i$; but then

$$\begin{aligned} |(\overline{u} - u_i)(x) - (\overline{u} - u_i)(y)| &\leq d(x, y)(\overline{g}(x) + \overline{g}(y) + 2i) \\ &\leq d(x, y)(\overline{g}(y) + 3i). \end{aligned}$$

By the convexity of Φ and by the Δ_2 condition we have

$$\begin{aligned}\int_X \Phi \circ h_i d\mu &\leq \int_{X\setminus F_i} \Phi \circ (\overline{g} + 3i)d\mu \\ &\leq C\Big[\int_{X\setminus F_i} \Phi \circ \overline{g} d\mu + \Phi(i)\mu(X \setminus F_i)\Big] \to 0 \text{ as } i \to \infty.\end{aligned}$$

This implies that $|||h_i|||_\Phi \to 0$ as $i \to \infty$ since Φ verifies the Δ_2 condition.

Now

$$|||\overline{u} - u_i|||_{\mathbf{L}^1_\Phi(X)} \leq |||h_i|||_\Phi \to 0 \text{ as } i \to \infty.$$

Thus $\overline{u} \in H^{1,0}_\Phi(O)$. The proof is complete. ∎

Definition 5. *A locally finite Borel measure μ is doubling if there is a positive constant C such that for every $x \in X$ and $r > 0$,*

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Definition 6. *A nonempty set $E \subset X$ is uniformly μ-thick if there are constants $C > 0$ and $0 < r_0 \leq 1$ such that*

$$\mu(B(x, r) \cap E) \geq C\mu(B(x, r)),$$

for every $x \in E$, and $0 < r < r_0$.

Now we give a Hardy type inequality in the context of Orlicz-Sobolev spaces.

Theorem 10. *Let Φ be an $\mathcal{N}$-function such that Φ^* satisfies the Δ_2 condition and suppose that μ is doubling. Let $O \subset X$ be an open set such that $X \setminus O$ is uniformly μ-thick. Then there is a constant $C > 0$ such that for every $u \in M_\Phi^{1,0}(O)$,*

$$|||v|||_{\Phi,O} \leq C|||u|||_{M_\Phi^{1,0}(O)},$$

where v is the function defined on O by $v(x) = \dfrac{u(x)}{dist(x, X \setminus O)}$. The constant C is independent of u.

Proof. Let $u \in M_\Phi^{1,0}(O)$ and $\widetilde{u} \in M_\Phi^1(O)$ be Φ-quasicontinuous such that $u = \widetilde{u}$ μ-a.e. in O and $\widetilde{u} = 0$ Φ-q.e. in $X \setminus O$. Let $g \in D(\widetilde{u}) \cap \mathbf{L}_\Phi(X)$ and set $O' = \{x \in O : \text{dist}(x, X \setminus O) < r_0\}$. For $x \in O'$, we choose $x_0 \in X \setminus O$ such that $r_x = \text{dist}(x, X \setminus O) = d(x, x_0)$. Recall that the Hardy-Littlewood maximal function of a locally μ-integrable function f is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)d\mu(y).$$

Using the uniform μ-thickness and the doubling condition, we get

$$\begin{aligned} \frac{1}{\mu(B(x_0,r_x) \setminus O)} \int_{B(x_0,r_x)\setminus O} g(y)d\mu(y) &\leq \frac{C}{\mu(B(x_0,r_x))} \int_{B(x_0,r_x)} g(y)d\mu(y) \\ &\leq \frac{C}{\mu(B(x,2r_x))} \int_{B(x,2r_x)} g(y)d\mu(y) \\ &\leq C\mathcal{M}g(x). \end{aligned}$$

On the other hand, for μ-a.e. $x \in O'$ there is $y \in B(x_0, r_x) \setminus O$ such that

$$\begin{aligned} |u(x)| &\leq d(x,y)(g(x) + \frac{1}{\mu(B(x_0,r_x) \setminus O)} \int_{B(x_0,r_x)\setminus O} g(y)d\mu(y)) \\ &\leq Cr_x(g(x) + \mathcal{M}g(x)) \\ &\leq C\text{dist}(x, X \setminus O)\mathcal{M}g(x). \end{aligned}$$

By [5], $\mathcal{M}$ is a bounded operator from $\mathbf{L}_\Phi(X)$ to itself since Φ^* satisfies the Δ_2 condition. Hence

$$|||v|||_{\Phi,O'} \leq C|||\mathcal{M}g|||_\Phi \leq C|||g|||_\Phi.$$

On $O \setminus O'$ we have

$$|||v|||_{\Phi,O\setminus O'} \leq r_0^{-1}|||u|||_{\Phi,O}.$$

Thus

$$|||v|||_{\Phi,O} \leq C(|||\widetilde{u}|||_{\Phi} + |||g|||_{\Phi}).$$

By taking the infimum over all $g \in D(\widetilde{u}) \cap \mathbf{L}_{\Phi}(X)$, we get the desired result. ∎

By Theorem 9 and Theorem 10 we obtain the following corollaries

Corollary 2. *Let Φ be an $\mathcal{N}$-function such that Φ and Φ^* satisfy the Δ_2 condition and suppose that μ is doubling. Let $O \subset X$ be an open set such that $X \setminus O$ is uniformly μ-thick. Then $M^{1,0}_{\Phi}(O) = H^{1,0}_{\Phi}(O)$.*

Corollary 3. *Let Φ be an $\mathcal{N}$-function such that Φ and Φ^* satisfy the Δ_2 condition and suppose that μ is doubling. Let $O \subset X$ be an open set such that $X \setminus O$ is uniformly μ-thick and let $(u_i)_i \subset M^{1,0}_{\Phi}(O)$ be a bounded sequence in $M^{1,0}_{\Phi}(O)$. If $u_i \to u$ μ-a.e., then $u \in M^{1,0}_{\Phi}(O)$.*

In the hypotheses of Corollary 3 we get $M^{1,0}_{\Phi}(O) = H^{1,0}_{\Phi}(O)$. Hence the following property $(\mathcal{P})$ is satisfied for sets E whose complement is μ-thick:
$(\mathcal{P})$ Let $(u_i)_i$ be a bounded sequence in $H^{1,0}_{\Phi}(E)$. If $u_i \to u$ μ-a.e., then $u \in H^{1,0}_{\Phi}(E)$.

Remark 3. *If $M^1_{\Phi}(X)$ is reflexive, then by Mazur's lemma closed convex sets are weakly closed. Hence every open subset O of X satisfies property $(\mathcal{P})$. But in general we do not know whether the space $M^1_{\Phi}(X)$ is reflexive or not.*

Recall that a space X is proper if bounded closed sets in X are compact.

Theorem 11. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and suppose that X is proper. Let O be an open set in X satisfying property $(\mathcal{P})$. Then $M^{1,0}_{\Phi}(O) = H^{1,0}_{\Phi}(O)$.*

Proof. It suffices to prove that $M^{1,0}_{\Phi}(O) \subset H^{1,0}_{\Phi}(O)$. Let $u \in M^{1,0}_{\Phi}(O)$ be a Φ-quasicontinuous function from $M^1_{\Phi}(X)$ such that $u = 0$ Φ-q.e. on $X \setminus O$. By using the property $(\mathcal{P})$, we deduce, by truncating and considering the positive and the negative parts separately, that we can assume that u is bounded and non-negative. If $x_0 \in O$ is a fixed point, define the sequence $(\eta_i)_i$ by

$$\eta_i(x) = \begin{cases} 1 & \text{if } d(x_0, x) \leq i - 1, \\ i - d(x_0, x) & \text{if } i - 1 < d(x_0, x) < i \\ 0 & \text{if } d(x_0, x) \geq i. \end{cases}$$

If we define the sequence $(v_i)_i$ by $v_i = u\eta_i$, then since $v_i \to u$ μ-a.e. in X and $|||v_i|||_{M^1_\Phi(X)} \leq 2|||u|||_{M^1_\Phi(X)}$, by the property $(\mathcal{P})$ it clearly suffices to show that $v_i \in H^{1,0}_\Phi(O)$. Remark that

$$\begin{aligned} |v_i(x) - v_i(y)| &\leq |u(x) - u(y)| + |\eta_i(x) - \eta_i(y)| \\ &\leq d(x, y)(g(x) + g(y) + u(x). \end{aligned}$$

Hence $v_i \in M^1_\Phi(X)$.

Now fix i and set $v = v_i$. Since v vanishes outside a bounded set, we can find a bounded open subset $U \subset O$ such that $v = 0$ Φ-q.e. in $X \setminus U$. We choose a sequence $(w_j) \subset M^1_\Phi(X)$ of quasicontinuous functions such that $0 \leq w_j \leq 1$, $w_j = 1$ on an open set O_j, with $|||w_j|||_{M^1_\Phi(X)} \to 0$, and so that the restrictions $v_{|X\setminus O_j}$ are continuous and $v = 0$ in $X \setminus (U \cup O_j)$. The sequence $(s_j)_j$, defined by $s_j = (1 - w_j)\max(v - \frac{1}{j}, 0)$, is bounded in $M^1_\Phi(X)$, and passing if necessary to a subsequence, $s_j \to v$ μ-a.e. Since $v_{|X\setminus O_j}$ is continuous, we get

$$\overline{\{x \in X : s_j(x) \neq 0\}} \subset \left\{x \in X : v(x) \geq \frac{1}{j}\right\} \setminus O_j \subset U.$$

This means that $\overline{\{x \in X : s_j(x) \neq 0\}}$ is a compact subset of O, whence by Theorem 9, $s_j \in H^{1,0}_\Phi(O)$. The property $(\mathcal{P})$ implies $v \in H^{1,0}_\Phi(O)$ and the proof is complete. ∎

Corollary 4. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and suppose that X is proper. Let O be an open set in X and suppose that $M^1_\Phi(X)$ is reflexive. Then $M^{1,0}_\Phi(O) = H^{1,0}_\Phi(O)$.*

Proof. By Remark 3, O satisfies property $(\mathcal{P})$, and Theorem 11 gives the result. ∎

4. Orlicz-Sobolev space with zero boundary values $N^{1,0}_\Phi(E)$

4.1. The Orlicz-Sobolev space $N^1_\Phi(X)$.

We recall the definition of the space $N^1_\Phi(X)$.

Let (X, d, μ) be a metric, Borel measure space, such that μ is positive and finite on balls in X.

If I is an interval in $\mathbb{R}$, a path in X is a continuous map $\gamma : I \to X$. By abuse of language, the image $\gamma(I) =: |\gamma|$ is also called a path. If $I = [a, b]$ is a closed interval, then the length of a path $\gamma : I \to X$ is

$$l(\gamma) = \text{length}(\gamma) = \sup \sum_{i=1}^{n} |\gamma(t_{i+1}) - \gamma(t_i)|,$$

where the supremum is taken over all finite sequences $a = t_1 \leq t_2 \leq ... \leq t_n \leq t_{n+1} = b$. If I is not closed, we set $l(\gamma) = \sup l(\gamma\,|_J)$, where the supremum is taken over all closed sub-intervals J of I. A path is said to be rectifiable if its length is a finite number. A path $\gamma : I \to X$ is locally rectifiable if its restriction to each closed sub-interval of I is rectifiable.

For any rectifiable path γ, there are its associated length function $s_\gamma : I \to [0, l(\gamma)]$ and a unique 1-Lipschitz continuous map $\gamma_s : [0, l(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The path γ_s is the arc length parametrization of γ.

Let γ be a rectifiable path in X. The line integral over γ of each non-negative Borel function $\rho : X \to [0, \infty]$ is $\int_\gamma \rho ds = \int_0^{l(\gamma)} \rho \circ \gamma_s(t) dt$.

If the path γ is only locally rectifiable, we set $\int_\gamma \rho ds = \sup \int_{\gamma'} \rho ds$, where the supremum is taken over all rectifiable sub-paths γ' of γ. See [5] for more details.

Denote by Γ_{rect} the collection of all non-constant compact (that is, I is compact) rectifiable paths in X.

Definition 7. *Let Φ be an $\mathcal{N}$-function and Γ be a collection of paths in X. The Φ-modulus of the family Γ, denoted $Mod_\Phi(\Gamma)$, is defined as*

$$\inf_{\rho \in \mathcal{F}(\Gamma)} |||\rho|||_\Phi,$$

where $\mathcal{F}(\Gamma)$ is the set of all non-negative Borel functions ρ such that $\int_\gamma \rho ds \geq 1$ for all rectifiable paths γ in Γ. Such functions ρ used to define the Φ-modulus of Γ are said to be admissible for the family Γ.

From the above definition the Φ-modulus of the family of all non-rectifiable paths is 0.

A property relevant to paths in X is said to hold for Φ-*almost all paths* if the family of rectifiable compact paths on which that property does not hold has Φ-modulus zero.

Definition 8. *Let u be a real-valued function on a metric space X. A non-negative Borel-measurable function ρ is said to be an upper gradient of u if for all compact rectifiable paths γ the following inequality holds*

$$|u(x) - u(y)| \leq \int_\gamma \rho ds, \tag{4.1}$$

where x and y are the end points of the path.

Definition 9. *Let Φ be an $\mathcal{N}$-function and let u be an arbitrary real-valued function on X. Let ρ be a non-negative Borel function on X. If there exists a family $\Gamma \subset \Gamma_{rect}$ such that $Mod_\Phi(\Gamma) = 0$ and the inequality (4.1) is true for all paths γ in $\Gamma_{rect} \setminus \Gamma$, then ρ is said to* *be a Φ-weak upper gradient of u. If inequality (4.1) holds true for Φ-modulus almost all paths in a set $B \subset X$, then ρ is said to be a Φ-weak upper gradient of u on B.*

Definition 10. *Let Φ be an $\mathcal{N}$-function and let the set $\widetilde{N^1_\Phi}(X, d, \mu)$ be the collection of all real-valued function u on X such that $u \in \mathbf{L}_\Phi$ and u have a Φ-weak upper gradient in $\mathbf{L}_\Phi$. If $u \in \widetilde{N^1_\Phi}$, we set*

$$|||u|||_{\widetilde{N^1_\Phi}} = |||u|||_\Phi + \inf_\rho |||\rho|||_\Phi, \tag{4.2}$$

where the infimum is taken over all Φ-weak upper gradient, ρ, of u such that $\rho \in \mathbf{L}_\Phi$.

Definition 11. *Let Φ be an $\mathcal{N}$-function. The Orlicz-Sobolev space corresponding to Φ, denoted $N^1_\Phi(X)$, is defined to be the space $\widetilde{N^1_\Phi}(X, d, \mu)/\backsim$, with norm $|||u|||_{N^1_\Phi} := |||u|||_{\widetilde{N^1_\Phi}}$.*

For more details and developments, see [3].

4.2. The Orlicz-Sobolev space with zero boundary values $N^{1,0}_\Phi(E)$.

Definition 12. *Let Φ be an $\mathcal{N}$-function. For a set $E \subset X$ define $Cap_\Phi(E)$ by*

$$Cap_\Phi(E) = \inf \left\{ |||u|||_{N^1_\Phi} : u \in \mathcal{D}(E) \right\},$$

where $\mathcal{D}(E) = \{u \in N^1_\Phi : u\,|_E \geq 1\}$.

If $\mathcal{D}(E) = \emptyset$, we set $Cap_\Phi(E) = \infty$. Functions belonging to $\mathcal{D}(E)$ are called admissible functions for E.

Definition 13. *Let Φ be an $\mathcal{N}$-function and E a subset of X. We define $\widetilde{N^{1,0}_\Phi}(E)$ as the set of all functions $u : E \to [-\infty, \infty]$ for which there exists a function $\widetilde{u} \in \widetilde{N^1_\Phi}(E)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ Cap_Φ-q.e. in $X \setminus E$; which means $Cap_\Phi(\{x \in X \setminus E : \widetilde{u}(x) \neq 0\}) = 0$.*

Let $u, v \in \widetilde{N^{1,0}_\Phi}(E)$. We say that $u \sim v$ if $u = v$ μ-a.e. in E. The relation $\sim$ is an equivalence relation and we set $N^{1,0}_\Phi(E) = \widetilde{N^{1,0}_\Phi}(E)/\backsim$. We equip this space with the norm $|||u|||_{N^{1,0}_\Phi(E)} := |||u|||_{N^1_\Phi(X)}$.

It is easy to see that for every set $A \subset X$, $\mu(A) \leq Cap_\Phi(A)$. On the other hand, by [3, Corollary 2] if $\widetilde{u}$ and $\widetilde{u}'$ both correspond to u in the above definition, then $|||\widetilde{u} - \widetilde{u}'|||_{N^1_\Phi(X)} = 0$. This means that the norm on $N^{1,0}_\Phi(E)$ is well defined.

Definition 14. *Let* Φ *be an* $\mathcal{N}$*-function and* E *a subset of* X*. We set*

$$Lip^{1,0}_{\Phi,N}(E) = \left\{ u \in N^1_\Phi(X) : u \textit{ is Lipschitz in } X \textit{ and } u = 0 \textit{ in } X \setminus E \right\},$$

and

$$Lip^{1,0}_{\Phi,C}(E) = \left\{ u \in Lip^{1,0}_{\Phi,N}(E) : u \textit{ has compact support} \right\}.$$

We let $H^{1,0}_{\Phi,N}(E)$ *be the closure of* $Lip^{1,0}_{\Phi,N}(E)$ *in the norm of* $N^1_\Phi(X)$*, and* $H^{1,0}_{\Phi,C}(E)$ *be the closure of* $Lip^{1,0}_{\Phi,C}(E)$ *in the norm of* $N^1_\Phi(X)$*.*

By definition $H^{1,0}_{\Phi,N}(E)$ and $H^{1,0}_{\Phi,C}(E)$ are Banach spaces. We prove that $N^{1,0}_\Phi(E)$ is also a Banach space.

Theorem 12. *Let* Φ *be an* $\mathcal{N}$*-function and* E *a subset of* X*. Then* $N^{1,0}_\Phi(E)$ *is a Banach space.*

Proof. Let $(u_i)_i$ be a Cauchy sequence in $N^{1,0}_\Phi(E)$. Then there is a corresponding Cauchy sequence $(\widetilde{u}_i)_i$ in $N^1_\Phi(X)$, where $\widetilde{u}_i$ is the function corresponding to u_i as in the definition of $N^{1,0}_\Phi(E)$. Since $N^1_\Phi(X)$ is a Banach space, see [3, Theorem 1], there is a function $\widetilde{u} \in N^1_\Phi(X)$, and a subsequence, also denoted $(\widetilde{u}_i)_i$ for simplicity, so that as in the proof of [3, Theorem 1], $\widetilde{u}_i \to \widetilde{u}$ pointwise outside a set T with $Cap_\Phi(T) = 0$, and also in the norm of $N^1_\Phi(X)$. For every i, set $A_i = \{x \in X \setminus E : \widetilde{u}_i(x) \neq 0\}$. Then $Cap_\Phi(\cup_i A_i) = 0$. Moreover, on $(X \setminus E) \setminus (\cup_i A_i \cup T)$, we have $\widetilde{u}(x) = \lim_{i\to\infty} \widetilde{u}_i(x) = 0$.

Since $Cap_\Phi(\cup_i A_i \cup T) = 0$, the function $u = \widetilde{u}_{|E}$ is in $N^{1,0}_\Phi(E)$. On the other hand we have

$$|||u - u_i|||_{N^{1,0}_\Phi(E)} = |||\widetilde{u} - \widetilde{u}_i|||_{N^1_\Phi(X)} \to 0 \text{ as } i \to \infty.$$

Thus $N^{1,0}_\Phi(E)$ is a Banach space and the proof is complete. ∎

Proposition 1. *Let* Φ *be an* $\mathcal{N}$*-function and* E *a subset of* X*. Then the space* $H^{1,0}_{\Phi,N}(E)$ *embeds isometrically into* $N^{1,0}_\Phi(E)$*, and the space* $H^{1,0}_{\Phi,C}(E)$ *embeds isometrically into* $H^{1,0}_{\Phi,N}(E)$*.*

Proof. Let $u \in H^{1,0}_{\Phi,N}(E)$. Then there is a sequence $(u_i)_i \subset N^1_\Phi(X)$ of Lipschitz functions such that $u_i \to u$ in $N^1_\Phi(X)$ and for each integer i, $u_{i|X\setminus E} = 0$. Considering if necessary a subsequence of $(u_i)_i$, we proceed as in the proof of [3, Theorem 1], we can consider the function $\widetilde{u}$ defined outside a set S with $Cap_\Phi(S) = 0$, by $\widetilde{u} = \frac{1}{2}(\limsup_i u_i + \liminf_i u_i)$. Then $\widetilde{u} \in N^1_\Phi(X)$ and $u_{|E} = \widetilde{u}_{|E}$ μ-a.e and $\widetilde{u}_{|(X\setminus E)\setminus S} = 0$. Hence $u_{|E} \in N^{1,0}_\Phi(E)$, with the two norms equal. Since $H^{1,0}_{\Phi,C}(E) \subset Lip^{1,0}_{\Phi,N}(E)$, it is easy to see that $H^{1,0}_{\Phi,C}(E)$ embeds isometrically into $H^{1,0}_{\Phi,N}(E)$. The proof is complete. ∎

When $\Phi(t) = \frac{1}{p}t^p$, there are examples of spaces X and $E \subset X$ for which $N^{1,0}_\Phi(E)$, $H^{1,0}_{\Phi,N}(E)$ and $H^{1,0}_{\Phi,C}(E)$ are different. See [13]. We give, in the sequel, sufficient conditions under which these three spaces agree. We begin by a definition and some lemmas.

Definition 15. *Let Φ be an $\mathcal{N}$-function. The space X is said to support a $(1,\Phi)$-Poincaré inequality if there is a constant $C > 0$ such that for all balls $B \subset X$, and all pairs of functions u and ρ, whenever ρ is an upper gradient of u on B and u is integrable on B, the following inequality holds*

$$\frac{1}{\mu(B)}\int_B |u - u_E| \leq C diam(B)\, |||g|||_{\mathbf{L}_\Phi(B)}\, \Phi^{-1}(\frac{1}{\mu(B)}).$$

Lemma 3. *Let Φ be an $\mathcal{N}$-function and Y a metric measure space with a Borel measure μ that is finite on bounded sets. Let $u \in N^1_\Phi(Y)$ be non-negative and define the sequence $(u_i)_i$ by $u_i = \min(u, i)$, $i \in \mathbb{N}$. Then $(u_i)_i$ converges to u in the norm of $N^1_\Phi(Y)$.*

Proof. Set $E_i = \{x \in Y : u(x) > i\}$. If $\mu(E_i) = 0$, then $u_i = u$ μ-a.e. and since $u_i \in N^1_\Phi(Y)$, by [3, Corollary 2] the $N^1_\Phi(Y)$ norm of $u - u_i$ is zero for sufficiently large i. Now, suppose that $\mu(E_i) > 0$. Since μ is finite on bounded sets, it is an outer measure. Hence there is an open set O_i such that $E_i \subset O_i$ and $\mu(O_i) \leq \mu(E_i) + 2^{-i}$.

We have

$$\frac{1}{i}\, |||u|||_{\mathbf{L}_\Phi(E_i)} \geq |||1|||_{\mathbf{L}_\Phi(E_i)} = \frac{1}{\Phi^{-1}(\frac{1}{\mu(E_i)})}.$$

Since Φ^{-1} is continuous, increasing and verifies $\Phi(x) \to \infty$ as $x \to \infty$, we get

$$\frac{1}{\Phi^{-1}(\frac{1}{\mu(O_i)-2^{-i}})} \leq \frac{1}{i}\, |||u|||_{\mathbf{L}_\Phi} \to 0 \text{ as } i \to \infty,$$

and

$$\mu(O_i) \to 0 \text{ as } i \to \infty.$$

Note that $u = u_i$ on $Y \setminus O_i$. Thus $u - u_i$ has $2g\chi_{O_i}$ as a weak upper gradient whenever g is an upper gradient of u and hence of u_i as well; see [3, Lemma 9]. Thus $u_i \to u$ in $N^1_\Phi(Y)$. The proof is complete. ∎

Remark 4. *By [3, Corollary 7], and in conditions of this corollary, if $u \in N^1_\Phi(X)$, then for each positive integer i, there is a $w_i \in N^1_\Phi(X)$ such that $0 \leq w_i \leq 1$, $|||w_i|||_{N^1_\Phi(X)} \leq 2^{-i}$, and $w_{i|F_i} = 1$, with F_i an open subset of X such that u is continuous on $X \setminus F_i$.*

We define, as in the proof of Theorem 11, for $i \in \mathbb{N}^*$, the function t_i by

$$t_i = (1 - w_i)\max(u - \tfrac{1}{i}, 0).$$

Lemma 4. *Let Φ be an $\mathcal{N}$-function satisfying the Δ' condition. Let X be a proper doubling space supporting a $(1,\Phi)$-Poincaré inequality, and let $u \in N^1_\Phi(X)$ be such that $0 \le u \le M$, where M is a constant. Suppose that the set $A = \{x \in X : u(x) \neq 0\}$ is a bounded subset of X. Then $t_i \to u$ in $N^1_\Phi(X)$.*

Proof. Set $E_i = \left\{x \in X : u(x) < \frac{1}{i}\right\}$. By [3, Corollary 7] and by the choice of F_i, there is an open set U_i such that $E_i \setminus F_i = U_i \setminus F_i$. Pose $V_i = U_i \cup F_i$ and remark that $w_{i|F_i} = 1$ and $u_{|E_i} < \frac{1}{i}$. Then $\{x \in X : t_i(x) \neq 0\} \subset A \setminus V_i \subset A$. If we set $v_i = u - t_i$, then $0 \le v_i \le M$ since $0 \le t_i \le u$. We can easily verify that $t_i = (1 - w_i)(u - 1/i)$ on $A \setminus V_i$, and $t_i = 0$ on V_i. Therefore

$$v_i = w_i u + (1 - w_i)/i \text{ on } A \setminus V_i, \tag{4.3}$$

and

$$v_i = u \text{ on } V_i. \tag{4.4}$$

Let $x, y \in X$. Then

$$\begin{aligned} |w_i(x)u(x) - w_i(y)u(y)| &\le |w_i(x)u(x) - w_i(x)u(y)| + |w_i(x)u(y) - w_i(y)u(y)| \\ &\le w_i(x)\,|u(x) - u(y)| + M\,|w_i(x) - w_i(y)|\,. \end{aligned}$$

Let ρ_i be an upper gradient of w_i such that $|||\rho_i|||_{\mathbf{L}_\Phi} \le 2^{-i+1}$ and let ρ be an upper gradient of u belonging to $\mathbf{L}_\Phi$. If γ is a path connecting two points $x, y \in X$, then

$$|w_i(x)u(x) - w_i(y)u(y)| \le w_i(x)\int_\gamma \rho ds + M\int_\gamma \rho_i ds.$$

Hence, if $z \in |\gamma|$, then

$$\begin{aligned} |w_i(x)u(x) - w_i(y)u(y)| &\le |w_i(x)u(x) - w_i(z)u(z)| + |w_i(z)u(z) - w_i(y)u(y)| \\ &\le w_i(z)\int_{\gamma_{xz}} \rho ds + M\int_{\gamma_{xz}} \rho_i ds + w_i(z)\int_{\gamma_{zy}} \rho ds + M\int_{\gamma_{zy}} \rho_i ds \\ &\le w_i(z)\int_\gamma \rho ds + M\int_\gamma \rho_i ds, \end{aligned}$$

where γ_{xz} and γ_{zy} are such that the concatenation of these two segments gives the original path γ back again. Therefore

$$|w_i(x)u(x) - w_i(y)u(y)| \le \int_\gamma \left(\inf_{z\in|\gamma|} w_i(z)\rho + M\rho_i\right) ds.$$

Thus

$$|w_i(x)u(x) - w_i(y)u(y)| \le \int_\gamma (w_i(z)\rho + M\rho_i)ds.$$

This means that $w_i\rho + M\rho_i$ is an upper gradient of $w_i u$. Since $|||w_i|||_{\mathbf{L}_\Phi} \le 2^{-i}$, we get that $w_i \to 0$ μ-a.e. On the other hand $w_i\rho \le \rho$ on X implies that $w_i\rho \in \mathbf{L}_\Phi$ and hence $\Phi \circ (w_i\rho) \in \mathbf{L}^1$ because Φ verifies the Δ_2 condition. Since Φ is continuous, $\Phi \circ (w_i\rho) \to 0$ μ-a.e. The Lebesgue dominated convergence theorem gives $\int_X \Phi \circ (w_i\rho)dx \to 0$ as $i \to \infty$. Thus $|||w_i\rho|||_{\mathbf{L}_\Phi} \to 0$ as $i \to \infty$ since Φ verifies the Δ_2 condition.

Let B be a bounded open set such that $A \subset BT$. Then $O_i = (A \cup F_i) \cap B$ is a bounded open subset of A and $O_i \subset A$. Therefore since $O_i \cap V_i \subset (E_i \cap A) \cup F_i$, we get

$$\begin{aligned}\mu(O_i \cap V_i) &\le \mu(E_i \cap A) + \mu(F_i)\\ &\le \mu\left(\left\{x \in X : 0 < u(x) < \frac{1}{i}\right\}\right) + Cap_\Phi(F_i).\end{aligned}$$

Hence $\mu(O_i \cap V_i) \to 0$ as $i \to \infty$, since bounded sets have finite measure and therefore $\mu\left(\left\{x \in X : 0 < u(x) < \frac{1}{i}\right\}\right) \to \mu(\emptyset) = 0$ as $i \to \infty$. Thus $|||\rho|||_{\mathbf{L}_\Phi(O_i\cap V_i)} \to 0$ as $i \to \infty$.

By [3, Lemma 8] and equations (4.3) and (4.4), we get

$$g_i := \left(w_i\rho + M\rho_i + \frac{1}{i}\rho_i\right)\chi_{O_i} + \rho\chi_{O_i\cap V_i}$$

is a weak upper gradient of v_i and since

$$|||g_i|||_{\mathbf{L}_\Phi} \le |||w_i\rho|||_{\mathbf{L}_\Phi} + (M + \tfrac{1}{i})\,|||\rho_i|||_{\mathbf{L}_\Phi} + |||\rho|||_{\mathbf{L}_\Phi(O_i\cap V_i)},$$

we infer that $|||g_i|||_{\mathbf{L}_\Phi} \to 0$ as $i \to \infty$.

On the other hand, we have

$$\begin{aligned}|||v_i|||_{\mathbf{L}_\Phi} &= |||u - t_i|||_{\mathbf{L}_\Phi} \le |||w_i u|||_{\mathbf{L}_\Phi(A\setminus V_i)} + \frac{1}{i}\,|||1 - w_i|||_{\mathbf{L}_\Phi(A\setminus V_i)} + |||u|||_{\mathbf{L}_\Phi(O_i\cap V_i)}\\ &\le M|||w_i|||_{N^1_\Phi(X)} + \frac{1}{i}\frac{1}{\Phi^{-1}(\frac{1}{\mu(A)})} + |||u|||_{\mathbf{L}_\Phi(O_i\cap V_i)}.\end{aligned}$$

Since $|||w_i|||_{N^1_\Phi(X)} \to 0$ and $|||u|||_{\mathbf{L}_\Phi(O_i\cap V_i)} \to 0$ as $i \to \infty$, we conclude that $|||v_i|||_{\mathbf{L}_\Phi} \to 0$ as $i \to \infty$, and hence $t_i \to u$ in $N^1_\Phi(X)$. The proof is complete. ∎

Theorem 13. *Let Φ be an $\mathcal{N}$-function satisfying the Δ' condition. Let X be a proper doubling space supporting a $(1, \Phi)$-Poincaré inequality and E an open subset of X. Then $N^{1,0}_\Phi(E) = H^{1,0}_{\Phi,N}(E) = H^{1,0}_{\Phi,C}(E)$.*

Proof. By Proposition 1 we know that $H^{1,0}_{\Phi,C}(E) \subset H^{1,0}_{\Phi,N}(E) \subset N^{1,0}_{\Phi}(E)$. It suffices to prove that $N^{1,0}_{\Phi}(E) \subset H^{1,0}_{\Phi,C}(E)$. Let $u \in N^{1,0}_{\Phi}(E)$, and identify u with its extension $\widetilde{u}$. By the lattice properties of $N^1_\Phi(X)$ it is easy to see that u^+ and u^- are both in $N^{1,0}_{\Phi}(E)$ and hence it suffices to show that u^+ and u^- are in $H^{1,0}_{\Phi,C}(E)$. Thus we can assume that $u \geq 0$. On the other hand, since $N^{1,0}_{\Phi}(E)$ is a Banach space that is isometrically embedded in $N^1_\Phi(X)$, if $(u_n)_n$ is a sequence in $N^{1,0}_{\Phi}(E)$ that is Cauchy in $N^1_\Phi(X)$, then its limit, u, lies in $N^{1,0}_{\Phi}(E)$. Hence by Lemma 3, it also suffices to consider u such that $0 \leq u \leq M$, for some constant M. By [3, Lemma 17], it suffices to consider u such that $A = \{x \in X : u(x) \neq 0\}$ is a bounded set. By Lemma 4, it suffices to show that for each positive integer i, the function $\varphi_i = (1 - w_i)\max(u - \frac{1}{i}, 0)$ is in $H^{1,0}_{\Phi,C}(E)$.

On the other hand, if O_i and F_i are open subsets of X and $Cap_\Phi(F_i) \leq 2^{-i}$, as in the proof of Lemma 4, we have $A \cup F_i = O_i \cup F_i$. Since u has bounded support, we can choose O_i as bounded sets contained in E. We have $w_i\,|_{F_i} = 1$ and hence $\varphi_i\,|_{F_i} = 0$. Set $E_i = \left\{x \in X : u(x) < \frac{1}{i}\right\}$. Then, as in the proof of Lemma 4, there is an open set $U_i \subset E$ such that $E_i \setminus F_i = U_i \setminus F_i$ and $\varphi_i\,|_{F_i \cup U_i} = 0$. Thus $\overline{\{x : \varphi_i(x) \neq 0\}} \subset \{x \in E : u(x) \geq 1/i\} \setminus F_i = O_i \setminus (E_i \cup F_i) \subset O_i \subset E$.

The support of φ_i is compact because X is proper, and hence $\delta =$dist(supp φ_i, $X \setminus E) > 0$. By [3, Theorem 5], φ_i is approximated by Lipschitz functions in $N^1_\Phi(X)$. Let g_i be an upper gradient of φ_i. By [3, Lemma 9] we can assume that $g_i\,\big|_{X\setminus O_i} = 0$. As in [3], define the operator $\mathcal{M}'$ by $\mathcal{M}'(f)(x) = \sup_B \dfrac{1}{\mu(B)}\Phi(|||f|||_{\mathbf{L}_\Phi(B)})$, where the supremum is taken over all balls $B \subset X$ such that $x \in B$. Then if $x \in X \setminus E$, we get

$$\mathcal{M}'(g_i)(x) = \sup_{x\in B,\ \mathrm{rad}B>\delta/2} \frac{1}{\mu(B)}\Phi(|||g_i|||_{\mathbf{L}_\Phi(B)}) \leq \frac{C'}{(\delta/2)^s}\Phi(|||g_i|||_{\mathbf{L}_\Phi}) < \infty,$$

where $s = \frac{LogC}{Log2}$, C being the doubling constant, and C' is a constant depending only on C and A. We know from [3, Proposition 4] that if $f \in \mathbf{L}_\Phi$, then $\lim_{\lambda\to\infty} \lambda\mu\,\{x \in X : \mathcal{M}'(f)(x) > \lambda\} = 0$. Hence in the proof of [3, Theorem 5], choosing $\lambda > \frac{C'}{(\delta/2)^s}\Phi(|||g_i|||_{\mathbf{L}_\Phi})$ ensures that the corresponding Lipschitz approximations agree with the functions φ_i on $X \setminus E$. Thus these Lipschitz approximations are in $H^{1,0}_{\Phi,N}(E)$, and therefore so is φ_i. Moreover, these Lipschitz approximations have compact support in E, and hence $\varphi_i \in H^{1,0}_{\Phi,C}(E)$. The proof is complete. ∎

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AN INTERACTING PARTICLES PROCESS FOR BURGERS EQUATION ON THE CIRCLE

ANTHONY GAMST

ABSTRACT. We adapt the results of Oelschläger (1985) to prove a weak law of large numbers for an interacting particles process which, in the limit, produces a solution to Burgers equation with periodic boundary conditions. We anticipate results of this nature to be useful in the development of Monte Carlo schemes for nonlinear partial differential equations.

A.M.S. (MOS) Subject Classification Codes. 35, 47, 60.
Key Words and Phrases. Burgers equation, kernel density, Kolmogorov equation, Brownian motion, Monte Carlo scheme.

1. INTRODUCTION

Several propagation of chaos results have been proved for the Burgers equation (Calderoni and Pulvirenti 1983, Osada and Kotani 1985, Oelschläger 1985, Gutkin and Kac 1986, and Sznitman 1986) all using slightly different methods. Perhaps the best result for the Cauchy free-boundary problem is Sznitman's (1986) result which describes the particle interaction in terms of the average 'co-occupation time' of the randomly diffusing particles. For various reasons, we follow Oelschläger and prove a Law of Large Numbers type result for the measure valued process (MVP) where the interaction is given in terms of a kernel density estimate with bandwidth a function of the number N of interacting diffusions.

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The heuristics are as follows: The (nonlinear) partial differential equation

$$u_t = \frac{u_{xx}}{2} - \left(u(x,t) \int b(x-y)u(y,t)\,dy\right)_x \tag{1}$$

is the Kolmogorov forward equation for the diffusion $X = (X_t)$ which is the solution to the stochastic differential equation

$$\begin{aligned} dX_t &= dW_t + \left\{\int b(X_t - y)u(y,t)\,dy\right\} dt & (2)\\ &= dW_t + E(b(X_t - \bar{X}_t))dt & (3) \end{aligned}$$

where $u(x,t)\,dx$ is the density of X_t, W_t is standard Brownian motion (a Wiener process), $\bar{X}$ is an independent copy of X, and E is the expectation operator. Note the change in notation: for a stochastic process X, X_t denotes its location at time t not a (partial) derivative with respect to t.

The law of large numbers suggests that

$$E(b(X_t - \bar{X}_t)) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} b(X_t - X_t^j)$$

where the X^j are independent copies of X and this empirical approximation suggests looking at the system of N stochastic differential equations given by

$$dX_t^{i,N} = dW_t^{i,N} + \frac{1}{N}\sum_{j=1}^{N} b(X_t^{i,N} - X_t^{j,N})dt, \quad i = 1,\ldots,N$$

where the $W^{i,N}$ are independent Brownian motions. Now if b is bounded and Lipschitz and the N particles are started independently with distribution μ_0, then the system of N stochastic differential equations will have a unique solution (Karatzas and Shreve 1991) and the measure valued process

$$\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{j,N}}$$

where δ_x is the point-mass at x will converge to a solution μ of (1) in the sense that for every bounded continuous function f on the real-line and every $t > 0$,

$$\int f(x)\mu_t^N(dx) = \frac{1}{N}\sum_{j=1}^{N} f(X_t^{j,N}) \to \int f(x)\mu_t(dx),$$

where μ_t has a density u so $\mu_t(dx) = u(x,t)dx$ and u solves (1).

By formal analogy, if we take $2b(x-y) = \delta_0(x-y)$, where δ_0 is the point-mass at zero, then

$$u_t = \frac{u_{xx}}{2} - \left(u(x,t) \int \frac{\delta_0(x-y)}{2} u(y,t)\, dy \right)_x \tag{4}$$

$$= \frac{u_{xx}}{2} - \left(\frac{u^2}{2} \right)_x \tag{5}$$

$$= \frac{u_{xx}}{2} - uu_x \tag{6}$$

which is the Burgers equation with viscosity parameter $\varepsilon = 1/2$. Unfortunately, δ_0 is neither bounded nor Lipschitz and a lot of work goes into dealing with this problem. This is covered in greater detail later in the paper.

Our interest in these models lies partially in their potential use as numerical methods for nonlinear partial differential equations. This idea has been the subject of a good deal of recent research, see Talay and Tubaro (1996). As noted there, and elsewhere, the Burgers equation is an excellent test for new numerical methods precisely because it does have an exact solution. In the next two sections, we prove the underlying Law of Large Numbers for the Burgers equation with periodic boundary conditions. Such boundary conditions seem natural for numerical work.

2. THE SETUP AND GOAL.

We are interested in looking at the dynamics of the measure valued process

$$\mu_t^N = \sum_{j=1}^N \delta_{Y_t^{j,N}} \tag{7}$$

with δ_x the point-mass at x,

$$Y_t^{j,N} = \varphi(X_t^{j,N}) \tag{8}$$

where $\varphi(x) = x - [x]$ and $[x]$ is the largest integer less than or equal to x, with the $X_t^{j,N}$ satisfying the following system of stochastic differential equations

$$dX_t^{j,N} = dW_t^{j,N} + F\left(\frac{1}{N} \sum_{l=1}^N b^N(X_t^{j,N} - X_t^{l,N}) \right) dt \tag{9}$$

where the $W_t^{j,N}$ are independent standard Brownian motion processes,

$$F(x) = \frac{x \wedge \|u_0\|}{2},$$

u_0 is a bounded measurable density function on $S = [0,1)$, $\|\cdot\|$ is the supremum norm, $\|f\| = \sup_S |f(x)|$, and $b^N(x) > 0$ is an infinitely-differentiable one-periodic function on the real line $\mathbb{R}$ such that

$$\int_0^1 b^N(x)\,dx = 1 \tag{10}$$

for all $N = 1, 2, \ldots$, and for any continuous bounded one-periodic function f

$$\int_{-1/2}^{1/2} f(x)b^N(x)\,dx \to f(0) \tag{11}$$

as $N \to \infty$. We call a function f on $\mathbb{R}$ one-periodic if $f(x) = f(x+1)$ for every $x \in \mathbb{R}$.

For any x and y in S, let

$$\rho(x,y) = |x-y-1| \wedge |x-y| \wedge |x-y+1| \tag{12}$$

and note that (S,ρ) is a complete, separable, and compact metric space. Let $C_b(S)$ denote the space of all continuous bounded functions on (S,ρ). Note that if f is a continuous one-periodic function on $\mathbb{R}$ and g is the restriction of f to S, then $g \in C_b(S)$. Additionally, for any one-periodic function f on $\mathbb{R}$ we have $f(Y_t^{j,N}) = f(X_t^{j,N})$ and therefore

$$\begin{aligned}
\langle \mu_t^N, f\rangle &= \int_S f(x)\mu_t^N(dx) \\
&= \frac{1}{N}\sum_{j=1}^N f(Y_t^{j,N}) \\
&= \frac{1}{N}\sum_{j=1}^N f(X_t^{j,N})
\end{aligned}$$

for any one-periodic function f on $\mathbb{R}$.

To study the dynamics of the process μ_t^N as $N \to \infty$ we will need to study, for any f which is both one-periodic and twice-differentiable with bounded first and second derivatives, the dynamics of the processes $\langle \mu_t^N, f\rangle$. These dynamics are obtained from (7), (9), and Itô's formula (see Karatzas and Shreve 1991, p.153)

$$\begin{aligned}
\langle \mu_t^N, f\rangle = \langle \mu_0^N, f\rangle &+ \int_0^t \langle \mu_s^N, F(g_s^N(\cdot))f' + \frac{1}{2}f''\rangle\,ds \\
&+ \frac{1}{N}\sum_{j=1}^N \int_0^t f'(X_s^{j,N})\,dW_s^{j,N}
\end{aligned} \tag{13}$$

where the use the notation

$$\langle \mu, f\rangle = \int_S f(x)\mu(dx)$$

with μ a measure on S,

$$g_t^N(x) = \frac{1}{N}\sum_{l=1}^{N} b^N(x - X_t^{l,N}) \tag{14}$$

and the fact that because b^N is one-periodic, $b^N(Y_t^{j,N} - Y_t^{l,N}) = b^N(X_t^{j,N} - X_t^{l,N})$.

Given any metric space (M, m), let $\mathcal{M}_1(M)$ be the space of probability measures on M equipped with the usual weak topology:

$$\lim_{k\to\infty} \mu^k = \mu$$

if and only if

$$\lim_{k\to\infty} \int_M f(x)\mu^k(dx) = \int_M f(x)\mu(dx)$$

for every f in $C_b(M)$, where $C_b(M)$ is the space of all continuous bounded and real-valued functions f on M under the supremum norm $\|f\| = \sup_M |f(x)|$.

On the space (S, ρ) the weak topology is generated by the bounded Lipschitz metric

$$\|\mu^1 - \mu^2\|_H = \sup_{f\in H} |\langle \mu^1, f\rangle - \langle \mu^2, f\rangle|$$

where

$$H = \{f \in C_b(S) \,:\, \|f\| \le 1,\ |f(x) - f(y)| < \rho(x,y) \text{ for all } x, y \in S\}$$

(Pollard 1984, or Dudley 1966).

Fix a positive $T < \infty$ and take $C([0,T], \mathcal{M}_1(S))$ to be the space of all continuous functions $\mu = (\mu_t)$ from $[0,T]$ to $\mathcal{M}_1(S)$ with the metric

$$m(\mu^1, \mu^2) = \sup_{0\le t\le T} \|\mu_t^1 - \mu_t^2\|_H,$$

then the empirical processes μ_t^N with $0 \le t \le T$ are random elements of the space $C([0,T], \mathcal{M}_1(S))$. Indeed, take any sequence $(t_k) \subset [0,T]$ with

$t_k \to t$, then for any f in H we have

$$\begin{aligned}|\langle \mu_t^N, f\rangle - \langle \mu_{t_k}^N, f\rangle| &= \left| \frac{1}{N}\sum_{j=1}^N f(Y_t^{j,N}) - f(Y_{t_k}^{j,N}) \right| \\ &\le \frac{1}{N}\sum_{j=1}^N |f(Y_t^{j,N}) - f(Y_{t_k}^{j,N})| \\ &\le \frac{1}{N}\sum_{j=1}^N \rho(Y_t^{j,N}, Y_{t_k}^{j,N}) \\ &\le \frac{1}{N}\sum_{j=1}^N |X_t^{j,N} - X_{t_k}^{j,N}| \\ &= \frac{1}{N}\sum_{j=1}^N \left| [W_t^{j,N} - W_{t_k}^{j,N}] + \int_{t_k}^t F(g_s^N(X_s^{j,N}))\, ds \right| \to 0\end{aligned}$$

because the $W_t^{j,N}$ are continuous in t and $\|F\| < \infty$. This means that the distributions $\mathcal{L}(\mu^N)$ of the processes $\mu^N = (\mu_t^N)$ can be considered random elements of the space $\mathcal{M}_1(C([0,T], \mathcal{M}_1(S)))$.

Our goal is to prove the following Law of Large Numbers type result.

Theorem 1. *Under the conditions that*

(i): b^N *is one-periodic, positive and infi nitely-differentiable with*

$$\int_0^1 b^N(x)\, dx = 1, \tag{15}$$

and

$$\int_{-1/2}^{1/2} f(x) b^N(x)\, dx \to f(0) \tag{16}$$

for every continuous, bounded, and one-periodic function f *on* $\mathbb{R}$,

(ii): $\|b^N\| \le AN^\alpha$ *for some* $0 < \alpha < 1/2$ *and some constant* $A < \infty$,

(iii): *there is a* β *with* $0 < \beta < (1 - 2\alpha)$ *such that*

$$\sum_\lambda |\tilde{b}^N(\lambda)|^2 (1 + |\lambda|^\beta) < \infty \tag{17}$$

where $\lambda = 2k\pi$, *with* $k \in \mathbf{Z}$, *and* $\tilde{b}^N(\lambda) = \int_0^1 e^{i\lambda x} b^N(x)\, dx$ *is the Fourier transform of* b^N,

(iv): u_0 *is a density function on* $[0,1)$ *with* $\|u_0\| < \infty$, *and*

(v): $\langle \mu_0^N, f\rangle = \frac{1}{N}\sum_{j=1}^N f(Y_0^{j,N}) = \frac{1}{N}\sum_{j=1}^N f(X_0^{j,N}) \to \int_0^1 f(x) u_0(x)\, dx$ *for every* $f \in C_b(S)$.

then there is a deterministic family of measures $\mu = (\mu_t)$ *on* $[0,1)$ *such that*

$$\mu^N \to \mu \tag{18}$$

in probability as $N \to \infty$*, for every* t *in* $[0,T]$*, with* $\mu^N = (\mu_t^N)$*,* μ_t *is absolutely continuous with respect to Lebesgue measure on* S *with density function* $g_t(x) = u(x,t)$ *satisfying the Burgers equation*

$$u_t + uu_x = \frac{1}{2}u_{xx} \tag{19}$$

with periodic boundary conditions.

The proof has three parts. First, we establish the fact that the sequence of probability laws $\mathcal{L}(\mu^N)$ is relatively compact in $\mathcal{M}_1(C([0,T],\mathcal{M}_1(S)))$ and therefore every subsequence of (μ^{N_k}) of (μ^N) has a further subsequence that converges in law to some μ in $C([0,T],\mathcal{M}_1(S))$. Second, we prove that any such limit process μ must satisfy a certain integral equation, and finally, that this integral equation has a unique solution. We follow rather closely the arguments of Oelschläger (1985) and apply his result (Theorem 5.1, p.31) in the final step of the argument.

3. The Law of Large Numbers.

Relative Compactness. The first step in the proof of Theorem 1 is to show that the sequence of probability laws $\mathcal{L}(\mu^N)$, $N = 1, 2, \dots$, is relatively compact in $\mathcal{M} = \mathcal{M}_1(C([0,T],\mathcal{M}_1(S)))$. Since S is a compact metric space $\mathcal{M}_1(S)$ is as well (Stroock 1983, p.122) and therefore for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathcal{M}_1(S)$ such that

$$\inf_N P\left(\mu_t^N \in K_\varepsilon, \forall t \in [0,T]\right) \geq 1 - \varepsilon; \tag{20}$$

in particular, we may take $K_\varepsilon = \mathcal{M}_1(S)$ regardless of $\varepsilon \geq 0$. Furthermore, for $0 \leq s \leq t \leq T$ and some constant $C > 0$ we have

$$\begin{aligned}
\|\mu_t^N - \mu_s^N\|_H^4 &= \sup_{f\in H}(\langle \mu_t^N, f\rangle - \langle \mu_s^N, f\rangle)^4 \\
&= \sup_{f\in H}\left(\frac{1}{N}\sum_{j=1}^N f(Y_t^{j,N}) - f(Y_s^{j,N})\right)^4 \\
&\leq \left(\frac{1}{N}\sum_{j=1}^N \rho(Y_t^{j,N}, Y_s^{j,N})\right)^4 \\
&\leq \left(\frac{1}{N}\sum_{j=1}^N |X_t^{j,N} - X_s^{j,N}|\right)^4 \\
&\leq \frac{1}{N}\sum_{j=1}^N |X_t^{j,N} - X_s^{j,N}|^4 \\
&= \frac{1}{N}\sum_{j=1}^N \left|(W_t^{j,N} - W_s^{j,N}) + \int_s^t F\left(g_u^N(X_u^{j,N})\right)\, du\right|^4 \\
&\leq C\left(\frac{1}{N}\sum_{j=1}^N \left|W_t^{j,N} - W_s^{j,N}\right|^4 + \frac{1}{N}\sum_{j=1}^N \left|\int_s^t F\left(g_u^N(X_u^{j,N})\right)\, du\right|^4\right)
\end{aligned}$$

and therefore

$$E\|\mu_t^N - \mu_s^N\|_H^4 \leq C(3(t-s)^2 + \|u_0\|^4(t-s)^4) < 3C\|u_0\|^4(t-s)^2 \tag{21}$$

for $t - s$ small. Together equations (20) and (21) imply that the sequence of probability laws $\mathcal{L}(\mu^N)$ is relatively compact (Gikhman and Skorokhod 1974, VI, 4) as desired.

Almost Sure Convergence. Now the relative compactness of the sequence of laws $\mathcal{L}(\mu^N)$ in $\mathcal{M}$ implies that there is an increasing subsequence $(N_k) \subset (N)$ such that $\mathcal{L}(\mu^{N_k})$ converges in $\mathcal{M}$ to some limit $\mathcal{L}(\mu)$ which is the distribution of some measure valued process $\mu = (\mu_t)$. For ease of notation, we assume at this point that $(N_k) = (N)$. The Skorokhod representation theorem implies now that after choosing the proper probability space, we may define μ^N and μ so that

$$\lim_{N\to\infty}\sup_{t\leq T}\|\mu_t^N - \mu_t\|_H = 0 \tag{22}$$

P-almost surely. This leaves us with the task of describing the possible limit processes, μ.

An Integral Equation. We know from Ito's formula that for any $f \in C_b^2(S)$, μ^N satisfies

$$\begin{aligned} \langle \mu_t^N, f\rangle - \langle \mu_0^N, f\rangle &- \int_0^t \langle \mu_s^N, F(g_s^N(\cdot))f' + \frac{1}{2}f''\rangle\, ds \\ &= \frac{1}{N}\sum_{j=1}^N \int_0^t f'(X_s^{j,N})\, dW_s^{j,N} \end{aligned} \tag{23}$$

where the right hand side is a martingale. Because $f \in C_b^2(S)$, the weak convergence of μ^N to μ gives us that

$$\langle \mu_t^N, f\rangle \to \langle \mu_t, f\rangle \tag{24}$$

as $N \to \infty$ for all $0 \le t \le T$ and we have

$$\langle \mu_0^N, f\rangle \to \langle \mu_0, f\rangle \tag{25}$$

as $N \to \infty$ by assumption. Furthermore, Doob's inequality (Stroock 1983, p.355) implies

$$\begin{aligned} E\left[\sup_{t\le T}\left(\frac{1}{N}\sum_{j=1}^N \int_0^t f'(X_s^{j,N})\, dW_s^{j,N}\right)^2\right] &\le 4E\left[\left(\frac{1}{N}\sum_{j=1}^N \int_0^T f'(X_s^{j,N})\, dW_s^{j,N}\right)^2\right] \\ &\le \frac{4}{N}T\|f'\|^2 \end{aligned}$$

and therefore the right hand side of (23) vanishes as $N \to \infty$. Clearly now, the integral term third in equation (23) must converge as well and the goal at present is to find out to what.

First, because $f \in C_b^2(S)$, the weak convergence of μ^N to μ gives us that

$$\frac{1}{2}\int_0^t \langle \mu_s^N, f''\rangle\, ds \to \frac{1}{2}\int_0^t \langle \mu_s, f''\rangle\, ds \tag{26}$$

as $N \to \infty$. Now only the $\int_0^t \langle \mu_s^N, F(g_s^N(\cdot))f'\rangle\, ds$-term remains and this is indeed the most troublesome because of the interaction between the μ_s^N and g_s^N terms. To study this term we will need to work out the convergence properties of the 'density' g_s^N. We start by working on some L^2 bounds.

The Convergence of the Density g_s^N. Note that

$$\langle g_s^N(\cdot), e^{i\lambda\cdot}\rangle = \langle \mu_s^N, e^{i\lambda\cdot}\rangle\, \tilde{b}^N(\lambda),$$

where $\tilde{b}^N$ is the Fourier transform of the interaction kernel b^N.

Ito's formula implies that for any $\lambda \in (2k\pi)$ with $k \in \mathbf{Z}$

$$
\begin{aligned}
|\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 e^{\lambda^2(t-\tau)} \; - \; & \int_0^t (((\langle \mu_s^N, e^{-i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(i\lambda)e^{i\lambda\cdot} - \frac{\lambda^2}{2}e^{i\lambda\cdot}\rangle \\
& +\langle \mu_s^N, e^{i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(-i\lambda)e^{-i\lambda\cdot} - \frac{\lambda^2}{2}e^{-i\lambda\cdot}\rangle)e^{\lambda^2(s-\tau)} \\
& +|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2\lambda^2 e^{\lambda^2(s-\tau)} + \frac{1}{N}\lambda^2 e^{\lambda^2(s-\tau)})\, ds \\
= \; |\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 e^{\lambda^2(t-\tau)} - & \int_0^t (((\langle \mu_s^N, e^{-i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(i\lambda)e^{i\lambda\cdot}\rangle \\
& +\langle \mu_s^N, e^{i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(-i\lambda)e^{-i\lambda\cdot}\rangle)e^{\lambda^2(s-\tau)} \\
& +\frac{\lambda^2}{N}e^{\lambda^2(s-\tau)})\, ds
\end{aligned} \tag{27}
$$

is a martingale.

Now take $\tau = t + h$ and

$$
k_h^N(\lambda, t) = |\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2 e^{-\lambda^2 h}
$$

then the martingale property above gives

$$
\begin{aligned}
E[k_h^N(\lambda, t)] \;&=\; E[k_{t+h}^N(\lambda, 0)] + \int_0^t E[\langle \mu_s^N, e^{-i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(i\lambda)e^{i\lambda\cdot}\rangle \\
&\qquad +\langle \mu_s^N, e^{i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(-i\lambda)e^{-i\lambda\cdot}\rangle \\
&\qquad +\frac{\lambda^2}{N}]e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2\, ds \\
&\leq\; E[k_{t+h}^N(\lambda, 0)] \\
&\quad + \int_0^t (E[2|\langle \mu_s^N, e^{i\lambda\cdot}\rangle||\langle \mu_s^N, F(g_s^N(\cdot))e^{i\lambda\cdot}\rangle \\
&\quad \cdot|\lambda|\, e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2] \\
&\quad +\frac{\lambda^2}{N}e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2)\, ds \\
&\leq\; E[k_{t+h}^N(\lambda, 0)] \\
&\quad + \int_0^t (2\|u_0\|E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\lambda|e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2] \\
&\quad +\frac{\lambda^2}{N}e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2)\, ds.
\end{aligned} \tag{28}
$$

Summing over $\lambda \in (\lambda_k)$ gives

$$\begin{aligned}\sum_\lambda E[k_h^N(\lambda,t)] &\le \sum_\lambda E[k_{t+h}^N(\lambda,0)] \\ &\quad +2\|u_0\|\sum_\lambda\int_0^t E\left[|\langle\mu_s^N, e^{i\lambda\cdot}\rangle|^2|\lambda|e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2\right]\,ds \\ &\quad +\sum_\lambda\int_0^t\left(\frac{\lambda^2}{N}e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2\right)\,ds \\ &= A_I + A_{II} + A_{III}.\end{aligned}$$

Now, of course,

$$\sum_\lambda k_{t+h}^N(\lambda,0) \le \sum_\lambda e^{-\lambda^2(t+h)} \le (t+h)^{-1/2}$$

and therefore

$$A_I = \sum_\lambda E[k_{t+h}^N(\lambda,0)] \le (t+h)^{-1/2}.$$

For A_{III}, using hypothesis (ii) from Theorem 1, we have

$$\begin{aligned}A_{III} &= \frac{1}{N}\sum_\lambda|\tilde{b}^N(\lambda)|^2\int_0^t\lambda^2e^{-\lambda^2(t+h-s)}\,ds = \frac{1}{N}\sum_\lambda|\tilde{b}^N(\lambda)|^2e^{-\lambda^2h} \\ &\le \frac{1}{N}\sum_\lambda|\tilde{b}^N(\lambda)|^2 = \frac{1}{N}\int_0^1(b^N(x))^2\,dx \\ &\le \frac{2N^{2\alpha}}{N}C \le 2C\end{aligned}$$

for some constant $C > 0$. Now

$$\begin{aligned}&2\|u_0\|\int_0^t E[|\langle\mu_s^N,e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2|\lambda|e^{-\lambda^2(t+h-s)}]\,ds \\ &= 2\|u_0\|\int_0^t E[|\langle\mu_s^N,e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2e^{-\lambda^2(t+h-s)/2}|\lambda|e^{-\lambda^2(t+h-s)/2}]\,ds \\ &\le 2\|u_0\|C\int_0^t E[|\langle\mu_s^N,e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2e^{-\lambda^2(t+h-s)/2}]\,ds \\ &= 2\|u_0\|C\int_0^t E[k_{(t+h-s)/2}^N(\lambda,s)]\,ds \\ &\le 2\|u_0\|C\int_0^t e^{-\lambda^2(t+h-s)/2}\,ds \\ &\le \frac{4\|u_0\|C}{\lambda^2}\end{aligned}$$

for some other constant $C > 0$ and therefore

$$A_{II} \le 4\|u_0\|C\sum_{\lambda\ne0}\lambda^{-2} \le 4\|u_0\|D$$

for some constant $D < \infty$. Hence

$$\begin{aligned}\sum_\lambda E[k_h^N(\lambda,t)] &= \sum_\lambda E[|\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2] e^{-\lambda^2 h} \\ &= A_I + A_{II} + A_{III} \\ &\le (t+h)^{-1/2} + C(\|u_0\|+1)\end{aligned}$$

uniformly in $h > 0$ for some constant $C < \infty$. Letting h go to zero gives

$$\begin{aligned}\sum_\lambda E|\tilde{g}_t^N(\lambda)|^2 &= \sum_\lambda E[k_0^N(\lambda,t)] \\ &= \lim_{h\to 0}\sum_\lambda E[k_h^N(\lambda,t)] \le t^{-1/2} + C(\|u_0\|+1).\end{aligned}$$

From the martingale property (27) we have

$$\begin{aligned}E[k_0^N(\lambda,t)] \le E[k_{t/2}^N(\lambda,t/2)] &+ 2\|u_0\|\int_{t/2}^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2]|\lambda| e^{-\lambda^2(t-s)}\,ds \\ &+ \frac{\lambda^2}{N}\int_{t/2}^t e^{-\lambda^2(t-s)}|\tilde{b}^N(\lambda)|^2\,ds\end{aligned}$$

and for $\beta \in (0, 1-2\alpha)$ we have

$$\begin{aligned}(1+|\lambda|^\beta)E[k_0^N(\lambda,t)] &\le (1+|\lambda|^\beta)E[k_{t/2}^N(\lambda,t/2)] \\ &\quad +2\|u_0\|\int_{t/2}^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2] \\ &\qquad \cdot |\lambda|(1+|\lambda|^\beta)e^{-\lambda^2(t-s)}\,ds \\ &\quad +(1+|\lambda|^\beta)\frac{\lambda^2}{N}\int_{t/2}^t e^{-\lambda^2(t-s)}|\tilde{b}^N(\lambda)|^2\,ds \\ &\le (1+|\lambda|^\beta)e^{-\lambda^2 t/2} \\ &\quad +2\|u_0\|C\int_{t/2}^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2]e^{-\lambda^2(t-s)/2}\,ds \\ &\quad +(1+|\lambda|^\beta)\frac{\lambda^2}{N}\int_{t/2}^t e^{-\lambda^2(t-s)}|\tilde{b}^N(\lambda)|^2\,ds\end{aligned}$$

for some constant $C < \infty$ and we know that

$$\sum_\lambda (1+|\lambda|^\beta)e^{-\lambda^2 t/2} < \infty,$$

$$2\|u_0\|C\int_{t/2}^t \sum_\lambda E[k_{(t-s)/2}^N(\lambda,s)]\,ds \le 2\|u_0\|C\int_{t/2}^t \sum_\lambda e^{-\lambda^2(t-s)/2}\,ds < \infty,$$

and, from hypothesis (iii) of Theorem 1,

$$\frac{1}{N}\sum_\lambda (1+|\lambda|^\beta)|\tilde{b}^N(\lambda)|^2 < \infty$$

and therefore

$$\sum_{\lambda}(1+|\lambda|^{\beta})E|\tilde{g}_t^N(\lambda)|^2 = \sum_{\lambda}(1+|\lambda|^{\beta})E[k_0^N(\lambda,t)] < \infty. \tag{29}$$

Finally, from (29) it is easy to work out the convergence properties of g^N. Indeed,

$$\begin{aligned}
\lim_{N,M\to\infty} \quad & E\left[\int_0^T \int_0^1 |g_t^N(x) - g_t^N(x)|^2 \, dx \, dt\right] \\
&= \lim_{N,M\to\infty} E\left[\int_0^T \sum_{\lambda} |\tilde{g}_t^N(\lambda) - \tilde{g}_t^M(\lambda)|^2 \, dt\right] \\
&\leq \lim_{N,M\to\infty} E\left[\int_0^T \sum_{|\lambda|\leq K} |\tilde{g}_t^N(\lambda) - \tilde{g}_t^M(\lambda)|^2 \, dt\right] \\
&\qquad + \lim_{N,M\to\infty} 2E\left[\int_0^T \sum_{|\lambda|>K} (|\tilde{g}_t^N(\lambda)|^2 + |\tilde{g}_t^M(\lambda)|^2) \, dt\right] \\
&\leq \lim_{N,M\to\infty} 4E\left[\int_0^T \sum_{|\lambda|\leq K} |\langle \mu_t^N - \mu_t^M, e^{i\lambda\cdot}\rangle|^2 \, dt\right] \\
&\qquad + 4(1+K^{\beta})^{-1} \sup_N E\left[\int_0^T \sum_{\lambda} |\tilde{g}_t^N(\lambda)|^2 (1+|\lambda|^{\beta}) \, dt\right] \\
&\leq C(1+K^{\beta})^{-1} T
\end{aligned}$$

for some constant $C < \infty$ and the right hand side of this last inequality can be made smaller than any given $\varepsilon > 0$ by the choice of K. So, by the completeness of L^2, we have proved the existence of a positive random function $g_t(x)$ such that

$$\lim_{N\to\infty} E\left[\int_0^T \int_0^1 |g_t^N(x) - g_t(x)|^2 \, dx \, dt\right] = 0. \tag{30}$$

Of course, this means that for any $f \in C_b(S)$ we have

$$\begin{aligned}
\int_0^1 f(x) g_t(x) \, dx &= \lim_{N\to\infty} \int_0^1 f(x) g_t^N(x) \, dx \;=\; \lim_{N\to\infty} \langle \mu_t^N * b^N, f\rangle \\
&= \lim_{N\to\infty} \langle \mu_t^N, f * b^N\rangle \;=\; \langle \mu_t, f\rangle \;=\; \int_0^1 f(x) \, \mu_t(dx)
\end{aligned}$$

and therefore μ_t is absolutely continuous with respect to Lebesgue measure on S with derivative g_t.

Conclusion. Finally, combining (23-26), and (30), implies

$$\langle \mu_t, f\rangle - \langle \mu_0, f\rangle = \int_0^t \langle \mu_s, F(g_s(\cdot))f' + \frac{1}{2}f''\rangle\, ds \tag{31}$$

and from Proposition 3.5 of Oelschläger (1985) we know that the integral equation (31) has a unique solution μ_t absolutely continuous with respect to Lebesgue measure on S with density g_t. We note also that the solution $g_t(x) = u(x,t)$ of the Burgers equation

$$u_t + uu_x = \frac{1}{2}u_{xx}$$

with periodic boundary conditions

$$u(x,t) = u(x+1,t),$$

for all real x, and all $t > 0$, and initial condition u_0, satisfies the integral equation

$$\langle g_t(\cdot), f\rangle - \langle u_0(\cdot), f\rangle = \int_0^t \langle g_s(\cdot), \frac{1}{2}g_s(\cdot)f' + \frac{1}{2}f''\rangle\, ds$$

and from the Hopf-Cole solution (II.67) we see that

$$\|g_t\| \leq \|u_0\|$$

and therefore $g_t(x)$ satisfies (31) as well. The uniqueness result for solutions to the periodic boundary problem for the Burgers equation then completes the proof of Theorem 1. ∎

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AN ITERATIVE METHOD FOR COMPUTING ZEROS OF OPERATORS SATISFYING AUTONOMOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. We use an iteration method to approximate zeros of operators satisfying autonomous differential equations. This iteration process has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, as the inverse of the operator involved is calculated once and for all. Our local and semilocal convergence results compare favorably with earlier ones under the same computational cost.

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Key Words and Phrases. Banach spaces, Newton's method, quadratic convergence, autonomous differential equation, local/semilocal convergence.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

We use the Newton-like method:

$$x_{n+1} = x_n - F'(y_n)^{-1} F(x_n) \quad (n \geq 0) \tag{2}$$

to generate a sequence approximating x^*.

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Here $F'(x) \in L(X,Y)$ denotes the Fréchet-derivative. We are interested in the case when:

$$y_n = \lambda_n x_n + (1-\lambda_n) z_n \qquad (n \geq 0) \tag{3}$$

where,

$$\lambda_n \in [0,1], \quad (n \geq 0) \tag{4}$$

$$z_n = x^* \tag{5}$$

or

$$z_n = x_n \qquad (n \geq 0), \tag{6}$$

or other suitable choice [1]-[4].

We provide a local and a semilocal convergence analysis for method (2) which compare favorably with earlier results [4], and under the same computational cost.

2. Convergence for method (2) for z_n given by (5) and $\lambda_n = 0$ $(n \geq 0)$

We can show the following local result:

Theorem 1. *Let $F: D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume:*
there exists a solution x^* of equation

$$F(x) = 0 \text{ such that } F'(x^*)^{-1} \in L(Y,X)$$

and

$$\left\| F'(x^*)^{-1} \right\| \leq b; \tag{7}$$

$$\|F'(x) - F'(x^*)\| \leq L_0 \|x - x^*\| \quad \text{for all} \quad x \in D, \tag{8}$$

and

$$\overline{U}(x^*, r_0) = \left\{ x \in X \,\middle|\, \|x - x^*\| \leq r_0 = \tfrac{2}{bL_0} \right\} \subseteq D. \tag{9}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton-like method (2) is well defined remains in $U(x^*, r_0)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in U(x^*, r_0)$.

Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_n - x^*\| \leq \theta_0^{2^n - 1} \|x_0 - x^*\| \quad (n \geq 1), \tag{10}$$

where

$$\theta_0 = \tfrac{1}{2} b L_0 \|x_0 - x^*\|. \tag{11}$$

Proof. By (2) and $F(x^*) = 0$ we get for all $n \geq 0$:
(12)

$$x_{n+1} - x^* = -F'(x^*)^{-1} \left[\int_0^1 \left(F'(x^* + t(x_n - x^*)) - F'(x^*) \right)(x_n - x^*) \right] dt$$

from which it follows

$$\|x_{n+1} - x^*\| \leq \tfrac{1}{2} b L_0 \|x_n - x^*\|^2 \tag{13}$$

from which (10) follows.

By (9) and (11) $\theta_0 \in [0, 1)$. hence it follows from (10) that $x_n \in U(x^*, r_0)$ $(n \geq 0)$ and $\lim_{n \to \infty} x_n = x^*$ (by using induction on the integer $n \geq 0$). □

Remark 1. *Method (2) has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, since the operator $F'(x^*)^{-1}$ is computed only once. It turns out that method (2) can be used for operators F which satisfy an autonomous differential equation*

$$F'(x) = G(F(x)), \tag{14}$$

where G is a known continuous operator on Y. As $F'(x^*) = G(0)$ can be evaluated without knowing the value of x^*.

Moreover in order for us to compare Theorem 1 with earlier results, consider the condition

$$\|F'(x) - F'(y)\| \leq L \|x - y\| \quad \textit{for all} \quad x \in D \tag{15}$$

used in [4] *instead of (8). The corresponding radius of convergence is given by*

$$r_R = \frac{2}{bL}. \tag{16}$$

since

$$L_0 \leq L \tag{17}$$

holds in general we obtain

$$r_R \leq r_0. \tag{18}$$

Furthermore in case strict inequality holds in (17), so does in (18). We showed in [1] *that the ration $\frac{L}{L_0}$ can be arbitrarily large. Hence we managed to enlarge the radius of convergence for method (2) under the same computational cost as in Theorem 1 in [4, p.113].*

This observation is very important in computational mathematics since a under choice of initial guesses x_0 can be obtained.

Below we give an example of a case where strict inequality holds in (17) and (18).

Example 1. *Let* $X = Y = R$, $D = U(0,1)$ *and define* F *on* D *by*

$$F(x) = e^x - 1. \tag{19}$$

Note that (19) satisfies (14) for $T(x) = x + 1$. *Using (7), (8), (9), (15) and (16) we obtain*

$$b = 1, L_0 = e - 1, L = e, \tag{20}$$

$$r_0 = 1.163953414 \tag{21}$$

and

$$r_R = .735758882. \tag{22}$$

In order to keep the iterates inside D *we can restrict* r_0 *and choose*

$$r_0 = 1. \tag{23}$$

In any case (17) and (18) holds as a strict inequalities.

We can show the following global result:

Theorem 2. *Let* $F : X \to Y$ *be Fréchet-differentiable operator, and* G *a continuous operator from* Y *into* Y. *Assume:*

condition (14) holds;

$G(0)^{-1} \in L(Y, X)$ *so that (7) holds;*

$$F(x) \leq c \quad \text{for all } x \in X; \tag{24}$$

$$\|G(0) - G(z)\| \leq a_0 \|z\| \quad \text{for all} \quad z \in Y \tag{25}$$

and

$$h_0 = \alpha_0 bc < 1. \tag{26}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ *generated by method (2) is well defined and converges to a unique solution* x^* *of equation* $F(x) = 0$.

Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_n - x^*\| \leq \frac{h_0^n}{1 - h_0} \|x_1 - x_0\| \quad (n \geq 0). \tag{27}$$

Proof. It follows from the contraction mapping principle [2] by using (25), (26) instead of

$$\|G(v) - G(z)\| \leq a \|v - z\| \quad \text{for all} \quad v, z \in Y \tag{28}$$

and

$$h = abc < 1 \tag{29}$$

respectively in the proof of Theorem 2 in [4, p.113]. □

Remark 2. *If F' is L_0 Lipschitz continuous in a ball centered at x^*, then the convergence of method (2) will be quadratic as soon as*

$$bL_0 \|x_0 - x^*\| < 2 \tag{30}$$

holds with x_0 replaced by an iterate x_n sufficiently close to x^*.

Remark 3. *If (25) is replaced by the stronger (28), Theorem 2 reduces to Theorem 2 in* [4]. *Otherwise our Theorem is weaker than Theorem 2 in* [4] *since*

$$a_0 < a \tag{31}$$

holds in general.

We note that if (25) holds and

$$\|F(x) - F(x_0)\| \le \gamma_0 \|x - x_0\| \tag{32}$$

then

$$\|F(x)\| \le \|F(x) - F(x_0)\| + \|F(x_0)\| \le \gamma_0 \|x - x_0\| + \|F(x_0)\|. \tag{33}$$

Let $r = \|x - x_0\|$, and define

$$P(r) = a_0 b (\|F(x_0)\| + \gamma_0 r). \tag{34}$$

If $P(0) = a_0 b \|F(x_0)\| < 1$, then as in Theorem 3 in [4, p.114] inequality (26) and the contraction mapping principle we obtain the following semilocal result:

Theorem 3. *If*

$$q = (1 - a_0 b \|F(x_0)\|)^2 - 4ba_0\gamma_0 \left\|G(0)^{-1} F(x_0)\right\| \ge 0, \tag{35}$$

then a solution x^* of equation

$$F(x) \text{ exists in } U(x_0, r_1),$$

and is unique in $U(x_0, r_2)$, where

$$r_1 = \frac{1 - a_0 b \|F(x_0)\| - \sqrt{q}}{2ba_0\gamma_0} \tag{36}$$

and

$$r_2 = \frac{1 - a_0 b \|F(x_0)\|}{ba_0\gamma_0}. \tag{37}$$

Remark 4. *Theorem 3 reduces to Theorem 3 in [4, p.114] if (25) and (32) are replaced by the stronger (28) and*

$$\|F(x) - F(y)\| \le \gamma \|x - y\| \tag{38}$$

respectively. Otherwise our Theorem is weaker than Theorem 3 in [4].

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PROVING MATRIX EQUATIONS

MICHAEL DEUTCH

Abstract. Students taking an undergraduate Linear Algebra course may face problems like this one (ref[1]):
Given $A_\lambda = (\lambda - A)^{-1}$ and $A_\mu = (\mu - A)^{-1}$
then prove

$$(\lambda - \mu)A_\lambda A_\mu = A_\mu - A_\lambda \tag{1}$$

where λ and μ are scalars and A_λ, A_μ and A are invertible $n \times n$ matrices.

The purpose of the note is to present a general method for determining the truth of symbolic matrix equations where 0 or more such equations are given as true. The idea behind the method is to write the equation to be proved in terms of independent variables only, removing all the dependent variables, effectively reducing the problem to the case of 0 equations given as true. It should then be a simple matter to determine the truth of the equation to be proved, as it must be true for all values of any variable in the equation.

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Key Words and Phrases. Symbolic matrix equation, dependent/independent variable, primitive number, normal form.

The Method.

- Determine dependent and independent variables in the given equations. In the example A_λ and A_μ are dependent variables and A, λ, and μ are independent.

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- Rewrite the given equations, if necessary, to express dependent variables in terms of only independent variables, for any dependent variables which appear in the equation to prove. In the example the dependent variables A_λ and A_μ are already expressed in terms of the independent variables λ, μ, and A. So the given equations need not be rewritten.
- Substitute independent variables for dependent variables in the equation to prove. Then we will have an equation that is totally expressed in independent variables, i.e. we have transformed the problem to the case of 0 equations given. In the example equation (1) is now
  $(\lambda-\mu)(\lambda-A)^{-1}(\mu-A)^{-1}=(\mu-A)^{-1}-(\lambda-A)^{-1}$
  It must prove true for any λ, μ, and A.
- Multiply and distribute as necessary to express the equation to prove in normal form (i.e. no parentheses) as follows: if an outer term has an exponent >0 then multiply and distribute the primitives. If the exponent is <0 then multiply the equation by the positive exponent of the same term to remove the negative exponent. For example $A(B+C)^2(A-C)^{-2}$ in an equation would be reduced to normal form by first distributing the $(B+C)^2$ to $(B^2+BC+CB+C^2)(A-C)^{-2}$. Then multiply the equation from the right by $(A-C)^2$ to remove the negative exponent. Continue to multiply and distribute terms as necessary to reduce the level (i.e. number of parentheses) of the equation until the equation is in normal form.
- Cancel terms until the resulting equation is $0=0$. If the resulting equation differs from $0=0$ then the equation to prove is not true.

The Example. Equation (1) in the example problem would be reduced as follows:

- $(\lambda-\mu)(\lambda-A)^{-1}(\mu-A)^{-1}=(\mu-A)^{-1}-(\lambda-A)^{-1}$

- Multiply from right by $(\mu-A)(\lambda-A)$ to achieve
  $(\lambda-\mu)=((\mu-A)^{-1}-(\lambda-A)^{-1})(\mu-A)(\lambda-A)$

- Distribute from right to achieve
  $(\lambda-\mu)=(\lambda-A)-(\lambda-A)^{-1}(\mu-A)(\lambda-A)$

- Multiply from left by $(\lambda-A)$ to achieve
  $(\lambda-A)(\lambda-\mu)=(\lambda-A)(\lambda-A)-(\mu-A)(\lambda-A)$

- At this point the nested inverses have been removed and the terms can simply be distributed to achieve
$\lambda^2 - \lambda\mu - A\lambda + A\mu = \lambda^2 - \lambda A - A\lambda + A^2 - (\mu\lambda - \mu A - A\lambda + A^2)$
- which reduces to normal form:
$\lambda^2 - \lambda\mu - A\lambda + A\mu = \lambda^2 - \lambda A - A\lambda + A^2 - \mu\lambda + \mu A + A\lambda - A^2$
- Finally cancel terms until the equation reduces to $0 = 0$.

It seems curious that textbooks for the introductory course in linear algebra do not include this simple but handy method.

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ABSOLUTELY CONTINUOUS MEASURES AND COMPACT COMPOSITION OPERATOR ON SPACES OF CAUCHY TRANSFORMS

Y. ABU MUHANNA AND YUSUF ABU MUHANNA

ABSTRACT. The analytic self map of the unit disk **D**, φ is said to induce a composition operator C_φ from the Banach space X to the Banach Space Y if $C_\varphi(f) = f \circ \varphi \in Y$ for all $f \in X$. For $z \in \mathbf{D}$ and $\alpha > 0$ the families of weighted Cauchy transforms F_α are defined by $f(z) = \int_{\mathbf{T}} K_x^\alpha(z) d\mu(x)$ where $\mu(x)$ is complex Borel measures, x belongs to the unit circle **T** and the kernel $K_x(z) = (1 - \overline{x}z)^{-1}$. In this paper we will explore the relationship between the compactness of the composition operator C_φ acting on F_α and the complex Borel measures $\mu(x)$.

A.M.S. (MOS) Subject Classification Codes. 30E20, 30D99.
Key Words and Phrases. Compact composition operator, Absolutly continuous measures, Cauchy transforms.

1. Background

Let **T** be the unit circle and **M** be the set of all complex–valued Borel measures on **T**. For $\alpha > 0$ and $z \in \mathbf{D}$, we define the space of weighted Cauchy transforms F_α to be the family of all functions $f(z)$ such that

$$f(z) = \int_{\mathbf{T}} K_x^\alpha(z) d\mu(x) \tag{1}$$

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where the Cauchy kernel $K_x(z)$ is given by

$$K_x(z) = \frac{1}{1 - \overline{x}z}$$

and where μ in (1) varies over all measures in $\mathbf{M}$. The class F_α is a Banach space with respect to the norm

$$\|f\|_{F_\alpha} = \inf \|\mu\|_{\mathbf{M}} \tag{2}$$

where the infimum is taken over all Borel measures μ satisfying (1). $\|\mu\|$ denotes the total variation norm of μ. The family F_1 has been studied extensively in the soviet literature. The generalizations for $\alpha > 0$, were defined by T. H. MacGregor [8]. The Banach spaces F_α have been well studied in [5, 8, 3, 4]. Among the properties of F_α we list the following:

- $F_\alpha \subset F_\beta$ whenever $0 < \alpha < \beta$.
- F_α is Möbius invariant.
- $f \in F_\alpha$ if and only if $f' \in F_{1+\alpha}$ and $\|f'\|_{F_{1+\alpha}} \leq \alpha \|f\|_{F_\alpha}$.
- If $g \in F_{\alpha+1}$ then $f(z) = \int_0^z g(w)dw \in F_\alpha$ and $\|f\|_{F_\alpha} \leq \frac{2}{\alpha}\|g\|_{F_{1+\alpha}}$.

The space F_α may be identified with $\mathbf{M}/\overline{H_0^1}$ the quotient of the Banach space $\mathbf{M}$ of Borel measures by $\overline{H_0^1}$ the subspace of L^1 consisting of functions with mean value zero whose conjugate belongs the Hardy space H^1. Hence F_α is isometrically isomorphic to $\mathbf{M}/\overline{H_0^1}$. Furthermore, $\mathbf{M}$ admits a decomposition $\mathbf{M} = L^1 \oplus \mathbf{M}_s$, where $\mathbf{M}_s$ is the space of Borel measures which are singular with respect to Lebesgue measure, and $\overline{H_0^1} \subset L^1$. According to the Lebesgue decomposition theorem any $\mu \in \mathbf{M}$ can be written as $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to the Lebesgue measure and μ_s is singular with respect to the Lebesgue measure $(\mu_a \perp \mu_s)$. Furthermore the supports $S(\mu_a)$ and $S(\mu_s)$ are disjoint. Since $|x| = 1$ in $\underline{(1)}$, if we let $x = e^{it}$ then $d\mu(e^{it}) = g_x(e^{it})dt + d\mu_s(e^{it})$ where $g_x(e^{it}) \in \overline{H_0^1}$. Consequently F_α is isomorphic to $L^1/\overline{H_0^1} \oplus \mathbf{M}_s$. Hence, F_α can be written as $F_\alpha = F_{\alpha a} \oplus F_{\alpha s}$, where $F_{\alpha a}$ is isomorphic to $L^1/\overline{H_0^1}$ the closed subspace of $\mathbf{M}$ of absolutely continuous measures, and $F_{\alpha s}$ is isomorphic to $\mathbf{M}_s$ the subspace of $\mathbf{M}$ of singular measures. If $f \in F_{\alpha a}$, then the singular part is nul and the measure μ for which (1) holds is such that $d\mu(x) = d\mu(e^{it}) = g_x(e^{it})dt$ where $g_x(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\mathbf{T}$, see [1]. Hence the functions in $F_{\alpha a}$ may be written as,

$$f(z) = \int_{-\pi}^{\pi} K_x^\alpha(z) g_x(e^{it})dt$$

Furthermore if $g_x(e^{it})$ is nonnegative then

$$\|f\|_{F_\alpha} = \inf_M \|\mu\| = \left\|g_x(e^{it})\right\|_{L^1}$$

Remark: For simplicity, we will adopt the following notation throughout the article. We will reserve μ for the Borel measures of $\mathbf{M}$, and since in (1) $|x| = 1$, we can write $x = e^{it}$ where $t \in [-\pi, \pi)$. We will reserve dt for the normalized Lebesgue of the unit circle $\mathbf{T}$, and $d\sigma$ for the singular part of $d\mu$. Hence instead of writing $d\mu(x) = d\mu(e^{it}) = d\mu_a(e^{it}) + d\mu_s(e^{it}) = g_x(e^{it})dt + d\mu_s(e^{it})$ we may simply write $d\mu(x) = g_x dt + d\sigma(t)$.

2. Introduction

If X and Y are Banach spaces, and L is a linear operator from X to Y, we say that L is bounded if there exists a positive constant A such that $\|L(f)\|_Y \leq A \|f\|_X$ for all f in X. We denote by $C(X, Y)$ the set of all bounded linear operators from X to Y. If $L \in C(X, Y)$, we say that L is a compact operator from X to Y if the image of every bounded set of X is relatively compact (i.e. has compact closure) in Y. Equivalently a linear operator L is a compact operator from X to Y if and only if for every bounded sequence $\{f_n\}$ of X, $\{L(f_n)\}$ has a convergent subsequence in Y. We will denote by $K(X, Y)$ the subset of $C(X, Y)$ of compact linear operators from X into Y.

Let $H(\mathbf{D})$ denote the set of all analytic functions on the unit disk $\mathbf{D}$ and map $\mathbf{D}$ into $\mathbf{D}$. If X and Y are Banach spaces of functions on the unit disk $\mathbf{D}$, we say that $\varphi \in H(\mathbf{D})$ induces a bounded composition operator $C_\varphi(f) = f(\varphi)$ from X to Y, if $C_\varphi \in C(X, Y)$ or equivalently $C_\varphi(X) \subseteq Y$ and there exists a positive constant A such that for all $f \in X$ and $\|C_\varphi(f)\|_Y \leq A \|f\|_X$. In case $X = Y$ then we say φ induces a composition operator C_φ on X. If $f \in X$, then $C_\varphi(f) = f(\varphi) \in X$. Similarly, we say that $\varphi \in H(\mathbf{D})$ induces a compact composition operator if $C_\varphi \in K(X, Y)$.

A fundamental problem that has been studied concerning composition operators is to relate function theoretic properties of φ to operator theoretic properties of the restriction of C_φ to various Banach spaces of analytic functions. However since the spaces of Cauchy transforms are defined in terms of Borel measures, it seems natural to investigate the relation between the behavior of the composition operator and the measure. The work in this article was motivated by the work of J. Cima and A. Matheson in [1], who showed that C_φ is compact on F_1 if and only if $C_\varphi(F_1) \subset F_{1a}$. In our work we will generalize this result for $\alpha > 1$.

Now if $C_\varphi \in C(F_\alpha, F_\alpha)$ then $C_\varphi(f) = (f \circ \varphi) = f(\varphi) \in F_\alpha$ for all $f \in F_\alpha$ and there exists a positive constant A such that

$$\|C_\varphi(f)\|_{F_\alpha} = \|f(\varphi)\|_{F_\alpha} = \|[\mu]\| \leq A \|f\|_{F_\alpha}$$

Since F_α can be identified with the quotient space $\mathbf{M}/\overline{H_0^1}$ we can view C_φ as a map:

$$C_\varphi : \mathbf{M}/\overline{H_0^1} \to \mathbf{M}/\overline{H_0^1}$$
$$f \mapsto f(\varphi)$$

The equivalence class of a complex measure μ will be written as:

$$[\mu] = \mu + \overline{H_0^1} = \{\mu + \overline{h} : h \in H_0^1\}$$

and

$$\|[\mu]\| = \inf_h \left\|\mu + \overline{h}\right\|$$

The space $C(F_\alpha, F_\alpha)$ has been studied by [6] where the author showed that:

(1) If $\alpha \geq 1$, then $C_\varphi \in C(F_\alpha, F_\alpha)$ for any analytic self map φ of the unit disc.
(2) $C_\varphi \in C(F_\alpha, F_\alpha)$ if and only if $\{K_x^\alpha(\varphi) : |x| = 1\}$ is a norm bounded subset of F_α.
(3) If $C_\varphi \in C(F_\alpha, F_\alpha)$ then $C_\varphi \in C(F_\beta, F_\beta)$ for $0 < \alpha < \beta$.
(4) If $C_\varphi \in C(F_\alpha, F_\alpha)$ then the operator $\varphi' C_\varphi \in C(F_{\alpha+1}, F_{\alpha+1})$.

In this article we will investigate necessary and sufficient conditions for C_φ to be compact on F_α for $\alpha \geq 1$ if and only if $C_\varphi(F_1) \subset F_{1a}$. Since F_α is Mobius invariant, then there is no loss of generality in assuming that $\varphi(0) = 0$.

3. Compactness and absolutely continuous measures

In this section we will show that compactness of the composition operator C_φ on F_α is strongly tied with the absolute continuity of the measure that supports it. First we state this Lemma due to [7].

Lemma 1. *If $0 < \alpha < \beta$ then $F_\alpha \subset F_{\beta a}$ and the inclusion map is a compact operator of norm one.*

Next we use the above result and the known fact that $H^\infty \subset F_{1a}$ to show that bounded function of F_α belong to $F_{\alpha a}$.

Proposition 1. $H^\infty \cap F_\alpha \subset F_{\alpha a}$ *for* $\alpha \geq 1$.

Proof. Let $f \in H^\infty \cap F_\alpha$, then using the previous lemma we get that for $\alpha \geq 1$ and any $z \in \mathbf{D}$, $f(z) \in H^\infty \subset H^1 \subset F_{1a} \subseteq F_{\alpha a}$, then $f(z) \in F_{\alpha a}$ for all $\alpha \geq 1$, which gives us the desired result. ☐

Theorem 1. *For a holomorphic self-map φ of the unit disc* $\mathbf{D}$ *and* $\alpha \geq 1$, *if* C_φ *is compact on* F_α *then* $(C_\varphi \circ K_x^\alpha)(z) \in F_{\alpha a}$ *and*

$$(C_\varphi \circ K_x^\alpha)(z) = \int_{-\pi}^{\pi} g_x\left(e^{it}\right) K_x^\alpha(z) dt \tag{3}$$

where $\|g_x\|_{L^1} \leq a < \infty$, g_x *is nonnegative and* L^1 *continuous function of* x .

Proof. Assume that C_φ is compact and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions such that

$$f_j(z) = K_x^\alpha(\rho_j z) = \frac{1}{(1 - \rho_j \overline{x} z)^\alpha}$$

where $0 < \rho_j < 1$ and $\lim_{j\to\infty} \rho_j = 1$. Then it is known from [4] that $f_j(z) \in F_\alpha$ for every j, and $\|f_j(z)\|_{F_\alpha} = 1$. Furthermore there exist $\mu_j \in \mathbf{M}$, such that $\|\mu_j\| = 1$, $d\mu_j >> 0$ and

$$\begin{aligned} f_j(z) &= \frac{1}{(1 - \rho_j \overline{x} z)^\alpha} \\ &= \int_{\mathbf{T}} K_x^\alpha(z)\, d\mu_j(x) \\ &= \int_{\mathbf{T}} \frac{1}{(1 - \overline{x} z)^\alpha} d\mu_j(x). \end{aligned}$$

Since C_φ is compact on F_α then $(C_\varphi \circ f_j) \in F_\alpha$ and $\|C_\varphi(f_j)\| \leq \|C_\varphi\| \, \|f_j\|_{F_\alpha} = \|C_\varphi\|$ for all j. Furthermore $C_\varphi \circ f_j \in H^\infty$, thus using the previous result, we get that $(C_\varphi \circ f_j) \in H^\infty \cap F_\alpha \subset F_{\alpha a}$ for every j. Therefore there exist L^1 nonnegative function g_x^j such that $d\mu_j(x) = g_x^j dt$, $\|g_x^j\|_{L^1} \leq \|C_\varphi\|$ and

$$\begin{aligned} (f_j \circ \varphi)(z) &= (K_x^\alpha \circ \varphi)(\rho_j z)) \\ &= \int_{-\pi}^{\pi} g_x^j\left(e^{it}\right) K_x^\alpha(\rho_j z)\, dt. \end{aligned}$$

Now because $F_{\alpha a}$ is closed and C_φ is compact, the sequence $\{f_j \circ \varphi\}_{j=1}^\infty$ has a convergent subsequence $\{f_{j_k} \circ \varphi\}$ that converges to $(K_x^\alpha \circ \varphi)(z) \in$

$F_{\alpha a}$. Therefore,

$$\begin{aligned}\lim_{k\to\infty}\left(f_{j_k}\circ\varphi\right)(z) &= \lim_{k\to\infty}\left(K_x^{\alpha}\circ\varphi\right)\left(\rho_{j_k}z\right)\\ &= \lim_{k\to\infty}\int_0^{2\pi} g_x^{j_k}\left(e^{it}\right)K_x^{\alpha}\left(\rho_{j_k}z\right)dt\\ &= \int_0^{2\pi} g_x\left(e^{it}\right)K_x^{\alpha}(z)\,dt\\ = \left(K_x^{\alpha}\circ\varphi\right)(z) &= \frac{1}{(1-\overline{x}\varphi(z))^{\alpha}}\in F_{\alpha a}\end{aligned}$$

where the function g_x is an L^1 nonnegative continuous function of x, and $\|g_x\|_{L^1}\leq\|C_\varphi\|$. For the continuity of g_x in L^1 with respect to x where $\|x\|=1$, we take a sequence $\{x_k\}$, such that $\|x_k\|=1$ and $x_k\to x$. Now since C_φ is compact then

$$\lim_{k\to\infty}\left(K_\alpha\circ\varphi\right)\left(\overline{x}_k z\right)=\left(K_\alpha\circ\varphi\right)\left(\overline{x}z\right)$$

which concludes the proof. □

Corollary 1. *Let $g_x(e^{it})$ be as in the last theorem then the operator $\int g_x\left(e^{it}\right)h(x)dx=u(e^{it})\in\overline{H_0^1}$, for $h(x)\in\overline{H_0^1}$ is bounded on $\overline{H_0^1}$.*

Proof. For the operator to be well defined, $\int\dfrac{h(x)dx}{(1-\overline{x}\varphi(z))^{\alpha}}=0$ for all $h(x)\in\overline{H_0^1}$. Hence, $\int g_x\left(e^{it}\right)h(x)dx=u(e^{it})\in\overline{H_0^1}$. □

The following lemmas are needed to prove the converse of Theorem 1.

Lemma 2. *Suppose $g_x\left(e^{it}\right)$ is a nonnegative L^1continuous function of x and let $\{\mu_n\}$ be a sequence of nonnegative Borel measures that are weak* convergent to μ. Define $w_n(t)=\int_{\mathbf{T}}g_x\left(e^{it}\right)d\mu_n(x)$ and $w(t)=\int_{\mathbf{T}}g_x\left(e^{it}\right)d\mu(x)$, then $\|w_n-w\|_{L^1}\longrightarrow 0$.*

Proof. Let

$$\begin{aligned}g_x(z) &= \int\operatorname{Re}\frac{(1+e^{-it}z)}{(1-e^{-it}z)}g_x\left(e^{it}\right)d(t)\ ,\\ w_n(z) &= \int g_x(z)\,d\mu_n(x)\ \text{ and}\\ w(z) &= \int g_x(z)\,d\mu_n(x)\end{aligned}$$

where$|z| < 1$. Notice that all functions are positive and harmonic in $\mathbf{D}$ and that the radial limits of $w_n(z)$ and $w(z)$ are $w_n(t)$ and $w(t)$ respectively. Then, for $|z| \leq \rho < 1$,

$$|g_x(z) - g_y(z)| \leq \frac{1}{1-\rho} \left\| g_x\left(e^{it}\right) - g_y\left(e^{it}\right) \right\|_{L^1}$$

Then the continuity condition implies that $g_x(z)$ is uniformly continuous in x for all $|z| \leq \rho$. Hence, weak star convergence, implies that $w_n(z) \to w(z)$ uniformly on $|z| \leq \rho$ and consequently the convergence is locally uniformly on $\mathbf{D}$. In addition, we have $\|w_n(\rho e^{it})\|_{L^1} \to \|w(\rho e^{it})\|_{L^1}$. Hence we conclude that

$$\left\| w_n(\rho e^{it}) - w(\rho e^{it}) \right\|_{L^1} \longrightarrow 0 \text{ as } n \to \infty.$$

Now using Fatou's Lemma we conclude that

$$\left\| w_n(e^{it}) - w(e^{it}) \right\|_{L^1} \longrightarrow 0.$$

□

Lemma 3. *Let $g_x(e^{it})$ be a nonnegative L^1 continuous function of x such that $\|g_x\|_{L^1} \leq a < \infty$ and $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$. If $f(z) = \int \frac{1}{(1-\overline{x}z)^\alpha} d\mu(x)$, let L be the operator given by*

$$L[f(z)] = \iint \frac{g_x(e^{it})}{(1-e^{-it}z)^\alpha} dt d\mu(x)$$

then L is compact operator on $F_\alpha, \alpha \geq 1$.

Proof. First note that the condition that $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$ implies that the L operator is a well defined function on $\mathbf{F}_\alpha$. Let $\{f_n(z)\}$ be a bounded sequence in F_α and let $\{\mu_n\}$ be the corresponding norm bounded sequence of measures in $\mathbf{M}$. Since every norm bounded sequence of measures in $\mathbf{M}$ has a weak star convergent subsequence, let $\{\mu_n\}$ be such subsequence that is convergent to $\mu \in \mathbf{M}$. We want to show that $\{L(f_n)\}$ has a convergent subsequence in F_α.

First, let us assume that $d\mu_n(x) >> 0$ for all n, and let $w_n(t) = \int g_x(e^{it}) d\mu_n(x)$ and $w(t) = \int g_x(e^{it}) d\mu(x)$,then we know from the previous lemma that $w_n(t), w(t) \in L^1$ for all n, and $w_n(t) \to w(t)$ in L^1. Now since $g_x(e^{it})$ is a nonnegative continuous function in x and

$\{\mu_n\}$ is weak star convergent to μ, then

$$L(f_n(z)) = \iint \frac{g_x(e^{it})\, d(t)}{(1-e^{-it}z)^{\alpha}} d\mu_n(x) = \int \frac{w_n(t)}{(1-e^{-it}z)^{\alpha}} dt$$

$$L(f(z)) = \iint \frac{g_x(e^{it})\, d(t)}{(1-e^{-it}z)^{\alpha}} d\mu(x) = \int \frac{w(t)}{(1-e^{-it}z)^{\alpha}} dt$$

Furthermore because $w_n(t)$ is nonnegative then

$$\|L(f_n)\|_{F_\alpha} = \|w_n\|_{L^1}$$
$$\|L(f)\|_{F_\alpha} = \|w\|_{L^1}$$

Now since$\|w_n - w\|_{L^1} \to 0$ then $\|L(f_n) - L(f)\|_{F_\alpha} \to 0$ which shows that $\{L(f_n)\}$ has convergent subsequence in F_α and thus L is a compact operator for the case where μ is a positive measure.
In the case where μ is complex measure we write $d\mu_n(x) = (d\mu_n^1(x) - d\mu_n^2(x)) + i(d\mu_n^3(x) - d\mu_n^4(x))$,
where each $d\mu_n^j(x) >> 0$ and define $w_n^j(t) = \int g_x(e^{it})\, d\mu_n^j(x)$ then $w_n(t) = \int g_x(e^{it})\, d\mu_n(x) = (w_n^1(t) - w_n^2(t)) + i\,(w_n^3(t) - w_n^4(t))$.
Using an argument similar to the one above we get that $w_n^j(t)$, $w^j(t) \in L^1$, and $\|w_n^j - w^j\|_{L^1} \longrightarrow 0$. Consequently, $\|w_n - w\|_{L^1} \longrightarrow 0$, where $w(t) = (w^1(t) - w^2(t)) + i\,(w^3(t) - w^4(t)) = \int g_x(e^{it})\, d\mu(x)$.
Hence, $\|L(f_n) - L(f)\|_{F_\alpha} \le \|w_n - w\|_{L^1} \longrightarrow 0$.
Finally, we conclude that the operator is compact. □

The following is the converse of Theorem 1.

Theorem 2. *For a holomorphic self-map φ of the unit disc $\mathbf{D}$, if*

$$\frac{1}{(1-\overline{x}\varphi(z))^{\alpha}} = \int \frac{g_x(e^{it})}{(1-e^{-it}z)^{\alpha}} dt$$

where $g_x \in L^1$, nonnegative, $\|g_x\|_{L^1} \le a < \infty$ for all $x \in \mathbf{T}$ and g_x is an L^1continuous function of x , then C_φ is compact on F_α.

Proof. We want to show that C_φ is compact on F_α. Let $f(z) \in F_\alpha$ then there exists a measure μ in $\mathbf{M}$ such that for every z in D

$$f(z) = \int \frac{1}{(1-\overline{x}z)^{\alpha}} d\mu(x)$$

Using the assumption of the theorem we get that

$$(f \circ \varphi)(z) = \int \frac{1}{(1-\overline{x}\varphi(z))^{\alpha}} d\mu_n(x) = \iint \frac{g_x(e^{it})}{(1-e^{-it}z)^{\alpha}} dt d\mu_n(x)$$

which by the previous lemma was shown to be compact on F_α. □

Now we give some examples:

Corollary 2. *Let* $\varphi \in H(\mathbf{D})$, *with* $\|\varphi\|_\infty < 1$. *Then* C_φ *is compact on* F_α, $\alpha \geq 1$.

Proof. $(C_\varphi \circ K_x^\alpha)(z) = \dfrac{1}{(1-\overline{x}\varphi(z))^\alpha} \in H^\infty \cap F_\alpha \subset F_{\alpha a}$ and is subordinate to $\dfrac{1}{(1-z)^\alpha}$, hence

$$(C_\varphi \circ K_x^\alpha)(z) = \int K_x^\alpha(z) g_x\left(e^{it}\right) dt$$

with $g_x\left(e^{it}\right) \geq 0$ and since $1 = (C_\varphi \circ K_x^\alpha)(0) = \displaystyle\int g_x\left(e^{it}\right) dt$ we get that $\|g_x\left(e^{it}\right)\|_1 = 1$. □

Remark 1. *In fact one can show that* C_φ, *as in the above corollary, is compact from* F_α, $\alpha \geq 1$ *into* F_1. *In other words a contraction.*

Corollary 3. *If* C_φ *is compact on* F_α, $\alpha \geq 1$ *and* $\lim_{r\to 1}\left|\varphi(re^{i\theta})\right| = 1$ *then* $\left|\dfrac{1}{\varphi'(e^{i\theta})}\right| = 0$.

Proof. If C_φ is compact then

$$(C_\varphi \circ K_x^\alpha)(z) = \int K_x^\alpha(z) g_x\left(e^{it}\right) dt$$

Hence, if $z = e^{i\theta}$ and $\varphi(e^{i\theta}) = x$ then

$$\lim_{r\to 1}\frac{(e^{i\theta} - re^{i\theta})^\alpha}{(1-\overline{x}\varphi(re^{i\theta}))^\alpha} = 0.$$

□

Corollary 4. *If* $C_\varphi \in K(F_\alpha, F_\alpha)$ *for* $\alpha \geq 1$, *then* C_φ *is contraction.*

4. Miscellaneous Results

We first start by giving another characterization of compactness on F_α.

Lemma 4. *Let* $\varphi \in C(F_\alpha, F_\alpha), \alpha > 0$ *then* $\varphi \in K(F_\alpha, F_\alpha)$ *if and only if for any bounded sequence* (f_n) *in* F_α *with* $f_n \to 0$ *uniformly on compact subsets of* $\mathbf{D}$ *as* $n \to \infty$, $\|C_\varphi(f_n)\|_{F_\alpha} \to 0$ *as* $n \to \infty$.

Proof. Suppose $C_\varphi \in K(F_\alpha, F_\alpha)$ and let (f_n) be a bounded sequence (f_n) in F_α with $\lim_{n\to\infty} f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. If the conclusion is false then there exists an $\epsilon > 0$ and a subsequence $n_1 < n_2 < n_3 < \cdots$ such that

$$\left\|C_\varphi(f_{n_j})\right\|_{F_\alpha} \geq \epsilon, \text{ for all } j = 1, 2, 3, \ldots$$

Since (f_n) is bounded and C_φ is compact, one can find a another subsequence $n_{j1} < n_{j2} < n_{j3} < \cdots$ and f in F_α such that

$$\lim_{k\to\infty} \left\| C_\varphi(f_{n_{j_k}}) - f \right\|_{F_\alpha} = 0$$

Since point functional evaluation are continuous in F_α then for any $z \in \mathbf{D}$ there exist $A > 0$ such that

$$\left| \left(C_\varphi(f_{n_{j_k}}) - f \right)(z) \right| \le A \left\| C_\varphi(f_{n_{j_k}}) - f \right\|_{F_\alpha} \to 0 \text{ as } k \to \infty$$

Hence

$$\lim_{k\to\infty} \left[C_\varphi(f_{n_{j_k}}) - f \right] \to 0$$

uniformly on compact subsets of $\mathbf{D}$. Moreover since $f_{n_{j_k}} \to 0$ uniformly on compact subsets of $\mathbf{D}$, then $f = 0$ i.e. $C_\varphi(f_{n_{j_k}}) \to 0$ on compact subsets of F_α. Hence

$$\lim_{k\to\infty} \left\| C_\varphi(f_{n_{j_k}}) \right\|_{F_\alpha} = 0$$

which contradicts our assumption. Thus we must have

$$\lim_{n\to\infty} \|C_\varphi(f_n)\|_{F_\alpha} = 0.$$

Conversely, let (f_n) be a bounded sequence in the closed unit ball of F_α. We want to show that $C_\varphi(f_n)$ has a norm convergent subsequence. The closed unit ball of F_α is compact subset of F_α in the topology of uniform convergence on compact subsets of $\mathbf{D}$. Therefore there is a subsequence (f_{n_k}) such that

$$f_{n_k} \to f$$

uniformly on compact subsets of D. Hence by hypothesis

$$\|C_\varphi(f_{n_k}) - C_\varphi(f)\|_{F_\alpha} \to 0 \text{ as } k \to \infty$$

which completes the proof. □

Proposition 2. *If $C_\varphi \in C(F_\alpha, F_\alpha)$ then $C_\varphi \in K(F_\alpha, F_\beta)$ for all $\beta > \alpha > 0$.*

Proof. Let (f_n) be a bounded sequence in the closed unit ball of F_α. Then $(f_n \circ \varphi)$ is bounded in F_α and since the inclusion map $i : F_\alpha \to F_{\beta a}$ is compact, $(f_n \circ \varphi)$ has a convergent subsequence in F_β. □

Proposition 3. *$C_\varphi(f) = (f \circ \varphi)$ is compact on F_α if and only if the operator $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is compact on $F_{\alpha+1}$.*

Proof. Suppose that $C_\varphi(f) = (f \circ \varphi)$ is compact on F_α. It is known from [6] that $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is bounded on $F_{\alpha+1}$. Let (g_n) be a bounded sequence in $F_{\alpha+1}$ with $g_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. We want to show that $\lim_{n\to\infty} \|\varphi'(g_n \circ \varphi)\|_{F_{\alpha+1}} = 0$. Let (f_n) be the sequence defined by $f_n(z) = \int_0^z g_n(w)dw$. Then $f_n \in F_\alpha$ and $\|f_n\|_{F_\alpha} \le \frac{2}{\alpha} \|g_n\|_{F_{\alpha+1}}$, thus (f_n) is a bounded sequence in F_α. Furthermore, using the Lebesgue dominated convergence theorem we get that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Thus

$$\begin{aligned} \|\varphi'(g_n \circ \varphi)\|_{F_{\alpha+1}} &= \|\varphi'(f_n' \circ \varphi)\|_{F_{\alpha+1}} \\ &= \|(f_n \circ \varphi)'\|_{F_{\alpha+1}} \\ &\le \alpha \|(f_n \circ \varphi)\|_{F_\alpha} \to 0 \text{ as } n \to \infty \end{aligned}$$

which shows that $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is compact on $F_{\alpha+1}$.
Conversely, assume that $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is compact on $F_{\alpha+1}$. Then in particular $\varphi' C_\varphi(f') = \varphi'(f' \circ \varphi) = (f \circ \varphi)'$ is a compact for every $f \in F_\alpha$. Now since $\|(f \circ \varphi)\|_{F_\alpha} \le \frac{2}{\alpha} \|(f \circ \varphi)'\|_{F_{\alpha+1}}$. Let (f_n) be a bounded sequence in F_α with $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. We want to show that $\lim_{n\to\infty} \|(f_n \circ \varphi)\|_{F_\alpha} = 0$. Since any bounded sequence of F_α is also a bounded sequence of $F_{\alpha+1}$, then $\|(f_n \circ \varphi)\|_{F_\alpha} \le \frac{2}{\alpha} \|(f_n \circ \varphi)'\|_{F_{\alpha+1}} \to 0$ as $n \to \infty$ and the proof is complete. □

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