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NON-ABELIAN COHOMOLOGY IN A TOPOS

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The following document was accepted as an AMS Memoir but was never published as I will now explain. When it was being typed in the final form for publication (before the day of TEX!) the secretary, who had never used the mathematical electronic text then required lost completely over one half of the manuscript. I had another student's thesis which needed typing and did not require the electronic text. I had her drop the paper and do the thesis, planning to come back to the paper at a later time. Much later I finally learned to type using TEX and planned to come back to the paper. Unfortunately, a stroke prevented my ever completing it myself. Recently, a former student of mine, Mohammed Alsani, an expert in TEX, offered to type the long manuscript and recently did so. The resulting paper, which I have left unchanged from its original form, except for minor changes made thanks to Mike Barr to make it compatible with *TAC*, is being presented here in the hope that it may still find some use in the mathematical community. The notion of morphism used here, which Grothendieck liked a lot, and its relation with that of Grothendieck, (see J. Giraud, Cohomologie non-Abélienne, Lect. Notes Math. Berlin-New York-Heidelberg: Springer, 1971), was explained in K.-H.Ulbrich, On the Correspondence between Gerbes and Bouquets, Math. Proc. Cam. Phil. Soc. (1990), Vol. 108, No. 1, pp 1–5. (Online 24 Oct 2008.)

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Introduction

If $\mathcal{S}\colon 0 \longrightarrow A \overset{u}{\hookrightarrow} B \xrightarrow{v} C \longrightarrow 0$ is a short exact sequence of abelian group objects of a topos $\mathbb{E}$, then it is well known that the global section functor $\Gamma(-) = \mathrm{Hom}_{\mathbb{E}}(\mathbb{1}, -)$ when applied to the sequence $\mathcal{S}$ yields, in general, only an exact sequence of ordinary abelian groups of the form

$$\mathbb{1} \longrightarrow \Gamma(A) \hookrightarrow \Gamma(B) \longrightarrow \Gamma(C) \ ,$$

since any global section of C can, in general, only be *locally* lifted past the epimorphism v. But since the sequence consists of abelian groups, it is a standard fact of the homological algebra of the abelian category $\mathrm{Grab}(\mathbb{E})$ that the deviation from exactness of Γ can be measured by the abelian group valued functor $H^1(-) = \mathrm{Ext}^1(\mathbb{Z}_{\mathbb{E}}, -)$ taken in $\mathrm{Grab}(\mathbb{E})$ since $\Gamma(C) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Grab}(\mathbb{E})}(\mathbb{Z}_{\mathbb{E}}, C)$ and "pull-back" along $s\colon \mathbb{Z}_{\mathbb{E}} \longrightarrow C$ provides a group homomorphism $\partial^1\colon \Gamma(C) \longrightarrow H^1(A)$ such that the extended sequence

$$0 \longrightarrow \Gamma(A) \hookrightarrow \Gamma(B) \longrightarrow \Gamma(C) \longrightarrow H^1(A) \longrightarrow H^1(B) \longrightarrow H^1(C)$$

is exact. Similarly, Yoneda splicing provides a homomorphism

$$\partial^2\colon H^1(C) \longrightarrow H^2(A) \ (= \mathrm{Ext}^2(\mathbb{Z}_{\mathbb{E}}, A))$$

which measures the deviation from exactness of $H^1(-)$ applied to the original sequence, and the same process may be continued with the definition of $H^n(-) = \mathrm{Ext}^n_{\mathrm{Grab}(\mathbb{E})}(\mathbb{Z}_{\mathbb{E}}, -)$. Moreover, if $\mathbb{E}$ is a Grothendieck topos, all of the groups in question are small since $H^n(-)$ may be computed by injective resolutions as the right derived functors of Γ, $H^n(-) \cong R^n\Gamma(\ )$. Given the fundamental nature of the functor Γ, the groups $H^n(A)$ are called *the cohomology groups of topos* $\mathbb{E}$ *with coefficients in* A, and for any object X in $\mathbb{E}$, the same process applied to the topos $\mathbb{E}/X$ yields the corresponding cohomology groups of the object X.

All these foregoing facts, however, depend heavily on the abelian nature of the given short exact sequence of groups and the question immediately poses itself of what, if anything, can be done if the original sequence does not consist of abelian groups but, for instance, consists of non-abelian groups, or just of a group, a subgroup, and the homogeneous space associated with the subgroup, or is even reduced to the orbit space under a principal group action? Primarily through the work of GROTHENDIECK (1953) and FRENKEL (1957), the answer in each of these cases was shown to be found through the use of the classical observation from fiber bundle theory that for an abelian coefficient group π, the group $H^1(\pi)$ was isomorphic to the group of isomorphism classes of "principal homogeneous spaces" of the topos on which the group π acted and that except for the absence of a group structure, by taking the *pointed set of isomorphism class of* *such principal homogeneous spaces as the definition* of $H^1(\pi)$ (for a non abelian π) one could recover much of what was possible in the abelian case. For instance, $H^1(\pi)$ is still functorial in π, is always supplied with a coboundary map $\partial^{\mathbb{1}} \colon \Gamma(C) \longrightarrow H^1(A)$ and in the case of an exact sequence of groups gives rise to an exact sequence

$$\begin{tikzcd}
\mathbb{1} \arrow[r] & \Gamma(A) \arrow[r, hook] & \Gamma(B) \arrow[r] & \Gamma(C) \arrow[dll, rounded corners, "\partial^{\mathbb{1}}"'] \\
& H^1(A) \arrow[r] & H^1(B) \arrow[r] & H^1(C)
\end{tikzcd}$$

of groups and pointed sets (along with a technique for recovering the information on the equivalence relations associated with these maps which would normally be lost in such a sequence in the absence of the group structures).

In and of themselves, the sets $H^1(\pi)$ are of considerable interest because of their ability to provide classification of objects which are locally isomorphic to objects of a given form. In outline this occurs as follows: one is given a fixed object on which π operates and a representative principal homogeneous space from $H^1(\pi)$; a simple construction is available which uses the principal space to "twist" the fixed object into a new one which is locally isomorphic to the original, and each isomorphism class of such objects is obtained in this fashion by an essentially uniquely determined principal homogeneous space. [Thus for example given any object T in a topos $H^1(\underline{\mathrm{Aut}}(T))$ classifies objects of $\mathbb{E}$ which are locally isomorphic to T, $H^1(G\ell_n(\Lambda))$ classifies isomorphism class of Λ-modules which are locally free of rank n . etc.] For this reason the (right-) principal homogeneous spaces under π of $\mathbb{E}$ which represent elements of $H^1(\pi)$ are called the *π-torsors of the topos* and we will follow this same terminology.

Given a homomorphism $v \colon B \longrightarrow C$, the pointed mapping $H^1(v) \colon H^1(B) \longrightarrow H^1(C)$ which establishes the functoriality of H^1 is obtained by a particular case of this just mentioned twisting construction which in fact defines a functor ($T \longmapsto v_T$) from the *groupoid* of B-torsors into the groupoid of C-torsors. A C-torsor so obtained from a B-torsor is said to be obtained by "*extending the structural group of the original torsor along v*". The problem of characterizing those C-torsors which can be obtained by extension of the structure group along an epimorphism $v \colon B \longrightarrow C$ or, more precisely, *given a C-torsor, to find an "obstruction" to its being "lifted" to a B-torsor*, thus becomes the fundamental problem to be solved for the continuation of the exact sequence to dimension 2.

However, unless the kernel A of the epimorphism is a central sub-group (in which case the abelian group $H^2(A)$ provides the solution) this problem appears considerably more difficult than any of those yet encountered in this "boot strap" approach. For instance, even if A is abelian, $H^2(A)$ does not work unless A is central and even then one still has the problem of a satisfactory definition of an H^2 for B and C.

In GIRAUD (1971), Giraud gave an extensive development of a 2-dimensional non-abelian cohomology theory devised by himself and Grothendieck intended to solve this problem for arbitrary topoi much as DEDECKER (1960, 1963) had been able to do in case of sheaves over a paracompact space.

The approach taken by Giraud was based on the following observation of Grothendieck: The "obstruction" to a "lifting" of any given C-torsor E to a torsor under B is "already found": Consider, for any C-torsor E, the following fibered category: for any object X in the topos let $R_X(E)$ be that category whose objects are the "local liftings of E to B", i.e. ordered pairs (T, α) consisting of a torsor T in $\mathrm{TORS}(\mathbb{E}/X; B|X)$ together with an isomorphism $\alpha\colon v_T \xrightarrow{\sim} E|X$, where $E|X$ and $B|X$ are the corresponding pull-backs of E and B in $\mathbb{E}/X$. With the natural definition of morphism, $R_X(E)$ becomes a groupoid for each X, and a pseudofunctor $R(E)$ on $\mathbb{E}$ by pull-back along any arrow $f\colon X \longrightarrow Y$. $R(E)$ is now said to be trivial provided $R_1(E) \neq \emptyset$, i.e., here if a global lifting is possible and $E \mapsto R(E)$ defines the desired obstruction. In the axiomatic version, such a fibered category was called by Giraud a *gerbe* (I5.0) but in order to recover some linkage with the coefficient group Giraud was forced to introduce the notion of a "*tie*" or "band" (fr. *lien*) (I3.1) which functions in the place of the coefficient group. Each gerbe has associated with it a tie and Giraud defined his $H^2_{\mathrm{GIR}}(L)$ as the set of cartesian equivalence classes of those gerbes of $\mathbb{E}$ which have tie L. The resulting H^2 is not, in general, functorial and gives rise only partially to the full 9-term exact sequence of pointed sets originally desired.

In addition to this just mentioned "snag" in the definition of the desired exact sequence, the difficulties of this approach are well known, not the least formidable of them being the extensive categorical background required for a full comprehension of the initial definitions.

This paper and its sequel attempt to ameliorate a number of the difficulties of this approach by replacing the externally defined gerbes of the Giraud theory with certain very simply defined internal objects of the topos which we will call the *bouquets* of $\mathbb{E}$. In analogy with the above cited example of Grothendieck we can motivate their presence in the theory as follows: For a given C-torsor E, in the place of the "gerbe of liftings of E to B", there is a much simpler object which we may consider: the B-object which we obtain by restricting the (principal) action of C on E to B via the epimorphism $v\colon B \longrightarrow C$. The resulting action is, of course, not principal so that E does not become a B-torsor. However, if we include the projection of $E \times B$ onto E along with the action map $\alpha|v\colon E \times B \longrightarrow E$ as parts of the structure, the resulting system forms the source (S) and target (T) arrows of an internal groupoid in $\mathbb{E}$

$$\begin{tikzcd} E \times B \arrow[r, shift left, "\alpha|v"] \arrow[r, shift right, "\mathrm{pr}"'] & E \end{tikzcd}$$

whose "objects of objects" is E and whose "object of arrows" is $E \times B$. Since E was a C-torsor and $v\colon B \longrightarrow C$ was an epimorphism, the resulting internal groupoid enjoys two essential properties

(a) it is (internally) *non empty*, i.e. the canonical $E \longrightarrow \mathbf{1}$ is an epimorphism, and

(b) it is (internally) *connected*, i.e., the canonical map

$$\langle T, S\rangle = \langle \mathrm{pr}, \alpha|v\rangle \colon E \times B \longrightarrow E \times E$$

is an epimorphism.

We will call any groupoid object of topos $\mathbb{E}$ which enjoys properties (a) and (b) above a *bouquet* of $\mathbb{E}$ with the above example called the 2-*coboundary bouquet* $\partial^2(E)$ *of the torsor* E. As we will show, it carries all of the obstruction information that the gerb of liftings of E does. Indeed, if we take the obvious generalization of principal homogeneous group actions to groupoid actions to define torsors under a groupoid (15.6), then we have an equivalence of categories

$$R_X(E) \xrightarrow{\approx} \underline{\underline{\mathrm{TORS}}}(X; \partial^2(E))\ .$$

Every bouquet $\underset{\sim}{G}$ of $\mathbb{E}$ has naturally associated with it a tie, defined through the a descent datum furnished by the canonical action (by "inner isomorphism") of $\underset{\sim}{G}$ on its internal subgroupoid of automorphisms (for $\partial^2(E)$ this subgroupoid is a locally given group, i.e. a group defined over a covering of $\mathbb{E}$, to which A is locally isomorphic) and in this way these somewhat mysterious "ghosts of departed coefficient groups" find a natural place in our version of the theory. (I.3).

We take for morphisms of bouquets the *essential equivalences*, i.e. internal versions of fully faithful, essentially epimorphic functors, and consider the equivalence relation generated by these functors (which because we are in a topos, do not necessarily admit quasi-inverses). For a given tie L of $\mathbb{E}$ we now define $H^2(\mathbb{E}; L)$ as connected component classes (under essential equivalence) of the bouquets of $\mathbb{E}$ which have their tie isomorphic to L.

The principal result of Part (I) of this paper is the following (Theorem (I5.21) and (I8.21)). The *assignment,* $\underset{\sim}{G} \longmapsto \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\sim}{G})$*, defines a neutral element* (I4.1) *preserving bijection*

$$\boxed{T\colon H^2(\mathbb{E}; L) \xrightarrow{\sim} H^2_{\mathrm{GIR}}(\mathbb{E}; L).}$$

Among other things, this result shows that this non-abelian H^2 (as we have already remarked for H^1) is also concerned with the classification of objects (the bouquets) which are internal to the topos. This is further reinforced by the simple observation that since a bouquet of $\mathbb{E}$ is just the internal version of the classical notion of a Brandt groupoid, Brandt's classical theorem characterizing such groupoids [BRANDT (1940)] still holds locally: *a bouquet of* $\mathbb{E}$ *is a category object of* $\mathbb{E}$ *which is locally essentially equivalent to a locally given group* (I2.5).

In the course of proving (I8.21) we also establish a number of relations between external and internal completeness (§I7, in particular I7.5 and I7.13) which are closely related to the work of JOYAL (1974), PENON and BOURN (1978), BUNGE and PARÉ (1979) as well as STREET (1980) and are of interest independently of their application here.

In Part (II) of this paper we introduce the notion of a 2-*cocycle defined on a hypercovering of* $\mathbb{E}$ *with coefficients in a locally given* group (II2.0) and by proving that every such cocycle factors through a bouquet of $\mathbb{E}$ show that $H^2(L)$ *may be computed by such cocycles* (II4.3)much as VERDIER (1972) showed was possible in the abelian case. We then use these cocycles to establish two results of Giraud (II5.2, II6.3) ("The Eilenberg-Mac Lane Theorems"): If $\mathbb{Z}(L)$ is the global abelian group which is the center of the tie L, then

(a) $H^2(L)$ *is a principal homogeneous space under the abelian group* $H^2(\mathbb{Z}(L))$, so that if $H^2(L) \neq \emptyset$ then $H^2(L) \xrightarrow{\sim} H^2(\mathbb{Z}(L))$.

(b) *There is an obstruction* $O(L) \in H^3(\mathbb{Z}(L))$ *which is null* if and only if $H^2(L) \neq \emptyset$.

(Whether every element of H^3 has such an "obstruction interpretation" remains an open question at the time of this writing).

In Part (III) of this paper we explore all of these notions in the "test topos" of G-sets. Here we show that since the notion of localization is that of passage to the underlying category of sets, a bouquet is just a G-groupoid which, on the underlying set level, is equivalent (as a category) to some ordinary group N. Moreover, since every such bouquet defines (and is defined by) an ordinary extension of G by the group N we obtain a new description of the classical $\mathrm{Ext}(G; N)$ (III4.2). We also show (III6.0) that, as Giraud remarked, a tie here is entirely equivalent to a homomorphism of G into the group $\mathrm{OUT}(N)\,[= \mathrm{AUT}(N)/\mathrm{INT}(N)]$ of automorphism classes of some group N, i.e. to an *abstract kernel* in the sense of ElLENBERG-MAC LANE (1947 II) and thus that the theorems of Part (II) are indeed generalizations of the classical results since our non-abelian 2-cocycles are shown to be here entirely equivalent to the classical "factor-systems" for group extensions of SCHREIER (1926).

An appendix is given which reviews the background of the formal "theory of descent" necessary for understanding many of the proofs which occur in the paper.

Since the subject of non-abelian cohomology has for such a long time remained an apparently obscure one in the minds of so many mathematicians, it will perhaps be worthwhile to make some further background comments on the results of Part III which may be taken as a guiding thread for a motivation of much that appears both here and the preceding fundamental work of Grotbendieck and Giraud:

On an intuitive basis the background for the definition of $H^2_{\mathrm{GIR}}(\mathbb{E};\, L)$ may be said to lie in sophisticated observations on the content of the seminal *Annals* papers of Eilenberg and Mac Lane on group cohomology [EILENBERG-MAC LANE 1947(I). 1947(II) particularly as presented in MAC LANE (1963). In these papers the extensions of a group G by a (non-abelian) kernel N were studied via the notion of an "abstract kernel", $\varphi\colon G \longrightarrow \mathrm{OUT}(N)$ - every extension induces one - and its relations with the groups $H^2(G;\, \mathbb{Z}(N))$ and $H^3(G;\, \mathbb{Z}(N))$ as defined by them.

Now from the point of view of topos theory. their groups $H^n(G; A)$ where A is a G-module may *quite literally* be taken to be (a cocycle computation of) the cohomology of the topos of G-sets with coefficients in A viewed as an internal abelian group in this topos, itself the topos of sheaves on the group G viewed as a site with the discrete topology (since a right G-set is just a functor from G^{op} into sets).

[This cohomological fact may easily be seen if one notes that the category of abelian group objects in G-sets is equivalent to the category of G-modules viewed as modules over the group ring. Alternatively it may be seen by using Verdier's theorem (cited above) which shows that for any topos $\mathbb{E}$ the "true" cohomology groups $H^n(\mathbb{E}, A)$ $(= \mathrm{Ext}^n(\mathbb{Z}_{\mathbb{E}}; A))$ may be computed in the simplicial Čech fashion as equivalence classes of n-cocycles under refinement provided that one replaces coverings by hypercoverings (essentially simplicial objects obtained from coverings by covering the overlaps) and then notes that in the topos of G-sets, every hypercovering may be refined by a single standard *covering* provided by the epimorphism $G_d \twoheadrightarrow \mathbb{1}$ (G operating on itself by multiplication). The n-cocycles (and coboundaries) on this covering then may easily be seen to be in bijective correspondence with the ordinary Eilenberg-Mac Lane ones.]

Furthermore, since any group object in this topos of G-sets is equivalent to an ordinary homomorphism $\psi\colon G \longrightarrow \mathrm{Aut}_{\mathrm{Gr}}(N)$ and if N is abelian, then an abstract kernel is just an abelian group object in the relations with $H^2(G; \mathbb{Z}(N))$ starting with Schreier's classical observation that every extension defines and is defined by a "factor set" and proceeded from there.

Now again from the point of view of topos theory a factor set is *almost* a sheaf of groups on G viewed as a category with a single object and hence *almost* a group object in the topos of G-sets, quite precisely, it is a pseudofunctor $F_{(\,)}\colon G^{\mathrm{op}} \rightsquigarrow \mathrm{Gr}$ (a functor up to coherent natural isomorphisms) with fibers in the (2-category of) groups and natural transformations of group homomorphisms. Every pseudo-functor on G defines and is defined by a Grothendieck fibration $\mathbb{F} \longrightarrow G$ and here the fibrations defined by factor sets are precisely the extensions of G, with those defined by actual functors corresponding to split extensions (hence the term *split* (fr. *scindé*)for those pseudo-functors which are actual functors). For an arbitrary site $\mathcal{S}$ this leads to the notion of a *gerbe* defined either as a particular sort of pseudofunctor defined on the underlying category of the site or as the particular sort of fibration over $\mathcal{S}$ which the pseudo functor defines. The required relationship with the topology of the site is that of "*completeness*", i.e. it is a "*stack*" (fr. *champ*) in the sense that every descent datum (= compatible gluing of objects or arrows in the fibers) over a covering of the topology on $\mathcal{S}$ is "effective", that is, produces an object or arrow at the appropriate global level. [This property of being complete for a fibration is precisely that same as that of being a sheaf for a presheaf when viewed as a discrete fibration (i.e., corresponding to a functor into sets).]

As every factor set defines canonically an abstract kernel, every gerbe canonically defines a tie and Grothendieck and Giraud thus define $H^2(\mathbb{E}; L)$ for the topos $\mathbb{E}$ of sheaves on $\mathcal{S}$ as the set of cartesian equivalence classes of gerbes on $\mathcal{S}$ which have tie isomorphic to L. The resulting theory, however, is "external to $\mathbb{E}$", taking place, at best, within the category of categories over $\mathbb{E}$ and not "within" the topos $\mathbb{E}$.

What has been added to their theory in this paper may be viewed as similar to what happened with the interpretation of $H^1(G; A)$ and its original definition as crossed homomorphisms modulo principal crossed homomorphisms: Viewed in the topos of G-sets, a crossed homomorphism is bijectively equivalent to a l-cocycle on the standard covering $G_d \longrightarrow\!\!\!\!\!\rightarrow \mathbf{1}$ with coefficients in the abelian group object A. Every such l-cocycle defines and can be defined by an object in this topos, namely an A-torsor (i.e., here, a nonempty G-set on which the G-module A operates equivariantly in a principal, homogeneous fashion). $H^1(G; A)$ is then seen to be isomorphism classes of such torsors [c.f. SERRE (1964)] Except for the abelian group structure that the resulting set inherits from A, as we have already remarked, this interpretation theory can be done for non-abelian A in any topos and the resulting set of isomorphism classes can be taken as the definition of $H^1(\mathbb{E}; A)$ and thus seen to be computable in the Čech fashion by refining cocycles on coverings.

What we have observed here is that there are similar internal objects available for the non-abelian H^2, the bouquets [internal Brandt groupoids, if one prefers, equivalent to $K(A, 2)$-torsors in the abelian case (DUSKIN (1979))] which play an entirely similar role provided that one relaxes the definition of equivalence to that of *essential equivalence* and takes the equivalence relation generated, much as one has to do in Yoneda-theory in dimensions higher than one. [If one wishes, this latter “problem” can be avoided by restricting attention to the “internally complete” bouquets, but those that seem to arise naturally such as the coboundary bouquet $\partial^2(E)$ of a torsor do not enjoy this property.] We have also observed that by using hypercoverings in place of coverings there is also internal to the topos always the notion of a non-abelian 2-cocycle (corresponding to a factor set in the case of G-sets) which may be used to compute H^2 and to prove at least two out of three of the classical theorems of Eilenberg and Mac Lane.

So far as the author knows, the observation that every Schreier factor set defines and is defined by a bouquet in G-sets, or put another way, that every extension of G by N corresponds to an internal nonempty connected groupoid in G-sets which is on the underlying set level (fully) equivalent to N, is new.

In the sequel to this paper we will use the results of I and II to explore the case of (global) group coefficients (where the tie is that of a globally given group) and to repair the “snag” in Giraud’s 9-term exact sequence, including there a much simpler proof of exactness in the case of H^1. We will also use these results to explain Dedecker’s results for paracompact spaces as well as those of DOUAI (1976). A brief introduction to this portion may be found in JOHNSTONE (1977) which is also a good introduction to the “yoga of internal category theory” used in our approach.

The results of Part (I) and a portion of Part (III) were presented, in outline form, at the symposium held in Amiens, France in honor of the work of the late Charles Ehresmann (DUSKIN (1982)). In connection with this it is fitting to note, as we did there, that it was EHRESMANN (1964) who first saw the importance of groupoids in the definition of the non-abelian H^2.

The results of Part II were written while the author was a Research Fellow of the School of Mathematics and Physics at Macquarie University, N.S.W. Australia in the summer of 1982. The author gratefully acknowledges the support of Macquarie University and its faculty at this time as well as the National Science Foundation which generously supported much of the original work.

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Buffalo, NY
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NON-ABELIAN COHOMOLOGY IN A TOPOS

Part (I): The theory of bouquets and gerbes

In what follows it will generally be assumed that the ambient category $\mathbb{E}$ is a Grothendieck topos, i.e. the category of sheaves on some U-small site. It will be quite evident, however, that a considerable portion of the theory is definable in any Barr-exact category [BARR (1971)] provided that the term "epimorphism" is always understood to mean "(universal) effective epimorphism".

1. THE CATEGORY OF BOUQUETS OF $\mathbb{E}$.

Recall that in sets a groupoid is a category in which every arrow is invertible. In any category E we make the following

Definition (1.0). By a *groupoid object* (or *internal groupoid*) of $\mathbb{E}$ we shall mean (as usual) an ordered pair $\underset{\approx}{G} = (\mathrm{Ar}(\underset{\approx}{G}), \mathrm{Ob}(\underset{\approx}{G}))$ of objects of $\mathbb{E}$ such that

(a) for each object U in $\mathbb{E}$, the sets $\mathrm{Hom}_{\mathbb{E}}(U, \mathrm{Ar}(\underset{\approx}{G}))$ and $\mathrm{Hom}_{\mathbb{E}}(U, \mathrm{Ob}(\underset{\approx}{G}))$ are the respective sets of arrows and objects of a groupoid (denoted by $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\approx}{G})$) such that

(b) for each arrow $f\colon U \longrightarrow V$, the mappings

$$\mathrm{Hom}(f, \mathrm{Ar}(\underset{\approx}{G}))\colon \mathrm{Hom}_{\mathbb{E}}(V, \mathrm{Ar}(\underset{\approx}{G})) \longrightarrow \mathrm{Hom}_{\mathbb{E}}(U, \mathrm{Ar}(\underset{\approx}{G})) \quad \text{and}$$
$$\mathrm{Hom}(f, \mathrm{Ob}(\underset{\approx}{G}))\colon \mathrm{Hom}_{\mathbb{E}}(V, \mathrm{Ob}(\underset{\approx}{G})) \longrightarrow \mathrm{Hom}_{\mathbb{E}}(U, \mathrm{Ob}(\underset{\approx}{G}))$$

defined by composition with f are the respective arrow and object mappings of a functor (i.e. morphism of groupoids)

$$\mathrm{Hom}_{\mathbb{E}}(f, \underset{\approx}{G})\colon \mathrm{Hom}_{\mathbb{E}}(V, \underset{\approx}{G}) \longrightarrow \mathrm{Hom}_{\mathbb{E}}(U, \underset{\approx}{G})\,.$$

Conditions (a) and (b), of course, simply state that the canonical functor

$$\langle \mathrm{Hom}_{\mathbb{E}}(\text{-}\,, \mathrm{Ar}(\underset{\approx}{G}))\,,\, \mathrm{Hom}_{\mathbb{E}}(\text{-}\,, \mathrm{Ob}(\underset{\approx}{G}))\rangle\colon \mathbb{E}^{\mathrm{op}} \longrightarrow (\mathrm{ENS}) \times (\mathrm{ENS})$$

factors through the obvious underlying set functor $U\colon \mathrm{GPD} \longrightarrow \mathrm{ENS} \times \mathrm{ENS}$.

As is well known for any such "essentially algebraic structure" the above definition (in the presence of fiber products) is equivalent to giving a system

$$\begin{tikzcd}
\underset{\approx}{G}\colon \mathrm{Ar}(\underset{\approx}{G})\ {}_{T}\!\times_{S}\ \mathrm{Ar}(\underset{\approx}{G}) \arrow[r, "\mu(\underset{\approx}{G})"] & \mathrm{Ar}(\underset{\approx}{G}) \arrow[r, shift left, "S(\underset{\approx}{G})"] \arrow[r, shift right, "T(\underset{\approx}{G})"'] & \mathrm{Ob}(\underset{\approx}{G}) \arrow[l, bend right=60, "\mathrm{I}(\underset{\approx}{G})"']
\end{tikzcd}$$

of objects and arrows of $\mathbb{E}$ which, in addition to satisfying in $\mathbb{E}$ the usual commutative diagram conditions expressing the properties of source (S), target (T), identity assignment (I) and composition (μ) of composable arrows that any category satisfies in sets, also has the commutative squares

(1.0.0)
$$\begin{tikzcd}
\mathrm{Ar}(\underset{\sim}{G}) \times_{\mathrm{Ob}(\underset{\sim}{G})} \mathrm{Ar}(\underset{\sim}{G}) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mu(\underset{\sim}{G})"'] & \mathrm{Ar}(\underset{\sim}{G}) \arrow[d, "T(\underset{\sim}{G})"] & \text{and} \\
\mathrm{Ar}(\underset{\sim}{G}) \arrow[r, "T(\underset{\sim}{G})"'] & \mathrm{Ob}(\underset{\sim}{G}) &
\end{tikzcd}$$

(1.0.1)
$$\begin{tikzcd}
\mathrm{Ar}(\underset{\sim}{G}) \times_{\mathrm{Ob}(\underset{\sim}{G})} \mathrm{Ar}(\underset{\sim}{G}) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mu(\underset{\sim}{G})"'] & \mathrm{Ar}(\underset{\sim}{G}) \arrow[d, "S(\underset{\sim}{G})"] \\
\mathrm{Ar}(\underset{\sim}{G}) \arrow[r, "S(\underset{\sim}{G})"'] & \mathrm{Ob}(\underset{\sim}{G})
\end{tikzcd}$$

cartesian (i.e., "pull-backs") as well since this latter condition will, in addition, guarantee that every arrow of the category is invertible. An object $u\colon U \to \mathrm{Ob}(\underset{\sim}{G})$ of the groupoid $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G})$ will sometimes be called a *U-object* of $\underset{\sim}{G}$. Similarly, a *U-arrow* of $\underset{\sim}{G}$ will then be an arrow $f\colon U \longrightarrow \mathrm{Ar}(\underset{\sim}{G})$ of $\mathbb{E}$. In $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G})$ its source is Sf and its target is Tf, $f\colon Sf \longrightarrow Tf$. Composition of composable U-arrows is given by composition in $\mathbb{E}$ with $\mu(G)$.

Remark. If we include in this system the canonical projections which occur in these diagrams, the resulting system defines a (truncated) simplicial object in $\mathbb{E}$

(1.0.2)
$$\begin{tikzcd}
\mathrm{Ar}(\underset{\sim}{G}) \times_{\mathrm{Ob}(\underset{\sim}{G})} \mathrm{Ar}(\underset{\sim}{G}) \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "\mu" description] \arrow[r, shift right=2, "\mathrm{pr}_2"'] \arrow[d, equal] & \mathrm{Ar}(\underset{\sim}{G}) \arrow[r, shift left=1, "S"] \arrow[r, shift right=1, "T"'] \arrow[l, bend left=40] \arrow[l, bend left=60] \arrow[d, equal] & \mathrm{Ob}(\underset{\sim}{G}) \arrow[l, bend left=50] \arrow[d, equal] \\
X_2 \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_0"'] & X_1 \arrow[r, shift left=1, "d_1"] \arrow[r, shift right=1, "d_0"'] \arrow[l, bend left=40, "s_1"] \arrow[l, bend left=60, "s_0"] & X_0 \arrow[l, bend left=50, "s_0"]
\end{tikzcd}$$

whose coskeletal completion to a full simplicial object (c.f. DUSKIN (1975, 1979)) for definitions of these terms) is called the *nerve of* $\underset{\sim}{G}$. We note that a simplicial object of

$\mathbb{E}$ is isomorphic to the nerve of a groupoid object in $\mathbb{E}$ if and only if for $i = 0, \cdots, n$ the canonical maps

$$(1.0.3) \qquad \langle d_0, \cdots, \widehat{d_i}, \cdots, d_n \rangle \colon X_n \longrightarrow \Lambda_i$$

into the "object of boundaries of n-simplices whose i^{th} face is missing (the "i-horn Λ_i") are isomorphisms for all $n \geq 2$. Thus viewed, groupoids in $\mathbb{E}$ may be identified with such "exact Kan-complexes" in SIMPL($\mathbb{E}$).

Definition (1.2). By an *(internal) functor* $\underset{\sim}{F} \colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ of *groupoid objects* we shall mean an ordered pair of arrows

$$\big(\mathrm{Ar}(\underset{\sim}{F}) \colon \mathrm{Ar}(\underset{\sim}{G}_1) \longrightarrow \mathrm{Ar}(\underset{\sim}{G}_2)\,,\, \mathrm{Ob}(\underset{\sim}{F}) \colon \mathrm{Ob}(\underset{\sim}{G}_1) \longrightarrow \mathrm{Ob}(\underset{\sim}{G}_2)\,\big)$$

such that for each object U of $\mathbb{E}$, the corresponding pair of mappings

$$\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{F}) \colon \mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_1) \longrightarrow \mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_2)$$

defines a functor in sets. Similarly an *(internal) natural transformation* $\varphi \colon \underset{\sim}{F}_1 \longrightarrow \underset{\sim}{F}_2$ of (internal) *functors* will be an arrow $\varphi \colon \mathrm{Ob}(\underset{\sim}{G}_1) \longrightarrow \mathrm{Ar}(\underset{\sim}{G}_2)$ such that the corresponding mapping $\mathrm{Hom}_{\mathbb{E}}(U, \varphi)$ defines a natural transformation of $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{F}_1)$ into $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{F}_2)$ for each object U of $\mathbb{E}$. A *(full) equivalence of groupoids* will be a pair of functors $\underset{\sim}{G} \colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$, $\underset{\sim}{H} \colon \underset{\sim}{G}_2 \longrightarrow \underset{\sim}{G}_1$ such that for each U in $\mathbb{E}$, the functor $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G})$ has $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{H})$ as a quasi-inverse.

Each of these terms has a corresponding equational statement in terms of commuting diagrams in $\mathbb{E}$, the (easy) formulation of which we leave to the reader.

We thus have defined over $\mathbb{E}$ the (2-category) $\underline{\underline{\mathrm{GPD}}}(\mathbb{E})$ of (internal) groupoids, functors and natural transformations (and note that it is fully imbedded in the (2-category) category SIMPL($\mathbb{E}$) of simplicial objects, simplicial maps, and homotopies of simplicial maps of $\mathbb{E}$ via the functor *Nerve*).

It is obvious from the form of the preceding definitions that because of the Yoneda lemma all of the "essentially algebraic" theorems about groupoids in sets (i.e. those that can be stated in terms of equations involving certain maps between finite inverse limits) transfer diagramatically to the corresponding statements in $\underline{\underline{\mathrm{GPD}}}(\mathbb{E})$ and that this portion of "internal category theory" is essentially identical to that found in sets. This is, of course, not true in $\mathbb{E}$ for all statements which commonly occur in category theory, in particular those that in sets assert (non unique) existence. For instance, in sets, a fully faithful functor $\underset{\sim}{F} \colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ which is essentially surjective (i.e. has the property that given any object G of $\underset{\sim}{G}_2$ there exists an object H of $\underset{\sim}{G}_1$ such that $\underset{\sim}{F}(H)$ is isomorphic to G) is a full equivalence since, using the axiom of choice, any such functor admits a quasi-inverse. In an arbitrary topos, epimorphisms replace surjective map but since not every epimorphism admits a section, the theorem fails. Nevertheless it is this notion of *essential equivalence* (and the equivalence relation which it generates) which we need in this paper. For groupoids in $\mathbb{E}$, this becomes the following diagrammatic

Definition (1.1). By an *essential equivalence* of $\underset{\sim}{G}_1$ with $\underset{\sim}{G}_2$ we shall mean a functor $\underset{\sim}{F} \colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ which satisfies the following two conditions:

(a) $\underset{\sim}{F}$ is *fully faithful* (i.e. the commutative diagram

(1.1.0)
$$\begin{tikzcd}
A_1 \arrow[r, "\mathrm{Ar}(\underset{\sim}{F})"] \arrow[d, "\langle T{,}S\rangle"'] & A_2 \arrow[d, "\langle T{,}S\rangle"] \\
O_1 \times O_1 \arrow[r, "\mathrm{Ob}(\underset{\sim}{F})\times\mathrm{Ob}(\underset{\sim}{F})"'] & O_2 \times O_2
\end{tikzcd}$$

is cartesian); and

(b) $\underset{\sim}{F}$ is *essentially epimorphic* (i.e., the canonical map $T \cdot \mathrm{pr}_{A_2} \colon O_1 \times_2 A_2 \longrightarrow O_2$ obtained by composition from the cartesian square

(1.1.1)
$$\begin{tikzcd}
O_1 \times_2 A_2 \arrow[r, "\mathrm{pr}_{A_2}"] \arrow[d, "\mathrm{pr}_{O_1}"'] & A_2 \arrow[r, "T"] \arrow[d, "S"] & O_2 \\
O_1 \arrow[r, "\mathrm{Ob}(\underset{\sim}{F})"'] & O_2 &
\end{tikzcd}$$

is an epimorphism).

<u>Remark</u> (1.2). The first of these conditions (a) is essentially algebraic and is equivalent to the assertion that for each U in $\mathbb{E}$, the functor

$$\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{F}) \colon \mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_1) \longrightarrow \mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_2)$$

is fully faithful. The second condition is "geometric", however, and does not guarantee that $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{F})$ is essentially surjective for each U. In fact, if it is, then the epimorphism $T \cdot \mathrm{pr}_{A_2} \colon O_1 \times_S A_2 \longrightarrow O_2$ is split by some section $s \colon O_2 \longrightarrow O_1 \times_2 A_2$ in $\mathbb{E}$, a condition much too strong for our intended applications. What *does* survive since we are in a topos is the notion that $\underset{\sim}{F}$ is *"locally" essentially surjective*, i.e. given any object $u \colon U \longrightarrow \mathrm{Ob}(\underset{\sim}{G}_2)$ in $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_2)$, there exists an epimorphism $c \colon C \twoheadrightarrow U$ and an object $v \colon C \twoheadrightarrow \mathrm{Ob}(\underset{\sim}{G}_1)$ in $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_1)$ such that $\underset{\sim}{F}(v) = \mathrm{Ob}(F)u$ is isomorphic to $uc \colon C \longrightarrow \mathrm{Ob}(\underset{\sim}{G}_2)$ in $\mathrm{Hom}_{\mathbb{E}}(C, \underset{\sim}{G}_2)$. In effect, just define c as the epimorphism obtained by pulling back the epimorphism $T \cdot \mathrm{pr}_{A_2}$ along u and define v as the composition of pr_2 and the projection pr_{O_1}. The terminology "local" is fully justified since this is equivalent to saying that for any object in the site of definition of the topos, there exists a covering of the object over which the restricted statement is indeed true. It thus coincides with the usual topological concept if the site consists of the open sets of a topological space.

We now are in a position to define the objects of the topos to which the relation of essential equivalence will be applied. They too have both an essentially algebraic and a "geometric" component to their

<u>Definition</u> (1.3). A groupoid object $\underset{\sim}{G} \colon A \overset{S}{\underset{T}{\rightrightarrows}} O$ in $\mathbb{E}$ will be called a *bouquet* of $\mathbb{E}$ provided it satisfies the additional two conditions

(a) $\underset{\sim}{G}$ is *(internally) non-empty* (i.e., the canonical map $\mathrm{Ob}(\underset{\sim}{G}) \longrightarrow \mathbb{1}$ into the terminal object of $\mathbb{E}$ is an epimorphism); and

(b) $\underset{\sim}{G}$ is (internally) *connected* (i.e., the canonical map

$$\langle T, S\rangle \cdot \mathrm{Ar}(\underset{\sim}{G}) \longrightarrow \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{Ob}(\underset{\sim}{G})$$

is an epimorphisim.)

As we have already remarked in (1.2), these conditions do not guarantee that for each U, the groupoid $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G})$ is nonempty and connected but rather only that these two properties are locally true: (a) for any object U in $\mathbb{E}$, there exists a covering (read epimorphism) $C \twoheadrightarrow U$ on which the groupoid $\mathrm{Hom}_{\mathbb{E}}(C, \underset{\sim}{G})$ is nonempty and (b) for any U-objects $x, y \colon U \rightrightarrows \mathrm{Ob}(\underset{\sim}{G})$ there exists a covering $d \colon D \twoheadrightarrow U$ on which the restrictions xd and yd are isomorphic in the groupoid $\mathrm{Hom}_{\mathbb{E}}(D, \underset{\sim}{G})$, i.e. *any two objects in* $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G})$ *are locally isomorphic.* Note that this does *not* imply that there exists a covering C of the entire site (read $C \twoheadrightarrow \mathbb{1}$) on which the groupoid $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G})$ is non empty and connected.

(1.4) Every functor $\underset{\sim}{F} \colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ of bouquets is necessarily essentially epimorphic, thus $\underset{\sim}{F}$ *is an essential equivalence of bouquets if and only if* $\underset{\sim}{F}$ *is fully faithful.* In effect, given any U-object $u \colon U \longrightarrow \mathrm{Ob}(\underset{\sim}{G}_2)$ of $\underset{\sim}{G}_2$, the fact that $\underset{\sim}{G}_1$ is locally non empty means that there exists an epimorphism $p \colon C \twoheadrightarrow U$ and a C-object $x \colon C \longrightarrow \mathrm{Ob}(\underset{\sim}{G}_1)$ and thus a pair $(up, \mathrm{Ob}(F)x) \colon C \rightrightarrows \mathrm{Ob}(\underset{\sim}{G}_2)$ of C objects in $\underset{\sim}{G}_2$. But since $\underset{\sim}{G}_2$ is locally connected, there exists an epimorphism $C' \xrightarrow{p'} C$ for which the restrictions upp' and $\mathrm{Ob}(\underset{\sim}{F})xp'$ are isomorphic in $\mathrm{Hom}_{\mathbb{E}}(U, \underset{\sim}{G}_2)$. Thus the composition $pp' \colon C' \twoheadrightarrow U$ and the C' object xp' of $\underset{\sim}{G}_1$ produce a cover of U on which the asserted property holds and F is thus essentially epimorphic.

We will designate by $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E})$ the 2-subcategory of $\underline{\underline{\mathrm{GPD}}}(\mathbb{E})$ whose objects are the bouquets of $\mathbb{E}$ and whose morphisms are essential equivalences of bouquets, with natural transformations of essential equivalences (necessarily all isomorphisms) for 2-cells.

(1.5) Examples of Groupoids and bouquets:

(1.5.0) Trivially, any epimorphism $X \twoheadrightarrow \mathbb{1}$ defines a bouquet, namely $X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$ viewed as a groupoid in which there is exactly one arrow connecting any two objects. Similarly the kernel pair $X \times_Y X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$ of any epimorphism $p \colon X \twoheadrightarrow Y$ defines a bouquet in the topos $\mathbb{E}/Y$ of objects of $\mathbb{E}$ above Y $\Big[$whose objects are arrows of $\mathbb{E}$ of the form $Z \longrightarrow Y$ and whose arrows are commutative triangles $\begin{tikzcd}[column sep=small, row sep=small] Z_1 \arrow[rr] \arrow[dr] & & Z_2 \arrow[dl] \\ & Y & \end{tikzcd}$]

since in $\mathbb{E}/Y$, Y is terminal and the product of p with itself in $\mathbb{E}/Y$ is just $X \times_Y X$.

(1.5.1) Every *group object of* $\mathbb{E}$ (i.e. groupoid $\utilde{G}$ of $\mathbb{E}$ for which the canonical map $\mathrm{Ob}(\utilde{G}) \longrightarrow \mathbb{1}$ into the terminal object is an isomorphism) is clearly a bouquet of $\mathbb{E}$ (and an essential equivalence of group objects is just an isomorphism). Thus any group object in the topos $\mathbb{E}/X$ is bouquet of the topos $\mathbb{E}/X$. Viewed in $\mathbb{E}$, a group object in $\mathbb{E}/X$ is simply a groupoid in $\mathbb{E}$ whose object of objects is X and whose source and target arrows coincide ($S = T$), thus in sets just a family of groups indexed by X. If the canonical map $X \longrightarrow \mathbb{1}$ is an epimorphism then such a groupoid will be called a *locally given group* since this amounts to a group object defined over a cover of $\mathbb{E}$. A group object of $\mathbb{E}$ itself will, in contrast, be often referred to as a *globally given group.*

(1.5.2) Every bouquet $\utilde{G}$ of $\mathbb{E}$ has canonically associated with it a locally given group (1.5.1), namely, its *subgroupoid* $\mathcal{E}(\utilde{G}) \longrightarrow \utilde{G}$ *of automorphisms of* $\utilde{G}$ defined through the cartesian square

(1.5.2a)

$$\begin{tikzcd}
\mathcal{E}(\utilde{G}) \arrow[r, hook] \arrow[d, "\mathrm{pr}"'] & \mathrm{Ar}(\utilde{G}) \arrow[d, "\langle T,S\rangle"] \\
\mathrm{Ob}(G) \arrow[r, "\Delta"] & \mathrm{Ob}(G) \times \mathrm{Ob}(\utilde{G})
\end{tikzcd}$$

For any object U of $\mathbb{E}$, $\mathrm{Hom}_{\mathbb{E}}(U, \mathcal{E}(\utilde{G}))$ represents the subgroupoid arrows of $\mathrm{Hom}_{\mathbb{E}}(U, \utilde{G})$ of the form $f\colon x \longrightarrow x$ for some U-object $x\colon U \longrightarrow \mathrm{Ob}(\utilde{G})$. Since $\mathrm{Ob}(\utilde{G}) \longrightarrow \mathbb{1}$ is epic, this is indeed a locally given group and thus may be viewed as a group object in the category $\mathbb{E}/\mathrm{Ob}(\utilde{G})$. In the next section we will show that every bouquet is "locally essentially equivalent" to this particular locally given group.

(1.5.3) The notion of a group G acting on a set E (on the right, say) can be easily axiomatized in an essentially algebraic fashion and thus defined in $\mathbb{E}$ via an action map $\alpha\colon E \times G \longrightarrow E$ which represents an action of $\mathrm{Hom}_{\mathbb{E}}(U, G)$ on $\mathrm{Hom}_{\mathbb{E}}(U, E)$ for each object U of $\mathbb{E}$. If one adds the projection $\mathrm{pr}_E\colon E \times G \longrightarrow E$ to this action one obtains the target (pr_E) and source (α) arrows of a groupoid $E \times G \rightrightarrows E$ in E whose composition is defined through the multiplication in G. Set-theoretically this amounts to viewing a right action on a set as defining the arrows $(x, g)\colon x^g \longrightarrow x$ of a groupoid whose objects consist of the elements of E.

In order that such a groupoid be a bouquet it is necessary and sufficient that the object on which the group acts be a *homogeneous space* under the group action, i.e., that the canonical maps $\langle \mathrm{pr}_E, \alpha\rangle\colon E \times G \longrightarrow E \times E$ and $E \longrightarrow \mathbb{1}$ be epimorphisms. It follows that any *torsor under* G (i.e., any principal homogeneous space under an action of G) is a bouquet of $\mathbb{E}$. [A group action is said to be principal if the canonical map $\langle \mathrm{pr}_E, \alpha\rangle$ is a monomorphism.]

(1.5.4) In particular, if $p\colon G_1 \longrightarrow G_2$ is an epimorphism of group objects of $\mathbb{E}$ and E is a torsor under G_2, then the restriction of the action of G_2 on E to G_1 via the epimorphism p makes E into a homogeneous space under G_1 which when viewed as a bouquet is called the 2-*coboundary bouquet* $\partial^2(E)$ *of the torsor* E *along the epimorphism* p. It will play a key role in the extension of the classical six term exact sequence of groups and pointed sets (referred to in the introduction) to dimension 2.

(1.5.5) If E is a homogeneous space under G which admits a global section $x\colon \mathbb{1} \longrightarrow E$, then the internal group G_x of automorphisms of x in the bouquet defined by E may be constructed via the cartesian square

(1.5.5*a*)
$$\begin{tikzcd}
G_x \arrow[r, hook] \arrow[d] & E \times G \arrow[d, "\langle \mathrm{pr}_E{,}\alpha \rangle"] \\
\mathbb{1} \arrow[r, "\langle x{,}x \rangle"'] & E \times E
\end{tikzcd}$$

and identified with the *isotropy subgroup* of x *in* G since it is isomorphic to the representation of the set of $g \in G$ such that $x^g = x$. The resulting inclusion *functor*

(1.5.5*b*)
$$\begin{tikzcd}
G_x \arrow[r, hook] \arrow[d] & E \times G \arrow[d, shift right, "\mathrm{pr}"'] \arrow[d, shift left, "\alpha"] \\
\mathbb{1} \arrow[r, "x"'] & E
\end{tikzcd}$$

furnishes an example of an essential equivalence of bouquets which clearly does not in general admit a quasi-inverse. Such is the case furnished by any short exact sequence $\mathbb{1} \longrightarrow A \xhookrightarrow{u} B \xrightarrow{v} C \longrightarrow \mathbb{1}$ of groups of $\mathbb{E}$ which is not split on the underlying object level: C itself becomes a homogeneous space under B via the epimorphism v and A may be identified with the isotropy subgroup of the unit section of C. Thus the short exact sequence gives rise to the essential equivalence

(1.5.5*c*)
$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & C \times B \arrow[d, shift right, "\mathrm{pr}"'] \arrow[d, shift left, "P_m"] \\
\mathbb{1} \arrow[r, "e"'] & C
\end{tikzcd}$$

of bouquets of E which admits no quasi-inverse.

2. BOUQUETS AND LOCALLY GIVEN GROUPS

(2.0) Localization. The process of “localization” to which we have alluded in the preceding sections is best viewed as taking place categorically “within the topos $\mathbb{E}$” in the following fashion: As we have used the term, given some property of objects and arrows of $\mathbb{E}$, or more generally of some diagram $\mathfrak{D}$ in $\mathbb{E}$, to say that the property "locally true" has not meant that for each object X in the site of definition of $\mathbb{E}$ the property holds in the corresponding diagram of sets $\mathfrak{D}(X)$ but rather that for each X there exists a covering $(X_\alpha \longrightarrow X)_{\alpha \in I}$ in the topology of the site such that the restricted diagram $\mathfrak{D}(X_\alpha)$ enjoys the property for each α. But since the topos $\mathbb{E}$ is the topos of sheaves on the site, the preceding notion is equivalent to the assertion of the existence of an epimorphism $c \colon C \twoheadrightarrow \mathbb{1}$ in $\mathbb{E}$ such that the diagram of sets $\mathfrak{D}(C)$ enjoys the property in question.

Now the topos $\mathbb{E}$ is isomorphic the topos $\mathbb{E}/\mathbb{1}$ and "pull back along c" defines a *functor of localization* $c^* \colon \mathbb{E}(\xrightarrow{\sim} \mathbb{E}/\mathbb{1}) \longrightarrow \mathbb{E}/C$ into the topos of objects above C. Its value at any X in $\mathbb{E}$ is just $X \times C \xrightarrow{\mathrm{pr}} C$ which we will denote by $X|C$ and refer to as "X localized over C". Since c^* has both left and right adjoints any categorical property of a diagram $\mathfrak{D}$ in $\mathbb{E}$ is preserved when localized to the corresponding (localized) diagram $\mathfrak{D}|C$ in $\mathbb{E}/C$. Moreover, since there is a one-to-one correspondence between arrows from C to X in $\mathbb{E}$ and arrows $C \longrightarrow X|C$ in $\mathbb{E}/C$ (and thus to global sections of $X|C$ since C is terminal $\mathbb{E}/C$), the diagram of sets $\mathfrak{D}(C)$ is just the diagram of global sections of the localized diagram $\mathfrak{D}|C$ in $\mathbb{E}/C$. Thus the local properties of $\mathfrak{D}$ have become the global properties of $\mathfrak{D}|C$ in $\mathbb{E}/C$.

This enables us to generalize the above informal notion of localization "within the topos $\mathbb{E}$" as follows: given any categorically stateable property of a diagram $\mathfrak{D}$ in $\mathbb{E}$, to say that $\mathfrak{D}$ enjoys the property *locally* will simply mean that *there exists an epimorphism* $c \colon C \twoheadrightarrow \mathbb{1}$ *such that the localized diagram* $\mathfrak{D}|C$ *enjoys the property in the topos* $\mathbb{E}/C$. Thus for example, objects X and Y of $\mathbb{E}$ are locally isomorphic will mean that there exists an epimorphism $C \twoheadrightarrow \mathbb{1}$ such that the objects $X|C$ and $Y|C$ are isomorphic in $\mathbb{E}/C$. This in turn is easily seen to be equivalent to the assertion that the canonical map $\underline{\mathrm{Iso}}(X, Y) \longrightarrow \mathbb{1}$ is an epimorphism since there is a one-to-correspondence between arrows $Z \longrightarrow \underline{\mathrm{Iso}}(X, Y)$ and isomorphisms $\alpha \colon X|Z \longrightarrow X|Z$ and to say for any given object T, that the canonical map $T \longrightarrow \mathbb{1}$ is an epimorphism is equivalent to saying that, locally, T admits a global section.

The functor of localization over an epimorphism c^* preserves and reflects limits and epimorphisms. It thus also preserves and reflects both monomorphisms and isomorphisms and commutativity of diagrams. Thus, for instance, a simplicial object $X_\bullet$ of $\mathbb{E}$ is the nerve of a category (resp., groupoid) object in $\mathbb{E}$ if and only if locally it is the nerve of a category (resp. groupoid) object in $\mathbb{E}$. Similarly a category $\underset{\approx}{G}$ is a bouquet of $\mathbb{E}$ if and only if locally it is a bouquet of $\mathbb{E}$.

A small amount of caution is necessary to make clear "where the localization is taking place". For instance an epimorphism $f \colon X \twoheadrightarrow Y$ need not be locally split (since the axiom of choice need not hold even locally there may be no epimorphism $C \twoheadrightarrow \mathbb{1}$ for which $f|C \colon X|C \longrightarrow Y|C$ admits a section in $\mathbb{E}/C$. However, considered as an *object* in the topos $\mathbb{E}/Y$ (where Y is terminal) there does exist an epimorphism in $\mathbb{E}/Y$

$$\left(\text{take}\quad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f"'] & & Y \arrow[dl, "\mathrm{id}"] \\ & Y & \end{tikzcd}\right)$$ such that pull back over it does produce a splitting since this is just saying in $\mathbb{E}/Y$ that the canonical map to the terminal object is an epimorphism.

In addition, it should also be clear that local existence of objects and arrows need not imply global existence. However, since an epimorphism $c\colon C \twoheadrightarrow \mathbb{1}$ is a morphism of effective descent (c.f. appendix), the functor of localization gives rise to an equivalence of the topos $\mathbb{E}$ with the category of algebras over the monad (triple) defined by the endofunctor $c_!c^*\colon \mathbb{E}/C \longrightarrow \mathbb{E}/C$. This category of algebras is easily seen to be equivalent to the category of objects of $\mathbb{E}/C$ supplied with a descent datum (i.e. a compatible gluing), thus local existence of objects and arrows of $\mathbb{E}$ only produces global existence (i.e. existence in $\mathbb{E}$) for "compatibly glued" objects or arrows in $\mathbb{E}/C$.

In the following paragraphs we shall make use of localization to give a characterization of the bouquets of $\mathbb{E}$ which should make their connection with group cohomology at least plausible.

(2.1) Bouquets and group objects.

First recall that our definition of a bouquet is just the "internal version" of the classical set theoretic notion of a "Brandt groupoid", i.e. a connected non empty category in which every arrow is an isomorphism. In BRANDT [1940], Brandt gave a structure theorem for such groupoids which showed that they are characterized by a group G and a non-empty set S in such a fashion that the set of arrows of the groupoid admitted a bijection onto the set $S \times S \times G$ [cf. BRUCK (1958)]. In present terminology he simply showed that any Brandt groupoid is equivalent (as a category) to a group (considered as a groupoid with a single object) which could be taken to be the subgroup(oid) of automorphisms of any chosen object of the groupoid. From a modern point of view the proof of the theorem is elementary: Simply pick an object x of the groupoid $\underset{\sim}{G}$ and look at the subgroupoid $\mathrm{aut}(x)$ of arrows of $\underset{\sim}{G}$ of the form $a\colon x \longrightarrow x$. The inclusion functor $\mathrm{aut}(x) \hookrightarrow \underset{\sim}{G}$ is fully faithful and since $\underset{\sim}{G}$ is connected it is essentially surjective (for every object of $\underset{\sim}{G}$ is isomorphic to x). Since the axiom of choice holds, this essential equivalence admits a quasi-inverse $P\colon \underset{\sim}{G} \longrightarrow \mathrm{aut}(x)$ which is also fully faithful, so that the square

(2.1.0)
$$\begin{tikzcd} \mathrm{Ar}(\underset{\sim}{G}) \arrow[r, "P"] \arrow[d, "\langle T,J\rangle"'] & \mathrm{aut}(x) \arrow[d] \\ \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{Ob}(\underset{\sim}{G}) \arrow[r] & \mathbb{1} \times \mathbb{1} (\xrightarrow{\sim} \mathbb{1}) \end{tikzcd}$$

is cartesian (1.1.a) and $\mathrm{Ar}(\underset{\sim}{G}) \xrightarrow{\sim} \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{aut}(x)$ as asserted. By taking any non-empty set S for $\mathrm{Ob}(\underset{\sim}{G})$ and defining the arrows of $\underset{\sim}{G}$ via the cartesian square (2.1.0) the converse of the theorem is established. Note also that since $\underset{\sim}{G}$ is connected for any choice of objects x and y the groups $\mathrm{aut}(x)$ and $\mathrm{aut}(y)$ are isomorphic by an isomorphism which is itself unique up to an inner automorphism.

If $\utilde{G}$ is a bouquet of a topos $\mathbb{E}$ for which the canonical maps $\mathrm{Ob}(\utilde{G}) \twoheadrightarrow \mathbb{1}$ and $\mathrm{Ar}(\utilde{G}) \xrightarrow{\langle T,S\rangle} \mathrm{Ob}(\utilde{G}) \times \mathrm{Ob}(\utilde{G})$ both admit sections $s_0\colon \mathbb{1} \longrightarrow \mathrm{Ob}(\utilde{G})$ and $t\colon \mathrm{Ob}(\utilde{G}) \times \mathrm{Ob}(\utilde{G}) \longrightarrow \mathrm{Ar}(\utilde{G})$, then Brandt's theorem holds without modification internally in $\mathbb{E}$ since for all objects U in $\mathbb{E}$, the groupoid $\mathrm{Hom}_{\mathbb{E}}(U, \utilde{G})$ is a Brandt groupoid (in sets) and the group object $\mathrm{aut}_{\utilde{G}}(s_0)$ defined by the cartesian square

(2.1.1)
$$\begin{tikzcd}
\mathrm{aut}_G(s_0) \arrow[r, hook] \arrow[d] & \mathrm{Ar}(\utilde{G}) \arrow[d, "\langle T,S\rangle"] \\
\mathbb{1} \arrow[r, "\langle s_0,s_0\rangle"] & \mathrm{Ob}(\utilde{G}) \times \mathrm{Ob}(\utilde{G})
\end{tikzcd}$$

together with its canonical inclusion functor

(2.1.2)
$$i_s\colon \mathrm{aut}_{\utilde{G}}(s_0) \hookrightarrow G$$

is an equivalence of groupoids in $\mathbb{E}$ with a quasi inverse defined using the section t to produce a choice of isomorphisms $s_1\colon \mathrm{Ob}(\utilde{G}) \longrightarrow \mathrm{Ar}(\utilde{G})$ internally connecting any object of $\utilde{G}$ to s_0. As in case of sets any two groups $\mathrm{aut}_{\utilde{G}}(s_0)$ and $\mathrm{aut}_{\utilde{G}}(s_0')$ are isomorphic in an essentially unique function.

Conversely, any category object $\utilde{G}$ which is equivalent to a group object G is a bouquet of $\mathbb{E}$ whose canonical epimorphism both admit sections of the form s_0 and t. Such a bouquet of $\mathbb{E}$ will be said to be *split by the group G*.

Note that from the simplicial point of view if $\utilde{G}$ is split, then the sections $s_0\colon \mathbb{1} \longrightarrow \mathrm{Ob}(\utilde{G})$ and $s_1\colon \mathrm{Ob}(\utilde{G}) \longrightarrow \mathrm{Ar}(\utilde{G})$ form the first two steps of a contracting homotopy on the 1-truncated nerve of $\utilde{G}$ since s_0 just internally picks an object s_0 of $\utilde{G}$ and s_1 just defines an isomorphism $s_1(x)\colon s_0 \longrightarrow x$ for each object x of $\utilde{G}$. But since any such contracting homotopy can be used to define a section $t(x,y) = s_1(y)s_1(x)^{-1}\colon x \longrightarrow y$ we see that *a bouquet $\utilde{G}$ is split* (by some group G) *if and only if* $\mathrm{Cosk}^1(\mathrm{Ner}(\utilde{G}))$ *admits a contracting homotopy.*

Of course, in an arbitrary topos a given bouquet $\utilde{G}$ may have one of its canonical epimorphisms split without the other one being split: For example let $v\colon B \longrightarrow C$ be an epimorphism of groups and let C_δ be the trivial torsor under C, (C acting on itself by multiplication) then the co boundary torsor $\partial^2(C)$ of C_δ along v (1.5.4) always has a canonical section for $\mathrm{Ob}(\partial^2(C_\delta)) \twoheadrightarrow \mathbb{1}$ furnished by the unit map for C but admits a section for $\mathrm{Ar}(\partial^2(C_\delta)) \xrightarrow{\langle T,S\rangle} \mathrm{Ob}(\partial^2(C_\delta)) \times \mathrm{Ob}(\partial^2(C_\delta))$ if and only if the epimorphism $v\colon B \longrightarrow C$ admits a section on the under lying object level. Similarly if $v\colon B \longrightarrow C$ admits a section on the underlying object level and E is a non trivial torsor under C, then $\partial^2(E)$ admits a section for $\langle T, S\rangle$ but not for $\mathrm{Ob}(\partial^2(E)) \twoheadrightarrow \mathbb{1}$. Thus these two possibilities must be considered separately in an arbitrary topos:

(2.2) Lemma. For any bouquet $\underset{\approx}{G}$ of $\mathbb{E}$, in order that there exist a group G and an essential equivalence $P\colon \underset{\approx}{G} \longrightarrow G$ (we shall say that $\underset{\approx}{G}$ *is split on the right by some group* G) it is necessary and sufficient that the canonical epimorphism $\langle T, S\rangle\colon \mathrm{Ar}(\underset{\approx}{G}) \longrightarrow \mathrm{Ob}(\underset{\approx}{G}) \times \mathrm{Ob}(\underset{\approx}{G})$ admits a section $t\colon \mathrm{Ob}(\underset{\approx}{G}) \times \mathrm{Ob}(\underset{\approx}{G}) \longrightarrow \mathrm{Ar}(\underset{\approx}{G})$ which has the following properties

(a) t is *normalized* (i.e., $t\Delta = I(\underset{\approx}{G}) \iff$ for all $x \in \mathrm{Ob}(\underset{\approx}{G})$, $t(x,x) = \mathrm{id}(x)$) and

(b) t is *multiplicative* (i.e., $\mu(\underset{\approx}{G})(t \times t) = t\mathrm{pr}_{13} \iff$ for all $x, y, z \in \mathrm{Ob}(\underset{\approx}{G})$ $t(x,y)t(y,z) = t(x,z)$).

Moreover, any two such groups G are *locally* isomorphic.

These two properties just say simplicially that the canonical simplicial map $p\colon \mathrm{Ner}(\underset{\approx}{G}) \longrightarrow \mathrm{Cosk}^0(\mathrm{Ner}(\underset{\approx}{G}))$ admits a simplicial section or, equivalently, that the canonical functor from $\underset{\approx}{G}$ to the groupoid $\mathrm{Ob}(\underset{\approx}{G}) \times \mathrm{Ob}(\underset{\approx}{G}) \rightrightarrows \mathrm{Ob}(\underset{\approx}{G})$ admits a *functorial* section. $\Big[$Note that any section t may be assumed to be normalized (since if it is not, the section $t'(x,y) = t(x,y)t(g,y)$ is) and that if $\mathrm{Ob}(\underset{\approx}{G}) \twoheadrightarrow \mathbb{1}$ also admits a section then any section t may be replaced with a functorial one. $\Big]$

In effect if such a functor P exists, then the square

(2.2.1)
$$\begin{tikzcd}
\mathrm{Ar}(\underset{\approx}{G}) \arrow[r, "P"] \arrow[d, "\langle T,S\rangle"] & G \arrow[d] \\
\mathrm{Ob}(\underset{\approx}{G}) \times \mathrm{Ob}(\underset{\approx}{G}) \arrow[r] \arrow[u, dotted, bend left, "e^{\#}"] & \mathbb{1} \times \mathbb{1} (\xrightarrow{\sim} \mathbb{1}) \arrow[u, bend left, "e"]
\end{tikzcd}$$

is cartesian and the unit section e of G defines a functorial section $e^{\#}$ of $\langle T, S\rangle$ which has the property that $Pe^{\#} = e$.

Conversely if $\langle T, S\rangle$ admits a functorial section,then the canonical action of $\underset{\approx}{G}$ on the subgroupoid $\mathcal{E}(\underset{\approx}{G}) \hookrightarrow G$ of automorphisms of $\underset{\approx}{G}$ (1.5.2) by inner isomorphisms may be combined with t to define a gluing t^* (c.f. Appendix) on $\mathcal{E}(\underset{\approx}{G})$ viewed as a locally given group defined on the covering $\mathrm{Ob}(\underset{\approx}{G}) \twoheadrightarrow \mathbb{1}$

(2.2.2)
$$\begin{tikzcd}
\mathrm{pr}_0^*(\mathcal{E}) \arrow[r, "t^*", "\sim"'] \arrow[d] & \mathrm{pr}_1(\mathcal{E}) \arrow[r] \arrow[dl] \arrow[r, bend right] & \mathcal{E}(G) \arrow[r, "P"] \arrow[d] & G \arrow[d] \\
\mathrm{Ob}(\underset{\approx}{G}) \times \mathrm{Ob}(\underset{\approx}{G}) \arrow[rr, shift left] \arrow[rr, shift right] & & \mathrm{Ob}(\underset{\approx}{G}) \arrow[r] & \mathbb{1}
\end{tikzcd}$$

via the group isomorphism

$$t^*\colon (x,y,a\colon x\to x)\longmapsto (x,y,t(x,y)^{-1}\quad \text{at } (x,y)\colon y\to y) \tag{2.2.3}$$

in $\underline{\underline{\mathrm{Gr}}}(\mathbb{E}/\,(\mathrm{Ob}(\underset{\sim}{G})\times \mathrm{Ob}(\underset{\sim}{G}))$.

Since t is functorial, t^* is a descent datum on $\mathcal{E}(\underset{\sim}{G})$ for this covering which, since we are in a topos, is effective and thus produces a group G on the global level which is locally isomorphic to the locally given group $\mathcal{E}(\underset{\sim}{G})$. $\underset{\sim}{G}$ is made essentially equivalent to G by the functor defined through the image of the internal composition $t(Sf,Tf)\ \ f$ under the canonical epimorphism of descent $p\colon \mathcal{E}(G)\twoheadrightarrow G$. Clearly, any two such groups are locally isomorphic to $\mathcal{E}(\underset{\sim}{G})$. $\Big[$ An alternative description of this construction may be found in (II 7) in connection with the notion of neutral cocycles.$\Big]$

On the other side of $\underset{\sim}{G}$, we have the following

(2.3) <u>Lemma</u>. For any bouquet $\underset{\sim}{G}$ of $\mathbb{E}$, in order that there exist a group G and an essential equivalence $\mathfrak{J}\colon G\longrightarrow \underset{\sim}{G}$ (we shall say that $\underset{\sim}{G}$ *is split from the left by* G) it is necessary and sufficient that the canonical epimorphism $\mathrm{Ob}(\underset{\sim}{G})\twoheadrightarrow \mathbb{1}$ admits a section $s_0\colon \mathbb{1}\longrightarrow \mathrm{Ob}(\underset{\sim}{G})$. Moreover, any two such groups are locally isomorphic.

In effect, the proof of Brandt's theorem involves no use of choice up to the point of construction of a quasi inverse. Thus given a section $s_0\colon \mathbb{1}\longrightarrow \mathrm{Ob}(\underset{\sim}{G})$, the cartesian square (2.1.1) now identifies G isomorphically with the group $\mathrm{aut}_{\underset{\sim}{G}}(s_0)$ and canonically defines an essential equivalence of G with the bouquet $\underset{\sim}{G}$. For any two sections s_0 and $s_0'\colon \mathbb{1}\longrightarrow \mathrm{Ob}(\underset{\sim}{G})$, the cartesian square

$$\begin{tikzcd} \mathrm{iso}_{\underset{\sim}{G}}(s_0',s_0) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Ar}(\underset{\sim}{G}) \arrow[d, "\langle T,S\rangle"] \\ \mathbb{1} \arrow[r, "\langle s_0,s_0'\rangle"'] & \mathrm{Ob}(\underset{\sim}{G})\times \mathrm{Ob}(\underset{\sim}{G}) \end{tikzcd} \tag{2.3.0}$$

furnishes an epimorphism $\mathrm{pr}_1\colon \mathrm{iso}_{\underset{\sim}{G}}(s_0',s_0)\twoheadrightarrow \mathbb{1}$ and a local isomorphism of s_0' and s_0 which defines a local isomorphism of $\mathrm{aut}_{\underset{\sim}{G}}(s_0)$ and $\mathrm{aut}_{\underset{\sim}{G}}(s_1)$. Thus any two such groups are locally isomorphic.

(2.4) What now survives of Brandt's theorem for a bouquet $\underset{\sim}{G}$ in an arbitrary topos, where the axiom of choice fails to hold? If the topos $\mathbb{E}$ is Boolean, then the axiom of choice holds locally (i.e. for any epimorphism $f\colon X\longrightarrow Y$ in $\mathbb{E}$, there exists an epimorphism $C\twoheadrightarrow \mathbb{1}$ on which the epimorphism $X|C\xrightarrow{f|C} Y|C$ admits a section in the topos $\mathbb{E}/C$). In this case for any bouquet $\underset{\sim}{G}$ one can find an epimorphism $C\twoheadrightarrow \mathbb{1}$ for which both of the canonical epimorphisms of $\underset{\sim}{G}|C$ admit sections and Brandt's theorem holds without modification; thus here, *in a Boolean topos a bouquet is simply a category which is locally* (fully) *equivalent to a group object defined over some covering of* $\mathbb{E}$, i.e. what we have called *a locally given group.*

In an arbitrary topos, the canonical epimorphism $\mathrm{Ob}(\underset{\sim}{G}) \twoheadrightarrow \mathbb{1}$ is always locally split since the epimorphism itself may be viewed as defining a covering $C = \mathrm{Ob}(\underset{\sim}{G}) \twoheadrightarrow \mathbb{1}$ over which the diagonal

$$\Delta\colon \mathrm{Ob}(\underset{\sim}{G}) \longrightarrow \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{Ob}(\underset{\sim}{G}) \xrightarrow{\sim} \mathrm{Ob}(\underset{\sim}{G}|C)$$

defines a canonical section of $\mathrm{Ob}(\underset{\sim}{G})|C \twoheadrightarrow \mathbb{1}$ in the topos $\mathbb{E}/C$. Thus in $\mathbb{E}/\mathrm{Ob}(\underset{\sim}{G})$ the Lemma (2.3) is applicable and then we have an *essential* equivalence $\mathfrak{J}_C\colon \mathrm{aut}_{G|C}(\Delta) \hookrightarrow \underset{\sim}{G}|C$. Thus any bouquet $\underset{\sim}{G}$ is locally essentially equivalent to locally given group. But this is characteristic since conversely, for any category object $\underset{\sim}{G}$ of $\mathbb{E}$ if there is a covering $C \twoheadrightarrow \mathbb{1}$ of $\mathbb{E}$ and a group object G of $\mathbb{E}/C$ for which there exists an essential equivalence $i\colon G \longrightarrow \underset{\sim}{G}|C$ in $\mathbb{E}/C$, then $\underset{\sim}{G}|C$ is a bouquet in $\mathbb{E}/C$ and thus must have been a bouquet in $\mathbb{E}$.

Thus any bouquet of $\mathbb{E}$ is just a category of $\mathbb{E}$ which is locally essentially equivalent to a locally given group which may be taken to be $\mathrm{aut}_{\underset{\sim}{G}|\mathrm{Ob}(\underset{\sim}{G})}(\Delta)$ in $\mathbb{E}/\mathrm{Ob}(\underset{\sim}{G})$. But what is this group object? A simple calculation reveals that it is canonically isomorphic to the subgroupoid $\mathcal{E}(\underset{\sim}{G})$ of automorphisms of $\underset{\sim}{G}$ (1.5.2) now viewed as a group object in $\mathbb{E}/\mathrm{Ob}(\underset{\sim}{G})$, i.e., in the topos $\mathbb{E}/\mathrm{Ob}(\underset{\sim}{G})$ we have a canonical essential equivalence

(2.4.0)

$$\begin{tikzcd}
\mathcal{E}(\underset{\sim}{G}) \arrow[rr, hook] \arrow[d] & & \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{Ar}(\underset{\sim}{G}) \arrow[d, shift right, "\mathrm{Ob}(\underset{\sim}{G})\times T"'] \arrow[d, shift left, "\mathrm{Ob}(\underset{\sim}{G})\times S"] \\
\mathrm{Ob}(\underset{\sim}{G}) \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{id}"] & & \mathrm{Ob}(\underset{\sim}{G}) \times \mathrm{Ob}(\underset{\sim}{G}) \arrow[dl, "\mathrm{pr}_1"] \\
& \mathrm{Ob}(\underset{\sim}{G}) &
\end{tikzcd}$$

and, in summary,we have proved

Theorem (2.5). *A bouquet of* $\mathbb{E}$ *is an internal category* $\underset{\sim}{G}$ *of* $\mathbb{E}$ *which is locally essentially equivalent to a locally given group* (*which may be taken to be the subgroupoid* $\mathcal{E}(\underset{\sim}{G})$ *of internal automorphisms of* $\underset{\sim}{G}$ *considered as a group object in the topos* $\mathbb{E}/\mathrm{Ob}(G)$).

3. THE TIE OF A BOUQUET

(3.0) Since every bouquet $\underset{\sim}{G}\colon A \overset{S}{\underset{T}{\rightrightarrows}} O$ of $\mathbb{E}$ is locally essentially equivalent to the locally given group $\mathcal{E}(\underset{\sim}{G})$ of its internal automorphisms on the covering $O \twoheadrightarrow \mathbb{1}$ the question immediately arises: Is there a globally given group $G \twoheadrightarrow \mathbb{1}$ and a covering $C\colon C \twoheadrightarrow \mathbb{1}$ over which one has locally an essential equivalence $G|C \longrightarrow \underset{\sim}{G}|C$? Suppose that G is such a group and C is such a covering. Then the product $C' = C \times O$ gives a covering of both 0 and C and $\mathbb{1}$ over which one has essential equivalences

$$\mathcal{E}(\underset{\sim}{G})|C' \hookrightarrow \underset{\sim}{G}|C' \quad \text{and} \quad G|C' \hookrightarrow \underset{\sim}{G}|C'$$

but then it follows that the groups $\mathcal{E}(\underset{\sim}{G})|C'$ and $G|C'$ are locally isomorphic over some cover $C'' \twoheadrightarrow C'$ (2.3) and thus that one has an epimorphism $p\colon C'' \twoheadrightarrow O$ over which the groups $G|O$ and $\mathcal{E}(\underset{\sim}{G})$ are isomorphic. Now given any epimorphism $p\colon C'' \twoheadrightarrow O$, the cartesian square

(3.0.0)
$$\begin{tikzcd}
A'' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & A \arrow[d, "\langle T,S\rangle"] \\
C'' \times C'' \arrow[r, "p\times p"'] & O \times O
\end{tikzcd}$$

defines a bouquet $\underset{\sim}{G}''\colon A'' \rightrightarrows C''$ and an essential equivalence

(3.0.1)
$$\begin{tikzcd}
A'' \arrow[r, "\mathrm{pr}_2"] \arrow[d, shift left] \arrow[d, shift right] & A \arrow[d, shift left] \arrow[d, shift right] \\
C'' \arrow[r, "p"'] & O
\end{tikzcd}$$

for which the groups $\mathcal{E}(\underset{\sim}{G})|C''$ and $\mathcal{E}(\underset{\sim}{G}'')$ are canonically isomorphic (as can be easily seen from reasoning on the cube

(3.0.2)
$$\begin{tikzcd}
 & A'' \arrow[rr, "\mathrm{pr}_2"] \arrow[dd] & & A \arrow[dd, "\langle T,S\rangle"] \\
\mathcal{E}(G'') \arrow[ur, hook] \arrow[rr, dashed] \arrow[dd] & & \mathcal{E}(G) \arrow[ur, hook] \arrow[dd] & \\
 & C'' \times C'' \arrow[rr, "p\times p"] & & O \times O \\
C'' \arrow[ur, "\Delta"'] \arrow[rr, "p"] & & O \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

where the dotted arrow is that unique one which makes the top and front squares commutative).

It follows that, up to essential equivalence, in this case *we may restrict our attention to bouquets* $\underset{\sim}{G}\colon A \rightrightarrows O$ *of* $\mathbb{E}$ *and group objects* G *of* $\mathbb{E}$ *which have the property that* *one has an isomorphism* $\mathcal{E}(\underset{\sim}{G}) \xrightarrow{\sim} G|O$ *of group objects in* $\mathbb{E}/O$. Since this means that we have a cartesian square

(3.0.3)
$$\begin{tikzcd}
\underset{\sim}{\mathcal{E}} \arrow[r] \arrow[d] & G \arrow[d] \\
O \arrow[r, two heads] & \mathbb{1}
\end{tikzcd}$$

in $\mathbb{E}$ which makes $\underset{\sim}{\mathcal{E}}(\underset{\sim}{G})$ a localization of G over O, it should follow that we could recover G from the group $\underset{\sim}{\mathcal{E}}(\underset{\sim}{G})$ by means of a descent datum somehow supplied intrinsically by the bouquet $\underset{\sim}{G}$. Now there is indeed a naturally occurring candidate for the provision of such a descent datum: the canonical action of $\underset{\sim}{G}$ on $\underset{\sim}{\mathcal{E}}(G)$ by inner isomorphisms used in (2.2) in conjunction with a functorial section t of $A \xrightarrow{\langle T,S\rangle} O \times O$. Thus consider the diagram

(3.0.4)
$$\begin{tikzcd}
O \times O \times O \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & O \times O \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\mathrm{pr}_1"'] & O \arrow[r, two heads] & \mathbb{1} \\
A \times_O A \arrow[u, two heads] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \arrow[u, two heads, "\langle T,S\rangle"] \arrow[r, shift left, "S"] \arrow[r, shift right, "T"'] & O \arrow[u, "\mathrm{id}"'] &
\end{tikzcd}$$

In the category $\mathbb{E}/A$, we have the group isomorphism

(3.0.5)
$$\begin{tikzcd}
\mathrm{int}(\underset{\sim}{G})^{-1}\colon & A \times_S \underset{\sim}{\mathcal{E}} \arrow[d, phantom, "\parallel"] & A \times_T \underset{\sim}{\mathcal{E}} \arrow[l, "\sim"'] \arrow[d, phantom, "\parallel"] \\
& S^*(\underset{\sim}{\mathcal{E}}) & T^*(\underset{\sim}{\mathcal{E}}) \arrow[l, "\sim"']
\end{tikzcd}$$

given by the assignment

$$(f\colon x \longrightarrow y,\ a\colon y \longrightarrow y) \quad \mapsto \quad (f\colon x \longrightarrow y,\ f^{-1}af\colon x \longrightarrow x)$$

As a "gluing" it is easily seen to satisfy the "cocycle condition" when restricted to the category $\mathbb{E}/A \times_O A$ (c.f. Appendix). Thus in order that it define a true descent datum $d\colon \mathrm{pr}_1^*(\mathcal{E}) \xrightarrow{\sim} \mathrm{pr}_2^*(\mathcal{E})$ in $\mathbb{E}/O \times O$ it is necessary and sufficient that it have the same restriction when pulled back along the two projections of the graph of the equivalence relation associated with the epimorphism $\langle T, S\rangle\colon A \twoheadrightarrow O \times O$. Since this equivalence relation consists of the object of internal ordered pairs of arrows $x \overset{f}{\underset{g}{\rightrightarrows}} y$ which have the same source and target, $\mathrm{int}(\underset{\sim}{G})^{-1}$ defines a descent datum on $\underset{\sim}{\mathcal{E}}(\underset{\sim}{G})$ if and only if for all $a\colon y \longrightarrow y$ and all pairs (f, g), $f^{-1}af = g^{-1}ag$, i.e. "inner isomorphism" be independent of choice of representative. But since $f^{-1}af = (g^{-1}f)^{-1}g^{-1}ag(g^{-1}f)$ and $g^{-1}f\colon x \longrightarrow x$ is an automorphism, this can occur only if $\mathrm{aut}(x)$ is abelian for all x, i.e. if and only if $\mathcal{E}(\underset{\sim}{G}) \longrightarrow O$ is an abelian group object in $\mathbb{E}/O$, a clearly untenable assumption if we wish to consider bouquets which are locally essentially equivalent to the localization of a given non-abelian group (where $\mathcal{E}(\utilde{G}) \longrightarrow O$ cannot be abelian for this would force G to be.)

(3.1) <u>The fibered category of ties of</u> $\mathbb{E}$. What does survive here even if $\utilde{\mathcal{E}}(\utilde{G}) \to O$ is not abelian is based on the observation that $\mathrm{int}(f)^{-1}$ and $\mathrm{int}(g)^{-1}$ while not identical for all f and g do differ by an inner automorphism of x (that defined by $g^{-1}f\colon x \longrightarrow x$) and thus are equal modulo an inner automorphism. This necessitates the replacement of the fibered category (Cf. Appendix) of locally given groups with a new fibered category called the *ties* (fr. *lien*) *of* $\mathbb{E}$.

This new fibered category is defined as follows: First we define the fibered category $\underline{\underline{\mathrm{Tie}}}(\mathbb{E})$ of *pre-ties* of $\mathbb{E}$. Its fiber at any object X of $\mathbb{E}$ has as objects the group objects of $\mathbb{E}/X$. Its morphisms, however, consist of the global sections (over X) of the coequalizer (i.e. orbit space)

$$(3.1.0) \qquad \underline{\mathrm{Hom}}_X(G_1, G_2) \times G_2 \rightrightarrows \underline{\mathrm{Hom}}_X(G_1, G_2) \longrightarrow \underline{\mathrm{Hom}}_X(G_1, G_2)$$

of the sheaf of group homomorphisms of G_1 into G_2 under the action of G_2 by composition with inner automorphisms of G_2. Under pullbacks this defines a fibered category over $\mathbb{E}$ in which morphisms still glue along a covering even though "true existence" may be only local over a some covering $C \twoheadrightarrow X$ of X. We now define the fibered category $\underline{\underline{\mathrm{TIE}}}(\mathbb{E})$ of *ties* of $\mathbb{E}$ by completing it to a stack so that every descent datum in $\underline{\underline{\mathrm{Tie}}}(\mathbb{E})$ over a covering $X \twoheadrightarrow Y$ is effective. An object of the fiber of $\underline{\underline{\mathrm{TIE}}}(\mathbb{E})$ over the terminal object $\mathbb{1}$ will be called a (global) $\underline{\underline{\mathrm{tie}}}$ of $\mathbb{E}$. As we shall see when we discuss the Grothendieck-Giraud theory later in this paper, it will be convenient to regard a tie of $\mathbb{E}$ as represented by an equivalence class under refinement of a descent datum in $\underline{\underline{\mathrm{Tie}}}(\mathbb{E})$ on some locally given group $\utilde{\mathcal{E}} \longrightarrow O$ over a covering $O \twoheadrightarrow \mathbb{1}$ of $\mathbb{E}$, i.e., by some given global section of $\underline{\mathrm{Hex}}(\mathrm{pr}_1^*(\utilde{\mathcal{E}}), \mathrm{pr}_2^*(\utilde{\mathcal{E}}))$ over $O \times O$.

(3.2) Clearly, from our preceding analysis, for any bouquet $\utilde{G}$, the canonical action by inner isomorphisms supplies $\utilde{\mathcal{E}}(\utilde{G}) \longrightarrow O$ with such a descent datum and hence defines, not a global group but rather a global tie which is unique up to a unique isomorphism. It will be called the *tie of the bouquet* $\utilde{G}$ and will be denoted by $\mathrm{tie}(\utilde{G})$. If $\utilde{\mathfrak{J}}\colon \utilde{G}_1 \longrightarrow \utilde{G}_2$ is an essential equivalence of bouquets, then $\mathrm{tie}(\utilde{G}_1) \xrightarrow{\sim} \mathrm{tie}(\utilde{G}_2)$. More details of an alternative description of this may be found in parts II2.1 and II2.4.

4. THE SECOND COHOMOLOGY CLASS $H^2(\mathbb{E}; L)$

We now give the following

<u>Definition</u> (4.0). Let L be a given global tie. We define the 2-category $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E}; L)$ as the 2-subcategory of $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E})$ consisting of the bouquets of $\mathbb{E}$ which have lien isomorphic to L (together with essential equivalences as morphisms). We will designate the class

of connected components of the underlying category by $H^2(\mathbb{E};L)$ and call it the *second cohomology class of* $\mathbb{E}$ *with coefficients in the tie* L. Note that under this relation, a bouquet $\utilde{G}_0$ is in the same connected component as $\utilde{G}_n$ provided there exists a finite sequence of bouquets and essential equivalences

(4.0.0)
$$\begin{tikzcd}
 & & \utilde{G}_{n-1} \arrow[r] \arrow[dl, dotted, dash] & \utilde{G}_n \\
 & {} \arrow[d] & & \\
\utilde{G}_1 \arrow[r] \arrow[d] & \utilde{G}_2 & & \\
\utilde{G}_0 & & &
\end{tikzcd}$$

which connects $\utilde{G}_0$ and $\utilde{G}_n$. Moreover, it can be shown that, in fact, *one step is sufficient*: In effect, for any essential equivalences of bouquets of the form

(4.0.1)
$$\begin{tikzcd}
 & \utilde{G}_2 \arrow[d, "\utilde{\mathfrak{J}}_2"] \\
\utilde{G}_1 \arrow[r, "\utilde{\mathfrak{J}}_1"'] & \utilde{G}
\end{tikzcd}$$

define the 2-*fibered product* $\utilde{G}_1 \overset{2}{\times}_{\utilde{G}} \utilde{G}_2$ *of* $\utilde{G}_1$ *with* $\utilde{G}_2$ over $\utilde{G}$ as the representation of the groupoid which has as objects triplets of the form $(A_1, A_2, \alpha)\colon \utilde{\mathfrak{J}}_1(A_1) \overset{\sim}{\longrightarrow} \utilde{\mathfrak{J}}_2(A_2)$ in $\mathrm{Ob}(\utilde{G}_1) \times \mathrm{Ob}(\utilde{G}_2) \times \mathrm{Ar}(\utilde{G})$ and as arrows, pairs $(f\colon A_1 \longrightarrow B_1\,,\, g\colon A_2 \longrightarrow B_2)$ in $\mathrm{Ar}(\utilde{G}_1) \times \mathrm{Ar}(\utilde{G}_2)$ such that $\utilde{\mathfrak{J}}_2(g)\alpha = \beta\utilde{\mathfrak{J}}_1(f)$ in $\utilde{G}$. Using the canonical projection functors one obtains a square of functors

(4.0.2)
$$\begin{tikzcd}
\utilde{G}_1 \overset{2}{\times}_{\utilde{G}} \utilde{G}_2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\curvearrowright"] & \utilde{G}_2 \arrow[d, "\utilde{\mathfrak{J}}_2"] \\
\utilde{G}_1 \arrow[r, "\utilde{\mathfrak{J}}_1"'] & \utilde{G}
\end{tikzcd}$$

which is commutative up to a natural isomorphism and for which it is easy to show (e.g. using Barr's embedding [BARR (1971)]) that $\utilde{G}_1 \overset{2}{\times}_{\utilde{G}} \utilde{G}_2$ is itself a bouquet with both projections essential equivalences.

If $\mathbb{E}$ is a site, *we define* $H^2(\mathbb{E};L)$ as $H^2(\mathbb{E}^{\sim};L)$ where $\mathbb{E}$ is the associated topos of the sheaves on the site and finally, if G is a sheaf of groups, then we define the *(unrestricted)* *second cohomology class of* $\mathbb{E}$ *with coefficients in the group* G as $H^2(\mathbb{E};\mathrm{tie}(G))$ and point this class by the class of the group G, considered as a bouquet whose tie is that of G. In this case the class of G will be called the *trivial* or *unit* class of $H^2(\mathbb{E};\mathrm{tie}(G))$.

(4.1) Neutral elements. In general, for a given global tie L there may be no globally given group G whose tie is isomorphic to L, indeed, $H^2(\mathbb{E};L)$ may be empty, there being no bouquet whatever in $\mathbb{E}$ whose tie is isomorphic to L. The obstruction to this is measured by an element of $H^3(\mathbb{E};\mathbb{Z}(L))$ where $\mathbb{Z}(L)$ is the "center of L" and will be discussed in (II 6.0) after we have shown how $H^2(\mathbb{E};L)$ may be computed using cocycles. If there exists a group G of $\mathbb{E}$ whose tie is isomorphic to L, then $H^2(\mathbb{E};L)$ is non-empty since any representative of the class of G in $H^2(\mathbb{E};L)$ including G itself will also have its tie isomorphic to L. Such a tie will be said to be *representable (by a group of* $\mathbb{E}$*)* and any bouquet which lies in such a class will be said to be *neutral.* The connected component classes of the neutral bouquets will then form a subset of $H^2(\mathbb{E};L)$ which we will denote by $H^2(\mathbb{E};L)'$ and call the *neutral elements* of $H^2(\mathbb{E};L)$. In the case of $H^2(\mathbb{E};G)$, where G is a global group, the (neutral) class of G will be called the *trivial* (or *unit*) *element* of $H^2(\mathbb{E};G)$.

Because of the nature of the equivalence relation (connected component class) which defines $H^2(\mathbb{E};L)$ the neutrality of a given bouquet may not be readily apparent. We have, however, the following recognition

Theorem (4.2). For any bouquet $\underset{\approx}{G}$ and group G of $\mathbb{E}$, the following statements are equivalent:

(a) $\underset{\approx}{G}$ lies in the same connected component as the group G (i.e., $\underset{\approx}{G}$ is *neutral*);

(b) $\underset{\approx}{G}$ lies in the same connected component as a bouquet $\underset{\approx}{T}$ which admits a global section $s\colon \mathbb{1} \longrightarrow \mathrm{Ob}(\underset{\approx}{T})$ for which $G \xrightarrow{\sim} \mathrm{aut}_{\underset{\approx}{T}}(s) \hookrightarrow \underset{\approx}{T}$ (i.e., $\underset{\approx}{T}$ is split from the left by G, (2.3));

(c) There exists an essential equivalence $\mathfrak{J}\colon \underset{\approx}{U} \longrightarrow \underset{\approx}{G}$ where $\underset{\approx}{U}$ is a bouquet of $\mathbb{E}$ whose object of arrows (and groupoid) structure is defined by an isomorphism $\mathrm{Ar}(\underset{\approx}{U}) \xrightarrow{\sim} \mathrm{Ob}(\underset{\approx}{U}) \times \mathrm{Ob}(\underset{\approx}{U}) \times G$ (i.e., $\underset{\approx}{U}$ is split from the right by G (2,2)).

In effect, if $\underset{\approx}{G}$ lies in the same connected component as a group G, then the identity map $\mathrm{id}(\mathbb{1})\colon \mathbb{1} \longrightarrow \mathbb{1} \cong \mathrm{Ob}(G)$ furnishes a splitting for $\underset{\approx}{G}$ and $\underset{\approx}{G}$ thus lies in the same connected component as a bouquet which is split from the left by G (take G for the bouquet $\underset{\approx}{T}$) so that, trivially, (a) implies (b). If $\underset{\approx}{G}$ lies in the same component as a bouquet $\underset{\approx}{T}$ which is split from the left then one has a diagram of essential equivalences

(4.2.0)
$$\begin{tikzcd}
\utilde{U} \arrow[r, "\utilde{P}_1"] \arrow[d, "\utilde{P}_0"'] & \mathrm{aut}_{\utilde{T}}(s)\ (\cong G) \arrow[d, hook] \\
\utilde{V}_1 \arrow[r, "\utilde{\mathfrak{J}}_1"'] \arrow[d, "\utilde{\mathfrak{J}}_0"'] & \utilde{T} \\
\utilde{G} &
\end{tikzcd}$$

where we have taken for $\utilde{U}$ the 2-fibered product (4.0.2) of $\utilde{\mathfrak{J}}_1$ with the inclusion defined by the splitting $s\colon \mathbb{1} \longrightarrow \utilde{T}$. But since $\utilde{P}_1$ is an essential equivalence of $\utilde{U}$ with the group G , the square

(4.2.1)
$$\begin{tikzcd}
\mathrm{Ar}(\utilde{U}) \arrow[r] \arrow[d, "\langle T,S\rangle"'] & G \arrow[d] \\
\mathrm{Ob}(\utilde{U}) \times \mathrm{Ob}(\utilde{U}) \arrow[r] & \mathbb{1}
\end{tikzcd}$$

is cartesian with $\utilde{\mathfrak{J}}_0\utilde{P}_0\colon \utilde{U} \longrightarrow \utilde{G}$ furnishing the desired essential equivalence for the implication of (c) by (b). Finally, since the condition on $\utilde{U}$ in (c) is just the provision of an essential equivalence $\utilde{U} \xrightarrow{\utilde{\mathfrak{J}}_1} \utilde{G}$, (c) implies (a) and the chain of equivalences is complete.

REMARK (4.2.1). In (5.6) we will define the notion of an $\mathbb{E}$-torsor under a groupoid $\utilde{G}$ and in (8.1.6) we will define for any bouquet $\utilde{G}$ (in a Grothendieck topos) its (*internal*) *completion* $\utilde{\widetilde{G}}$, supplied with an essential equivalence $\mathfrak{k}\colon \utilde{G} \longrightarrow \utilde{\widetilde{G}}$. With these objects in hand, we may add to the above chain of equivalences the following two equivalent conditions:

(d) there exists an $\mathbb{E}$-torsor $\utilde{E}$ under $\utilde{G}$ (whose sheaf of automorphisms is isomorphic to G), i.e. $\mathrm{TORS}(\mathbb{E};\utilde{G}) \neq \emptyset$;

(e) the completion $\utilde{\widetilde{G}}$ of $\utilde{G}$ is split from the left by the group G.

(4.2) $H^2(\mathbb{E}; L)$ is not, in general, functorial on morphisms of ties and $H^2(\mathbb{E}; \mathrm{tie}(G))$ does not, in general, lead to a continuation of the cohomology exact sequence of groups and pointed sets associated with a short exact sequence of groups in $\mathbb{E}$. We will rectify this at a later point by considering a certain class of members of $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E})$ associated with a given group G and related to the sequence in question.

For now, we will define a neutral element preserving bijection of the above defined $H^2(\mathbb{E}; L)$ with the set of the same name defined in GIRAUD (1971).

5. THE FUNCTOR FROM BOUQUETS TO GERBES

Recall the following

Definition (5.0). If $\mathbb{E}$ is a U-small site for some universe U, then a *gerbe* (over $\mathbb{E}$) is a fibered category $\mathbb{F}$ over $\mathbb{E}$ (c.f. Appendix) which satisfies the following conditions:

(a) $\mathbb{F}$ is a *stack* (fr. *champ*), i.e., both objects and arrows in the fiber over any covering glue (c.f. Appendix);

(b) $\mathbb{F}$ is *fibered in* U-small *groupoids*; i.e., for each object X in $\mathbb{E}$, the category fiber $\mathbb{F}_X$ is a U-small groupoid;

(c) there is a covering of $\mathbb{E}$ such that each of the fibers over that covering are non empty; and

(d) any objects x and y of a fiber $\mathbb{F}_X$ are locally isomorphic.

From (a) it follows that for any object x in $\mathbb{F}_X$, the presheaf $\underline{\mathrm{Aut}}_X(x)$ on $\mathbb{E}/X$ is, in fact, a sheaf and that for any isomorphism $f\colon x \longrightarrow y$ in $\mathbb{F}_X$, the induced group isomorphism of $\underline{\mathrm{Aut}}_X(x)$ with $\underline{\mathrm{Aut}}_X(y)$ is unique up to an inner automorphism. Thus conditions (c) and (d) define the existence of a global tie over $\mathbb{E}$ (which is unique up to a unique isomorphism) and is called the *tie of the gerbe* $\mathbb{F}$. For any tie L , Grothendieck and Giraud make the following

Definition (5.1). The *second cohomology set of* $\mathbb{E}$ *with coefficients in the tie* L, $H^2_{\mathrm{Gir}}(L)$ is the set of equivalence classes (under cartesian equivalence of fibered categories) of those gerbes of $\mathbb{E}$ which have tie L. If G is a sheaf of groups, then $H^2_{\mathrm{Gir}}(\mathrm{tie}(G))$ is pointed by class of the gerbe $\underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(G)$ of G-*torsors* over $\mathbb{E}$, whose fiber at any X in $\mathbb{E}$ is the groupoid of torsors (i.e., principal homogenous spaces) in $\mathbb{E}/X$ under the group $G|X$.
(5.2) Torsors under a groupoid. We now intend to establish a mapping (Theorem (5.21)) of our $H^2(\mathbb{E}; L)$ into $H^2_{\mathrm{Gir}}(\mathbb{E}; L)$ which will turn out to be a bijection. To do this we must extend the standard definition of torsor under a sheaf of groups to that of a torsor under a sheaf of groupoids. Recall that this is done as follows:

Definition (5.3). Let $\underset{\approx}{G}\colon A \rightrightarrows O$ be a groupoid in a topos $\mathbb{E}$. By an *internal (Contravariant) functor from* $\underset{\approx}{G}$ *into* $\mathbb{E}$ (or a (right-) *operation* of $\underset{\approx}{G}$ on an object E of $\mathbb{E}$, or more simply, a $\underset{\approx}{G}$-*object of* $\mathbb{E}$) we shall mean an object E of $\mathbb{E}$ supplied with an arrow $P_0\colon E \longrightarrow O$ together with an action $\xi\colon E \times_{P_0,T} A \longrightarrow E$ such that for all

$T \in \mathrm{Ob}(\mathbb{E})$, $\mathrm{Hom}_{\mathbb{E}}(T, \alpha)$ defines a right action of the groupoid $\mathrm{Hom}_{\mathbb{E}}(T, \underset{\approx}{G})$ on the set $\mathrm{Hom}_{\mathbb{E}}(T, E) \longrightarrow \mathrm{Hom}_{\mathbb{E}}(T, O)$ in the usual operator set theoretic sense,

(5.3.0)

$$\begin{tikzcd}
E \times_T A \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, "\mathrm{pr}"'] \arrow[d, shift left, "\xi"] & A \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
E \arrow[r, "P_0"'] \arrow[d] & O \\
\mathbb{1} &
\end{tikzcd}$$

Using the obvious definition of $\underset{\approx}{G}$*-equivariant map of such* $\underset{\approx}{G}$*-objects* (or *internal natural transformation*) we obtain the category $\underline{\underline{\mathrm{OPER}}}(\mathbb{E}; \underset{\approx}{G})$ (also denoted by $\mathbb{E}^{\underset{\approx}{G}^{\mathrm{op}}}$) of $\underset{\approx}{G}$ objects of $\mathbb{E}$ and equivariant maps of $\underset{\approx}{G}$-objects. For each object X of $\mathbb{E}$ we also have the corresponding category of $\underset{\approx}{G}$*-objects of* $\mathbb{E}$ *above* X, defined by $\underline{\underline{\mathrm{OPER}}}(\mathbb{E}/X; \underset{\approx}{G}_X)$. Under pull-backs this defines the corresponding *fibered category* $\underline{\underline{\mathrm{OPER}}}_{\mathbb{E}}(\underset{\approx}{G})$ of $\underset{\approx}{G}$ *objects over* $\mathbb{E}$.

(5.4) For each global section $\mathbb{1} \xrightarrow{\ulcorner x \urcorner} O$ of O, we have the corresponding *internal representable* functor defined by $\lceil x \rceil$ which is defined using the fibered product $\mathbb{1} \times_T A \xrightarrow{S\,\mathrm{pr}_A} O$ for "total space" and the composition in $\underset{\approx}{G}$ to define the action. In sets this just gives the category $\underset{\approx}{G}/\lceil x \rceil$ of $\underset{\approx}{G}$ objects above the object $\lceil x \rceil$. We have the following immediate result:

<u>THEOREM</u> (5.5). *In order that an internal functor ($\underset{\approx}{G}$-object) be representable*, i.e. isomorphic to $\underset{\approx}{G}/\lceil x \rceil$ for some global section $\lceil x \rceil \colon \mathbb{1} \longrightarrow O$, *it is necessary and sufficient that it satisfy the following* two conditions

(a) the canonical map $\langle \mathrm{pr}_E, \xi \rangle \colon E \times_T A \longrightarrow E \times E$ is an isomorphism (i.e., the action ξ is a *principal homogeneous action*); and

(b) the canonical mapping $E \longrightarrow \mathbb{1}$ admits a *splitting* $s \colon \mathbb{1} \longrightarrow E$ (i.e. E is globally non empty).

In effect, since $\underset{\approx}{G}$ is a groupoid, the operation of $\underset{\approx}{G}$ on $\underset{\approx}{G}/\lceil x \rceil$ by composition is always principal and the composition of the arrows $x \colon \mathbb{1} \longrightarrow O \xrightarrow{\mathrm{id}} A$ canonically defines a splitting of $\underset{\approx}{G}/\lceil x \rceil \longrightarrow \mathbb{1}$. Conversely, if $s_0 \colon \mathbb{1} \longrightarrow E$ is a splitting of $E \longrightarrow \mathbb{1}$ and $\underset{\approx}{G}$ operates principally, then the endomorphism of E with itself defined by s_0 lifts to $s_1 \colon E \longrightarrow E \times_O A$ and the square

(5.5.0)

$$\begin{tikzcd}
E \arrow[r, "\mathrm{pr}_A\, s_1"] \arrow[d] & A \arrow[d, "T"] \\
\mathbb{1} \arrow[r, "P_0\, s_0"] & O
\end{tikzcd}$$

is cartesian.

We now make the following

Definition (5.6). By a $\underset{\sim}{G}$*-torsor of* $\mathbb{E}$ *above* X we shall mean an $\underset{\sim}{G}_X$-object of $\mathbb{E}/X$ which is *locally representable*, i.e. becomes representable when restricted to some covering of $\mathbb{E}/X$. For the canonical topology on $\mathbb{E}$, this is entirely equivalent to the following two conditions on the defining diagram in $\mathbb{E}$

(5.6.0)
$$\begin{tikzcd}
E \times_T A \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, "\mathrm{pr}"'] \arrow[d, shift left, "\xi"] & A \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
E \arrow[r] \arrow[d, "P"'] & O \\
X &
\end{tikzcd}$$

(a) *the canonical map* $E \times_T A \xrightarrow{\langle \mathrm{pr}_E, \xi \rangle} E \times_X E$ *is an isomorphism*, and

(b) *the canonical map* $p\colon E \longrightarrow X$ *is an epimorphism.*

A torsor is thus *representable, if and only if it is split*, i.e. p *admits a section* $s\colon X \longrightarrow E$.

(5.7) Again, since this definition is stable under pull backs we have the corresponding fibered category $\underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\sim}{G})$ of $\underset{\sim}{G}$*-torsors of* $\mathbb{E}$ whose fiber at any X is $\underline{\mathrm{TORS}}(\mathbb{E}/X; \underset{\sim}{G}|X)$. If G is a group object in $\mathbb{E}$, we immediately recover the usual definition of G-objects of $\mathbb{E}$ and G-torsors of $\mathbb{E}$. Note also that a group, as a category, has up to isomorphism only one representable functor, which is just G_d, i.e. G acting on itself on the right by multiplication.

(5.8) As with the case of group objects, $\underline{\mathrm{OPER}}(\mathbb{E}, \underset{\sim}{G})$ is functorial on functors $\mathfrak{J}\colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ of groupoids $\underline{\mathrm{OPER}}(\mathbb{E}, \mathfrak{J})\colon \underline{\mathrm{OPER}}(\mathbb{E}, \underset{\sim}{G}_2) \longrightarrow \underline{\mathrm{OPER}}(\mathbb{E}, \underset{\sim}{G}_1)$ is just defined by "restricting" the $\underset{\sim}{G}_2$-action on to the object $E \xrightarrow{P_0} O_2$ to that of $\underset{\sim}{G}_1$ on $E_{\mathfrak{J}_0} \times_{P_0} O_1 \xrightarrow{\mathrm{pr}} O_1$. Also, as with groups, $\underline{\mathrm{OPER}}(\mathbb{E}, \mathfrak{J})$ has a left exact left adjoint which carries torsors under $\underset{\sim}{G}_1$ to torsors (above the same base) under $\underset{\sim}{G}_2$. Its restriction to the corresponding categories of torsors will be denoted by

(5.8.0)
$$\underline{\mathrm{TORS}}_X(\mathfrak{J})\colon \underline{\mathrm{TORS}}(\mathbb{E}/X, \underset{\sim}{G}_1) \longrightarrow \underline{\mathrm{TORS}}(\mathbb{E}/X, \underset{\sim}{G}_2)\,.$$

In (5.16) we will give this construction in detail as it is used in the proof of the theorem which follows.

We may now state the principal result of this section:

Theorem (5.9). *Let* $\mathbb{E}$ *be a Grothendieck topos* (over some small U-site) *and* $\underset{\sim}{G}$ *a groupoid object of* $\mathbb{E}$*. Then*

1° *the fibered category* $\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\underset{\approx}{G})$ *is a* U*-small stack of groupoids* over $\mathbb{E}$ (i.e., each of the fibers is a U-small groupoid and descent data on $\underset{\approx}{G}$-torsors defined over a covering is always effective (on both objects and arrows.)

2° *if* $\underset{\approx}{G}$ *is a bouquet, then* $\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\underset{\approx}{G})$ *is a gerbe whose tie is isomorphic to the tie of* $\underset{\approx}{G}$.

3° *If* $\mathfrak{J}\colon \underset{\approx}{G}_1 \longrightarrow \underset{\approx}{G}_2$ *is an essential equivalence, then*

$$\underline{\underline{\underline{\mathrm{TORS}}}}(\mathfrak{J})\colon \underline{\underline{\underline{\mathrm{TORS}}}}(\underset{\approx}{G}_1) \longrightarrow \underline{\underline{\underline{\mathrm{TORS}}}}(\underset{\approx}{G}_2)$$

is a (full) cartesian equivalence of fibered categories.

In effect, for 1° note that every morphism of $\underset{\approx}{G}$-torsors over an arrow $f\colon X \to Y$ in $\mathbb{E}$ is cartesian, i.e. the commutative square in the diagram of morphisms of $\underset{\approx}{G}$-torsors

(5.9.0)
$$\begin{tikzcd}
 & & O \\
E_1 \arrow[r, "f_1"] \arrow[urr] \arrow[d, two heads, "P_X"'] & E_2 \arrow[ur] \arrow[d, two heads, "P_Y"] & \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

is cartesian (since the action is principal and P_X is a universal effective epimorphism, the fact that the square

(5.9.1)
$$\begin{tikzcd}
E_1 \times_O A \arrow[r, "f_1 \times A"] \arrow[d, "\mathrm{pr}_{E_1}"'] & E_2 \times_O A \arrow[d, "\mathrm{pr}_{E_2}"] \\
E_1 \arrow[r, "f_1"'] & E_2
\end{tikzcd}$$

is necessarily cartesian makes the Grothendieck lemma [GROTHENDIECK (1962)] applicable or, equivalently Barr's embedding theorem for which this lemma is a principal example). Thus if f is the identity (so that one has a morphism of $\underset{\approx}{G}$-torsors over X) $f_1\colon F_1 \longrightarrow F_2$ is an isomorphism and each fiber is a groupoid. That torsors defined over a covering and supplied with descent data necessarily glue is a similar diagram chasing exercise from the theory of Barr-exact categories whose proof we leave to the reader.

That each fiber is equivalent to a category whose objects and arrows are members of the universe U is based on the following

Lemma (5.10). Let $p\colon C \twoheadrightarrow X$ be an epimorphism and

$$\begin{tikzcd} C_\bullet\colon C\times_X C\times_X C \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C\times_X C \arrow[r, shift left] \arrow[r, shift right] & C \arrow[r, two heads] & X \end{tikzcd} \tag{5.10.0}$$

its (truncated) nerve. The groupoid $\underline{\mathrm{TORS}}(C_\bullet/X;\underset{\sim}{G})$ of $\underset{\sim}{G}$-torsors above X which are split (i.e. representable with a given representation) when restricted along the epimorphism p is equivalent to the groupoid $\underline{\mathrm{Simpl}}_{\mathbb{E}}(C_\bullet,\underset{\sim}{G})$ of simplicial homotopies of simplicial mappings of $C_\bullet$ into (the nerve of) $\underset{\sim}{G}$. (This latter groupoid is, of course. isomorphic to the groupoid of internal functors and natural transformations of the internal groupoid whose structural maps are given by $C\times C \rightrightarrows C$ into the internal groupoid $\underset{\sim}{G}$.)

Using this lemma and passing to the filtered limit under refinement we see that the set of isomorphism classes of the $\underline{\mathrm{TORS}}(X;G)$ is equivalent to that of the groupoid $\varinjlim_{C\in E_{pi}/X} \underline{\mathrm{TORS}}(C/X;G)$. Since every epimorphism is refined by a representable covering (i.e. an epimorphism of the form $\coprod_{x\in I} a(x) \longrightarrow X$, where x is an object of the original site, and $a(x)$ is its associated sheaf) we have that

$$\underline{\mathrm{TORS}}(X;\underset{\sim}{G}) \approx \varinjlim_{C_\bullet\in\mathrm{Cov}(X)} \underline{\mathrm{Simpl}}_{\mathbb{E}}(C_\bullet,\underset{\sim}{G}) \tag{5.10.1}$$

and thus is a U-groupoid since it is equivalent to a groupoid whose objects and arrows are members of the universe U.

The proof of Lemma (5.10) is a modification of the classical proof which classifies torsors under a group by means of Čech-cocycles with coefficients in a group and will be given elsewhere since it is properly a part of the general theory of monadic ("triple") cohomology combined with a generalization of Grothendieck's notion of $\varinjlim$.

(5.11) for the proof of part 2° of Theorem (5.9) we first note that if $\underset{\sim}{G}\colon A \rightrightarrows O$ is a bouquet then the epimorphism $O \twoheadrightarrow 1$ supplies us with a covering of $\mathbb{E}$ for which $\underline{\mathrm{TORS}}(O;\underset{\sim}{G})$ is non-empty. In effect, the canonical split $\underset{\sim}{G}$-torsor above O which represents the identity arrow $\mathrm{id}(O)\colon O \longrightarrow O$ is always present:

$$\begin{tikzcd}[row sep=large] A\times_O A \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & A \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\ A \arrow[r, "S"'] \arrow[d, "T"] & O \arrow[u, bend left=50, "\mathrm{id}"] \arrow[d, two heads] \\ O \arrow[u, bend left=50, "\mathrm{id}"] \arrow[ur, dashed, "\mathrm{id}(O)"'] \arrow[r, two heads] & \mathbb{1} \end{tikzcd} \tag{5.11.0}$$

We now must show that any two $\underset{\sim}{G}$-torsors above X are locally isomorphic. For this we will use the following lemma and its immediate corollary (the proofs of which we leave to the reader).

<u>Lemma</u> (5.12). (<u>Internal Yoneda Lemma</u>): Let E be a split torsor above X which represents the X-object $x\colon X \longrightarrow O$ and let $\underset{\approx}{F} = (F \xrightarrow{f_0} O, \alpha)$ be any $\underset{\approx}{G}$-object above X. There is a bijective correspondence between the set of operator maps of E into F above X (i.e., internal natural transformations) and the set of arrows $f\colon X \longrightarrow F$ such that $f_0 f = x$, i.e. the value of x at F).

$$(5.12.0) \qquad \mathrm{OPER}_X(E, \underset{\approx}{F}) \xrightarrow{\sim} \Gamma_X(F \times_O X) \overset{\mathrm{def}}{=} \Gamma_X(F(x))$$

<u>Corollary</u>(5.13). *Let E_1 and E_2 be split $\underset{\approx}{G}$-torsors above X, with E_1 representing $x_l\colon X \longrightarrow O$ and E_2 representing $x_2\colon X \longrightarrow O$. The set of $\underset{\approx}{G}$-torsor maps above X of E_1 into E_2 is in bijective correspondence with the set all arrows $f\colon X \longrightarrow A$ such that $Sf = x_l$ and $Tf = x_2$.*

$$(5.13.0) \qquad \mathrm{TORS}_X(E_1, E_2) \xrightarrow{\sim} \Gamma_X(\underline{\mathrm{Hom}}_X(x_1, x_2))\,.$$

(5.14) Now let E_1 and E_2 be arbitrary $\underset{\approx}{G}$-torsors above X. Since E_1 is split when pulled back along the epimorphism $p_1\colon E_1 \longrightarrow X$ where it represents $f_0^1\colon E_1 \longrightarrow O$, and E_2 is split when pulled back along the epimorphism $P_2\colon E_2 \longrightarrow X$ where it represents $f_0^2\colon E_2 \longrightarrow O$, both are split when pulled back along $E_l \times_X E_2 \xrightarrow{p_1\,\mathrm{pr}} X$. Now since $\underset{\approx}{G}$ is a bouquet, so that $\langle T, S\rangle\colon A \twoheadrightarrow O \times O$ is an epimorphism, we may continue to pull them back along the epimorphism $\mathrm{pr}_1\colon A^{\#} \longrightarrow E_1 \times_X E_2$ defined through the cartesian square

(5.14.0)

$$\begin{tikzcd}
A^{\#} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & A \arrow[d, "\langle T,S\rangle"] \\
E_1 \times_X E_2 \arrow[r, "\langle f_0^1\mathrm{pr}_2\,,\, f_0^2\mathrm{pr}_2\rangle"'] & O \times O
\end{tikzcd}$$

where they remain split and represent, respectively, $f_0^1\mathrm{pr}_1$ and $f_0^2\mathrm{pr}_1$. But there, the arrow $\mathrm{pr}_2\colon A^{\#} \longrightarrow A$ represents an isomorphism between these two split torsors. Thus both of the original torsors become isomorphic when localized over the epimorphism $p_1\mathrm{pr}_1\mathrm{pr}_1\colon A^{\#} \longrightarrow X$.

We have now shown that $\underline{\underline{\mathrm{TORS}}}(\underset{\approx}{G})$ is a gerbe. Let us look at its tie which is calculated by gluing (the tie of a representable localization of) the sheaf of groups $\underline{\mathrm{Aut}}(A) \longrightarrow O$ where A is the split torsor defined by $\begin{tikzcd} A \arrow[r, "T"] & O \arrow[l, bend left, "\mathrm{id}"] \end{tikzcd}$ which represents the identity map $\mathrm{id}(O)\colon O \longrightarrow O$. But for any $t\colon U \longrightarrow O$, $\underline{\mathrm{Aut}}(A)(U) = \mathrm{TORS}_T(A|U, A|U)$ which by Corollary 6, is just bijectively equivalent to the set of all arrows $f\colon u \longrightarrow A$ such that $Sf = t$ and $Tf = t$ which, in turn, is bijectively equivalent to the set of all arrows from U into the group object $\mathcal{E}(\underset{\approx}{G}) \longrightarrow O$ of internal automorphisms of $\underset{\approx}{G}$. Thus $\underline{\mathrm{Aut}}(A) \longrightarrow O$ is just isomorphic to the locally given group $\mathcal{E}(\underset{\approx}{G}) \longrightarrow O$ which we used to define the tie of $\underset{\approx}{G}$ and thus the two liens are isomorphic.

(5.15) In order to prove part 3° of Theorem (5.9), note that any functor $\underset{\sim}{\mathfrak{J}}\colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ gives rise to the following augmented bisimplicial diagram in $\mathbb{E}$ (called the *internal profunctor from* $\underset{\sim}{G}_2$ to $\underset{\sim}{G}_1$ *adjoint to* $\underset{\sim}{\mathfrak{J}}$).

(5.15.0)

$$\begin{tikzcd}
(O_1 \times_T A_2) \times A_2 \times A_1 \arrow[r, shift left, "\mathrm{pr}"] \arrow[r, shift right, "\mathscr{H}"'] \arrow[d, shift right, "\mathrm{pr}"'] \arrow[d, shift left, "\mathscr{H}(F)^{-1}"] & (O_1 \times_T A_2) \times_T A_2 \arrow[r, "\mathrm{pr}"] \arrow[d, shift right] \arrow[d, shift left, "\mathscr{H}(F)^{-1}"] & A_1 \arrow[l, dashed, bend left] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
(O_1 \times_T A_2) \times_T A_2 \arrow[r, shift left, "\mathrm{pr}"] \arrow[r, shift right, "\mathscr{H}"'] \arrow[d, "\mathrm{pr}"'] & O_1 \times_T A_2 \arrow[r, "\mathrm{pr}"'] \arrow[d, "S\mathrm{pr}"] & O_1 \arrow[l, dashed, bend left, "F"] \\
A_2 \arrow[r, shift left, "T"] \arrow[r, shift right, "S"'] & O_2 &
\end{tikzcd}$$

in which the object $O_1 \times_T A_2 = \{(X,\ a\colon A \longrightarrow F(x))\}$ plays two roles: first, supplied with the arrow $S\mathrm{pr}\colon (X,\ A \longrightarrow F(X)) \mapsto A$, simple composition with an arrow $B \xrightarrow{u} A$ in A_2 defines an action of $\underset{\sim}{G}_2$ on $O_1 \times_T A_2$ which is compatible with $\mathrm{pr}\colon O_1 \times_T A_2 \longrightarrow O_1$ and makes the lower horizontal augmented complex into a torsor under $\underset{\sim}{G}_2$ which is split by the arrow
$F\colon X \mapsto (x, \mathrm{id}\colon (F(X) \longrightarrow F(X))$. It thus is the $\underset{\sim}{G}_2$ torsor above O_1 which represents the object mapping $\mathrm{Ob}(\underset{\sim}{F})\colon \mathrm{Ob}(\underset{\sim}{G}_1) \longrightarrow \mathrm{Ob}(\underset{\sim}{G}_2)$. Second, supplied with the arrow $\mathrm{pr}\colon O_1 \times_T A_2 \longrightarrow O_1$, the groupoid $\underset{\sim}{G}_1$ acts on it via the mapping
$\mathscr{H}(F)^{-1}\colon (X,\ A \xrightarrow{a} F(X),\ Y \xrightarrow{f} X) \mapsto (Y,\ A \xrightarrow{F(f)^{-1}a} F(Y))$. This action is moreover clearly compatible with $S\mathrm{pr}\colon O_1 \times_T A_2 \longrightarrow O_2$ and is a *principal action under* $\underset{\sim}{G}_1$ *if and only if* $\underset{\sim}{\mathfrak{J}}\colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ *is fully faithful and thus is a torsor under* $\underset{\sim}{G}_1$ *above* O_2 *if and only if* $\underset{\sim}{\mathfrak{J}}\colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ *is an essential equivalence.* It follows then that if $\underset{\sim}{\mathfrak{J}}$ is an essential equivalence, then $(O_1 \times_T A_2) \times_T A_2 \xrightarrow{\mathrm{pr}} A_2$ is also a torsor under $\underset{\sim}{G}_1$ (above A_2) as is its pull back over $A_2 \times_{T,S} A_2$ along $\mathrm{pr}\colon A_2 \times_{T,S} A_2 \longrightarrow A_2$. In any case, a similar observation shows that $\mathrm{pr}\colon (O_1 \times_T A_2) \times_T A_2 \longrightarrow A_1$ is a split torsor under $\underset{\sim}{G}_2$ above A_l and so is its pull back to $A_1 \times_{T,S} A$. Thus, in summary, we see that any functor $\underset{\sim}{\mathfrak{J}}\colon \underset{\sim}{G}_1 \longrightarrow \underset{\sim}{G}_2$ gives rise to a canonical simplicial complex of split $\underset{\sim}{G}_2$-torsors whose augmentation is the nerve of the category $\underset{\sim}{G}$, and if $\underset{\sim}{\mathfrak{J}}$ is an essential equivalence, to a simplicial complex of $\underset{\sim}{G}_1$-torsors whose augmentation is the nerve of $\underset{\sim}{G}_2$.

(5.16) Using this pair of complexes (5.15) we now note the functor

$$\underline{\mathrm{TORS}}(X;\underset{\sim}{\mathfrak{J}})\colon \underline{\mathrm{TORS}}(X;\underset{\sim}{G}_1) \longrightarrow \underline{\mathrm{TORS}}(X;\underset{\sim}{G}_2)$$

is obtained as follows: given any $\underset{\sim}{G}_1$-torsor above X, its structural maps into $\underset{\sim}{G}_1$ may be used to pull back the simplicial complex of split $\underset{\sim}{G}_2$-torsors to obtain a simplicial complex of $\underset{\sim}{G}_2$-torsors over the nerve of the torsor. But when viewed as the nerve of a covering of X this complex may be viewed as defining a descent datum on the $\underset{\approx}{G}_2$-torsor $U = (O_1 \times_T A_2) \times_O E \xrightarrow{\mathrm{pr}} E$, where E is the total space of the $\underset{\approx}{G}_1$-torsor $E \longrightarrow X$. Since $\underset{\approx}{G}_2$-torsors defined on a covering of X glue to produce an $\underset{\approx}{G}_2$-torsor above X, we define the image of E under $\underline{\mathrm{TORS}}(X;\mathfrak{J})$ to be the so glued $\underset{\approx}{G}_2$-torsor T above X.

(5.16.0)

$$\begin{tikzcd}
 & (O_1 \times_T A_2) \times A_2 \times A_1 \arrow[rr, shift left, "\mathrm{pr}"] \arrow[rr, shift right, "\mathscr{H}"'] \arrow[dd, shift right, "\mathrm{pr}"'] \arrow[dd, shift left, "\mathscr{H}(F)^{-1}"] & & (O_1 \times_T A_2) \times_T A_2 \arrow[rr, "\mathrm{pr}"] \arrow[dd, shift right] \arrow[dd, shift left, "\mathscr{H}(F)^{-1}"] & & A_1 \arrow[dd, shift right, "T"'] \arrow[dd, shift left, "S"] \\
 & & U \times A_1 \arrow[ur] \arrow[rr, "\mathrm{pr}"] \arrow[dd, shift right] \arrow[dd, shift left] & & E \times A_1 \arrow[ur] \arrow[dd, shift right] \arrow[dd, shift left] & \\
 & (O_1 \times_T A_2) \times_T A_2 \arrow[rr, shift left, "\mathrm{pr}"] \arrow[rr, shift right, "\mathscr{H}"'] \arrow[dd, "\mathrm{pr}"] & & O_1 \times_T A_2 \arrow[rr, "\mathrm{pr}"] \arrow[dd, "S\mathrm{pr}"] & & O_1 \\
U \times_T A_2 \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & U \arrow[ur] \arrow[rr] \arrow[dd] & & E \arrow[ur, "f_0"] \arrow[dd, "P_E"] & \\
 & A_2 \arrow[rr, shift left, "T"] \arrow[rr, shift right, "S"'] & & O_2 & & \\
T \times_T A_2 \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] & & T \arrow[ur, "g_0"] \arrow[rr, "\mathrm{pr}"] & & X &
\end{tikzcd}$$

In more conventional terms. this defines T as the quotient of U under the "diagonal" action of $\underset{\approx}{G}_1$:

$$(e, f_0(e),\ A \xrightarrow{h} F(F_0(e))\ ,\ Y \xrightarrow{u} f_0(e)) \mapsto (e^u, Y,\ A \xrightarrow{F(u)^{-1}h} F(Y))\ .$$

(5.17) We now may construct a quasi-inverse for $\underline{\mathrm{TORS}}(X;\mathfrak{J})$ given that $\mathfrak{J}$ is an essential equivalence which, as we have already noted, is itself equivalent to the assertion that $S\mathrm{pr}\colon O_1 \times_T A_2 \longrightarrow O_2$ is a $\underset{\approx}{G}_1$-torsor above O_2. If so, then given an $\underset{\approx}{G}_2$-torsor T above X, $S\mathrm{pr}_1$ (and its associated simplicial system) may be pulled back to T to give a descent datum on this $\underset{\approx}{G}_1$-torsor on the nerve of T viewed as a covering of X. Consequently, it may be glued to give an $\underset{\approx}{G}_1$-torsor E above X whose image under $\underline{\mathrm{TORS}}(X;\mathfrak{J})$ is easily seen to be isomorphic to the original T. Since it is clear that $\underline{\mathrm{TORS}}(X;\mathfrak{J})$ is fully faithful, this completes the proof of 3° of Theorem (5.9).

(5.18) Example. Let $p\colon A \longrightarrow B$ be an epimorphism of groups and T a torsor under B. If $\partial^2(T)$ is the coboundary bouquet associated with T along p (1.5), then the gerbe $\underline{\underline{\mathrm{TORS}}}(\partial^2(T))$ is cartesian equivalent to the *gerbe of liftings of* T *to* A as defined in GIRAUD (1971). This will be discussed in detail when we discuss separately the cohomology of groups.

(5.19) Now let $\mathbb{E}$ be a U-small site and $\mathbb{E}^{\sim}$ its associated category of sheaves. From Theorem (5.9) it follows that for any bouquet $\underset{\approx}{G}$ with tie L, the restriction $\underline{\underline{\mathrm{TORS}}}(\mathbb{E};\underset{\approx}{G})$ of the gerbe $\underline{\underline{\mathrm{TORS}}}(\mathbb{E}^{\sim};\underset{\approx}{G})$ to $\mathbb{E}$ (whose fiber for any representable X is just $\underline{\underline{\mathrm{TORS}}}(\mathbb{E}^{\sim}/a(X);\underset{\approx}{G})$ where $a(X)$ is the associated sheaf) is a gerbe over $\mathbb{E}$ whose tie is also L. Moreover. if $\underset{\approx}{G}_1$ and $\underset{\approx}{G}_2$ lie in the same connected component of $\underline{\mathrm{BOUQ}}(\mathbb{E}^{\sim}, L)$, then $\underline{\underline{\mathrm{TORS}}}(\mathbb{E};\underset{\approx}{G}_1)$ is cartesian equivalent to $\underline{\underline{\mathrm{TORS}}}(\mathbb{E};\underset{\approx}{G}_2)$. Thus we have

Theorem (5.20). *The assignment*

$$\underset{\approx}{G} \longmapsto \underline{\underline{\mathrm{TORS}}}(\mathbb{E};\underset{\approx}{G})$$

defines a functor

$$\boxed{\underline{\mathrm{TORS}}_{\mathbb{E}}\colon \underline{\mathrm{BOUQ}}(\mathbb{E}^{\sim};L) \longrightarrow \underline{\mathrm{GERB}}(\mathbb{E};L)} \tag{5.20.0}$$

which, in turn, induces a mapping

$$\boxed{T\colon H^2(\mathbb{E};L) \longrightarrow H^2_{\mathrm{Gir}}(\mathbb{E};L)} \tag{5.20.1}$$

which preserves the base point (*if* $L \xrightarrow{\sim} \mathrm{tie}(G)$) *and also neutral classes.* (A gerbe is said to be *neutral* if and only if it admits a cartesian section).

We now may state the main result of this paper,

Theorem (5.21). *The mapping*

$$T\colon H^2(\mathbb{E};L) \longrightarrow H^2_{\mathrm{Gir}}(\mathbb{E};L)$$

is a bijection which carries the set of equivalence class of neutral elements, $H^2(\mathbb{E};L)'$, *bijectively onto* $H^2_{\mathrm{Gir}}(\mathbb{E};L)'$, *the equivalence classes of the neutral elements of* $H^2_{\mathrm{Gir}}(\mathbb{E};L)$.

6. THE FUNCTOR FROM GERBES TO BOUQUETS.

We will establish this theorem (5.21) by defining a functor from $\underline{\mathrm{GERB}}(\mathbb{E};L)$ to $\underline{\mathrm{BOUQ}}(\mathbb{E};L)$ which will induce an inverse for T. Background information for the constructions used in this and the following section may be found in the Appendix.

(6.0) To this end recall [GIRAUD (1962)] that if $\mathbb{F}$ is a category over $\mathbb{E}$, then for each object X in $\mathbb{E}$ we may consider the category $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F})$ whose objects are cartesian $\mathbb{E}$ functors from the category $\mathbb{E}/X$ of objects of $\mathbb{E}$ above X into $\mathbb{F}$ and whose morphisms are $\mathbb{E}$-natural transformations of such $\mathbb{E}$-functors. If $\mathbb{F}_X$ denotes the category fiber of $\mathbb{F}$ at X (whose arrows are those of $\mathbb{F}$ which project onto the identity of X), then evaluation of such a cartesian functor at the terminal object $X \xrightarrow{\mathrm{id}} X$ of $\mathbb{E}/X$ defines a functor $\mathrm{ev}_X\colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F}) \longrightarrow \mathbb{F}_X$ which is an equivalence of categories provided $\mathbb{F}$ is a fibration. If $\mathbb{E}$ is U-small and $\mathbb{F}$ is fibered in U-small categories, then the assignment $X \mapsto \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F})$ defines a presheaf of U-small categories and thus a category object of $\mathbb{E}^{\wedge}$ which will be denoted by $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$. The split fibration over $\mathbb{E}$ which it determines is denoted by $\mathcal{S}\mathbb{F}$ and is called the (right) *split fibration* $\mathbb{E}$*-equivalent to* $\mathbb{F}$.

(6.1) Now suppose that $\mathbb{E}$ is a U-small site. Since the associated sheaf functor $a\colon \mathbb{E}^{\wedge} \longrightarrow \mathbb{E}^{\sim}$ is left-exact, we may apply the functor to the nerve of category object $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ and obtain a sheaf of categories $a\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ together with a canonical *functor* (in $\mathrm{CAT}(\mathbb{E}^{\wedge})$)

(6.1.0)
$$\begin{tikzcd}
\mathrm{Ar}(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[r, "a_1"'] \arrow[d, shift right, "\mathbb{T}"'] \arrow[d, shift left, "\mathcal{S}"] & a\,\mathrm{Ar}(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[d, shift right, "a(T)"'] \arrow[d, shift left, "a(S)"] \\
\mathrm{Ob}(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[r, "a_0"'] & a\,\mathrm{Ob}(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))
\end{tikzcd}$$

$$\overline{\underline{a}\colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}) \longrightarrow a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})\,.}$$

(The split fibration over $\mathbb{E}$ determined by $a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ is denoted by $\mathbb{K}\mathcal{S}\mathbb{F}$). We may now state

Theorem (6.2).

1° *If* $\mathbb{F}$ *is fibered in groupoids, then* $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ [*as well as* $a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$] *is a groupoid* (*and conversely*)*;*

2° *if* $\mathbb{F}$ *is a stack then* $\underline{a}\colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}) \longrightarrow a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ *is an equivalence of categories in* $\mathbb{E}^{\wedge}$ (*and conversely*)*;*

3° *if* $\mathbb{F}$ *is a gerbe with tie* L*, then* $a\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ *is a bouquet with tie* L *in* $\mathbb{E}^{\sim}$ (*and conversely*)*.*

On the basis of Theorem (6.2) we see that the assignment

$$\mathbb{F} \mapsto a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$$

defines a functor

$$\boxed{a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,-) \colon \underline{\mathrm{GERB}}(\mathbb{E};L) \longrightarrow \underline{\mathrm{BOUQ}}(\mathbb{E};L)} \tag{6.2.1}$$

and thus a mapping

$$\boxed{a \colon H^2_{\mathrm{Gir}}(\mathbb{E};L) \longrightarrow H^2(\mathbb{E};L)} \tag{6.2.2}$$

which we claim provides an inverse for $T \colon H^2(\mathbb{E};L) \longrightarrow H^2_{\mathrm{Gir}}(\mathbb{E};L)$ [(5.20.1)].

In order to prove Theorem (6.2) and complete the proof of Theorem (5.21), we shall now discuss certain relations between "internal and external completeness" for fibrations and category objects.

(6.3) Remark. We note that a portion of the "internal version" of what follows (specifically Theorem (7.5) 1°) was enunciated by JOYAL (1974) and later, but independently, by PENON and BOURN (1978). It is also closely related to work of BUNGE and PARÉ (1979) establishing a conjecture of LAWVERE (1974) as well as that of STREET (1980). Our principal addition here is the relation between external and internal completeness.

7. EXTERNAL AND INTERNAL COMPLETENESS.

(7.0) The informal and intuitive definition of a stack which we have used so far ("every descent datum on objects or arrows is effective" or "objects and arrows from the fibers over a covering glue") GROTHENDIECK (1959) implicitly uses the respective notions of *fibered category* (c.f. Appendix) (*defined as a pseudo functor* $\mathbb{F}_{(\ )} \colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{CAT}$ as formalized by Grothendieck in 1960 [GROTHENDIECK (1971)] and *covering* as defined in the original description of Grothendieck topologies formalized in ARTIN (1962).(loc. cit.)

For the formal development of the theory, however, both of these notions proved cumbersome and were replaced by the intrinsic formulation of GIRAUD (1962, 1971) which replaces "*covering*" by "*covering seive*" (which we shall view as a subfunctor $R \hookrightarrow X$ of a representable X in $\mathbb{E}\hat{\ }$ as in DEMAZURE (1970)) and notes that every descent datum over a covering of X corresponds to an $\mathbb{E}$-*cartesian functor*

$$\begin{tikzcd} \mathbb{E}/R \arrow[r, "d"] \arrow[d] & \mathbb{F} \arrow[dl] \\ \mathbb{E} & \end{tikzcd} \tag{7.0.0}$$

from the category $\mathbb{E}/R$ *of representables of* $\widehat{\mathbb{E}}$ *above the covering presheaf* R (the corresponding covering seive of X) *into the fibration* ($\mathbb{E}$-category) defined by the pseudofunctor $\mathbb{F}_{(\,)}$. [c.f. Appendix].

Thus *every descent datum on arrows, respectively, on both objects and arrows is effective if and only if the canonical restriction functor*

$$i\colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F}) \longrightarrow \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/R,\mathbb{F}) \tag{7.0.1}$$

is fully faithful, respectively, *an equivalence of categories for every covering subfunctor* $R \hookrightarrow X$ *in the topology of the site.*

The corresponding terminology for a fibration $\mathbb{F} \longrightarrow \mathbb{E}$ over a site $\mathbb{E}$ is then

Definition (7.1). A fibration $\mathbb{F}$ is said to be *precomplete*, respectively, *complete*, provided that for every covering seive $R \hookrightarrow X$ of a representable X, the canonical functor $i\colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F}) \longrightarrow \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/R,\mathbb{F})$ is fully faithful, respectively, an equivalence of categories.

Since every fibration is determined up to equivalence by its associated pseudo-functor we see that "*a stack is just a complete fibration*".

(7.2) We now look at the "internal version" of this notion. Let $\mathbb{E}$ be a topos and $\underset{\sim}{G}\colon A \rightrightarrows O$ a category object in $\mathbb{E}$. The presheaf of categories defined by the assignment $X \mapsto \mathrm{HOM}_{\mathbb{E}}(X,\underset{\sim}{G})$ ($= \mathrm{Hom}_{\mathbb{E}}(X,A) \rightrightarrows \mathrm{Hom}_{\mathbb{E}}(X,O)$) considered as a pseudofunctor defines a *split* fibration, its "*externalization*" $\mathbb{EXX}(\underset{\sim}{G}) \longrightarrow \mathbb{E}$, which has as objects the arrows $x\colon X \longrightarrow O$ of $\mathbb{E}$, and for which an arrow $\alpha\colon x \longrightarrow y$ of projection $f\colon X \longrightarrow Y$ in $\mathbb{EXX}(\underset{\sim}{G})$, is just an arrow $\alpha\colon X \longrightarrow A$ in $\mathbb{E}$

(7.2.0)

$$\begin{tikzcd}
& A \arrow[dd, shift right, "T"'] \arrow[dd, shift left, "S"] \\
X \arrow[ur, "\alpha"] \arrow[dr, "x"] \arrow[dd, "f"'] & \\
& O \\
Y \arrow[ur, "y"'] &
\end{tikzcd}$$

such that $S\alpha = x$ and $T\alpha = yf$. Such an arrow is cartesian if and only if α is an isomorphism in the category $\mathrm{Hom}_{\mathbb{E}}(X,\underset{\sim}{G}) = \mathbb{EXX}(\underset{\sim}{G})_X$, the fiber at X.

(7.3) Now let $C_\bullet/X\colon C \times_X C \times_X C \Rrightarrow C \times_X C \rightrightarrows C \twoheadrightarrow X$ be the nerve of a covering $p\colon C \twoheadrightarrow X$. It is not difficult to see that *a descent datum on* $\mathrm{Hom}_{\mathbb{E}}(-,\underset{\sim}{G})$ over the covering $C \twoheadrightarrow X$ is nothing more than a simplicial map $d\colon C_\bullet/X \longrightarrow \mathrm{Nerve}(\underset{\sim}{G})$

(7.3.0)
$$\begin{tikzcd}
C \times_X C \times_X C \arrow[r, "d_2"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & A \times_O A \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
C \times_X C \arrow[r, "d_1"'] \arrow[d, shift right, "\mathrm{pr}_2"'] \arrow[d, shift left, "\mathrm{pr}_1"] & A \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
C \arrow[r, "d_0"'] \arrow[d, two heads] & O \\
X &
\end{tikzcd}$$

i.e., an internal functor from the groupoid $C \times_X C \rightrightarrows C$ into $\underset{\approx}{G}$. Similarly, a morphism of such descent data is nothing more than a homotopy of such simplicial maps, i.e. an internal natural transformation of such internal functors. Consequently, a datum is effective if and only if there exists an arrow $x\colon X \longrightarrow O$ such that the trivial functor defined through $xp\colon C \longrightarrow O$ is isomorphic to d. Thus $\mathbb{EX}(\underset{\approx}{G}) \longrightarrow \mathbb{E}$ is *complete* (for the canonical topology) *if and only if for every epimorphism* $C \twoheadrightarrow X$, *the canonical restriction functor*

$$P_{\underset{\approx}{G}} : \underline{\mathrm{Hom}}_{\mathbb{E}}(X, \underset{\approx}{G}) \longrightarrow \underline{\mathrm{Simpl}}_{\mathbb{E}}(C_\bullet/X, \underset{\approx}{G})$$

is an equivalence of categories. But if $\underset{\approx}{G}$ is a groupoid, we have already noted (5.10) that $\underline{\mathrm{Simpl}}_{\mathbb{E}}(C_\bullet/X, \underset{\approx}{G})$ is equivalent to the category of $\underset{\approx}{G}$-torsors above X which are split when restricted along $C \twoheadrightarrow X$, while $\underline{\mathrm{Hom}}_{\mathbb{E}}(X, \underset{\approx}{G})$ is equivalent to the category of split $\underset{\approx}{G}$-torsors above X. Under refinement, we thus obtain that $\underline{\underline{\underline{\mathbb{EX}}}}(\underset{\approx}{G}) \longrightarrow \mathbb{E}$ is complete if and only if the canonical functor

$$\underline{\mathrm{Spl\ TORS}}(X; \underset{\approx}{G}) \longrightarrow \underline{\mathrm{TORS}}(X; \underset{\approx}{G}) \tag{7.3.1}$$

is an equivalence of categories, i.e. *every torsor splits.*

(7.4) If every $\underset{\approx}{G}$-torsor is split, then $\underset{\approx}{G}$ may be seen to possess an additional property of "*injectivity*" in GPD($\mathbb{E}$): If $H\colon \underset{\approx}{\mathfrak{J}}_1 \longrightarrow \underset{\approx}{\mathfrak{J}}_2$ *is any essential equivalence, then the restriction functor*

$$\mathrm{CAT}_{\mathbb{E}}(H, \underset{\approx}{G})\colon \mathrm{CAT}_{\mathbb{E}}(\underset{\approx}{\mathfrak{J}}_2, \underset{\approx}{G}) \longrightarrow \mathrm{CAT}_{\mathbb{E}}(\underset{\approx}{\mathfrak{J}}_1, \underset{\approx}{G}) \tag{7.4.0}$$

is an equivalence of categories.

In effect, let us first show that if H *is any essential equivalence, then* $\mathrm{CAT}_{\mathbb{E}}(H, \underset{\approx}{G})$ *is always fully faithful* (*for any groupoid* $\underset{\approx}{G}$). To this end recall that as we have already noted, H is an essential equivalence if and only if in the diagram (5.15.0) which defines the internal profunctor adjoint to H, the vertical columns H_{A_2} and H_{D_2} are each torsors under $\underset{\sim}{\mathfrak{J}}_1\colon A_1 \rightrightarrows O_1$. Thus, in particular

(7.4.1)
$$\begin{tikzcd}
(D \times_T A_2) \times_T A_1 \arrow[r, shift left, "\mathcal{H}H^{-1}"] \arrow[r, shift right, "\mathrm{pr}"'] & O_1 \times_T A_2 \arrow[r, "\mathcal{S}\mathrm{pr}"] & O_2
\end{tikzcd}$$

is left exact with $\mathcal{S}$pr defining its quotient. Now let $\alpha, \beta\colon O_2 \rightrightarrows A$ be a pair of arrows of $\mathbb{E}$ which define internal natural transformations of functors $F, G\colon \underset{\sim}{\mathfrak{J}}_2 \rightrightarrows \underset{\sim}{G}$ which have the property that $\alpha{*}H = \beta{*}H\colon FH \longrightarrow GH$ in CAT($\mathbb{E}$). We will show that it then follows that $\alpha\,\mathcal{S}\mathrm{pr} = \beta\,\mathcal{S}\mathrm{pr}$ in $\mathbb{E}$ so that $\alpha = \beta$. For this it suffices to make the verification in sets. Here let $(X, x\colon A \longrightarrow H(X))$ be an element of $O_1 \times_T A_2$; its image under $\alpha\,\mathcal{S}\mathrm{pr}$ is just $\alpha_A\colon F(A) \longrightarrow G(A)$, and under $\beta\,\mathcal{S}\mathrm{pr}$, $\beta_A\colon F(A) \longrightarrow G(A)$. Since we have natural transformations, each of the pair of squares in

(7.4.2)
$$\begin{tikzcd}
F(A) \arrow[r, shift left, "\alpha_A"] \arrow[r, shift right, "\beta_A"'] \arrow[d, "F(x)"'] & G(A) \arrow[d, "G(x)"] \\
FH(X) \arrow[r, shift left, "\alpha H(X)"] \arrow[r, shift right, "\beta H(X)"'] & GH(X)
\end{tikzcd}$$

is commutative. By hypothesis $\alpha H(x) = pH(x)$, but then $\alpha_A = \beta_A$ since $F(x)$ and $G(x)$ are isomorphisms. Thus CAT($H, \underset{\sim}{G}$) is faithful. That it is full may be seen as follows: let $\theta\colon FH \longrightarrow GH$ be a natural transformation. We will define an arrow $\bar{\alpha}\colon O_1 \times_T A_2 \longrightarrow A$ which coequalizes pr and $\mathcal{H}H^{-1}$. By passage to quotient this will define an arrow $\alpha\colon O_2 \longrightarrow A$ in $\mathbb{E}$ which will define the desired internal natural transformation from F into G. Thus let $(X, x\colon A \longrightarrow H(X))$ be given along with $\theta\colon FH \longrightarrow GH$. Define the image of $(X, x\colon A \longrightarrow H(X))$ as $\bar{\alpha}(X, x) = G(x)^{-1}\theta_X F(x)\colon F(A) \longrightarrow G(A)$. Since for each $f\colon Y \longrightarrow X$, the two squares in the diagram

(7.4.3)
$$\begin{tikzcd}
F(A) \arrow[r, "\alpha(X{,}x)"] \arrow[d, "F(x)"'] & G(A) \arrow[d, "G(x)"] \\
F(H(X)) \arrow[r, "\theta_X"'] \arrow[d, "FH(f^{-1})"'] & GH(X) \arrow[d, "GH(f^{-1})"] \\
F(H(Y)) \arrow[r, "\theta_Y"'] & GH(Y)
\end{tikzcd}$$

are commutative with $FH(f^{-1})F(x) = F(H(f^{-1})x)$ and $GH(f^{-1})G(x) = G(H(f^{-1})x)$, we see that $\bar{\alpha}$ coequalizes pr and $\mathcal{H}H^{-1}$ and the desired natural transformation is produced in CAT($\mathbb{E}$).

Now suppose that any torsor under $\underset{\approx}{G}$ is split and let $F\colon \underset{\sim}{\mathfrak{J}}_1 \longrightarrow \underset{\approx}{G}$ be given, we wish to produce a functor, $\tilde{F}\colon \underset{\sim}{\mathfrak{J}}_2 \longrightarrow \underset{\approx}{G}$ such $\tilde{F}H \xrightarrow{\sim} F$.

(7.4.4)
$$\begin{tikzcd}
\underset{\sim}{\mathfrak{J}}_1 \arrow[r, "H"] \arrow[d] & \underset{\sim}{\mathfrak{J}}_2 \arrow[dl, dashed, "\tilde{F}"] \\
\underset{\approx}{G} \arrow[ur, phantom, "\overset{\sim}{\longleftarrow}"] &
\end{tikzcd}$$

But since H_{O_2} is an $\underset{\sim}{\mathfrak{J}}_1$-torsor above O_2, $\mathrm{TORS}(\underset{\sim}{\mathfrak{J}})(H_{O_2})$ is a torsor under $\underset{\approx}{G}$ above O_2 and $\mathrm{TORS}(H)(H_{A_2})$ is a torsor under $\underset{\approx}{G}$ above A_2, since both of these split we obtain by composition arrows $\tilde{F}_0\colon O_2 \longrightarrow O$ and $\tilde{F}_1\colon A_2 \longrightarrow A$ in $\mathbb{E}$ which are easily seen to define the desired functor $\tilde{F}\colon \underset{\sim}{\mathfrak{J}}_2 \longrightarrow \underset{\approx}{G}$.

Finally, suppose the $\mathrm{CAT}(-, \underset{\approx}{G})$ carries essential equivalences into equivalences. Since, for any covering $C \times_X C \rightrightarrows C \longrightarrow X$, the canonical functor

(7.4.5)
$$\begin{tikzcd}
C \times_X C \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & X \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] \\
C \arrow[r, two heads] & X
\end{tikzcd}$$

is an essential equivalence, we see that this condition immediately implies that $\underline{\underline{\underline{\mathbb{EX}}}}(\underset{\approx}{G})$ is complete.

In summary, we have established the following

Theorem (7.5). *For any topos $\mathbb{E}$, and any groupoid $\underset{\approx}{G}$ in $\mathbb{E}$, the following statements are equivalent*:

1° $\mathbb{EX}(\underset{\approx}{G}) \longrightarrow \mathbb{E}$ *is a complete fibration* (i.e. *every descent datum is effective*),

2° *For any covering* $C \times_X C \rightrightarrows C \longrightarrow X$ of X, *the fully faithful functor*

$$\underline{\mathrm{Hom}}_{\mathbb{E}}(X, \underset{\approx}{G}) \longrightarrow \underline{\mathrm{Simpl}}_{\mathbb{E}}(C_\bullet/X, \underset{\approx}{G})$$

is an equivalence of categories (i.e. Čech-cohomology is essentially trivial);

3° *For any object X in $\mathbb{E}$, the fully faithful functor*

$$\underline{\mathrm{Spl\ TORS}}_{\mathbb{E}}(X; \underset{\approx}{G}) \longrightarrow \underline{\mathrm{TORS}}_{\mathbb{E}}(X; \underset{\approx}{G})$$

is an equivalence of categories (i.e. every torsor under $\underset{\approx}{G}$ is split);

4° *For any essential equivalence of groupoids* $H\colon \underset{\sim}{\mathfrak{J}}_1 \longrightarrow \underset{\sim}{\mathfrak{J}}_2$ *the fully faithful functor*

$$\mathrm{CAT}_{\mathbb{E}}(H, \underset{\sim}{G})\colon \mathrm{CAT}_{\mathbb{E}}(\underset{\sim}{\mathfrak{J}}_2, \underset{\sim}{G}) \longrightarrow \mathrm{CAT}_{\mathbb{E}}(\underset{\sim}{\mathfrak{J}}_1, \underset{\sim}{G})$$

is essentially surjective and thus a full equivalence (i.e. *given any functor* $F\colon \underset{\sim}{\mathfrak{J}}_1 \longrightarrow \underset{\sim}{G}$*, there exists a functor* $\tilde{F}\colon \underset{\sim}{\mathfrak{J}}_2 \longrightarrow \underset{\sim}{G}$ *such that* $\tilde{F}H \xrightarrow{\sim} F$*)*.

Remark. A similar theorem holds for category objects, locally representable functors, and existence of adjoints.

Definition (7.6). A groupoid which satisfies any one and hence all of the equivalent conditions of Theorem (7.5) will be said to be (internally) *complete*. A functor $c\colon \underset{\sim}{G} \longrightarrow \underset{\sim}{\widetilde{G}}$ will be called a *completion* of $\underset{\sim}{G}$ provided any other functor $\underset{\sim}{G} \longrightarrow \underset{\sim}{L}$ into a complete groupoid $\underset{\sim}{L}$ factor essentially uniquely through $\underset{\sim}{\widetilde{G}}$. Such a completion is essentially unique and we have the following

Corollary (7.7). *If* $\underset{\sim}{G}$ *is complete, then*

(a) *any essential equivalence* $H\colon \underset{\sim}{G} \longrightarrow \underset{\sim}{T}$ *admits a quasi-inverse* $H'\colon \underset{\sim}{T} \longrightarrow \underset{\sim}{G}$*;*

(b) $c\colon \underset{\sim}{G} \longrightarrow \underset{\sim}{\widetilde{G}}$ *ia a completion of* $\underset{\sim}{G}$ *if and only if* $\underset{\sim}{\widetilde{G}}$ *is complete and* c *is an essential equivalence.*

(7.8) We are now in a position to return to our original situation where $\mathbb{E}$ is a U-small site and $\mathbb{F} \longrightarrow \mathbb{E}$ is a fibration fibered in U-small groupoids. We have the following

Lemma (7.9).

(a) *For any presheaf* P *in* $\mathbb{E}\hat{}$*, one has a natural equivalence of groupoids*

$$\mathrm{Nat}(P, \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \xrightarrow{\approx} \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/P, \mathbb{F})\,;$$

(b) *as a groupoid object in* $\mathbb{E}\hat{}$*,* $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ *is complete in the canonical topology on* $\mathbb{E}\hat{}$*.*

Here $\mathbb{E}/P \longrightarrow \mathbb{E}$ denotes the fibered category of representables above P. It is, in fact the restriction to $\mathbb{E}$ of the fibration $\mathbb{E}\mathbb{X}\,(\overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} P) \longrightarrow \mathbb{E}\hat{}$ where $P \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} P$ is the discrete groupoid object defined by the object P of $\mathbb{E}\hat{}$. Of course $\mathrm{Nat}(P, Q) = \mathrm{Hom}_{\mathbb{E}\hat{}}(P, Q)$.

In effect, a natural transformation from P into $\mathrm{Ob}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$ is entirely equivalent to an internal functor from the discrete internal groupoid $P \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} P$ into the category

$\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ and thus defines a cartesian functor from the restrictions to $\mathbb{E}$ of the corresponding externalizations $\mathbb{E}\mathbb{X}_{\mathbb{E}}(P \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} P)$ and $\mathbb{E}\mathbb{X}_{\mathbb{E}}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))$. But $\mathbb{E}\mathbb{X}_{\mathbb{E}}(P \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} P)$ is isomorphic to the split fibration $\mathbb{E}/P \xrightarrow{S} \mathbb{E}$ and $\mathbb{E}\mathbb{X}_{\mathbb{E}}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))$ is, by definition $\mathcal{S}\mathbb{F} \longrightarrow \mathbb{E}$, the split fibration $\mathbb{E}$-equivalent to $\mathbb{F}$. Hence such a natural transformation defines a cartesian functor from $\mathbb{E}/P$ into $\mathbb{F}$. Inversely given a cartesian functor $C\colon \mathbb{E}/P \longrightarrow \mathbb{F}$, define the transformation $C'_T\colon F(T) \longrightarrow \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/T,\mathbb{F})$ via $C'_T(T \xrightarrow{t} F) = Ct_*\colon \mathbb{E}/T \longrightarrow \mathbb{F}$ where $t_*\colon \mathbb{E}/T \longrightarrow \mathbb{E}/F$ is the cartesian functor defined by composition with $t\colon T \longrightarrow F$. That this is indeed natural follows from the fact that for any $f\colon U \longrightarrow T$ in $\mathbb{E}$, the triangle

(7.9.0)
$$\begin{tikzcd}
U \arrow[rr, "f"] \arrow[dr, "P(f)(t)"'] & & T \arrow[dl, "t"] \\
& P &
\end{tikzcd}$$

is commutative in $\mathbb{E}^\wedge$ (when we have identified objects and arrows of $\mathbb{E}$ with the presheaves of representable functors which they define in $\mathbb{E}^\wedge$).

For part (b), let $p\colon E \xrightarrow{P}$ be the projection map for a torsor in $\mathbb{E}^\wedge$ under $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ and let $q\colon \coprod_{\alpha\in I} X_\alpha \longrightarrow F$ be the representation of F as a full quotient of representables $x_\alpha\colon X_\alpha \longrightarrow F$. Since $E(X) \longrightarrow P(X)$ is surjective for each $X \in \mathrm{Ob}(\mathbb{E})$, q lifts to a map $q\colon \coprod X_\alpha \longrightarrow E$ such that $pq' = q$ and hence to functor of

$$\rho\colon \left(\coprod X_\alpha \times X_\rho \rightrightarrows \coprod X_\alpha\right) \longrightarrow \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})\,.$$

Moreover, since the torsor is locally split when restricted along q, it is split in $\mathbb{E}^\wedge$ if and only if its "defining cocycle" P is isomorphic to the restriction of a functor

$$\tilde{s}\colon \big(F \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} F\big) \longrightarrow \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$$

along the canonical functor

(7.9.1)
$$\begin{tikzcd}
\coprod X_\alpha \times_F X_\beta \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & F \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] \\
\coprod X_\alpha \arrow[r, "q"'] & F
\end{tikzcd}$$

$$q_\bullet\colon C_\bullet \longrightarrow F$$

But this canonical functor is always fully faithful with $q(T)$ surjective on each representable T. Consequently, the restriction to $\mathbb{E}$ of the externalization of $q_\bullet$, $\mathbb{E}\mathbb{X}(q_\bullet)\colon \mathbb{E}\mathbb{X}(C_\bullet) \longrightarrow \mathbb{E}\mathbb{X}(E)$ is a cartesian equivalence of fibered categories over $\mathbb{E}$ and we may define the desired natural transformation $\tilde{s}\colon F \longrightarrow \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ as that essentially unique transformation $\tilde{s}$ which corresponds under part (a) to the functor $\tilde{\tilde{s}}$ which makes the diagram

(7.9.2)

$$\begin{tikzcd}
\mathbb{E}\mathbb{X}(C_0) \arrow[r, "\mathbb{E}\mathbb{X}(p)"] \arrow[d, "\wr"'] & S\mathbb{F} \arrow[d, "\wr"] \\
\mathbb{E}\mathbb{X}(F) \arrow[r, "\tilde{\tilde{s}}"] \arrow[d, phantom, "\wr"] & \mathbb{F} \\
\mathbb{E}/F &
\end{tikzcd}$$

commutative in $\mathrm{Cart}_{\mathbb{E}}$.

Corollary (7.10). I*f* $\mathbb{E}$ *is a site and* $\mathbb{F} \longrightarrow \mathbb{E}$ *is fibration, then* $\mathbb{F}$ *is precomplete, respectively, complete if and only if the following two equivalent conditions hold*:

(a) *For every covering subfunctor R of a representable X, the canonical restriction functor* $\mathrm{Nat}(X, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \longrightarrow \mathrm{Nat}(R, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$ *is fully faithful, respectively, an equivalence of categories*;

(b) *For every covering* $(X_\alpha \longrightarrow X)_{\alpha\in I}$ *(in the topology of* $\mathbb{E}$*), the canonical restriction functor* $\mathrm{Nat}(X, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \longrightarrow \mathrm{Simpl}_{\mathbb{E}^\wedge}(C_\bullet/X, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$ *defined by the projection of the nerve* $C_\bullet\colon \coprod X_0 \times_X X_\beta \rightrightarrows \coprod X_\alpha \xrightarrow{P} X$ *into X is fully faithful, respectively, an equivalence of categories.*

Both of these observations are immediate since $(X_\alpha \longrightarrow X)_{\alpha\in I}$ is a covering if and only if the image of the projection $p\colon \coprod X_\alpha \longrightarrow X$ is a covering subfunctor of X. We may now return to the proof of Theorem (6.2) part 2°. We shall split it into several parts.

Lemma (7.11). $\mathbb{F} \longrightarrow \mathbb{E}$ *is precomplete if and only if any one and hence all of the following equivalent conditions are satisfied*:

(a) *for any* $X \in \mathrm{Ob}(\mathbb{E})$ *and arrow*

$$X \xrightarrow{\langle x,y\rangle} \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \times \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$$

the presheaf above X defined by the cartesian square

(7.11.0)

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_X(x,y) \arrow[r, "\mathrm{pr}"] \arrow[d] & \mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \arrow[d, "\langle T,S\rangle"] \\
X \arrow[r, "\langle x,y\rangle"] & \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \times \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))
\end{tikzcd}$$

is a sheaf (above X).

(b) *for any $X \in \mathrm{Ob}(\mathbb{E})$ and any pair of objects x, y in $\mathbb{F}_X$, the presheaf $\underline{\mathrm{Hom}}_X(x, y)$ on $\mathbb{E}/X$ defined by $\underline{\mathrm{Hom}}_X(x,y)(T \xrightarrow{f} X) = \mathrm{Hom}_{\mathbb{F}_T}(\mathbb{F}_f(x), \mathbb{F}_f(y))$ is a sheaf (in the induced topology on $\mathbb{E}/X$);*

(c) *if for any presheaf P, $L(P)$ designates the presheaf whose value at X is given by $L(X) = \varinjlim_{R \in \mathrm{Cov}(X)} \mathrm{Nat}(R, P)$, so that $LL(P)$ is the associated sheaf functor aP, and $\ell \colon P \longrightarrow L(P)$ is the canonical map, then the canonical functor*

(7.11.1)
$$\begin{tikzcd}
A \arrow[d, shift left] \arrow[d, shift right] & {= \mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))} \arrow[r, "\ell_A"] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] & L(\mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))) \arrow[d, shift right, "LT"'] \arrow[d, shift left, "LS"] \\
O & {= \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))} \arrow[r, "\ell_0"'] & L(\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})))
\end{tikzcd}$$

$$\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \xrightarrow{\ \ell\ } L\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$$

is fully faithful;

(d) *the canonical functor*

(7.11.2)
$$\begin{tikzcd}
\mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \arrow[r, "a_A"'] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] & a\,\mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \arrow[d, shift right, "a(T)"'] \arrow[d, shift left, "a(S)"] \\
\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \arrow[r, "a_0"'] & a\,\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))
\end{tikzcd}$$

$$a \colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow a\,\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$$

is fully faithful.

In effect, the equivalence of precomp1eteness with (a) and (b) is a literal translation of the definition since the presheaf of sections of $\underline{\mathrm{Hom}}_X(x, y) \longrightarrow X$ in $\mathbb{E}^{\frown}$ is isomorphic to the presheaf defined in (b). The equivalence of (c) with (d) may be established by noting that the canonical functor a is the composite of the two canonical functors

(7.11.3)
$$\begin{tikzcd}
A \arrow[r, "\ell_A"] \arrow[d, shift right] \arrow[d, shift left] & L(A) \arrow[r, hook, "\ell_{SA}"] \arrow[d, shift right] \arrow[d, shift left] & LL(A) \arrow[d, shift right] \arrow[d, shift left] \\
O \arrow[r, "\ell_O"'] & L(O) \arrow[r, hook, "\ell_{LO}"'] & LL(O)
\end{tikzcd}$$

where $LP(X) = \varinjlim_{R \in \mathrm{Cov}(X)} \mathrm{Hom}_{\mathbb{E}\hat{}}(R, P)$ for any presheaf P. Now ℓ is fully faithful if and only if the square

(7.11.4)
$$\begin{tikzcd} A \arrow[r, "\ell_A"] \arrow[d, "\langle T,S\rangle"'] & L(A) \arrow[d] \\ O \times O \arrow[r, "\ell_0 \times \ell_0"'] & LO \times LO \end{tikzcd}$$

is cartesian. Now if (7.11.4) is cartesian then so is the square

(7.11.5)
$$\begin{tikzcd} LA \arrow[r, "L(\ell_A)"] \arrow[d] & LL(A) \arrow[d] \\ LO \times LO \arrow[r, "L(\ell_0)\times L(\ell_0)"'] & LLO \times LLO \end{tikzcd}$$

since L is left exact. But since $L(\ell_P) = \ell_{L(P)}$ for any presheaf P, this now implies that ℓ_L is fully faithful and hence that a is fully faithful. But if a is fully faithful then so is ℓ since ℓ_{LA} is always a monomorphism. Thus (c) and (d) are equivalent.

We now show that (d) is equivalent to (b). But this immediate since $\underline{\mathrm{Hom}}_X(x, y) \longrightarrow X$ is a sheaf over X if and only if the commutative square

(7.11.6)
$$\begin{tikzcd} \underline{\mathrm{Hom}}_X(x,y) \arrow[r] \arrow[d] & a\,\underline{\mathrm{Hom}}_X(x,y) \arrow[d] \\ X \arrow[r] & a(X) \end{tikzcd}$$

is cartesian (by a standard lemma from the theory of Grothendieck topologies [c.f. DEMAZURE (1970)]). Thus, by composition of commutative squares, (7.11.6) is cartesian if and only if a is fully faithful.

Lemma (7.12). If $\mathbb{F}$ is precomplete, then the following statements are equivalent:

(a) $\ell\colon \underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow L\underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is essentially epimorphic;

(b) $\ell\colon \underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow L\underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ admits a quasi inverse in $\mathrm{CAT}(\hat{\mathbb{E}})$;

(c) $a\colon \underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow a\,\underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is an equivalence of categories in $\mathrm{CAT}(\hat{\mathbb{E}})$;

If ℓ is essentially epimorphic and fully faithful, then by Theorem (7.5), ℓ admits a quasi inverse since $\underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is complete in the canonical topology on $\hat{\mathbb{E}}$ (i.e. every torsor splits).

But if ℓ admits a quasi-inverse, then so does $L(\ell)$ and hence also $a = (\ell L(\ell)\ell)$. Similarly, if a is an equivalence the ℓ is essentially epimorphic since $L(\ell)$ is fully faithful. Consequently, the three statements are equivalent.

Remark. This Lemma and the following Theorem hold without the assumption that $\mathbb{F}$ is fibered in groupoids, the proof will be given elsewhere.

Lemma (7.13). Let $\mathbb{F} \longrightarrow \mathbb{E}$ be a U-fibration on a U-small site $\mathbb{E}$, fibered in groupoids. Then $\mathbb{F}$ is complete if and only if any one (and hence all) of the following equivalent conditions are satisfied:

(a) $\ell\colon \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow L\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is an essential equivalence;

(b) $\ell\colon \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow L\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is an equivalence;

(c) $a\colon \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}) \longrightarrow a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is an equivalence.

In effect, let us show that for any covering $c\colon R \hookrightarrow X$ of a representable, $\mathrm{Hom}(X, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \longrightarrow \mathrm{Hom}(R, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$ is essentially surjective if and only if ℓ is essentially surjective under the assumption that ℓ is fully faithful. Thus given any $t\colon R \longrightarrow O$, we know that $v(t)\colon X \longrightarrow LO$ is such that that square

(7.13.1)
$$\begin{tikzcd}
O \arrow[r, "\ell"] & L(O) \\
R \arrow[u, "t"] \arrow[r, hook, "c"'] & X \arrow[u, "v(t)"']
\end{tikzcd}$$

is commutative when $v(t)$ is the corresponding equivalence class which defines $L(O)(x)$. If now we assume that ℓ is essentially surjective then there exists an arrow $\tilde{v}(t)\colon X \longrightarrow O$ such that $f\colon \ell\widetilde{v(t)} \xrightarrow{\sim} v(t)$ in the category $\mathrm{Hom}(X, L(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})))$. Now consider the objects $\widetilde{v(t)}c$ and t in $\mathrm{Hom}(R, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$. Since $fc\colon \ell\widetilde{v(t)}c \longrightarrow v(t)c$ is then an isomorphism and $v(t)c = \ell t$, it follows that $fc\colon \ell(\widetilde{v(t)}c) \xrightarrow{\sim} \ell(t)$ and hence that $\widetilde{v(t)}c \xrightarrow{\sim} t$ since ℓ is fully faithful. Thus $\mathrm{Hom}(c, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$ is essentially surjective.

Now suppose that $\mathrm{Hom}(i, \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F}))$ is essentially surjective (and fully faithful) and let $x\colon X \longrightarrow L(O)$ be an object of $\mathrm{Hom}(X, L(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})))$.

(7.13.2)
$$\begin{tikzcd}
A \arrow[r, "\ell_A"] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] & L(A) \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\
O \arrow[r, "\ell_O"] & L(O) \\
R \arrow[u, "v"] \arrow[uu, bend left=60, "i"] \arrow[uur, "\ell_1"] \arrow[r, hook, "c"'] & T \arrow[u, shift right, "x"] \arrow[u, shift left] \arrow[ul, dashed] \arrow[uu, bend right=60, "v_R(i)"']
\end{tikzcd}$$

From the fundamental properties of $\ell_0\colon O \longrightarrow LO$ we know that there exists a covering seive $c\colon R \hookrightarrow X$ and a map $u\colon R \longrightarrow O$ such that $\ell_0 u = xc$. Consequently there exists an $\bar{x}\colon X \longrightarrow O$ and an isomorphism $i\colon R \overset{\sim}{\longrightarrow} A$ in $\mathrm{Hom}(R, A)$ with $Ti = u$ and $Si = C$. Let $v_R(i)\colon X \longrightarrow LA$ be the equivalence class of i in $LA(x)$ so that $\ell_A i = v_R(i)c$. We claim that $v_R(i)$ is an isomorphism of x with $\ell_0(\bar{x})$ so that ℓ will be essentially surjective. But $Tv_R(i)_c = T\ell_A i = \ell_0 Ti = \ell_0 u = xc$ so that $x = Tv_R(i)$ and $SV_R(i)_c = S\ell_A i = \ell_0 Si = \ell_0 \bar{x}c$ so that $\ell_0\bar{x} = Sv_R(i)$, since LA is a separated presheaf. But since $v_R(i)c = i\ell_A$ is an isomorphism, $V_R(\tau)\colon \ell_0(\bar{x}) \overset{\sim}{\longrightarrow} x$ is the desired isomorphism and the Theorem is established (along with Theorem (6.2(2°)).

To complete the proof of Theorem (6.2) it now only remains to show that if $\mathbb{F}$ is a gerbe with tie L, then $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})$ is a bouquet with tie L (and conversely). For this, in the light of what we have already shown, it suffices to observe that in the canonical equivalence of groupoids

(7.13.3)

$$\begin{tikzcd}
\mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[rr, "a_A"] \arrow[dd, shift right, "T"'] \arrow[dd, shift left, "S"] \arrow[dr, "\langle T,S\rangle"] & & a\,\mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[dd, shift right, "a(T)"'] \arrow[dd, shift left, "a(S)"] \arrow[dr, "\langle a(T),a(S)\rangle = a\langle T,S\rangle"] & \\
& \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \times \mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[rr] \arrow[dl, shift right, "\mathrm{pr}"'] \arrow[dl, shift left, "\mathrm{pr}"] & & a(O) \times a(O) \arrow[dl, shift right] \arrow[dl, shift left] \\
\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[rr, "a_0"'] \arrow[d] & & a\,\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) \arrow[d] & \\
\mathbb{1} \arrow[rr] & & a(\mathbb{1}) \overset{\sim}{\rightarrow} \mathbb{1} &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}) \arrow[r, "a"] & a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})
\end{tikzcd}$$

the conditions (c) and (d) of the definition of a gerbe (5.0) immediately translate via representability into the respective assertions (c′): *the canonical map* $\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-, \mathbb{F})) \longrightarrow \mathbb{1}$ *is a covering* (in the induced topology on $\mathbb{E}\hat{\ }$) and (d′): *the canonical map* $\langle T, S\rangle\colon A \longrightarrow O \times O$ *is a covering* (in the induced topology on $\mathbb{E}\hat{\ }$). But from a standard theorem on Grothendieck topologies [DEMAZURE (1970)], such a natural transformation is covering if and only if its image under the associated sheaf functor is an epimorphism in $\mathbb{E}^{\sim}$, thus here (since a is left exact) if and only if $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ is a bouquet. Since ties are preserved under equivalences and the tie of $\mathbb{F}$ is the same as the tie of $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ the entirety of Theorem (6.2) has now been established.

The preceding lemma combined with (6.2) now give two corollaries:

<u>Corollary</u> (7.14). *If $\mathbb{F}$ is (externally) complete in the topology of $\mathbb{E}$ then $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ is (internally) complete in $\mathbb{E}^{\sim}$, i.e. in $\mathbb{E}^{\sim}$ every torsor under $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ splits.*

In effect let $p\colon E \longrightarrow X$ be the projection map of a torsor under $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ above X in $\mathbb{E}^{\sim}$. Since p is an epimorphism of sheaves, its image $p(E) \hookrightarrow X$ in $\mathbb{E}^{\wedge}$ is a covering seive of X and $E \twoheadrightarrow P(E)$ becomes a torsor under $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ above $P(E)$ in $\mathbb{E}^{\wedge}$. If a^* designates a quasi-inverse for the functor a in $\mathrm{CAT}(\mathbb{E}^{\wedge})$ we have that the torsor $a^*(\mathbb{E} \longrightarrow p(E))$ is split under $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ since
$\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ is complete in the canonical topology of $\mathbb{E}^{\wedge}$. Thus
$a(a^*(E \longrightarrow p(E)) \xrightarrow{\sim} E \longrightarrow p(E)$ is split with a splitting $s\colon p(E) \longrightarrow E$. But X is a sheaf and $p(E) \hookrightarrow X$ is a covering seive, hence s extends uniquely to a morphism of sheaves $\tilde{s}\colon X \to E$ which is immediately seen to be a splitting for p.

<u>Corollary</u> (7.15). If $\mathbb{F}$ is complete, then one has a chain of cartesian equivalence of fibered categories over $\mathbb{E}$

(7.15.0)
$$\mathbb{F} \xrightarrow{\approx} \mathcal{S}\mathbb{F} \xrightarrow{\approx} \mathbb{K}\mathcal{S}\mathbb{F} \xrightarrow{\approx} \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))$$

where $\mathcal{S}\mathbb{F}_X = \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F})$, $\mathbb{K}\mathcal{S}\mathbb{F}_X = \mathrm{Hom}_{\mathbb{E}^{\sim}}(a(X),\, a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))$, and $\underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})) = \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}^{\sim}}(a(X),\, a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))$ for each representable X in $\mathbb{E}^{\wedge}$.

8. COMPLETION OF THE PROOF OF THEOREM (5.21) AND RESTATEMENT OF MAIN RESULT.

(8.0) Recall that we have let $\underline{\underline{\mathrm{GERBE}}}(\mathbb{E};L)$ designate the 2-category whose objects are the gerbes of $\mathbb{E}$ with tie L and whose 1-morphisms are cartesian equivalences of $\mathbb{E}$-gerbes. Similarly $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E};L)$ designates the 2-category whose objects are bouquets of $\mathbb{E}$ with tie L and whose 1-morphisms are essential equivalences of bouquets so that the class of connected components of (the respective underlying category) of $\underline{\underline{\mathrm{GERBE}}}(\mathbb{E};L)$ and $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E};L)$ are $H^2_{\mathrm{Ger}}(\mathbb{E};L)$ and $H^2(\mathbb{E};L)$. Now let $\underline{\underline{\mathrm{CBOUQ}}}(\mathbb{E};L)$ be the full 2-subcategory of $\underline{\underline{\mathrm{BOUQ}}}(\mathbb{E};L)$ whose objects are those bouquets which are internally complete (i.e. every torsor under the bouquet splits) with $\pi_0(\underline{\underline{\mathrm{CBOUQ}}}(\mathbb{E};L))$ the class of connected components of this latter category.

From Theorem (6.2) and Corollary (7.14) we see that we have defined a 2-functor,

$$(8.0.0)\qquad a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,-)\colon \underline{\underline{\underline{\mathrm{GERBE}}}}(\mathbb{E};L)\longrightarrow \underline{\underline{\underline{\mathrm{CBOUQ}}}}(\mathbb{E};L)\xhookrightarrow{\mathrm{in}}\underline{\underline{\underline{\mathrm{BOUQ}}}}(\mathbb{E};L)$$

$$\langle\langle \mathbb{F}\longmapsto a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})\rangle\rangle\ ,$$

which assigns to any gerbe with tie L the (internally) complete sheaf of groupoids which is associated to the presheaf of groupoids $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$.

From Theorem (5.20) we see that we have defined a 2-functor,

$$(8.0.1)\qquad \underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(-)\colon \underline{\underline{\underline{\mathrm{BOUQ}}}}(\mathbb{E};L)\longrightarrow \underline{\underline{\underline{\mathrm{GERBE}}}}(\mathbb{E};L)$$

$$\langle\langle \utilde{G}\longmapsto \underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\utilde{G})\rangle\rangle\ ,$$

which assigns to any bouquet with tie L the restriction to $\mathbb{E}$ of fibered category of $\mathbb{E}^{\sim}$-torsors under $\utilde{G}$. We may now state and prove the following refinement of Theorem (5.20).

Theorem (8.1). The 2-functor $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,-)$ defines a weak 2-equivalence of the category $\underline{\underline{\underline{\mathrm{GERBE}}}}(\mathbb{E};L)$ with $\underline{\underline{\underline{\mathrm{BOUQ}}}}(\mathbb{E};L)$ and a full 2-equivalence of the category $\underline{\underline{\underline{\mathrm{GERBE}}}}(\mathbb{E};L)$ with $\underline{\underline{\underline{\mathrm{CBOUQ}}}}(\mathbb{E};L)$. From these one deduces bijections on the classes of connected components

$$(8.1.0)\qquad \boxed{H^2_{\mathrm{Gir}}(\mathbb{E};L)\xrightarrow{\sim}\pi_0(\underline{\underline{\underline{\mathrm{CBOUQ}}}}(\mathbb{E};L))\xrightarrow{\sim}H^2(\mathbb{E};L)}$$

In effect, Corollary (7.15) establishes for each gerbe $\mathbb{F}$ a cartesian $\mathbb{E}$-equivalence

$$(8.1.1)\qquad \mathbb{F}\xrightarrow{\approx}\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F}))\,,$$

which takes care of one composition. For the other composition let $\utilde{G}$ be a bouquet of $\mathbb{E}$ and $\mathbb{E}\mathbb{X}_{\mathbb{E}}(\utilde{G})\longrightarrow\mathbb{E}$ be the restriction to $\mathbb{E}$ of external fibration which it defines.

The assignment to any object $X\xrightarrow{x}O$ of $\mathbb{E}\mathbb{X}_{\mathbb{E}}(\utilde{G})$ of the split torsor above X under $\utilde{G}$ defines a fully faithful cartesian functor

$$(8.1.2)\qquad \mathrm{Spl}\colon \mathbb{E}\mathbb{X}_{\mathbb{E}}(\utilde{G})\longrightarrow\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\utilde{G})$$

which is covering since any torsor under $\utilde{G}$ is locally split. One thus obtains by functoriality a fully faithful covering functor in $\mathbb{E}^{\wedge}$

$$(8.1.3)\qquad \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathrm{Spl})\colon\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \mathbb{E}\mathbb{X}_{\mathbb{E}}(\utilde{G}))\longrightarrow\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\utilde{G}))\,.$$

Since for any presheaf of categories $\utilde{G}$ one has an *essential equivalence*(!) (in $\mathbb{E}^{\wedge}$)

$$\mathrm{sub}\colon \utilde{G}\longrightarrow\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \mathbb{E}\mathbb{X}_{\mathbb{E}}(\utilde{G}))$$

one has by composition fully faithful covering functor

$$\Sigma_{\underset{\approx}{G}} \colon \underset{\approx}{G} \longrightarrow \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\approx}{G})) \tag{8.1.4}$$

and a commutative diagram in $\mathrm{CAT}(\mathbb{E}^{\widehat{\ }})$

$$\begin{tikzcd}
\underset{\approx}{G} \arrow[r, "\Sigma_{\underset{\approx}{G}}"] \arrow[d, "a_{\underset{\approx}{G}}"', "\wr"] & \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\approx}{G})) \arrow[d, "a_C", "\wr"'] \\
a\underset{\approx}{G} \arrow[r, "a\Sigma_{\underset{\approx}{G}}"] & a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\approx}{G}))
\end{tikzcd} \tag{8.1.5}$$

in which $a_{\underset{\approx}{G}}$ is an isomorphism and a_C is an equivalence. But $\Sigma_{\underset{\approx}{G}}$ is covering if and only if $a\Sigma_{\underset{\approx}{G}}$ is essentially epimorphic; thus we have a canonical essential equivalence in $\mathrm{CAT}(\mathbb{E}^{\sim})$

$$C \colon \underset{\approx}{G} \longrightarrow a\,\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\approx}{G})) \tag{8.1.6}$$

which thus admits a quasi-inverse if and only if $\underset{\approx}{G}$ is internally complete and our principal result is established.

Remark (8.1.6). Since $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\approx}{G}))$ is internally complete and C is an essential equivalence, $a\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\ \underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\underset{\approx}{G}))$ may be taken as the (internal) *completion* of $\underset{\approx}{G}$, for any functor from $\underset{\approx}{G}$ into a complete groupoid must factor essentially uniquely through C. Thus in any Grothendieck topos any groupoid has a completion. If we only have an elementary topos (without generators), the fibered category $\underline{\underline{\mathrm{TORS}}}(\underset{\approx}{G})$ furnishes an *external* completion for $\underset{\approx}{G}$ which may be too large to be internalized.

Remark (8.1.7). Any sheaf of groups G is pre-complete since any sheaf of categories is pre-complete. Since a sheaf of groups has only one X-object it is complete if and only if every torsor under G over X is isomorphic to the split torsor G_d pulled back to X, i.e. if and only if the set of isomorphisms classes of $\underline{\underline{\mathrm{TORS}}}(X,G)$ has a single element. But this set is, by definition $H^1(X;G)$; thus *a group object G in $\mathbb{E}^{\sim}$ is complete if and only if $H^1(X;G)$ is trivial* for all X.

NON-ABELIAN COHOMOLOGY IN A TOPOS

Part (II): The calculation of $H^2(L)$ by cocycles

(1) Hypercoverings. We briefly recall the following definitions and results modified from VERDIER (1972). The relevant simplicial limits used are explained in DUSKIN (1975, 1979).
(1.0) Definition. Let $\mathbb{E}$ be a Grothendieck topos and $X_\bullet \longrightarrow \mathbb{1}$ a simplicial object of $\mathbb{E}$ augmented over the terminal object of $\mathbb{E}$. $X_\bullet$ is called a *hypercovering of* $\mathbb{E}$ provided that for each $k \geq -1$, the canonical map

$$\text{(1.0.0)} \qquad \langle d_n \rangle \colon X_k \longrightarrow (\mathrm{Cosk}^{k-1}(X_\bullet))_k$$

is an epimorphism. It is said to be *semi-representable* if for each $k \geq 0$, one has an isomorphism $X_k \xrightarrow{\sim} \coprod_{i \in I} S_i$, where S_i is a member of the family of generators for $\mathbb{E}$. For any $n \geq -1$, $X_\bullet$ is said to be of *type* n, provided that the canonical simplicial map

$$\text{(1.0.1)} \qquad \langle d \rangle \colon X_\bullet \longrightarrow \mathrm{Cosk}^n(X_\bullet)$$

is an isomorphism.

A simplicial map $f_\bullet \colon X_\bullet \longrightarrow Y_\bullet$ of hypercoverings will be said to be a *refinement* of $Y_\bullet$. Clearly, a hypercovering of type -1 is just the constant complex $K(\mathbb{1}, 0)$ which has the terminal object in every dimension, while one of type 0 is a covering of $\mathbb{E}$ in the usual sense. If the site of definition of $\mathbb{E}$ has finite limits and colimits then every hypercovering may be refined by a semi-representable one. The dual of category of hypercoverings and homotopy classes of simplicial maps is filtering and Verdier has shown that if one defines for any abelian group object A and hypercovering $R_\bullet$ of $\mathbb{E}$, the abelian group

$$\text{(1.0.2)} \qquad \check{H}^n(R_\bullet\,; A) = \mathrm{Hom}_{\mathrm{Simpl}_{\mathbb{E}}}[R_\bullet\,,\ K(A, n)]\,,$$

where $\mathrm{Hom}[\ ,\ ]$ here denotes homotopy classes of simplicial maps, then one has an isomorphism

$$\text{(1.0.3)} \qquad \varinjlim_{R_\bullet \in \mathrm{HO\,Cov}(\mathbb{E})} \check{H}^n(R_\bullet\,; A) \xrightarrow{\sim} H^n(\mathbb{E}; A)$$

for all $n \geq 0$ Thus by replacing coverings by hyper-coverings, ordinary cohomology may be computed in the Čech fashion (as is classically the case for H^1 using coverings). Moreover, since every n-dimensional cocycle on a hypercovering $R_\bullet$ defines an n-dimensional cocycle on $\mathrm{Cosk}^{n-1}(R_\bullet)$ to which it is equivalent, it suffices to take hypercoverings of type $n-1$ for the computation of H^n.
(2.0) Definition. Let $X_\bullet$ be a hypercovering of a topos $\mathbb{E}$. By a (*normalized*) 2-*cocycle on* $X_\bullet$ *with coefficients in a locally given group* we shall mean a system consisting of

1° a locally given group $\begin{tikzcd} N \arrow[r, "p"] & X_0 \arrow[l, bend left, "e"] \end{tikzcd}$ (i.e., an object of $\mathbb{G}r(\mathbb{E}/X_0)$ called the *local coefficient group*, 2° a global section $\alpha \in \Gamma(\mathrm{Iso}_{\mathbb{G}r/X_1}(d_0^*(N), d_1^*(N)))$ (i.e., an isomorphism

$$\begin{tikzcd}[column sep=small]
\alpha\colon & d_0^*(N) \arrow[rr, "\sim"] \arrow[d, equal] & & d_1^*(N) \arrow[d, equal] \\
& X_1 \times_{d_0} N \arrow[dr, "\mathrm{pr}_1"'] & & X_1 \times_{d_1} N \arrow[dl, "\mathrm{pr}_1"] \\
& & X_1 &
\end{tikzcd}$$

in $\mathbb{G}r/X_1$) called the *gluing*, and

3° an arrow $\chi\colon X_2 \longrightarrow N$, called the 2-*cocycle*, subject to the following conditions

(a) $\begin{cases} \chi s_0 &= \chi s_1 = e d_1 \\ p\chi &= d_1 d_2 \ (= d_1 d_1)\,; \end{cases}$

(b) $\begin{cases} s_0^*(\alpha) = \mathrm{id}_N \ \text{ in } \mathbb{G}r/X_0 \\ d_1^*(\alpha) d_0^*(\alpha)^{-1} d_2^*(\alpha)^{-1} = \mathrm{int}(\chi^*) \text{ in } \mathbb{G}r/X_2, \end{cases}$ (i.e., α is a "pseudo-descent datum"), and

(c) in the group $\mathrm{Hom}_{\mathbb{E}/X_0}(X_3, N)$, $(\chi d_3)^{-1}(\chi d_1)^{-1}(\chi d_2) f_{01}^{\alpha}(\chi d_0) = e$ (the "non-abelian 2-cocycle condition"), where $f_{01}^{\alpha}(\chi d_0)$ is an abbreviation for the map $\mathrm{pr}_N \circ \alpha \circ \langle d_2 d_3, \chi d_0 \rangle \colon X_3 \to N$, and $\chi^*\colon X_2 \to (d_1 d_1)^*(N)$ is the global section defined by $\chi\colon X_2 \to N$. The d_i^*, of course, denote the pull back functors in the fibered category of groups above objects of $\mathbb{E}$ and follow the standard conventions.

(2.0.0)

$$\begin{tikzcd}
& & d_0^*(N) \arrow[r, bend left, "\alpha", "\sim"'] \arrow[dd, "\mathrm{pr}_{X_1}"'] \arrow[dr] & d_1^*(N) \arrow[d, "\mathrm{pr}_N"] \arrow[ddl] & \\
& & & N \arrow[d, "p"] & \\
X_3 \arrow[r, shift left=3, "d_3"] \arrow[r, shift left=1, "d_2"] \arrow[r, shift right=1, "d_1"] \arrow[r, shift right=3, "d_0"] & X_2 \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"] & X_1 \arrow[r, shift left=1, "d_1"] \arrow[r, shift right=1, "d_0"] \arrow[ur, bend left] \arrow[l, bend left=40, "s_1"] \arrow[l, bend left=60, "s_0"'] & X_0 \arrow[r, two heads] \arrow[u, bend right, "e"'] \arrow[l, bend left=40, "s_0"] & \mathbb{1}
\end{tikzcd}$$

In intuitive set-theoretic terms (or even precisely by transfer to the category of sets by "homing" from an arbitrary test object in $\mathbb{E}$) the algebraic portion of such a system amounts to giving for each vertex $x_0 \in X_0$ a group N_{x_0}, for each edge $f\colon x_0 \longrightarrow x_1 \in X_1$, a group isomorphisms $f^{\alpha}\colon N_{x_1} \longrightarrow N_{x_0}$, and for each 2-simplex t a natural transformation of group isomorphisms $\chi(t)\colon f_2^\alpha \circ f_0^\alpha \longrightarrow f_1^\alpha$,

(2.0.1)
$$\begin{tikzcd}
x_0 \arrow[r, "f_2"] \arrow[dr, "f_1"'] & x_1 \arrow[d, "f_0"] & \longmapsto & N_{x_0} & N_{x_1} \arrow[l, "f_2^\alpha"'] \\
& x_2 & & & N_{x_2} \arrow[u, "f_0^\alpha"', ""{name=A}] \arrow[ul, "f_1^\alpha", ""'{name=B}] \arrow[from=A, to=B, bend right, "\chi(t)"']
\end{tikzcd}$$

i.e., an element $\chi(t) \in N_{x_0}$ such that $\mathrm{int}(\chi(t)) \circ f_2^\alpha \circ f_0^\alpha = f_1^\alpha$. The cocycle condition then says that for any 3-simplex u, the elements $\chi(d_2(u))^{-1}$, $\chi(d_1(u))^{-1}$, $\chi(d_2(u))$, and

(2.0.2)
$$\begin{tikzcd}
& x_1 \arrow[dr, "f_{13}"] \arrow[dd, "f_{12}"] & \\
x_0 \arrow[ur, "f_{01}"] \arrow[rr, crossing over] \arrow[dr, "f_{02}"'] & & x_3 \\
& x_2 \arrow[ur, "f_{23}"'] &
\end{tikzcd}$$

$f_{01}^\alpha(\chi(d_0(n)))$ of N_{x_0} have product equal to the unit of N_{x_0}.

(2.1) Example. *The canonical cocycle associated with a bouquet.* Let $\underset{\sim}{G}$ be a bouquet of $\mathbb{E}$. The 1-coskeleten of the nerve of $\underset{\sim}{G}$ is a hypercovering of $\mathbb{E}$ which is canonically supplied with a 2-cocycle with coefficients in the subgroupoid $\mathcal{E}$ of internal automorphisms of $\underset{\sim}{G}$ (considered as a group object in $\mathbb{G}r/\mathrm{Ob}(\underset{\sim}{G})$) as follows: The canonical "action" $d_0^*(\mathcal{E}) = T^*(\mathcal{E}) \xrightarrow{\mathrm{int}(\underset{\sim}{G})^{-1}} S^*(\mathcal{E}) = d_1^*(\mathcal{E})$ defined by "inner morphisms"

(2.1.0) $$\mathrm{int}(f^{-1})\colon (f\colon x_0 \longrightarrow x_1,\ a\colon x_1 \longrightarrow x_1) \mapsto (f\colon x_0 \longrightarrow x_1,\ f^{-1}af\colon x_0 \longrightarrow x_0)$$

provides the canonical gluing when coupled with the assignment

(2.1.1)
$$\begin{tikzcd}
\chi\colon & x_0 \arrow[r, "f_2"] \arrow[dr, "f_1"'] & x_1 \arrow[d, "f_0"] & \longmapsto & \chi(f_0, f_1, f_2)\colon f_1^{-1}f_0f_2\colon x_0 \longrightarrow x_0 \\
& & x_2 & &
\end{tikzcd}$$

for the canonical 2-cocycle. That χ and α define a pseudo descent datum is an immediate consequence of the commutativity of the square

(2.1.2)
$$\begin{tikzcd}
x_0 \arrow[r, "f_2"] \arrow[d, "f_1^{-1}f_0f_2"'] & x_1 \arrow[d, "f_0"] \\
x_0 \arrow[r, "f_1"'] & x_2
\end{tikzcd} ,$$

while the 2-cocycle condition follows from the equality

(2.1.3)
$$(f_{01}^{-1}f_{12}^{-1}f_{02})(f_{02}^{-1}f_{23}^{-1}f_{03})(f_{03}^{-1}f_{13}f_{01})f_{01}^{-1}(f_{13}^{-1}f_{23}f_{12})f_{01} \;=\; \mathrm{id}(x_0)\,,$$

which holds for any 3-simplex (2.0.2) in $\mathrm{Cosk}^1(\underset{\sim}{G})$.

(2.2) <u>Definition</u>. Let (χ_1, α_1, N_1) and (χ_2, α_2, N_2) be 2-cocycles on a fixed hypercovering $X_\bullet$ of $\mathbb{E}$. By a *morphism* of (χ_1, α_1, N_1) into (χ_2, α_2, N_2) we shall mean a pair (θ, ξ) consisting of a homomorphism $\xi\colon N_1 \longrightarrow N_2$ in $\mathbb{G}r/X_0$ together with a global section $\theta^* \in \Gamma(d_1^*(N_2))$ (or equivalently, an arrow $\theta\colon X_1 \longrightarrow N_2$ such that $p_2\theta = d_1$) which satisfies the following two conditions:

(a) In $\mathbb{G}r/X_1$, $\mathrm{int}(\theta^*)d_1^*(\xi)\alpha_1 = \alpha_2 d_0^*(\xi)\,,$

(b) In the group $\mathrm{Hom}_{\mathbb{E}/X_0}(X_2, N_2)$,
$\chi_2[d_2^{\alpha_2}(\theta d_0)]\cdot\theta d_2 = (\theta d_1)(\xi\chi_1)$ where $\alpha_2(\theta d_0) = \mathrm{pr}_{N_2}\alpha_2 \circ \theta d_0^*$ with $(\theta d_0)^* \in \Gamma(d_0^*(N_2))$ being the global section defined by $\theta d_0\colon X_2 \longrightarrow N_2$

(2.2.0)
$$\begin{tikzcd}
 & N_1 \arrow[r, bend left, "\xi"] \arrow[dr] & N_2 \arrow[d] \\
X_2 \arrow[ur, bend left, "\chi_1"] \arrow[urr, "\chi_2"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[ur, "\theta"] \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[l, bend left, "s_0"]
\end{tikzcd}$$

The basic data for a morphism of 2-cocycles thus provide in $\mathbb{G}r/X_1$ a natural isomorphism $\theta\colon d_1^*(\xi)\alpha_1 \xrightarrow{\sim} \alpha_2 d_0^*(\xi)$

(2.2.1)
$$\begin{tikzcd}
d_0^*(N_1) \arrow[r, "d_0^*(\xi)"] \arrow[d, "\alpha_1"'] & d_0^*(N_2) \arrow[d, "\alpha_2", ""{name=B}] \\
d_1^*(N_1) \arrow[r, "d_1^*(\xi)"', ""{name=A}] & d_1^*(N_2)
\arrow[from=A, to=B, bend left, "\theta", "\sim"']
\end{tikzcd}$$

such that in $\mathbb{G}r/X_2$, the prism

(2.2.2)
$$\begin{tikzcd}
d_0^*(N_1) \arrow[rr, "d_0^*(\xi)"] & & d_0^*(N_2) & \\
d_1^*(N_1) \arrow[u, "d_2^*(\alpha_1)"] \arrow[rr, "d_1^*(\xi)"] \arrow[rru, phantom, "\theta d_2" description] & & d_1^*(N_2) \arrow[u, "d_2^*(\alpha_2)"] \arrow[r, phantom, "\theta d_1" description] & \\
& d_2^*(N_1) \arrow[ul, "d_0^*(\alpha_1)"] \arrow[uul, "d_1^*(\alpha_1)"'] \arrow[rr, "d_2^*(\xi)"] \arrow[ur, phantom, "\theta d_0" description] \arrow[u, phantom, "\chi_1" description] & & d_2^*(N_2) \arrow[ul, "d_0^*(\alpha_2)"] \arrow[uul, "d_1^*(\alpha_2)"'] \arrow[u, phantom, "\chi_2" description]
\end{tikzcd}$$

is commutative, i.e., one has the equality

(2.2.3)
$$[\chi_2 * d_2^*(\xi)] \cdot [d_2^*(\alpha_2) * \theta d_0] \cdot [\theta d_2 * d_0^*(\alpha_1)] = \theta d_1 \circ (d_0^*(\xi) * \chi_1) \ ,$$

where $d_0^*(\xi) = d_1^* d_1^*(\xi) = d_2^* d_1^*(\xi)$, $d_1^*(\xi) = d_2^* d_0^*(\xi) = d_0^* d_1^*(\xi)$, and $d_2^*(\xi) = d_0^* d_0^*(\xi) = d_1^* d_0^*(\xi)$, following the standard conventions of the fibered category of groups.

(2.3) For any given hypercovering $X_\bullet$ of $\mathbb{E}$ *the 2-cocycles and their morphisms form a category*, $\underline{\mathbb{Z}}^2(X_\bullet; \mathbb{G}r)$, whose composition law we leave to the reader. If $f_\bullet \colon Y_\bullet \longrightarrow X_\bullet$ is a simplicial map of hypercoverings (so that $Y_\bullet$ refines $X_\bullet$) and (χ, α, N) is a 2-cocycle defined on $X_\bullet$, then the triplet $f_\bullet^*(\chi, \alpha, N) = ((\chi f_2)^*, f_1^*(\alpha), f_0^*(N))$ obtained by pull-back along $f_\bullet$ is clearly a 2-cocycle on $Y_\bullet$ called the *inverse image of* (χ, α, N) under $f_\bullet$. Under this functor, the *2-cocycles and their morphisms become a fibered category over the category of hypercoverings of* $\mathbb{E}$ which we will denote by $\underline{\mathbb{Z}}^2(\mathbb{G}r)$.

(2.4) Lemma. *If* $h_\bullet \colon f_\bullet \longrightarrow g_\bullet$ *is a simplicial homotopy of simplicial maps* $f_\bullet, g_\bullet \colon Y_\bullet \rightrightarrows X_\bullet$ *and* (χ, α, N) *is a 2-cocycle on* $X_\bullet$, *then we may define an isomorphism* $h_\bullet^*$ *of* $g_\bullet^*(\chi, \alpha, N)$ *with* $f_\bullet^*(\chi, \alpha, N)$ *via the pair* $((\chi h_1)^{-1}(\chi h_0),\ h_0^*(\alpha))$, *where* $h_0 \colon Y_0 \longrightarrow X_1$ *and* $h_0, h_1 \colon Y_1 \rightrightarrows X_2$ *are the first two structural maps of the homotopy.*

In effect, since $f_0^*(N) \cong h_0^* d_1^*(N)$ and $g_0^*(N) \cong h_0^* d_0^*(N)$, $h_0^*(\alpha) \colon g_0^*(N) \xrightarrow{\sim} f_0^*(N)$

provides the isomorphism while the inverse images of the 2-cocycle

(2.4.0)
$$\begin{tikzcd}
d_2^*(N) \arrow[d, "d_0^*(\alpha)"'] \arrow[dr, "d_1^*(\alpha)", ""'{name=D}] & \\
d_1^*(N) \arrow[r, "d_2^*(\alpha)"', ""{name=B}] & d_0^*(N)
\arrow[from=B, to=D, bend left, "\chi"]
\end{tikzcd}$$

under h_0 and h_1 give the square

(2.4.1)
$$\begin{tikzcd}
d_0^*(g_0^*(N)) \arrow[r, "d_0^*(h_0^*(\alpha))"] \arrow[d, "g_1^*(\alpha)"'] \arrow[dr, ""{name=D}, ""'{name=E}] & d_0^*(f_0^*(N)) \arrow[d, "f_1^*(\alpha)", ""'{name=F}] \\
d_1^*(g_0^*(N)) \arrow[r, "d_1^*(h_0^*(\alpha))"', ""{name=B}] & d_1^*(f_0^*(N))
\arrow[from=F, to=D, bend right, "(\chi h_1)^*"', "\sim"]
\arrow[from=B, to=E, bend left, "(\chi h_0)^*", "\wr"']
\end{tikzcd}$$

as use of the simplicial identities will easily show.

(2.5) Definition. Let (χ_1, α_1, N_1) be a 2-cocycle on $X_\bullet$ and (χ_2, α_2, N_2) be a 2-cocycle on $Y_\bullet$. We will say that (χ_1, α_1, N_1) is *equivalent* to (χ_2, α_2, N_2) provided that there exists a common refinement $f_\bullet\colon S_\bullet \longrightarrow X_\bullet$, $g_\bullet\colon S_\bullet \longrightarrow Y_\bullet$ of $X_\bullet$ and $Y_\bullet$ such that $f_\bullet(\chi_1, \alpha_1, N_1)$ is isomorphic to $g_\bullet(\chi_1, \alpha_1, N_1)$.

Since the category of hypercoverings is, up to homotopy, filtering, the result of (2.4) shows this relation is indeed an equivalence relation and divides the category of 2-cocycles over variable hypercoverings into equivalence classes which we will denote by $\mathbb{H}^2(\mathbb{E}; \mathbb{G}r)$.

(2.6) Lemma. Let (χ, α, N) be a 2-cocycle on a hypercovering $X_\bullet$. There exists a cocycle (χ', α, N) on $\mathrm{Cosk}^1(X_\bullet)$ which is equivalent to (χ, α, N).

In effect, first note that since the canonical map $\langle \alpha \rangle_2\colon X_3 \longrightarrow K_3^2 = (\mathrm{Cosk}^2(X_\bullet))_3$ is an epimorphism, the cocycle condition holds on $X_\bullet$ if and only if it holds on the 2-coskeleton thus it is immediately clear that 2-coskeleta suffice. Now suppose that one has a 2-cocycle on a hypercovering of type 2. We must produce a 2-cocycle $\chi'\colon K_2^1 \longrightarrow N$ whose restriction along the epimorphism
$\langle d \rangle_2\colon X_2 \twoheadrightarrow K_2^1 = (\mathrm{Cosk}^1(X_\bullet))_2$ is equal to χ. To do this, simply look at the equivalence relation $\Re$ associated with $\langle d \rangle_2$. Set theoretically it consists of those ordered pairs (t_1, t_2) of 2-simplices whose corresponding 0,1, and 2 faces are identical. Since X_3 is

isomorphic to K_3^2, we now have a canonical embedding of $\Re$ into K_3^2 via the assignment

$$(t_1, t_2) \mapsto \big(t_1, t_2, s_0(d_1(t_1)),\ s_0(d_2(t_1))\big) \qquad = \tag{2.6.0}$$

$$\begin{tikzcd}
 & x_0 \arrow[dr, "f_1"] \arrow[dd, "f_2"] & \\
x_0 \arrow[ur, "s_0(x_0)"] \arrow[rr, "f_1"'] \arrow[dr, "f_2"'] & & x_2 \\
 & x_1 \arrow[ur, "f_0"'] &
\end{tikzcd}$$

But χ is a 2-cocycle and thus sends the faces of this 3-simp1ex into the product

$$(\chi s_0 d_2(t_1))^{-1}(\chi t_2)^{-1}(\chi s_0 d_1(t_2)) s_0^{\alpha}(x_0)(\chi t_1) = e(\chi t_2)^{-1} e(\chi t_1) = e$$

so that $\chi t_1 = \chi t_2$ and χ equalizes the two projections of $\Re$. Since $\langle d\rangle_2$ is effective, by passage to the quotient, we obtain $\chi^1 \colon K_2^1 \longrightarrow N$ such that $\chi' \langle d\rangle_2 = \chi$. Since the canonical map $\langle d\rangle_3 \colon X_3 \cong K_3^2 \longrightarrow K_3^1$ is also an epimorphism, it follows that χ' is a cocycle and the result is established.

(2.7) Corollary. In terms of elements, a 2-cocycle (χ, α, N) on a hypercovering of type 1 verifies the following conditions:

(a) for any vertex $\chi \in X_0$ and any $n \in N_x$

$$\boxed{(s_0(x_1))^{\alpha}(n) = n} \tag{2.7.0}$$

(b) for any 1-simplex $f \colon x_0 \longrightarrow x_1$ in X_1

$$\boxed{\chi(f, f, s_0(x_0)) = e = \chi(s_0(x_1), f, f)} \tag{2.7.1}$$

(c) for any 2-simplex

$$\begin{tikzcd}
x_0 \arrow[rr, "f_2"] \arrow[dr, "f_1"'] & & x_1 \arrow[dl, "f_0"] \\
 & x_2 &
\end{tikzcd}$$

in X_2 ($\cong \mathrm{Cosk}^1(X_\bullet)_2$) and $n \in N_{x_0}$,

$$\boxed{\chi(f_0, f_1, f_2) f_2^{\alpha}(f_0^{\alpha}(n)) \chi(f_0, f_1, f_2)^{-1} = f_1^{\alpha}(n)} \tag{2.7.2}$$

(d) for any 3-simplex

$$\begin{tikzcd}
 & x_1 \arrow[dr, "f_{13}"] \arrow[dd, "f_{12}"] & \\
x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{23}"'] \arrow[dr, "f_{02}"'] & & x_3 \\
 & x_2 \arrow[ur, "f_{23}"'] &
\end{tikzcd}$$

in X_3 ($\cong \mathrm{Cosk}^1(X_\bullet)_3$) ,

$$\boxed{\chi(f_{13}, f_{03}, f_{01}) f_{01}^{\alpha}(\chi(f_{23}, f_{13}, f_{12})) = \chi(f_{23}, f_{03}, f_{02})\chi(f_{12}, f_{02}, f_{01})} \tag{2.7.3}$$

(2.8) A similar transcription for a morphism of 2-cocycles on a type 1 hypercover gives the pair of equalities

$$\boxed{\theta_{s_0(x)} = e} \tag{2.8.0}$$

$$\boxed{\xi_{x_0}(f^{\alpha_1}(n)) = \theta_f^{-1}(f^{\alpha_2}(\xi_{x_1}(n)))\theta_f\ ,} \tag{2.8.1}$$

and

$$\boxed{\xi(\chi_1(f_0, f_1, f_2)) = \theta_{f_1}^{-1}\chi_2(f_0, f_1, f_2) f_2^{\alpha_2}(\theta_{f_0})\theta_{f_2}\ .} \tag{2.8.2}$$

(3) The bouquet $\mathbb{B}(\chi, \alpha, N)$ associated with a 2-cocycle (χ, α, N).

In this section we will show that every 2-cocycle on a hypercovering $X_\bullet$ is equivalent under refinement to a canonical one-defined on (the l-coskeleton of) a bouquet. The resulting bouquet may be viewed as realizing the cocycle. From this result it will follow that H^2 may be computed using these non-abelian 2-cocycles.

(3.0) Let $X_\bullet$ be a hypercover of $\mathbb{E}$ (which we may take to be of type 1 using Lemma (2.6)) and $K_1^0 = (\mathrm{Cosk}^0(X_\bullet))_1 \cong X_0 \times X_0$. Consider K_1^0 as an object above X_0 via the arrow $\mathrm{pr}_1\colon K_1^0 \longrightarrow X_0$ and X_1 as an object above X_0 via the arrow $d_1\colon X_1 \longrightarrow X_0$. Let $\langle d_0, d_1\rangle\colon X_1 \twoheadrightarrow K_1^0$ be the canonical epimorphism and consider the covering of K_1^0 which this epimorphism defines,

$$\begin{tikzcd} X_1 \times_{K_1} X_1 \times_{K_1} X_1 \arrow[r, shift left=2, "\mathrm{pr}_2"] \arrow[r, "\mathrm{pr}_1" description] \arrow[r, shift right=2, "\mathrm{pr}_0"'] & X_1 \times_{K_1^0} X_1 \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_0"'] & X_1 \arrow[r, two heads] & K_1^0 \cong X_0 \times X_0\ . \end{tikzcd} \tag{3.0.0}$$

This covering is supplied canonically with maps $i_1\colon X_1 \times_{K_1} X_1 \longrightarrow X_2$ ($\cong K_2^1$) and $i_2\colon X_1 \times_{K_1} X_1 \times_{K_1} X_1 \longrightarrow X_3$ ($\cong K_3^1$) ("the 0-*coboundary system*" c.f. DUSKIN (1975)) defined by the assignments

$$i_1\colon \begin{tikzcd} x_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & x_1 \end{tikzcd} \quad \rightarrow \quad \begin{tikzcd} x_0 \arrow[rr, "s_0(x_0)"] \arrow[dr, "f_0"'] & & x_0 \arrow[dl, "f_1"] \\ & x_1 & \end{tikzcd} \quad = (f_1, f_0, s_0(x_0)) \tag{3.0.1}$$

and

(3.0.2)
$$\begin{tikzcd}
i_1\colon x_0 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & x_1 & \longmapsto & & 1 & \\
& & & 0 \; x_0 \arrow[r, "s_0(x_0)"] \arrow[rr, "f_0"'] \arrow[dr, "s_0(x_0)"'] & x_0 \arrow[r, "f_1"] \arrow[d, "s_0(x_0)"] & x_1 \; 3 \\
& & & & x_0 \arrow[ur, "f_2"'] & \\
& & & & 2 &
\end{tikzcd}$$

$$= \ [(f_2, f_1, s_0(x_0)),\ (f_2, f_0, s_0(x_0)),\ (f_1, f_0, s_0(x_0)),\ (s_0(x_0), s_0(x_0), s_0(x_0))]\ .$$

We have the following key

(3.1) Lemma. *Let* (χ, α, N) *be a 2-cocycle on* $X_\bullet$. *In* $\mathbb{E}/X_0$ *the mapping* $\widetilde{\chi} = \chi i_1 \colon X_1 \times_{K_1^0} X_1 \longrightarrow N$ *is a (non-abelian) 1-cocycle above* $X_0 \times X_0$ *with coefficients in the group* N *(i.e. an ordinary 1-cocycle with coefficients in* $\mathrm{pr}_1^*(N)$ *in the category* $\mathbb{E}/X_0{\times}X_0$*)*.

(3.1.0)
$$\begin{tikzcd}
 & & X_1 \times X_1 \times X_1 \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[dddll, bend right, "i_2"'] & N \times_{X_0} N \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \\
 & & X_1 \times_X X_1 \arrow[r, "\widetilde{\chi}"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, bend right, "i_1"'] & N \arrow[d] & \\
K_3^1 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & K_2^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[urr, "\chi"] & X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, two heads] & X_0 \arrow[r, two heads] & \mathbb{1} \\
 & & X_0 \times X_0 \arrow[ur, shift left, "\mathrm{pr}_1"] \arrow[ur, shift right, "\mathrm{pr}_0"'] & &
\end{tikzcd}$$

In effect since it is clear that $\widetilde{\chi} s_0 = \widetilde{\chi}\Delta = e$, it only remains to look at χ applied to the 3-simp1ex of (3.0.2). The 2-cocyc1e identity (2.7.3) applied here then gives the identity

$$\chi(s_0(x_0), s_0(x_0)s_0(x_0))^{-1}\chi(f_2, f_1, s_0(x_0))^{-1}\chi(f_0, f_1, s_0(x_0))s_0(x_0^\alpha)(\chi(f_2, f_1, s_0(x_0)) \\ = e(\bar{\chi}p_1)^{-1}(\bar{\chi}p_2)(\bar{\chi}p_0) = (\bar{\chi}p_1)^{-1}(\bar{\chi}p_2)(\bar{\chi}p_0) = e, \quad \text{or equivalently,}$$

$(\bar{\chi}p_2)(\bar{\chi}p_0) = \bar{\chi}p_1$, and χ is a l-cocyc1e as asserted.

(3.2) Using the l-cocycle $\bar{\chi}$ we will construct a bouquet following the following outline: Associated with $\bar{\chi}$ there is a canonical torsor (principal homogeneous space) $X_1 \times_{\bar{\chi}} N$ above $X_0 \times X_0$ which realizes the l-cocycle $\bar{\chi}$. Using its canonical epimorphism to $X_0 \times X_0$ and composing with the projections we will obtain a truncated complex

$$\begin{tikzcd} X_1 \times_{\bar{\chi}} N \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[l, bend right, "s_0"'] \end{tikzcd}$$

in $\mathbb{E}$ whose l-coskeleton will be shown to be supplied with a 2-cocycle equivalent to the original one on $X_\bullet$. The desired bouquet will then be defined using $\begin{tikzcd} X_1 \times_{\bar{\chi}} N \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \end{tikzcd}$ for objects and arrows with composition defined through the "fiber" (= Ker(e)) of the new cocycle.

Thus let (χ, α, N) be a 2-cocycle on $X_\bullet$. From Lemma (3.1) we know that the mapping $\widetilde{\chi}(f_0, f_1) = \chi(f_0, f_1, s_0(x_0))$ is a l-cocycle. It thus has canonically associated with it a torsor above $X_0 \times X_0$ under N which may be described as the quotient of the fiber product

(3.2.0)
$$\begin{tikzcd} N \times_{X_0} X_1 \arrow[r, "\mathrm{pr}_{x_1}"] \arrow[d, "\mathrm{pr}_N"'] & X_1 \arrow[d, "d_1"] \\ N \arrow[r, "p"'] & X_0 \end{tikzcd}$$

under the equivalence relation defined by the 1-cocycle:

(3.2.1)
$$\boxed{(a, f) \sim (b, g) \iff d_0(f) = d_0(g),\ d_1(f) = d_1(g), \text{ and } \widetilde{\chi}(f, g)a = b.}$$

i.e., as the coequalizer $N \times_{\widetilde{\chi}} X_1$ of the equivalence pair of the diagram

(3.2.2)
$$\begin{tikzcd} N \times_N X_1 \times_{K_1} X_1 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] & N \times X_1 \arrow[r, two heads, "q"] & N \times_{\widetilde{\chi}} X_1 \end{tikzcd}$$

where $p_0(a, f, g) = (\widetilde{\chi}(f, g)a, g)$ and $p_1(a, f, g) = (a, f)$ as they appear in the simplicial diagram

(3.2.3)
$$\begin{tikzcd}
 & N \times N \arrow[rr, "\mathrm{pr}_0"] \arrow[dd, shift left] \arrow[dd, shift right] & & N \arrow[dd, "p"] \\
N \times (X_1 \times X_1) \arrow[ur] \arrow[rr, "\mathrm{pr}_1"] \arrow[dd, shift right, "p_0"'] \arrow[dd, shift left, "p_1"] & & X_1 \times_{K_1} X_1 \arrow[ur, "\widetilde{\chi}"'] \arrow[dd, shift left] \arrow[dd, shift right] & \\
 & N \arrow[rr, "p"] & & X_0 \arrow[ll, bend left, "e"] \\
N \times_{X_1} X_1 \arrow[ur] \arrow[rr, "\mathrm{pr}_{X_1}"] \arrow[d, two heads, "q"] & & X_1 \arrow[ll, bend left, "e^{\#}"] \arrow[ur, "d_1"'] \arrow[d, two heads, "\langle d_0{,}d_1\rangle"'] & \\
N \times_{\widetilde{\chi}} X_1 \arrow[rr, two heads, "\overline{\mathrm{pr}}_{x_1}"'] & & X_0 \times X_0 \arrow[uur, shift left, "\mathrm{pr}_0"] \arrow[uur, shift right, "\mathrm{pr}_1"'] &
\end{tikzcd}$$

in which the top, back and bottom squares of the cube are cartesian and $\overline{\mathrm{pr}}_{x_1}\colon N \times_{\widetilde{\chi}} X_1 \twoheadrightarrow X_0 \times X_0$ is deduced from pr_{X_1} so that it is an epimorphism. From this it follows that the lower most square is also cartesian and thus that, locally $\big($over the epimorphism $\langle d_0, d_1\rangle\big)$, $N \times_{\widetilde{\chi}} X_1$ is isomorphic to $d_1^*(N)$.

We now define a truncated 1-complex $\overline{X}_\bullet\big]_0^1$ as follows: Its object of 1-simplices is $N \times_{\widetilde{\chi}} X_1$ and its object of 0-simplices is X_0. Its target map $\overline{d}_0$ is defined by $\overline{d}_0 = \mathrm{pr}_0\, \overline{\mathrm{pr}}_{X_1}$ which its source map is defined by $\overline{d}_1 = \mathrm{pr}_1\, \overline{\mathrm{pr}}_{X_1}$. Its degeneracy is given by $\overline{s}_0 = qe^{\#}s_0\colon X_0 \longrightarrow N \times_{\widetilde{\chi}} X_1$, where s_0 is the degeneracy of $X_\bullet$ and $e^{\#}$ is the section of $N \times_{X_0} X_1$ defined by the unit map of N.

(3.2.4)
$$\begin{tikzcd}
N \times_{\widetilde{\chi}} X_1 \arrow[r, shift left, "\overline{d}_1"] \arrow[r, shift right, "\overline{d}_0"'] & X_0 \arrow[l, bend left=60, "\overline{s}_0"] \arrow[r, two heads] & \mathbb{1}.
\end{tikzcd}$$

By construction, $\overline{X}_\bullet\big]_0^1$ is connected and locally non-empty and is supplied with a truncated simplicial map from $\overline{X}_\bullet\big]_0^1$

(3.2.5)
$$\begin{tikzcd}
X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "qe^{\#}"'] & X_0 \arrow[d, "\mathrm{id}(X_0)"] \\
N \times_{\widetilde{\chi}} X_1 \arrow[r, shift left, "\overline{d}_1"] \arrow[r, shift right, "\overline{d}_0"'] & X_0 \arrow[l, bend left=60, "\overline{s}_0"]
\end{tikzcd}$$

(3.3) Theorem. $\overline{X}_\bullet\big]_0^1$ *is supplied with the structure of a bouquet* $\mathbb{B}(\chi, \alpha, N)$ *whose canonical 2-cocycle is equivalent under refinement to* (χ, α, N) *on* $X_\bullet$ *.*

We will prove this claim by showing that $\operatorname{cosk}^1\big(\overline{X}_\bullet\big]_0^1\big)$ is supplied canonically with a 2-cocycle whose fiber gives the appropriate law of composition for the bouquet. This in turn will be done by defining a 2-cocycle structure on a complex $\widetilde{X}_\bullet$ which will become the desired one on $\operatorname{cosk}^1\big(\overline{X}_\bullet\big]_0^1\big)$ after passage to quotients.

Thus define the truncated complex $\widetilde{X}_\bullet\big]_0^1$ with $N \times_{d_1} X_1$ for 1-simplices, $\tilde{d}_i = d_i \mathrm{pr}_{X_1}$ for faces, and $\tilde{s}_0 = e^{\#}s_0$ for degeneracy. The 2-cocycle structure will be defined on its 1-coskeleton using algebraic operations and the 2-cocycle structure of $X_\bullet$.

(3.3.0)
$$\begin{tikzcd} N^3 \times_{X_1} K_2^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & N \times_{d_1} N_1 \arrow[r, shift left, "\tilde{d}_1"] \arrow[r, shift right, "\tilde{d}_0"'] & X_0 \arrow[l, bend left=50, "\tilde{s}_0"] \end{tikzcd}$$

We first define a gluing: $\tilde{\alpha}\colon \tilde{d}_0^*(N) \longrightarrow \tilde{d}_1^*(N)$ via the mapping

(3.3.1)
$$\boxed{(a, f)^{\tilde{\alpha}}(n) = a^{-1} f^{\alpha}(n) a \qquad \text{(product in } N)}$$

and a 2-cocycle $\widetilde{\chi}$: $N^3 \times_{X_1} K_2^1 \longrightarrow N$ via the mapping

(3.3.2)
$$\boxed{\widetilde{\chi}(a_0, a_1, a_2, f_0, f_1, f_2) = a_1^{-1}\chi(f_0, f_1, f_2) f_2^{\alpha}(a_0) a_2 \qquad \text{(product in } N).}$$

and verify that these are the structural maps of a 2-cocycle $(\widetilde{\chi}, \tilde{\alpha}, N)$ on $\operatorname{cosk}^1\big(\widetilde{X}_\bullet\big]_0^1\big)$ through use of the equalities of Corollary (2.7).

In effect, let us first check normalization: for this we have the chains of equalities

(3.3.3)
$$(e, s_0(x))^{\tilde{\alpha}}(n) = e^{-1} s_0(x)^{\alpha}(n) e = s_0(x)^{\alpha}(n) = n\,,$$

as well as

(3.3.4)
$$\widetilde{\chi}(a, a, e, f, f, s_0(x_0)) = a^{-1}\chi(f, f, s_0(x_0)) s_0(x_0)^{\alpha}(a) e = a^{-1} e a e = e$$

and

(3.3.5)
$$\widetilde{\chi}(e, a, a, s_0(x_1), f, f) = a^{-1}\chi(s_0(x_1), f, f) f^{\alpha}(e) a = a^{-1} e e a = e\,,$$

so that we have normalization.

We now verify that we have a pseudo descent datum: for this we have by definition the equality

(3.3.6)
$$\begin{aligned} &\widetilde{\chi}(a_0, a_1, a_2, f_0, f_1, f_2)[a_2, f_2]^{\tilde{\alpha}}([a_0, f_0]^{\tilde{\alpha}}(n))\widetilde{\chi}(a_0, a_1, a_2, f_0, f_1, f_2)^{-1} \; = \\ &(a_1^{-1}\chi(f_0, f_1, f_2) f_2^{\alpha}(a_0) a_2)(a_2^{-1} f_2^{\alpha}(a_0^{-1} f_0^{\alpha}(n) a_0) a_2)(a_2^{-1} f_2^{\alpha}(a_0^{-1})\chi(f_0, f_1, f_2)^{-1} a_1)\,. \end{aligned}$$

But since f_2^α is a homomorphism, this gives, after cancelation and use of condition (c) of Corollary (2.7),

$$a_1^{-1}\chi(f_0,f_1,f_2)f_2^\alpha(f_0^\alpha(n))\chi(f_0,f_1,f_2)^{-1}a_1 = a_1^{-1}f_1^\alpha(n)a_1 = (a_1,f_1)^{\tilde{\alpha}}(n)\,, \tag{3.3.7}$$

so that we have a pseudo descent datum.

We now verify the cocycle condition: For this, use of (c) and (d) of Corollary (2.7) gives the chain of equalities:

$$\begin{aligned}
&\widetilde{\chi}(a_{13},a_{03},a_{01})[a_{01},f_{01}]^{\tilde{\alpha}}(\widetilde{\chi}(a_{23},a_{13},a_{12},f_{23},f_{13},f_{12})) \ = \\
&\big[a_{03}^{-1}\chi(f_{13},f_{03},f_{01})f_{01}^\alpha(a_{13})a_{01}\big]\ \big[[a_{01},f_{01}]^{\tilde{\alpha}}(a_{13}^{-1}\chi(f_{23},f_{13},f_{12})f_{12}^\alpha(a_{23})a_{12})\big] \ = \\
&a_{03}^{-1}\chi(f_{13},f_{03},f_{01})f_{01}^\alpha(a_{13})a_{01}a_{01}^{-1}f_{01}^\alpha(a_{13}^{-1})f_{01}^\alpha(\chi(f_{23},f_{13},f_{12})f_{01}^\alpha(f_{12}^\alpha(a_{23}))f_{01}^\alpha(a_{12})a_{01}) \ = \\
&a_{03}^{-1}\chi(f_{23},f_{03},f_{02})f_{02}^\alpha(a_{23})\chi(f_{12},f_{02},f_{01})f_{01}^\alpha(a_{12})a_{01} \ = \\
&\big[a_{03}^{-1}\chi(f_{23},f_{03},f_{02})f_{02}^\alpha(a_{23})a_{02}\big]\ \big[a_{02}^{-1}\chi(f_{12},f_{02},f_{01})f_{01}^\alpha(a_{12})a_{01})\big] \ = \\
&\widetilde{\chi}(a_{23},a_{03},a_{02},f_{23},f_{03},f_{02})\widetilde{\chi}(a_{12},a_{02},a_{01},f_{12},f_{02},f_{01})
\end{aligned}$$

and we have completed the verification that $(\widetilde{\chi},\tilde{\alpha},N)$ is a 2-cocycle on the hypercovering $\widetilde{X}_\bullet$.

We now verify that the definition of $\widetilde{\chi}$ and $\tilde{\alpha}$ is compatible with the equivalence relation of (3.2.1). From this it will immediately follow by passage to quotients that we have defined a 2-cocycle structure $(\overline{\chi},\bar{\alpha},N)$ on $\overline{X}_\bullet$ which is equivalent to the original 2-cocycle (χ,α,N) on X.

We first show that the gluing $\tilde{\alpha}$ descends: thus let (a,f) and (b,g) be equivalent 1-simplices in $N\times_{d_1}X_1$. This means that f and g have the same respective source and target and

$$\chi(f,g,s_0(x_0))a = b \tag{3.3.8}$$

Application of condition (c) of Corollary (2.7) to the 2-simplex

$$\begin{tikzcd}
x_0 \arrow[rr, "s_0(x_0)"] \arrow[dr, "g"'] & & x_0 \arrow[dl, "f"] \\
& x_1 &
\end{tikzcd} \tag{3.3.9}$$

gives the equalities

$$\begin{aligned}
g^\alpha(\eta) = \chi(f,g,s_0(x_0))s_0(x_0)^\alpha(f^\alpha(n))\chi(f,g,s_0(x_0))^{-1} \ = \\
\chi(f,g,s_0(x_0))f^\alpha(n)\chi(f,g,s_0(x_0))^{-1}\,.
\end{aligned} \tag{3.3.10}$$

for al $n\in N_{x_1}$. We thus compute

$$\begin{aligned}
&(b,g)^{\tilde{\alpha}}(n) = b^{-1}g^\alpha(n)b = [\chi(f,g,s_0(x_0))a]^{-1}\,g^\alpha(n)\chi(f,g,s_0(x_0))a \ = \\
&a^{-1}\chi(f,g,s_0(x_0))^{-1}\chi(f,g,s_0(x_0))f^\alpha(n)\chi(f,g,s_0(x_0))^{-1}\chi(f,g,s_0(x_0))a \ = \\
&a^{-1}f^\alpha(n)a = (a,f)^{\tilde{\alpha}}(n)
\end{aligned}$$

and $\tilde{\alpha}$ descends.

We now show that $\widetilde{\chi}$ descends: Thus we consider a pair of 2-simplicies, $(a_0, a_1, a_2, f_0, f_1, f_2)$ and $(b_0, b_1, b_2, g_0, g_1, g_2)$, whose faces are equivalent, i.e. are such that we have the equalities

$$\chi(f_0, g_0, s_0(x_1))a_0 = b\,,\ \chi(f_1, g_1, s_0(x_0))a_1 = b_1\,,\ \text{and}\ \ \chi(f_2, g_2, s_0(x_0))a_2 = b_2\,.$$

Then

$$\begin{aligned}&\widetilde{\chi}(b_0, b_1, b_2, g_0, g_1, g_2) = b_1^{-1}\chi(g_0, g_1, g_2)g_2^{\alpha}(b_0)b_2 \ = \\ &[\chi(f_1, g_1, s_0(x_0))a_1]^{-1}\,\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1))a_0\chi(f_2, g_2, s_0(x_0))a_2 \ = \\ &a_1^{-1}\chi(f_1, g_1, s_0(x_0))^{-1}\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1))g_2^{\alpha}(a_0)\chi(f_2, g_2, s_0(x_0))a_2\end{aligned}$$

but since $g_2^{\alpha}(a_0) = \chi(f_2, g_2, s_0(x_0))f_2^{\alpha}(a_0)\chi(f_2, g_2, s_0(x_0))^{-1}$, substitution and cancellation in the last equality becomes

$$= \ a_1^{-1}\chi(f_1, g_1, s_0(x_0))^{-1}\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1)))\chi(f_2, g_2, s_0(x_0))f_2^{\alpha}(a_0)a_2\,.$$

The proof will be complete provided that we can show that

$$\chi(f_1, g_1, s_0(x_0))^{-1}\chi(g_0, g_1, g_2)g_2^{\alpha}(f_0, g_0, s_0(x_1))\chi(f_2, g_2, s_0(x_0)) = \chi(f_0, f_1, f_2)\,. \tag{3.3.11}$$

For this, observe that the two 3-simplices

(3.3.12)

$$\begin{tikzcd}
 & x_0 \arrow[dr, "f_1"] \arrow[dd, "f_2"] & \\
x_0 \arrow[ur, "s_0(x_0)"] \arrow[rr, "g_1"] \arrow[dr, "g_2"'] & & x_2 \\
 & x_1 \arrow[ur, "f_0"'] &
\end{tikzcd}$$

and

(3.3.13)

$$\begin{tikzcd}
 & x_1 \arrow[dr, "g_0"] \arrow[dd, "s_0(x_0)"] & \\
x_0 \arrow[ur, "g_2"] \arrow[rr, "g_1"] \arrow[dr, "g_2"'] & & x_2 \\
 & x_1 \arrow[ur, "f_0"'] &
\end{tikzcd}$$

have a common face (f_0, g_1, g_2). The cocycle identity applied here gives the equalities

$$\chi(f_1, g_1, s_0(x_0))s_0^{\alpha}(x_0)(\chi(f_0, f_1, f_2)) = \chi(f_0, g_1, g_2)\chi(f_2, g_2, s_0(x_0)) \quad \text{and} \tag{3.3.14}$$

$$\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1)) = \chi(f_0, g_1, g_2)\chi(s_0(x_1), g_2, g_2) = \chi(f_0, g_1, g_2) \tag{3.3.15}$$

and thus the equality

$$\chi(f_1, g_1, s_0(x_0))\chi(f_0, f_1, f_2)\chi(f_2, g_2, s_0(x_0))^{-1} = \chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_0)) \tag{3.3.16}$$

from which (3.3.11) immediately follows.

We have now shown that the quotient complex $\overline{X}_\bullet$ is supplied with the structure of a 2-cocycle $(\overline{\chi}, \bar{\alpha}, N)$ which is clearly equivalent to the original cocycle (X, α, N) on $X_\bullet$ via the simplicial map (3.2.5). We now observe the fundamental property of the constructed 2-cocycle $(\overline{\chi}, \bar{\alpha}, N)$.

(3.4) Lemma. *For $i = 0, 1, 2$ the commutative squares*

(3.4.0) $\left(\overline{K}_2^1 \xrightarrow{\sim}\right)$

$$\begin{tikzcd}
\overline{X}_2 \arrow[r, "\overline{\chi}"] \arrow[d, "\langle \bar{d}_i \rangle"'] & N \arrow[d, "p"] \\
\bar{\Lambda}_i^1 \arrow[r, "d_1\,\mathrm{pr}"'] & X_0
\end{tikzcd}$$

are cartesian.

In effect, using Barr's embedding it will suffice to verify this in (ENS). For this it will clearly be sufficient to verify that the commutative squares

(3.4.1)

$$\begin{tikzcd}
\widetilde{X}_2 \arrow[r, "\overline{\chi}"] \arrow[d, "\langle d_i \rangle"'] & N \arrow[d, "p"] \\
\tilde{\Lambda}_i^1 \arrow[r, "d_1\,\mathrm{pr}"'] & X_0
\end{tikzcd}$$

(which have (3.4.0) as their quotient) are, up to equivalence, cartesian. We will show this for the 1-horn $\tilde{\Lambda}_1^1$, the proof for the other two cases being entirely similar. Thus let $(a_0, f_0 \colon x_1 \longrightarrow x_2, \quad a_2, f_2 \colon x_0 \longrightarrow x_1)$ be an element of $\tilde{\Lambda}_i$ and $a \in N_{x_0}$. Since $\widetilde{X}_\bullet\big]_0^1$ is connected, we may choose a 1-simplex $f_1 \colon x_0 \longrightarrow x_2$ and define $a_1 \in N_{x_0}$ by

$$a_1 = \chi(f_0, f_1, f_2) f_2^{\alpha}(a_0) a_2 a^{-1}\,, \tag{3.4.2}$$

so that $\widetilde{\chi}(a_0, a_1, a_2, f_0, f_1, f_2) = a$ and $\langle d_0, d_2 \rangle(a_0, a_1, a_2, f_0, f_1, f_2) = (a_0, f_0, a_2, f_2)$.

Now suppose that we have 2-simplices $(a_0, a_1, a_2, f_0, f_1, f_2)$ and $(b_0, b_1, b_2, g_0, g_1, g_2)$ such that $(a_2, f_2) \sim (b_2, g_2)$, $(a_0, f_0) \sim (b_0, g_0)$ and $\widetilde{\chi}(a_0, a_1, a_2, f_0, f_1, f_2) = \widetilde{\chi}(b_0, b_1, b_2, g_0, g_1, g_2)$; we will show that $(a_1, f_1) \sim (b_1, g_1)$.

This means that given the equations

(a) $\chi(f_2, g_2, s_0(x_0))a_2 = b_2$,

(b) $\chi(f_0, g_0, s_0(x_1))a_0 = b_0$, and

(c) $a_1^{-1}\chi(f_0, f_1, f_2)f_2^{\alpha}(a_0)a_2 = b_1^{-1}\chi(g_0, g_1, g_2)g_2^{\alpha}(b_0)b_2,$

we must show that
(d) $\chi(f_1, g_1, s_0(x_0))a_1 = b_1.$
We thus start with (c) and make substitutions dictated by (a) and (b). This leads to the equality

$$\begin{aligned} &a_1^{-1}\chi(f_0, f_1, f_2)f_2^{\alpha}(a_0)a_2 \ = \\ &b_1^{-1}\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1))a_0)\chi(f_2, g_2, s_0(x_0))a_2\,, \end{aligned} \tag{3.4.3}$$

so that

$$\begin{aligned} b_1 = \chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1))g_2^{\alpha}(a_0)\chi(f_2, g_2, s_0(x_0))f_2^{\alpha}(a_0^{-1}) \\ \times\chi(f_0, f_1, f_2)^{-1}a_1\,. \end{aligned} \tag{3.4.4}$$

Since $g_2^{\alpha}(a_0) = \chi(f_2, g_2, s_0(x_0))f_2^{\alpha}(a_0)\chi(f_2, g_2, s_0(x_0))^{-1}$, (3.4.4) becomes the equation

$$b_1 = \chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1))\chi(f_2, g_2, s_0(x_0))\chi(f_0, f_1, f_2)^{-1}a_1 \tag{3.4.5}$$

and the result will follow if we can show that

$$\begin{aligned} &\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_1(x_1))\chi(f_2, g_2, s_0(x_0))\chi(f_0, f_1, f_2)^{-1} \ = \\ &\chi(f_1, g_1, s_0(x_0))\,. \end{aligned} \tag{3.4.6}$$

holds.

Now from the 3-simplex

$$\begin{tikzcd} & x_1 \arrow[dr, "g_0"] \arrow[dd, "s_0(x_1)"] & \\ x_0 \arrow[ur, "g_2"] \arrow[rr, "g_1"] \arrow[dr, "f_2"'] & & x_2 \\ & x_1 \arrow[ur, "f_0"'] & \end{tikzcd} \tag{3.4.7}$$

the cocycle identity gives the equality

$$\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_0(x_1)) = \chi(f_0, g_1, f_2)\chi(s_0(x_1), f_2, g_2)\,, \tag{3.4.8}$$

while the 3-simplex

$$\begin{tikzcd} & x_0 \arrow[dr, "f_2"] \arrow[dd, "g_2"] & \\ x_0 \arrow[ur, "s_0(x_0)"] \arrow[rr, "f_2"] \arrow[dr, "f_2"'] & & x_1 \\ & x_1 \arrow[ur, "s_0(x_1)"'] & \end{tikzcd}$$

gives the equality

(3.4.9) $$\chi(f_2, g_2, s_0(x_0))^{-1} = \chi(s_0(x_1), f_2, g_2)\,.$$

Hence after substitution and cancellation the left hand side of (3.4.6) becomes

(3.4.10) $$\begin{aligned}&\chi(g_0, g_1, g_2)g_2^{\alpha}(\chi(f_0, g_0, s_1(x_1))\chi(f_2, g_2, s_0(x_0))\chi(f_0, f_1, f_2)^{-1} \;= \\ &\chi(f_0, g_1, f_2)\chi(f_0, f_1, f_2)^{-1}\,.\end{aligned}$$

But the cocycle identity applied to the 3-simplex

(3.4.11) $$\begin{tikzcd} & x_0 \arrow[dr, "g_1"] \arrow[dd, "f_2"] & \\ x_0 \arrow[ur, "s_0(x_0)"] \arrow[rr, "f_1"'] \arrow[dr, "f_2"'] & & x_2 \\ & x_1 \arrow[ur, "f_0"'] & \end{tikzcd}$$

gives

(3.4.12) $$\chi(f_0, g_1, f_2)\chi(f_0, f_1, f_2)^{-1} = \chi(f_1, g_1, s_0(x_0))$$

and the result is established.

To complete the proof of Theorem (3.3) it only remains to show that $\widetilde{X}_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} X_0$ has the structure of a bouquet. This is a consequence of the following

(3.5) <u>Lemma</u>. *Let (χ, α, N) be a 2-cocycle on a hypercovering of type 1 for which the commutative squares*

(3.5.0) $$\begin{tikzcd} K_2^1 \arrow[r, "\chi"] \arrow[d, "\langle d_i \rangle"'] & N \arrow[d, "p"] \\ \Lambda_i \arrow[r] & X_0 \end{tikzcd}$$

are cartesian for $i = 0, 1, 2$. Let $e^{\#}\colon \Lambda_1 \longrightarrow K_2^1$ be the section of $\langle d_0, d_1 \rangle$ defined by the unit section of p and define a law of composition $\mu\colon \Lambda_1 \longrightarrow X_1$ as $p_1 e^{\#} = \mu$. Then

(3.5.1) $$\begin{tikzcd} \Lambda_1 \arrow[r, shift left=2] \arrow[r, "\mu" description] \arrow[r, shift right=2] & X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[l, bend left=50] \end{tikzcd}$$

is a bouquet whose subgroupoid of automorphisms is isomorphic to N and whose canonical cocycle structure (2.1) is isomorphic to (χ, α, N).

In effect we have defined a non-abelian version of a "2-dimensional Kan-action" of N on the complex $\begin{tikzcd} X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[l, bend left=50, "s_0"] \end{tikzcd}$ (DUSKIN (1979)) and the proof that the law of composition defined on the "fiber" is indeed that of a groupoid is similar to that of the abelian case. Intuitively, the law of composition for a composable pair $x_0 \xrightarrow{f_2} x_1 \xrightarrow{f_0} x_2$ is defined as the 1-face of the unique element of the simplicial kernel

$$\begin{tikzcd} x_0 \arrow[r, "f_2"] \arrow[dr, "f_1"'] & x_1 \arrow[d, "f_0"] \\ & x_2 \end{tikzcd} \tag{3.5.2}$$

which has the property that $\chi(f_0, f_1, f_2) = e \in N_{\chi_0}$. (Such a 2-simplex is said to be *commutative*). Inverses are given using the horns Λ_0 and Λ_1, while associativity is an immediate consequence of the cocycle identity applied to a tetrahedron all of whose faces except for one are allowed to be commutative. (Associativity follows from the fact that the remaining face must also be commutative.) Identification of N with the subgroupoid of automorphisms follows from the assignment of any $a \in N_{x_0}$ to that unique 2-simplex

$$\begin{tikzcd} x_0 \arrow[rr, "s_0(x_1)"] \arrow[dr, "s_0(x_0)"'] & & x_0 \arrow[dl, "\tilde{a}"] \\ & x_0 & \end{tikzcd} \tag{3.5.3}$$

which has the property that $\chi(\tilde{a}, s_0(x_0), s_0(x_0)) = a$. The cocycle identity now applied to the 3-simplex.

$$\begin{tikzcd} & x_0 \arrow[dr, "\widetilde{ba}"] \arrow[dd, "s_0(x_0)"] & \\ x_0 \arrow[ur, "s_0(x_0)"] \arrow[rr, "\tilde{a}"'] \arrow[dr, "s_0(x_0)"'] & & x_0 \\ & x_0 \arrow[ur, "\tilde{b}"'] & \end{tikzcd} \tag{3.5.4}$$

now gives the equation

$$ba\chi(\tilde{b}, ba, \tilde{a}) = ba\,, \tag{3.5.5}$$

so that $\chi(\tilde{b}, \widetilde{ba}, \tilde{a}) = e$ and hence that $\tilde{b}\tilde{a} = \widetilde{ba}$ and hence that $a \mapsto \tilde{a}$ is an isomorphism (since clearly $\tilde{e} = s_0(x_0)$).

It only remains to show that (χ, α) may be identified with canonical pair $(f^{-1}af, f_1^{-1}f_0f_2)$.

We first identify the glueing by considering the cocycle identity applied to the three tetrahedra

(3.5.6)

$$\begin{tikzcd}
x_0 \arrow[d, "f"'] \arrow[ddr, "f"] \arrow[drr, bend left, "f"] & & \\
x_1 \arrow[rr, "s_0(x_1)"] \arrow[dr, "s_0(x_1)"'] & & x_1 \\
& x_1 \arrow[ur, "\tilde{a}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)"] \arrow[ddr, bend right, "f"'] \arrow[drr, "f"] & x_0 \arrow[dd, "f"] \arrow[dr, bend left, "\tilde{a}f"] & \\
& & x_1 \\
& x_1 \arrow[ur, "\tilde{a}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "s_0(x_0)"] \arrow[rr, bend left, "s_0(x_0)"] \arrow[dr, bend right, "f"'] & x_0 \arrow[r, "f^{-1}\tilde{a}f"] \arrow[d, "\tilde{a}f"] & x_0 \arrow[dl, bend left, "f"] \\
& x_1 &
\end{tikzcd}$$

This then gives immediately the identity

$$f^\alpha(a) = \chi(\tilde{a}, f, f) = \chi(f^{-1}\tilde{a}f, s_0(x_0), s_0(x_0)) \tag{3.5.7}$$

and thus that the gluing is the canonical one.

A similar computation using the definition of $f_1^{-1}f_2f_0$ gives

$$(f_1^{-1}f_0f_2)^\alpha(\chi(f_0, f_1, f_2)) = \chi(f_1^{-1}f_0f_2, s_0(x_0), s_0(x_0))\,, \tag{3.5.8}$$

so that we have

$$\begin{aligned}\chi(f_1^{-1}f_1f_2, s_0(x_0), s_0(x_0)) &= \chi(\chi(\widetilde{f_0, f_1,} f_2), (f_1^{-1}f_0f_2), (f_1^{-1}f_0f_2)) \\ &= \chi((f_1^{-1}f_0f_2)^{-1}\chi(\widetilde{f_0, f_1,} f_0)(f_1^{-1}f_0f_2), s_0(x_0), s_0(x_0))\,.\end{aligned} \tag{3.5.9}$$

Thus

$$(f_1^{-1}f_0f_2)^{-1}\chi(\widetilde{f_0, f_1,} f_2)(f_1^{-1}f_0f_2) = f_1^{-1}f_0f_2$$

and we have

$$\chi(\widetilde{f_0, f_1,} f_2) = f_1^{-1}f_0f_2$$

as desired.

This completes the proof of Lemma (3.5) and hence of Theorem (3.3).

(3.6) Remark. If $\underset{\approx}{G}$ is a bouquet and (χ, α, ξ) is its canonical cocycle, then the proceeding construction applied here simply yields a bouquet which is canonically isomorphic to $\underset{\approx}{G}$. In effect, the equivalence relation on $\xi \times_{d_1} A$ defined by the cocycle is simply the commutativity of the square

$$\begin{tikzcd}
x_0 \arrow[r, "b"] \arrow[d, "a"'] & x_0 \arrow[d, "g"] \\
x_0 \arrow[r, "f"'] & x_1
\end{tikzcd} \tag{3.6.0}$$

The mapping defined by $(a, f) \mapsto fa$ then defines $\mathrm{Ar}(\underset{\approx}{G})$ as the quotient, since it is canonically supplied with a section via the assignment $f \mapsto (\mathrm{id}(x_0), f)$.

(3.7) <u>Theorem</u>.

(a) Let $(\theta,\xi)\colon(\chi_1,\alpha_1,N_1)\longrightarrow(\chi_2,\alpha_2,N_2)$ be a morphism of 2-cocycles on a hypercovering $X_\bullet$. Then the arrow defined by the assignment

(3.7.0) $$\boxed{(a,f)\longmapsto(\theta_f\xi(a),f)}$$

descends and defines a functor

(3.7.1) $$\boxed{\mathbb{B}(\theta,f)\colon\mathbb{B}(\chi_1,\alpha_1,N_1)\longrightarrow\mathbb{B}(\chi_2,\alpha_2,N_2)}$$

on the associated bouquets which is an isomorphism provided (θ,ξ) is an isomorphism.

(b) Let $(\theta,\xi)\colon(\chi_x,\alpha_x,N_x)\longrightarrow F_\bullet^*(\chi_y,\alpha_y,N_y)$ be an $F_\bullet\colon X_\bullet\longrightarrow Y_\bullet$ morphism of cocycles above $X_\bullet$ and $Y_\bullet$. Then the arrow defined by the assignment

(3.7.2) $$\boxed{(a,f)\mapsto(\theta_f,\xi(a),F_1(f))}$$

together with the arrow $F_0\colon X_0\longrightarrow Y_0$ defines a functor

(3.7.3) $$\boxed{\mathbb{B}(\theta,\xi)\colon\mathbb{B}(\chi_x,\alpha_x,N_x)\longrightarrow\mathbb{B}(\chi_y,\alpha_y,N_y)}$$

which is an essential equivalence if and only if (θ,ξ) is an isomorphism.

In effect a morphism of 2-cocycles above $X_\bullet$ defines a homotopy of the corresponding 1-cocycles which define the arrows of the corresponding bouquets. For an $F_\bullet$ morphism this gives rise to the commutative diagram of 1-torsors

(3.7.4)
$$\begin{tikzcd}
 & N_x \arrow[rr, "\xi"] \arrow[dd] & & N_y \arrow[dd] \\
\overline{X}_1\times_{x_0}N_x \arrow[ur] \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] & & \overline{Y}_1\times_{y_0}N_y \arrow[ur] \arrow[dd, shift left] \arrow[dd, shift right] & \\
 & X_0 \arrow[rr, "F_0"] & & Y_0 \\
\overline{X}_1 \arrow[ur] \arrow[rr] \arrow[dd, two heads] & & \overline{Y}_1 \arrow[ur] \arrow[dd, two heads] & \\
X_0\times X_0 \arrow[uur, "\mathbb{B}(\theta,\xi)"'] \arrow[rr, "F_0\times F_0"'] & & Y_0\times Y_0 \arrow[uur, "\mathrm{pr}"'] &
\end{tikzcd}$$

in which it is easily checked that the lower square is cartesian (i.e., $\mathbb{B}(\theta,\xi)$ is fully faithful) if and only if the rear square is cartesian (i.e., (θ,ξ) is an equivalence under refinement). Other details of the verification are left to the reader as an exercise in computation with 2-cocycles.

(3.8) Corollary. Let BOUQ[$\mathbb{E}$] be the class of connected component classes of the category $\overline{\mathrm{BOUQ}(\mathbb{E})}$ of bouquets of $\mathbb{E}$ and essential equivalences and $H^2_{\mathbb{Z}}(\mathbb{E})$ be the class of equivalence classes under refinements of the fibered category of 2-cocycles of $\mathbb{E}$. Then the assignment $(\chi,\alpha,N)\mapsto\mathbb{B}(\chi,\alpha,N)$ of any 2-cocycle to its associated bouquet defines a bijection

$$\boxed{\mathbb{B}\colon H^2_{\mathbb{Z}}(\mathbb{E})\xrightarrow{\sim} BOUQ[\mathbb{E}]\,.} \tag{3.8.0}$$

(4) THE TIE OF A 2-COCYCLE.

(4.0) If $\mathbb{F}$ is a fibered category over $\mathbb{E}$ which is a category of descent (i.e., such that morphisms glue) then, following a technique similar to that used in Lemma (2.6), ARTIN and MAZUR (1969) have shown that there is a one-to-one correspondence (that is, an equivalence under refinement) between descent data on a hypercovering $X_\bullet$ and descent data on $\mathrm{Cosk}^0(X_\bullet)$, the covering of $\mathbb{E}$ which is associated with $X_\bullet$.

This fact has several immediate consequences for us here:

(4.1) *The abelian case.* First, observe that if $(X',\quad,A')$ is a 2-cocycle on $X_\bullet$ for which the coefficient group A' is abelian, then condition (b) of the definition of the 2-cocycle (2.0) asserts that the gluing α is a descent datum on A' in the fibered category of groups over the hypercovering $X_\bullet$. Since this fibered category is a category of effective descent (i.e., both morphisms and objects glue), this gluing defines an effective descent datum on A' over the *covering* $\mathrm{Cosk}^0(X_\bullet)$ of $\mathbb{E}$. Consequently, there exists a global abelian group A whose localization over X_0 is isomorphic to A' and whose restriction to X_1 has its projection equalize the gluing α.

$$\begin{tikzcd}
d_0^*(A') \arrow[rr] \arrow[dr, bend left, "\alpha"'] \arrow[dd] & & \mathrm{pr}_0^*(A') \arrow[rr] \arrow[dr, bend left, "\alpha'"'] \arrow[dd] & & A' \arrow[r, "\mathrm{pr}_A"] \arrow[dd] & A \arrow[dd] \\
& d_1^*(A') \arrow[rr] \arrow[dl] & & \mathrm{pr}^*(A') \arrow[ur] \arrow[dl] & & \\
X_1 \arrow[rr, two heads, "\langle d_0,d_1\rangle"'] & & X_0\times X_0 \arrow[rr, shift left, "\mathrm{pr}_1"] \arrow[rr, shift right, "\mathrm{pr}_0"'] & & X_0 \arrow[r] & \mathbb{1}
\end{tikzcd} \tag{4.1.0}$$

The cocycle $\chi\colon X_2\longrightarrow A'$ may now be composed with pr_A to produce a mapping $\chi\colon X_2\longrightarrow A$ and the cocycle condition (c) then becomes

$$(\chi d_0)-(\chi d_1)+(\chi d_2)-(\chi d_3)=0 \tag{4.1.1}$$

(after using commutativity) and thus, quite simply, an ordinary normalized 2-cocycle on the hypercovering $X_\bullet$ which coefficients in the abelian group object A of $\mathbb{E}$, i.e. a simplicial mapping $\chi_\bullet \colon X_\bullet \longrightarrow K(A,2)$. Clearly, every ordinary 2-cocycle on a hypercovering produces, by localization, a 2-cocycle with coefficients in the locally given group $A|X0) \longrightarrow X_0$. Moreover, equivalence under refinement becomes ordinary homotopy and the abelian theory is immediately recoverable from our non-abelian theory.

Thus, in particular, one obtains that H^2 with abelian coefficients may be computed using $K(A,2)$ torsors (c.f. DUSKIN [1977]) and using VERDIER [1972] or GLENN [1982], one obtains its coincidence with the classical (derived functor) definition.

There is a cautionary note, however, which we will discuss when we discuss cohomology with coefficients in a (globally given) group.

(4.2) *The tie of a 2-cocycle.* If the coefficient group is non-abelian, condition (b) still asserts that the gluing α defines a descent datum on N over the hypercovering $X_\bullet$ in the fibered category of *pre-ties* of $\mathbb{E}$, i.e., the fibered category of groups *modulo* global sections of the *sheaf* of *inner automorphisms.* Thus, while not a category of effective descent, is a category of descent and thus defines a descent datum on N over $\mathrm{Cosk}^0(X_\bullet)$. The category of *ties of* $\mathbb{E}$ is the completion of this fibered category and thus what we have at the global level is, not a group, but a tie, which we will call the *tie of the cocycle* (χ, α, N) and denote by $\mathbb{L}(\chi, \alpha, N)$. Note that *if* $\underset{\approx}{G}$ *is a bouquet, then the tie of the canonical 2-cocycle associated with* $\underset{\approx}{G}$, (2.1) *is precisely the tie of* $\underset{\approx}{G}$ *as defined in* (I 3.2). Clearly if two cocycles are equivalent, then their ties are isomorphic. Thus we obtain using Corollary (2.8) the following

(4.3) Theorem. (calculation of the non-abelian H^2 by cocycles)
Let $H^2_{\mathbb{Z}}(\mathbb{E}; L)$ *be the class of equivalence classes of 2-cocycles of* $\mathbb{E}$ *which have their tie isomorphic to L, and let* $\mathrm{BOUQ}[\mathbb{E}; L]$ *be the class of equivalence classes of* $\mathbb{E}$*-bouquets of* $\mathbb{E}$ *which have their tie isomorphic to L. Finally let* $H^2_{\mathrm{Gir}}(\mathbb{E}; L)$ *be the class of cartesian equivalence classes of gerbes with tie L. Then one has bijections*

$$\mathrm{BOUQ}[\mathbb{E}; L] \xrightarrow{\sim} H^2_{\mathrm{Gir}}(\mathbb{E}; L) \xrightarrow{\sim} H^2_{\mathbb{Z}}(\mathbb{E}; L) \tag{4.3.0}$$

(5) THE EILENBERG-MAC LANE THEOREMS FOR A TOPOS.

(5.0) We now show that $H^2(\mathbb{E}; L)$ is a principal homogeneous space under the abelian group $H^2(\mathbb{E}; \mathbb{Z}(L))$, where $\mathbb{Z}(L)$ is the center of the tie (defined below). It will follow that $H^2(\mathbb{E}; L)$ is a set, which if non-empty, admits a (non-canonical) bijection onto $H^2(\mathbb{E}; \mathbb{Z}(L))$. We will then show that every tie has an obstruction to its being represented as the tie of a bouquet so that $H^2(\mathbb{E}; L)$ would then be non-empty. This obstruction defines a cocycle of $Z^3(\mathbb{E}; \mathbb{Z}(L))$ and $H^2(\mathbb{E}; L)$ will be non empty if and only if one of those obstruction cocycles is identically 0. Whether or not every element of H^3 can be so represented as an obstruction to the representability of a tie by a bouquet in an arbitrary topos is still unclear at the time of this writing.

(5.1) *The center of a tie.* Since every tie L of $\mathbb{E}$ has a representative which may be identified with a descent datum on a locally given group in the fibered category of pre-ties of $\mathbb{E}$, we may choose some locally given group $N \longrightarrow X_0$ on a covering $X_0 \twoheadrightarrow \mathbb{1}$ and look at the center $\mathbb{Z}(N) \longrightarrow N$ of N in $\mathbb{G}r/X_0$. Since $\mathbb{Z}(N)$ is central the descent datum on N in the category of pre-ties must define a descent datum on $\mathbb{Z}(N)$ (over the same covering) in the fibered category of locally given abelian groups of $\mathbb{E}$ and thus by descent, produce a global abelian group which is locally isomorphic to $\mathbb{Z}(N)$. This global abelian group is clearly independent of the choice of representative for the tie L and unique up to isomorphism. It will be called the *center of the tie* L and be denoted by $\mathbb{Z}(L)$. It is through the use of this abelian group that the link between abelian and non-abelian theory is obtained.

(5.2) Theorem. *For any tie L of $\mathbb{E}$, $H^2(\mathbb{E}; L)$ is a principal homogenous space under the abelian group $H^2(\mathbb{E}; \mathbb{Z}(L))$. That is, $H^2(\mathbb{E}; \mathbb{Z}(L))$ has a principal, transitive, action on $H^2(\mathbb{E}; L)$.*

We first define the action: Thus represent any element of $H^2(\mathbb{E}; L)$ by a non-abelian cocycle on some hypercovering of $\mathbb{E}$ and any element of $H^2(\mathbb{E}; \mathbb{Z}(L))$ by an ordinary abelian 2-cocycle on another hypercovering. Using the filtration (up to homotopy) of the category of hypercovers of $\mathbb{E}$, both of these 2-cocycles are equivalent to ones of the same type on a common hypercovering $X_\bullet$ which we may take to be of type 1.

(5.2.0)
$$\begin{tikzcd}
 & & & & \mathbb{Z}(N) \arrow[r, "\mathrm{pr}"] \arrow[d, hook] & \mathbb{Z}(L) \arrow[dd] \\
 & d_0^*(N) \arrow[rr, "\alpha"] \arrow[dr] & & d_1^*(N) \arrow[dl] & N \arrow[d] & \\
K_2^1 \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] \arrow[rrrru, "\chi"] \arrow[rrrrruu, bend left, "c"] & & X_1 \arrow[rr, shift left] \arrow[rr, shift right] & & X_0 \arrow[r] & \mathbb{1}
\end{tikzcd}$$

Thus let (χ, α, N) be the non abelian cocycle and $c\colon K_2^1 \longrightarrow \mathbb{Z}(L)$ the abelian one. Since the tie of (χ, α, N) is L and $\mathbb{Z}(L)$ is the center of L, $\mathbb{Z}(L)$ is locally isomorphic to $\mathbb{Z}(N)$ supplied with the descent datum $\alpha| \, \mathbb{Z}(N)\colon d_0^*(\mathbb{Z}(N)) \xrightarrow{\sim} d_1^*(\mathbb{Z}(N))$ which is the restriction of α to the center of its respective source and target. Now designate by c' the map $\langle d_1 d_1, c\rangle\colon K_2^1 \longrightarrow \mathbb{Z}(N)$ and by $c^\#$ the map $\mathrm{in}_{\mathbb{Z}(N)} c'\colon K_2^1 \longrightarrow N$. Since c is a 2-cycle, c' has the property that

$$f_{01}^{\alpha| \, \mathbb{Z}(N)}(c' d_0) - (c' d_1) + (c' d_2) - (c' d_3) = 0 \tag{5.2.1}$$

and its image in N that of

$$f^{\alpha}_{01}(c^{\#}d_1)^{-1}(c^{\#}d_2)(c^{\#}d_3)^{-1} = e\,. \tag{5.2.2}$$

We claim that we may define an action of $H^2(\mathbb{E};\mathbb{Z}(L))$ on $H^2(\mathbb{E};L)$ by the class of the product $\chi c^{\#}$ in N. For this we need to show that $(\chi c^{\#}, \alpha, N)$ is again a non-abelian 2-cocycle. But

$$d_1^*(\alpha)d_0^*(\alpha)^{-1}d_0^*(\alpha)^{-1} = \mathrm{int}(\chi^*) = \mathrm{int}(\chi^*)\mathrm{id} = \mathrm{int}(\chi^*)\mathrm{int}(c^{\#}) = \mathrm{int}(\chi^* c^{\#})\,, \tag{5.2.3}$$

since $c^{\#}$ is central. Similarly we have

$$\begin{aligned} &(\chi c^{\#}d_3)^{-1}(\chi c^{\#}d_2)^{-1}(\chi c^{\#}d_2)f^{\alpha}_{01}(\chi c^{\#}d_0)\\ &= (\chi d_3)^{-1}(\chi d_1)^{-1}(\chi d_2)f^{\alpha}_{01}(\chi d_0)\left[f^{*}_{01}(c^{\#}d_0)(c^{\#}d_1)^{-1}(c^{\#}d_2)(c^{\#}d_3)^{-1}\right]\\ &= e\cdot e = e,\ \text{since } d_i \text{ is a homomorphism and } c^{\#} \text{ is central.}\end{aligned} \tag{5.2.4}$$

We leave it to the reader to verify that the resulting definition is independent of the choice of representatives and thus define a principal action. That it is transitive now will follow from the following observations:

(5.3) Using the homotopy filtration of the category of hypercoverings of $\mathbb{E}$, any two cocycles may be replaced with equivalent ones defined on the same hypercovering. Thus it suffices to only consider the case of cocycles (χ_1, α_1, N_1) and (χ_2, α_2, N_2) defined on $X_{\bullet}$ whose ties are isomorphic to the given tie L. Since this means that one has an isomorphism of descent data in the category of pre-ties, after a possible further refinement of $X_{\bullet}$, one then has an *isomorphism* $\xi\colon N_1 \xrightarrow{\sim} N_2$ in $\mathbb{G}r/X_0$ and a normalized arrow $\theta\colon X_1 \longrightarrow N_2$ such that

$$\mathrm{int}(\theta)d_1^*(\xi)\alpha_1 = \alpha_2 d_0^*(\xi) \tag{5.3.0}$$

in $\mathbb{G}r/X_1$. Using this data, we now have the following

(5.4) <u>Lemma</u>. Let (χ_1, α_1, N_1) be a 2-cocycle on $X_{\bullet}$ and $\alpha_2\colon d_0^*(N_2) \longrightarrow d_1^*(N_2)$ a gluing of a group N_2. Then given any isomorphism $\xi\colon N_1 \longrightarrow N_2$ and normalized arrow $\theta\colon X_1 \longrightarrow N_2$ such that

$$\mathrm{int}(\theta)d_1^*(\xi)\alpha_1 = \alpha_2 d_0^*(\xi)\,, \tag{5.4.0}$$

the arrow $\chi_1'\colon K_2' \longrightarrow N_2$ defined by

$$\boxed{\chi_1'(f_0, f_1, f_2) = \theta_{f_1}\xi(\chi_1(f_0, f_1, f_2))\theta_{f_2}^{-1}f_2^{\alpha_2}(\theta_{f_0}^{-1})} \tag{5.4.1}$$

gives a 2-cocycle (χ_1', α_2, N_2) on $X_{\bullet}$ which is isomorphic to (χ_1, α_1, N_1).

In effect, we must show that this definition of χ_1' defines a cocycle since it is clearly isomorphic to the original one. Thus write for any $n \in N_2,\ n = \xi(\xi^{-1}(n))$ and calculate

$$\begin{aligned} &\left[\theta_{f_1}\xi(\chi_1(f_0, f_1, f_2))\theta_{f_2}^{-1}f_2^{\alpha_2}(\theta_{f_0}^{-1})\right] f_2^{\alpha_2}(f_0^{\alpha_2}(n))\\ &\quad\times\left[f_2^{\alpha_2}(\theta_{f_0})\theta_{f_2}\xi(\chi_1(f_0, f_1, f_2)^{-1}\theta_{f_1}^{-1}\right]\end{aligned} \tag{5.4.2}$$

which we must show is equal to $f_1^{\alpha_2}(n)$.

Now since

$$\theta_2^{-1} f_2^{\alpha_2}(\theta_{f_0}^{-1}) f_2^{\alpha_2}(f_0^{\alpha_2}(n)) f_2^{\alpha_2}(\theta_{f_0})\theta_{f_2} = \theta_{f_2}^{-1} f_2^{\alpha_2}(\theta_{f_0}^{-1} f_0^{\alpha_2}(n)\theta_{f_0})\theta_{f_2}$$

and

$$\theta_{f_0}^{-1} f_0^{\alpha_2}(n)\theta_{f_0} = \theta_{f_0}^{-1} f_0^{\alpha_2}(\xi(\xi^{-1}(n)))\theta_{f_0} = \xi(f_0^{\alpha_1}(\xi^{-1}(n))) \ ,$$

we have that

$$\begin{aligned} \theta_2^{-1} f_2^{\alpha_2}(\theta_{f_0}^{-1}) f_2^{\alpha_2}(f_0^{\alpha_2}(n)) f_2^{\alpha_2}(\theta_{f_0})\theta_{f_2} &= \xi(f_2^{\alpha_1}(f_0^{\alpha_1}(\xi^{-1}(n))) \\ &= \xi(\chi_1(f_0, f_1, f_2)^{-1} f_1^{\alpha_1}(\xi_1^{-1}(n))\chi_1(f_0, f_1, f_2)) \\ &= \xi(\chi_1(f_0, f_1, f_2))^{-1}\xi(f_1^{\alpha_1}(\xi^{-1}(n)))\xi(\chi_1(f_0, f_1, f_2)) \ , \end{aligned}$$

so that (5.4.2) becomes

$$\begin{aligned} &\theta_{f_1}\xi(\chi_1(f_0, f_1, f_2))\xi(\chi(f_0, f_1, f_2)^{-1}\xi(f_1^{\alpha_1}(\xi^{-1}(n)))\xi(\chi_1(f_0, f_1, f_2))\xi(\chi_1(f_0, f_1, f_2)^{-1}\theta_{f_1}^{-1} \\ &= \theta_{f_1}\xi(f_1^{\alpha_1}(\xi^{-1}(n)))\theta_{f_1}^{-1} \\ &= f_1^{\alpha_2}(\xi(\xi^{-1}(n))) = f_1^{\alpha_2}(n) \end{aligned}$$

as desired for the gluing condition.

We similarly calculate the cocycle identity:

$$\begin{aligned} &\chi_1'(f_{13}, f_{03}, f_{01}) f_{01}^{\alpha_2}(\chi_1'(f_{23}, f_{13}, f_{12})) \ = \\ &\theta_{f_{03}}\xi(\chi_1(f_{13}, f_{03}, f_{01}))\theta_{f_{01}}^{-1} f_{01}^{\alpha_2}(\theta_{f_{13}}^{-1}) f_{01}^{\alpha_2}(\theta_{f_{13}}) f_{01}^{\alpha_2}(\xi(\chi_1(f_{23}, f_{13}, f_{12}))) \\ &\qquad\qquad \times\ f_{01}^{\alpha_2}(\theta_{f_{12}}^{-1}) f_{01}^{\alpha_2}(f_{12}^{\alpha_2}(\theta_{f_{23}}^{-1}))\,, \end{aligned} \tag{5.4.3}$$

where $f_{01}^{\alpha_2}(\theta_{f_{12}}^{-1}) f_{01}^{\alpha_2}(f_{12}^{\alpha_2}(\theta_{f_{23}}^{-1})) = f_{01}^{\alpha_2}(\theta_{f_{12}}^{-1})\chi_1'(f_{12}, f_{02}, f_{01})^{-1} f_{02}^{\alpha_2}(\theta_{f_{23}}^{-1})\chi_1'(f_{12}, f_{02}, f_{01})$ using the previous result.

Now since

$$\begin{aligned} \chi_1'(f_{23}, f_{03}, f_{02})\chi_1'(f_{12}, f_{02}, f_{01}) &= \theta_{f_{03}}\xi(\chi_1(f_{23}, f_{03}, f_{02}))\theta_{f_{02}}^{-1} f_{02}^{\alpha_2}(\theta_{f_{23}}^{-1}) \\ &\times\ \theta_{f_{02}}\xi(\chi_1(f_{12}, f_{02}, f_{01}))\theta_{f_{01}}^{-1} f_{01}^{\alpha_2}(\theta_{f_{12}}^{-1}) \end{aligned}$$

we will have the desired identity provided we can show that

$$\begin{aligned} &\xi(\chi_1(f_{13}, f_{03}, f_{01}))\theta_{f_{01}}^{-1} f_{01}^{\alpha_2}(\chi_1(f_{23}, f_{13}, f_{12})) f_{01}^{\alpha_2}(\theta_{f_{12}}^{-1})\chi_1'(f_{12}, f_{02}, f_{01})^{-1} f_{02}^{\alpha_2}(\theta_{f_{23}}^{-1}) \\ &= \xi(\chi_1(f_{23}, f_{03}, f_{02}))\theta_{f_{02}}^{-1} f_{02}^{\alpha_2}(\theta_{f_{23}}^{-1})\,. \end{aligned} \tag{5.4.4}$$

But since

$$\chi_1'(f_{12}, f_{02}, f_{01})^{-1} = f_{01}^{\alpha_2}(\theta_{f_{12}})\theta_{f_{01}}\xi(\chi_1(f_{12}, f_{02}, f_{01}))^{-1}\theta_{f_{02}}^{-1}\,,$$

the left hand side of (5.4.4) becomes

$$\xi(\chi_1(f_{13}, f_{03}, f_{01}))\theta_{f_{01}}^{-1} f_{01}^{\alpha_2}(\xi(\chi_1(f_{23}, f_{13}, f_{12}))\theta_{f_{01}}(\xi(\chi_1(f_{12}, f_{02}, f_{01}))\theta_{f_{02}}^{-1} f_{02}^{\alpha_2}(\theta_{f_{23}}^{-1})$$

and since $\theta_{f_{01}}^{-1} f_{01}^{\alpha_2}(\xi(\chi_1(f_{23}, f_{13}, f_{12}))\theta_{f_{01}} = \xi(f_{01}^{\alpha_2}(\chi_1(f_{23}, f_{13}, f_{12})))$, the desired equality follows from the application of the isomorphism ξ to the cocycle identity for χ_1 .

(5.5) Corollary. Any two cocycles with isomorphic ties may be replaced with respectively equivalent ones (χ, α, N), (ρ, α, N) which are defined on the same hypercovering $X_\bullet$ with the same coefficient group and the same gluing.

The completion of the proof of Theorem (5.2) now immediately follows from the following

(5.6) Corollary. Let (χ, α, N) and (ρ, α, N) be cocycles defined on $X_\bullet$ with the same local group and gluing. Then $\chi^{-1}\rho\colon K_2^1 \longrightarrow N$ factors through the center of N and defines an abelian 2-cocycle with coefficients in the center of L.

In effect,

$$\operatorname{int}(\chi\rho^{-1})\colon \operatorname{int}(\chi)\operatorname{in}(\rho^{-1}) = \left[d_1^*(\alpha)d_0^*(\alpha)^{-1}d_2^*(\alpha)^{-1}\right]\left[d_1^*(\alpha)d_0^*(\alpha)^{-1}d_2^*(\alpha)\right]^{-1} = \mathrm{id}$$

so that $\chi\rho^{-1}$ is central. But since

$$(\chi d_3)^{-1}(\chi d_1)^{-1}(\chi d_2)f_{01}^{\alpha}(\chi d_0) = e = (\rho d_3)^{-1}(\rho d_1)^{-1}(\rho d_2)f_{01}^{\alpha}(\rho d_0) \tag{5.6.0}$$

and $\chi\rho^{-1}$ is central, (5.6.0) becomes

$$(\chi\rho^{-1}d_3)^{-1}(\chi\rho^{-1}d_1)^{-1}(\chi\rho^{-1}d_2)f_{01}^{\alpha}(\chi\rho^{-1}d_0) = e \tag{5.6.1}$$

and the proof is completed.

(6) The obstruction to the realization of a tie by a bouquet.

(6.0) As we have already noted, $H^2(\mathbb{E}; L)$ may be empty, what is there may not be a bouquet $\underset{\sim}{G}$ in $\mathbb{E}$ which has the given global tie L as its tie. We will now show that there is an obstruction to this lying in the abelian group $H^3(\mathbb{E}; \mathbb{Z}(L))$ which is 0 if and only if $H^2(\mathbb{E}; L) \neq \emptyset$.

To see this recall that any tie L has a representative which may be taken as a descent datum on a locally given group over a covering of $\mathbb{E}$ in the fibered category of pre-ties of $\mathbb{E}$. By a suitable choice of refinements, this covering may be refined to produce a type-2 *hypercovering* $X_\bullet$ of $\mathbb{E}$ supplied with the basic structural ingredients of a normalized non-abelian 2-cocycle

(6.0.0)

$$\begin{tikzcd}
 & & d_0^*(N) \arrow[r, "\alpha", "\sim"'] \arrow[dr] & d_1^*(N) \arrow[d] & N \arrow[d] \\
K_3^2 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[urrr, "a"] \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] & & X_1 \arrow[r, shift left=1, "d_1"] \arrow[r, shift right=1, "d_0"'] & X_0
\end{tikzcd}$$

for which the relation

$$d_1^*(\alpha)d_0^*(\alpha)^{-1}d_2^*(\alpha)^{-1} = \mathrm{int}(a^*) \tag{6.0.1}$$

holds in $\mathbb{G}r/X_2$ and defines the original descent datum on N over its 0-coskeleton.

In outline, this hypercovering is constructed as follows: over a representative cover of $\mathbb{1}$,

$$\begin{tikzcd} X_0 \times X_0 \times X_0 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_0 \times X_0 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r] & \mathbb{1} \end{tikzcd}$$

one is given a descent datum on a group $N \longrightarrow X_0$ in category of pre-ties. The gluing of this datum is a global section $g \in \Gamma\underline{\mathrm{Hex}}\,\underline{\mathrm{Iso}}(\mathrm{pr}_0^*(N), \mathrm{pr}_1^*(N))$ above $X_0 \times X_0$. Since $\underline{\mathrm{Hex}}\,\underline{\mathrm{Iso}}(\mathrm{pr}_0^*(N), \mathrm{pr}_1^*(N))$ is the quotient of $\underline{\mathrm{Iso}}(\mathrm{pr}_0^*(N), \mathrm{pr}_1^*(N))$ by the section of $\underline{\mathrm{Int}}(\mathrm{pr}^*(N))$, we may find a covering $p\colon C \twoheadrightarrow X_0 \times X_0$ and global section

$$\tilde{\alpha} \in \Gamma\underline{\mathrm{Iso}}((\mathrm{pr}_0 p)^*(N), (\mathrm{pr}_1 p)^*(N))$$

which realizes g. Since this global section is an isomorphism $\tilde{\alpha}\colon (\mathrm{pr}_0 p)^*(N) \longrightarrow (\mathrm{pr}_1 p)^*(N)$ in $\mathbb{G}r/C$, it may be used to define an isomorphism $\alpha\colon d_0^*(N) \xrightarrow{\sim} d_1^*(N)$ over the object $X_l = C \amalg X_0$ supplied with its arrows $d_0 = \mathrm{pr}_0 p \amalg \mathrm{id}\colon X_1 \longrightarrow X_0$ and $d_1 = \mathrm{pr}_1 p \amalg \mathrm{id}\colon X_1 \longrightarrow X_0$ which is clearly equivalent to the original g. We now take the simplicial kernel $K_2^1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \rightrightarrows X_0$ of this truncated complex and note that over it we are given a global section of $\mathrm{Int}(\alpha^*(N))$ which defines the "cocycle condition" of the original datum. But since $\mathrm{Int}(d^*(N))$ is the quotient of $d^*(N)$ by its center, we may find a covering C_2 of K_2^1 on which we have a global section of $d^*(N)$ which realizes the original one. We now define X_2 as the coproduct of C_2 with the co-simplicial kernel of $s_0\colon X_0 \longrightarrow X_1$ (to restore the degeneracies) and transform the global section of $d^*(N)$ back to X_2. The desired hypercovering may now be considered as the 2-coskeleton of the so defined truncated complex. Supplied with the so constructed pseudo-descent datum, it will be said to be *a pseudo descent datum which represents the tie* L and we have the following

(6.1) Theorem. Let (a, α, N) be a pseudo descent datum which represents a tie L over a hypercovering $X_\bullet$ of $\mathbb{E}$. Define the map $k\colon K_2^2 \longrightarrow N$ by

$$k = (ad_3)^{-1}(ad_1)^{-1}(ad_2)f_{01}^{\alpha}(ad_0) \qquad \text{(product in } N\text{)}. \tag{6.1.0}$$

The following statements hold:

(a) k factors through $\mathbb{Z}(N)$, i.e., $\mathrm{int}(k) = \mathrm{id}$;

(b) considered as a map into $\mathbb{Z}(N)$, k defines a 3-cocycle on $X_\bullet$ with coefficients in $Z(L)$, i.e.,

$$f_{01}^{\alpha}(kd_0) - (kd_1) + (kd_2) - (kd_3) + (kd_4) = 0 \quad \text{in } \mathbb{Z}(N);$$

(c) if $k = e$, then (a, α, N) is a non-abelian 2-cocycle whose tie (and hence the tie of its associated bouquet) is isomorphic to L.

We first show that k is in the center. Now in the category $\mathbb{G}r/X_2$, we have the 2-simplex of group isomorphisms

(6.1.1)
$$\begin{tikzcd}
d_2^*(N) \arrow[r, "d_0^*(\alpha)"] \arrow[dr, "d_1^*(\alpha)"', ""{name=D}] & d_1^*(N) \arrow[d, "d_2^*(\alpha)", ""'{name=V}] \\
& d_0^*(N)
\arrow[from=V, to=D, bend right, "a"']
\end{tikzcd}$$

so that $d_1^*(\alpha)d_0^*(\alpha)^{-1}d_2^*(\alpha)^{-1} = \operatorname{int}(a^*)$. If we pull this triangle back into $\mathbb{G}r/K_2^2$ along the faces $d_i\colon K_2^2 \longrightarrow X_2$, the simplicial identities give a 3-simplex of groups and isomorphisms

(6.1.2)
$$\begin{tikzcd}
& d_2^*(N) \arrow[dr, "d_{20}^*(\alpha)"] \arrow[dd, "d_{21}^*(\alpha)"'] & \\
d_3^*(N) \arrow[ur, "d_{32}^*(\alpha)"] \arrow[rr, crossing over, "d_{30}^*(\alpha)"] \arrow[dr, "d_{31}^*(\alpha)"'] & & d_0^*(N) \\
& d_1^*(N) \arrow[ur, "d_{10}^*(\alpha)"'] &
\end{tikzcd}$$

whose respective faces have the property that

(6.1.3)
$$\begin{aligned}
&d_i^*(d_1^*(\alpha)d_0^*(\alpha)^{-1}d_2^*(\alpha)^{-1}) \\
&= (d_1d_i)^*(\alpha)(d_0d_i)^*(\alpha)^{-1}(d_2d_i)^*(\alpha)^{-1} = \operatorname{int}(ad_i^*)
\end{aligned}$$

for $i = 0, 1, 2, 3$.

But for any such 3-simplex of isomorphisms (in any category) one always has the equality

(6.1.4)
$$\begin{aligned}
&\left[d_{20}(\alpha)d_{21}^{-1}(\alpha)d_{10}(\alpha)^{-1}\right]^{-1}\left[d_{30}(\alpha)d_{32}^{-1}(\alpha)d_{20}^{-1}(\alpha)\right]^{-1} \\
&\quad\times\left[d_{30}(\alpha)d_{31}^{-1}(\alpha)d_{10}^{-1}(\alpha)\right]d_{10}(\alpha)\left[d_{31}(\alpha)d_{32}(\alpha)^{-1}d_{21}(\alpha)^{-1}\right]d_{10}(\alpha)^{-1} = \mathrm{id}
\end{aligned}$$

and hence on substitution of (6.1.3) one obtains

$$\begin{aligned}
&\operatorname{int}(ad_3)^{-1}\operatorname{int}(ad_1)^{-1}\operatorname{int}(ad_2)d_{10}(\alpha)\operatorname{int}(ad_0)d_{10}(\alpha)^{-1} \\
&= \operatorname{int}\left[(ad_3)^{-1}(ad_1)^{-1}(ad_2)f_{01}^{\alpha}(ad_0)\right] = \mathrm{id}
\end{aligned}$$

and thus k is in the center as asserted.

To show (b) we must show that the 3-cocycle identity still holds in this case when written multiplicatively. For this we follow the method of ElLENBERG-MAC LANE (1954) and we write the definition of k in the form

(6.1.5)
$$(ad_1)(ad_3)k = (ad_2)f_{01}^{\alpha}(ad_0)\,.$$

This gives for the five faces of a 4-simplex the identities

$$\begin{aligned}(ad_1d_0)(ad_3d_0)(kd_0) &= (ad_2d_0)f^\alpha_{12}(ad_0d_0)\\ (ad_1d_1)(ad_3d_1)(kd_1) &= (ad_2d_1)f^\alpha_{02}(ad_0d_1)\\ (ad_1d_2)(ad_3d_2)(kd_2) &= (ad_2d_2)f^\alpha_{01}(ad_0d_2)\\ (ad_1d_3)(ad_3d_3)(kd_3) &= (ad_2d_3)f^\alpha_{01}(ad_0d_3)\,, \text{ and}\\ (ad_1d_4)(ad_3d_4)(kd_4) &= (ad_2d_4)f^\alpha_{01}(ad_0d_4)\,.\end{aligned} \tag{6.1.6}$$

We will now calculate the expression

$$L = (ad_2d_3)f^\alpha_{01}\left((ad_0d_3)f^\alpha_{12}(ad_0d_1)\right) \tag{6.1.7}$$

in two ways and then equate the result. To aid in following this calculation we define the following (using the simplicial identities) for the 3-faces of a 4-simplex

$$\begin{array}{ll|ll} d_3d_0 = d_0d_4 = a & (1,2,3) & d_1d_2 = d_1d_2 = f & (0,3,4)\\ d_1d_0 = d_0d_2 = b & (1,3,4) & d_3d_1 = d_1d_4 = g & (0,2,3)\\ d_2d_0 = d_0d_3 = c & (1,2,4) & d_3d_2 = d_2d_4 = h & (0,1,3)\\ d_2d_0 = d_0d_1 = d & (2,3,4) & d_2d_2 = d_2d_3 = i & (0,1,4)\\ d_2d_1 = d_1d_3 = e & (0,2,4) & d_3d_3 = d_3d_4 = j & (0,1,2)\end{array}$$

L then becomes for one calculation using (6.1.6) above,

$$\begin{aligned}L &= if^\alpha_{01}(cf^\alpha_{12}(d)) = if^\alpha_{01}(ba(kd_0)) = if^\alpha_{01}(b)f^\alpha_{01}(a)f^\alpha_{01}(kd_0)\\ &= i(i^{-1}fk(kd_2))h^{-1}gj(kd_4)f^\alpha_{01}(kd_0) = (kd_2)(kd_4)f^\alpha_{01}(kd_0)fgj\end{aligned}$$

where we have collected the central terms in front and cancelled.

For the second calculation, we first apply the isomorphism f^α_{01} and then substitute using the relation (6.1.6). Thus

$$\begin{aligned}L &= if^\alpha_{01}(cf^\alpha_{12}(d)) = if^\alpha_{01}(c)f^\alpha_{01}(f^\alpha_{12}(d)) = if^\alpha_{01}(c)j^{-1}f^\alpha_{02}j\\ &= ej(kd_3)j^{-1}(e^{-1}fg(kd_1))j = (kd_3)(kd_1)ejj^{-1}e^{-1}fgj = (kd_3)(kd_1)fg\,.\end{aligned}$$

We now equate the two expressions and cancel the common factor fgj to obtain the identity

$$(kd_3)(kd_1) = (kd_2)(kd_4)f^\alpha_{01}(kd_0) \tag{6.1.8}$$

which is just the 3-cocycle identity written multiplicatively. Thus (b) is established.

(c) is obvious from the original definition of a non-abelian 2-cocycle and its associated tie and we can make the following

(6.2) Definition. The class $\mathcal{O}(k)$ in $H^3(\mathbb{E}, \mathbb{Z}(L))$ defined by the 3-cocycle k (6.1.0) will be called the *obstruction to the realization of the tie L by a bouquet of* $\mathbb{E}$.

(6.3) Corollary. $\mathcal{O}(k) = 0$ iff the tie L is representable by a bouquet of $\mathbb{E}$.

In effect, the proof of Theorem (4.3) shows that the obstruction cocycle is independent of choice of representatives and gives the desired result.

(7) Neutral Co-cycles

(7.0) If a 2-cocycle on a hypercover $X_\bullet$ has the form (e, α, N) where $e\colon X_2 \longrightarrow N$ is the unit element of N, then the cocycle reduces to a simple descent datum on N over the covering $\mathrm{Cosk}^0(X_\bullet)$ and thus produces, by descent, a global group N' which is locally isomorphic to N. The bouquet produced by such a cocycle is easily seen to be split on the right by N' (I 4.1) and is thus neutral. We will thus define a cocycle to be *neutral* if it is equivalent under refinement to a cocycle of the form (e, α, N) and with this definition our computation theorem (3.8) is easily seem to preserve neutral elements. This will be discussed in more detail in the sequel in connection with global group coefficients.

(7.1) Remark. If G is a global group and $O \twoheadrightarrow \mathbb{1}$ is an epimorphism, then the canonical cocycle on the right split bouquet whose arrows A are $O \times O \times G$ is isomorphic to the trivial cocycle $(e, \mathrm{id}, O \times G)$ via the arrow $\theta\colon A \longrightarrow O \times G$ given by $(x, y, n) \mapsto n^{-1}$ since the equalities

$$(7.1.0)\qquad \begin{aligned} &(\mathcal{O}^{-1})^{-1}\,\underbrace{\mathcal{O}^{-1}mn}\,\underbrace{n^{-1}(m^{-1})n}\,n^{-1} = e \\ &(m^{-1})^{-1}\,\underbrace{m^{-1}nm}\,m^{-1} = n \end{aligned}$$

of (2.8) always hold trivially.

NON-ABELIAN COHOMOLOGY IN A TOPOS

Part (III): Bouquets and group extensions

As we remarked in the introduction, observations about the topos of $\mathbb{E}$-sets where $\mathbb{E}$ is a group form a "guiding thread" for motivation of much of what occurs in both the Grothendieck-Giraud theory and what is presented in this paper. It is thus incumbent on us to effectively translate into the move familiar terms of $\mathbb{E}$-sets the notions which occur in this paper. As we have noted before, the background definitions of "fibration", "cartesian functor", "pseudo-functor" etc. have been given in the Appendix.

(1.0) If $F\colon \mathbb{F} \longrightarrow \mathbb{E}$ is a homomorphism of groups, then, viewed as groupoids with a single object, it is easy to see that since every arrow of $\mathbb{F}$ is cartesian (being an isomorphism), $\mathbb{F}\colon \mathbb{F} \longrightarrow \mathbb{E}$ *is a fibration if and only if F is surjective*, that is, defines an extension of $\mathbb{E}$ by the kernel of F, which categorically is just the fiber $\mathbb{F}_e$ above the single object e of $\mathbb{E}$. A normalized cleavage of this fibration (i.e. a choice of inverse images which preserves identities) is then nothing more than a set theoretic section C of F which has the property that $C(e) = e$ (i.e., a transversal in the terminology of group theorists) and is a splitting if and only if the section is a homomorphism of groups (so that $\mathbb{F}$ is a split extension of $\mathbb{E}$ by the kernel of F).

The pseudo-functor $\mathbb{F}^C_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ which is defined by such a cleavage is easily seen to be entirely equivalent to the Schreier factor system defined by the transversal as we now show.

(1.1) <u>The pseudo-functor defined by a group extension</u>.

If $C\colon \mathbb{E} \longrightarrow \mathbb{F}$ is a normalized section for $F\colon \mathbb{F} \longrightarrow \mathbb{E}$, as a cleavage it defines for each $f\colon e \longrightarrow e$ an inverse image functor $\mathbb{F}^C_f\colon \mathbb{F}_e \longrightarrow \mathbb{F}_e$ as follows: by definition, for any $x\colon e \longrightarrow e$ in $\mathbb{N} = \mathbb{F}_e = \mathrm{Ker}(F)$, $\mathbb{F}^C_f(x)$ is that unique element of $\mathbb{N}$ such that the square

(1.1.0)
$$\begin{tikzcd}
e \arrow[r, "C(f)"] \arrow[d, dashed, "\mathbb{F}^C_f(x)"'] & e \arrow[d, "x"] & & \mathbb{F} \arrow[d, "F"] \\
e \arrow[r, "C(f)"'] & e & & \mathbb{E} \arrow[u, dashed, bend left, "C"] \\
e \arrow[r, "f"'] & e
\end{tikzcd}$$

is commutative in $\mathbb{F}$. Thus $\mathbb{F}^C_f(x) = c(f)^{-1}xC(f)$, or in other words, $\mathbb{F}^C_f$ is just the restriction to $\mathbb{N}$ of the inner automorphism $\mathrm{int}(C(f)^{-1})\colon \mathbb{F} \longrightarrow \mathbb{F}$ defined by the element $C(f)^{-1}$ of $\mathbb{F}$. Similarly, for each ordered pair (f, g) in $\mathbb{E} \times \mathbb{E}$, the natural transformation, $C(f, g)\colon \mathbb{F}^C_f \circ \mathbb{F}^C_g \longrightarrow \mathbb{F}^C_{gf}$, by definition has as its value at the single object e of $\mathbb{N}$ that unique arrow in $\mathbb{N}$ which makes the diagram

(1.1.1)
$$\begin{tikzcd}
e \arrow[dr] \arrow[dd, dashed, "C(f{,}g)"'] & & & \\
& e \arrow[dr] & & \mathbb{F} \arrow[dd, "F"] \\
e \arrow[rr, "C(gf)"'] & & e & \\
e \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & e \arrow[r, "g"] & e & \mathbb{E}
\end{tikzcd}$$

commutative in $\mathbb{F}$, i.e. $C(f,g) = C(gf)^{-1}C(g)C(f)$. Thus $C(f,g)$ has precisely the property that

$$\mathbb{F}^C_{gf} = \operatorname{int}(C(f,g^{-1})) \circ \mathbb{F}^C_f \circ \mathbb{F}^C_g \tag{1.1.2}$$

holds for all (f,g) in $\mathbb{E}\times\mathbb{E}$. Moreover, since these natural transformations always satisfy the equalities

$$\begin{gathered} C(f,hg)\circ(\mathbb{F}^C_f * C(g,h)) = C(gf,h)\circ(C(f,g)*\mathbb{F}^C_k) \\ \text{and} \quad C(e,f) = C(f,e) = e\,, \end{gathered} \tag{1.1.3}$$

with $\mathbb{F}^C_f * (C(g,h)) = \mathbb{F}^C_f(C(g,h))$ and $C(f,g)*\mathbb{F}^C_k = C(f,g)$, we have that the pseudo functor defined by the cleavage C is entirely equivalent to giving mappings

$$\mathbb{F}^C\colon \mathbb{E} \longrightarrow \operatorname{Aut}(\mathbb{N})^{\mathrm{op}}\,,\ f \longmapsto \mathbb{F}^a_f\,, \quad \text{and} \quad C\colon \mathbb{E}\times\mathbb{E} \longrightarrow \mathbb{N}\,,\ (f,g) \longmapsto C(f,g)$$

subject to the following conditions:

(1.1.4)

(a) $\operatorname{int}(C(f,g)^{-1}) \circ \mathbb{F}^C_f \circ \mathbb{F}^C_g = \mathbb{F}^C_{gf}\,,$ for any pair (f,g) in $\mathbb{E}\times\mathbb{E}$

(b) $C(f,e) = C(e,f) = e$ for all $f \in \mathbb{E}$; and

(c) $C(f,hg)\mathbb{F}^C_f(C(g,h)) = C(gf,h)C(f,g)$ for all $(f,g,h) \in \mathbb{E}\times\mathbb{E}\times\mathbb{E}\,,$

or, in other words, just a *Schreier factor system* for the extension with $\mathbb{F}^C_f$ as "automorphisms" and $C(f,g)$ as "factors" $\big($c.f., SCHREIER (1926), KUROSH (1955) or SCOTT (1964)$\big)$.

(1.2) Remark. If $\mathbb{N}$ is abelian, then $\mathbb{F}^C\colon \mathbb{E} \longrightarrow \operatorname{Aut}(\mathbb{N})^{\mathrm{op}}$ becomes a homomorphism of groups and the equality (1.1.4(c)) may be written as

$$\mathbb{F}^C_f(C(g,h)) - C(gf,h) + C(f,hg) - C(f,g) = 0\;; \tag{1.2.0}$$

thus $C(-,-)$ is a normalized 2-cocycle of $\mathbb{E}$ with coefficients in the $\mathbb{E}$-module $\mathbb{N}$ and the Schreier factor system may be viewed as the non-abelian version of such a 2-cocycle.

As with 2-cocycles with coefficients in an $\mathbb{E}$-module, the non-abelian factor system may also be described in simplicial terms as a simplicial map from the nerve of $\mathbb{E}^{\mathrm{op}}$ into the nerve of the 2-groupoid defined by $\mathbb{N}$. We will show this in detail when we discuss group cohomology in a general topos, but will briefly discuss it in(6.0).

(1.3) The Grothendieck theory associates with any pseudo-functor $\mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ a fibration above $\mathbb{E}$ which realizes the pseudo-functor in such a fashion that fibrations are determined, up to equivalence, by their associated pseudo-functors. Here, in the case of a group extension and any pseudo-functor with values in the 2-category of groups, homomorphisms of groups, and natural transformations of group homomorphisms, the Grothendieck construction for the associated fibration just gives the familiar description of multiplication on the set $\mathbb{E} \times \mathbb{N}$ which the factor set determines, viz.,

$$(g,n)\cdot(f,m) = (gf, c(f,g)n^f m)\ , \qquad \text{where } n^f = \mathbb{F}^C_f(n). \tag{1.3.0}$$

(2.) The split fibration and bouquet of $\mathbb{E}$-sets determined by a group extension

(2.0) Since not every group extension can be split, the theory of group extensions provides a convincing example that not every fibration can be replaced (up to isomorphism) with a split fibration and thus that pseudofunctors cannot be naively replaced with functors. Never-the-less, as we have noted (I(6.0) and Appendix), the Grothendieck-Giraud theory does allow every fibration $\mathbb{F}$ to be replaced with split fibration $\mathcal{S}\mathbb{F}$ which is $\mathbb{E}$-cartesian equivalent to $\mathbb{F}$. In the case of a group extension $\mathbb{F} \longrightarrow \mathbb{E}$, we will see that $\mathcal{S}\mathbb{F} \longrightarrow \mathbb{E}$ is a split fibration of the group $\mathbb{E}$, fibered in *groupoids* rather than groups. As a split fibration, $\mathcal{S}\mathbb{F}$ is determined as the externalization of a presheaf of groupoids on $\mathbb{E}$ (specifically, by $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$) that is, in more familiar terms by a groupoid object in the topos of (right) $\mathbb{E}$-sets. But since the cartesian $\mathbb{E}$- equivalence $\mathbb{F} \xrightarrow{\approx} \mathcal{S}\mathbb{F}$ (which in general depends on the choice of a cartesian section of $\mathbb{F}$) is fully faithful, this means that the group extension $\mathbb{F}$ is recoverable up to *isomorphism* as the group of automorphisms of any selected object of $\mathcal{S}\mathbb{F}$ and consequently that all information about the group extension $\mathbb{F}$ is encoded in the category of $\mathbb{E}$-sets via the $\mathbb{E}$-groupoid $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$. On the set theoretic level, $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ turns out to be a connected non-empty groupoid, equivalent as a category, to the group $\mathrm{Ker}(F)$, that is, to *a (complete) bouquet of* $\mathbb{E}$*-sets locally equivalent to the group* $\mathrm{Ker}(F)$.

Since every bouquet of $\mathbb{E}$-sets turns out to determine (up to isomorphism) an extension of $\mathbb{E}$, our theory will show that *the study if all possible extensions of* $\mathbb{E}$ *by a group* $\mathbb{F}$ *is equivalent to the study of the category of category objects in the topos of* $\mathbb{E}$*-sets which are locally* (i.e., on the underlying set level) *equivalent to ordinary groups.*

We will now proceed to describe the structure of $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ in the more conventional terms of $\mathbb{E}$-sets.

(2.1) THEOREM. *Let* $\mathbb{F} \xrightarrow{F} \mathbb{E}$ *be an extension of a group* $\mathbb{E}$. *The presheaf of groupoids* $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ *is isomorphic to the bouquet of* $\mathbb{E}$*-sets* $\Gamma(\mathbb{F}/\mathbb{E})$ *whose* $\mathbb{E}$*-set of objects is the set* $\Gamma(\mathbb{F}/\mathbb{E})$ *of all normalized (set-theoretic) sections* $s\colon \mathbb{E} \longrightarrow \mathbb{F}$ *of* $\mathbb{F} \xrightarrow{F} \mathbb{E}$ *with right* $\mathbb{E}$*-action defined by* $s^g(x) = s(g)^{-1}s(gx)$ (product in $\mathbb{F}$) *and whose* $\mathbb{E}$*-set of arrows is the set* $\{n\colon s \longrightarrow t \in \mathrm{Ker}(F) \times \Gamma(\mathbb{F}/\mathbb{E}) \times \Gamma(\mathbb{F}/\mathbb{E})\}$ *with right* $\mathbb{E}$*-action defined by* $n^g = t(g)^{-1}ns(g)\colon s^g \longrightarrow t^g$ (product in $\mathbb{F}$) *and composition defined by the product in* $\mathrm{Ker}(f)$. *Moreover, the extension* $\mathbb{F}$ *is split if and only if the bouquet* $\Gamma(\mathbb{F}/\mathbb{E})$ *has an invariant object* s (i.e., $s^g = s$ for all $g \in \mathbb{E}$).

In effect, let $e\ (= e_{\mathbb{E}})$ represent the single object of $\mathbb{E}$, so that the groupoid $\mathbb{E}/e$ has as arrows $f\colon x \longrightarrow y$ the commutative triangles

(2.1.0)
$$\begin{tikzcd}
e \arrow[rr, "f"] \arrow[dr, "x"'] & & e \arrow[dl, "y"] \\
& e &
\end{tikzcd}$$

$(yf = x)$ of $\mathbb{E}$ fibered over $\mathbb{E}$ via the fibration $f\colon x \longrightarrow y \mapsto f\colon e \longrightarrow e$. Now let $c\colon \mathbb{E}/e \longrightarrow \mathbb{F}$ be any $\mathbb{E}$-functor (necessarily cartesian since all arrows of $\mathbb{F}$ are invertible) and for any $x\colon e \longrightarrow e$ in $\mathbb{E}$ define $s^C(x)$ as the image in $\mathbb{F}$ under c of the arrow $x\colon x \longrightarrow e$ in $\mathbb{E}/e$ given by commutative triangle

(2.1.2)
$$\begin{tikzcd}
e \arrow[rr, "x"] \arrow[dr, "x"'] & & e \arrow[dl, "e"] \\
& e &
\end{tikzcd}$$

Since C is an $\mathbb{E}$-functor $C(x)\colon e \longrightarrow e$ must project under F onto x and carry $e\colon e_{\mathbb{E}} \longrightarrow e_{\mathbb{E}}$ onto $e_{\mathbb{F}}$.

Thus the assignment $x \mapsto s^C(x)$ defines a normalized section of F.

Inversely, let $s\colon \mathbb{E} \longrightarrow \mathbb{F}$ be a normalized section for F. Define $C^s(f)\colon e \longrightarrow e$ for any arrow $\begin{tikzcd}[column sep=small] e \arrow[rr, "f"] \arrow[dr, "x"'] & & e \arrow[dl, "y"] \\ & e & \end{tikzcd}$ as the product $C^s(f) = s(y)^{-1}s(x)$ in $\mathbb{F}$. As is easily seen, C^s is an $\mathbb{E}$-functor which is inverse to s since the tetrahedron

(2.1.3)
$$\begin{tikzcd}
e \arrow[rr, "f"] \arrow[dr, "x"] \arrow[ddr, "x"'] & & e \arrow[dl, "y"'] \arrow[ddl, "y"] \\
& e \arrow[d, "e"] & \\
& e &
\end{tikzcd}$$

is always commutative in $\mathbb{E}$.

Now suppose that $\varphi\colon C_1 \longrightarrow C_2$ is an $\mathbb{E}$-natural transformation of cartesian $\mathbb{E}$-functors from $\mathbb{E}/e \longrightarrow \mathbb{F}$, then by definition, for each object $x\colon e \longrightarrow e$ in $\mathbb{E}/e$, $\varphi(x)\colon \begin{matrix} C_1(e) \\ \| \\ e \end{matrix} \longrightarrow \begin{matrix} C_2(e) \\ \| \\ e \end{matrix}$ projects onto $e = \mathrm{id}(e)$ in $\mathbb{E}$ and is thus an element of $\mathrm{Ker}(F)$. Moreover, since for each $f\colon x \longrightarrow y$ in $\mathbb{E}/e$, the prism

(2.1.4)
$$\begin{tikzcd}
 & e \arrow[dd, "\varphi(e)"'] & \\
C \arrow[ur, "s(x)"] \arrow[rr, "C_1(f)"'] \arrow[dd, "\varphi(x)"'] & & e \arrow[ul, "s(y)"'] \arrow[dd, "\varphi(y)"] \\
 & e & \\
e \arrow[ur, "t(x)"] \arrow[rr, "C_2(f)"'] & & e \arrow[ul, "t(y)"']
\end{tikzcd}$$

is commutative, with C_l and C_2 determining and being determined by the respective normalized sections s and t, we see that φ is completely determined by $\varphi(e) \in \mathrm{Ker}(f)$ with

(2.1.5)
$$\varphi(x) = t(x)^{-1}\varphi(e)s(x)\,.$$

We thus have established a bijection between $\mathbb{E}$-natural transformations and ordered triplets $(s, t, u) \in \Gamma(\mathbb{F}/\mathbb{E})^2 \times \mathrm{Ker}(F)$

(2.1.6)
$$\varphi\colon C_1 \longrightarrow C_2 \;\mapsto\; \left(s^{C_1}, s^{C_2}, \varphi(e)\right).$$

Under this bijection composition of natural transformations clearly becomes multiplication in $\mathrm{Ker}(f)$. Notice that in $\mathbb{ENS}$, we have now established an isomorphism of groupoids together with a fully faithful functor

$$\begin{tikzcd}
\mathrm{Ar}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/e, \mathbb{F})) \arrow[r, "\sim"] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] & \Gamma(\mathbb{F}/\mathbb{E}) \times (\Gamma(\mathbb{F}/\mathbb{E}) \times \mathrm{Ker}(F) \arrow[r] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mathrm{pr}_2"] & \mathrm{Ker}(F) \arrow[d] \\
\mathrm{Ob}(\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/e, \mathbb{F})) \arrow[r, "\sim"] & \Gamma(\mathbb{F}/\mathbb{E}) \arrow[r] & \mathbb{1}
\end{tikzcd}$$

(2.1.7)
$$\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/e, \mathbb{F}) \xrightarrow{\sim} \mathbb{\Gamma}(\mathbb{F}/\mathbb{E}) \longrightarrow \mathrm{Ker}(F)$$

which is, in fact, an equivalence of categories in $\mathbb{ENS}$. *Thus the* $\mathbb{E}$*-groupoid* $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ *is a groupoid object in* $\mathbb{E}$*-sets which is locally essentially equivalent* (in fact, here, fully equivalent) *to a locally given group, viz.* $\mathrm{Ker}(F)$. This terminology is consistent with that already used (I.2.) since, as is well known (and will shortly be reviewed in detail (5.0)) the underlying set functor $\mathbb{E}\text{-sets} \longrightarrow \mathbb{ENS}$ is equivalent to that of localization (i.e. "pull back") along the epimorphism $\underline{e} \twoheadrightarrow \mathbb{1}$ whose source is the $\mathbb{E}$-set $\underline{e} = \mathbb{E}_{\delta}$ ($\mathbb{E}$ operating on itself on the right by multiplication.)

We now look at the $\mathbb{E}$-set structure on $\Gamma(\mathbb{F}/\mathbb{E})$. For any $g\colon e \longrightarrow e$ in $\mathbb{E}$ the $\mathbb{E}$-functor $\mathbb{E}/e \longrightarrow \mathbb{E}/e$ which it defines is given by the assignment

(2.1.8)
$$\begin{tikzcd} e \arrow[rr, "f"] \arrow[dr, "x"'] & & e \arrow[dl, "y"] & \longmapsto & e \arrow[rr, "f"] \arrow[dr, "gx"'] & & e \arrow[dl, "gy"] \\ & e & & & & e & \end{tikzcd}$$

and thus the presheaf structure on $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/e,\mathbb{F})$ just becomes on objects the automorphism $\langle\langle c \mapsto c \circ g_{!} \rangle\rangle$. Translating this into its effect on the section $s\colon \mathbb{E} \longrightarrow \mathbb{F}$ defined by C, we see that the result of the action of g on s is just given by

$$s^g(x) = s(g)^{-1}s(gx) \qquad (\text{product in } \mathbb{F})\,, \tag{2.1.9}$$

while the corresponding effect on natural transformations $\langle\langle \varphi \mapsto \varphi * g_{!} \rangle\rangle$ just becomes on arrows between sections the assignment

$$n\colon s \longrightarrow t \quad \mapsto \quad n^g\colon s^g \longrightarrow t^g \tag{2.1.10}$$

given by

$$n^g = t(g)^{-1}ns(g) \in \mathrm{Ker}(F) \qquad (\text{product in } \mathbb{F})\,. \tag{2.1.11}$$

Finally, note that a section s is invariant under the action of $\mathbb{E}$, if and only if for all $x, g \in \mathbb{E}$, $s^g(x) = s(x)$, i.e. if and only if $s(g)^{-1}s(gx) = s(x)$, or in other words, $s(gx) = s(g)s(x)$ and the section defines a splitting of the original extension $\mathbb{F} \longrightarrow \mathbb{E}$.

(2.2) Definition. $\Gamma(\mathbb{F}/\mathbb{E})$ will be called the *bouquet of sections of the extension* of $\mathbb{F}$.

In the canonical topology on the topos of $\mathbb{E}$-sets, $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ is complete (I.(7.1)) and thus so is the bouquet of $\mathbb{E}$-sets $\Gamma(\mathbb{F}/\mathbb{E})$ determined by the group extension. Moreover, the assignment $\mathbb{F} \longrightarrow \mathbb{E} \ \mapsto \ \Gamma(\mathbb{F}/\mathbb{E})$ is obviously functorial on $\mathbb{E}$-morphisms of group extensions and defines a functor from the category $\mathrm{Ext}(\mathbb{E})$ of extensions of $\mathbb{E}$ into the category of $\mathrm{BOUQ}(\mathbb{E})$ of bouquets of $\mathbb{E}$-sets. We now look in the opposite direction at

(3.) The group extension determined by a bouquet of $\mathbb{E}$-sets.

(3.0) Let $\underset{\approx}{G}\colon A \overset{S}{\underset{T}{\rightrightarrows}} O$ be a bouquet of $\mathbb{E}$-sets. As a presheaf of groupoids on $\mathbb{E}^{\mathrm{op}}$, $\underset{\approx}{G}$ determines a split fibration $\mathbb{F}\colon \mathbb{E}\mathbb{X}(\underset{\approx}{G}) \xrightarrow{F} \mathbb{E}$, the externalization of $\underset{\approx}{G}$ restricted to $\mathbb{E}$ (I(7.2)), whose subgroup of automorphisms $\underline{\mathrm{Aut}}(\alpha)_{\underset{\approx}{G}} \longrightarrow \mathbb{E}\mathbb{X}(\underset{\approx}{G})$ for any object α in $\mathbb{E}\mathbb{X}(\underset{\approx}{G})$ determines, we claim, an ordinary group extension $\mathrm{Aut}_{\underset{\approx}{G}}(\alpha) \xrightarrow{F} \mathbb{E}$ of $\mathbb{E}$. This group may be described directly as follows:

Since $\underset{\approx}{G}$ is locally non empty, we may pick an object $\alpha \in O = \mathrm{Ob}(U(\underset{\approx}{G}))$ from the underlying set of objects of $\underset{\approx}{G}$ and make the following definition: for any $g \in \mathbb{E}$, an *arrow of projection* g is an arrow of the form $f\colon \alpha \longrightarrow \alpha^g$ in $A = \mathrm{Ar}(U(\underset{\approx}{G}))$, i.e. an element m of A such that $S(\xi) = \alpha$ and $T(\xi) = \alpha^g$. Since $A \xrightarrow{\langle T,S\rangle} O \times O$ is an epimorphism, on the underlying object level $\underset{\approx}{G}$ is connected so that the set of arrows of projection g is never empty. $\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha)$ is then defined to be the disjoint union over $\mathbb{E}$ of the set of arrows of projection g,

$$\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha) = \{(g,\xi) \mid \xi\colon \alpha \longrightarrow \alpha^g \in A,\ g \in \mathbb{E}\}\,, \tag{3.0.0}$$

and is supplied with a surjective mapping $\langle\langle (g,\xi) \mapsto g\rangle\rangle$ onto $\mathbb{E}$.

Composition in $\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha)$ is defined by

$$\boxed{(h, \rho\colon \alpha \longrightarrow \alpha^h) \cdot (g, \xi\colon \alpha \longrightarrow \alpha^g) = (hg, \rho^g\xi\colon \alpha \longrightarrow \alpha^{kg})}\,, \tag{3.0.1}$$

where $\rho^g\xi$ is given by the composition

$$\begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[dr, "\rho^g\xi"'] & \alpha^g \arrow[d, "\rho^g"] & \alpha \arrow[d, "\rho"] \\ & \alpha^{kg} & \alpha^k \end{tikzcd} \tag{3.0.2}$$

in $\underset{\approx}{G}$. Since $\underset{\approx}{G}$ is a category, this composition is always associative with

$$(e, \mathrm{id}_\alpha\colon \alpha \longrightarrow \alpha^e = \alpha) \tag{3.0.3}$$

as identity element. Moreover, since both $\underset{\approx}{G}$ and $\mathbb{E}$ are groupoids, $\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha)$ is a group with $(g^{-1}, \alpha \xrightarrow{(\xi^{-1})^{g^{-1}}} \alpha^{g^{-1}})$ providing an inverse for $(g, \alpha \xrightarrow{\xi} \alpha^g)$ in $\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha)$

$$\alpha \xrightarrow{\xi} \alpha^g \mapsto \alpha \xrightarrow{\xi^{-1}} \alpha^g \mapsto \alpha^{g^{-1}} \xleftarrow{(\xi^{-1})^{g^{-1}}} \alpha^{gg^{-1}=e} = \alpha\ . \tag{3.0.4}$$

Thus $\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha) \longrightarrow \mathbb{E}$ is indeed a group extension whose kernel is precisely the subgroup $N = \underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha) = \{n\colon \alpha \longrightarrow \alpha^e\ (= \alpha) \mid n \in A\} \hookrightarrow G$ of automorphisms of α in $\underset{\approx}{G}$. It will be called the g*roup extension defined by the* $\mathbb{E}$*-bouquet* $\underset{\approx}{G}$ *at the object* α *of* $\underset{\approx}{G}$.

(3.1) Theorem. The group extension $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ is, up to isomorphism, independent of the choice of object $\alpha \in \mathrm{ob}(\underset{\sim}{G})$. Moreover, if bouquets, $\underset{\sim}{G}$ and $\underset{\sim}{H}$ lie in the same connected component under the relation (generated by) essential equivalence, then $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ is isomorphic to $\underline{\mathrm{Aut}}_{\underset{\sim}{H}}(\beta)$ for any choice of objects α and β.

In effect since $\underset{\sim}{G}$ is connected, there is an isomorphism $f\colon \alpha \longrightarrow \beta$ in $\underset{\sim}{G}$. We claim that the assignment $(g, \xi) \mapsto (g, f^g \xi f^{-1})$ defined by the commutative square (in $\underset{\sim}{G}$)

(3.1.0)
$$\begin{tikzcd}
\alpha \arrow[r, "f"] \arrow[d, "\xi"'] & \beta \arrow[d, "f^g \xi f^{-1}"] \\
\alpha^g \arrow[r, "f^g"'] & \beta^g
\end{tikzcd}$$

defines an isomorphism $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(f)\colon \underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha) \xrightarrow{\sim} \underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\beta)$. Similarly, $\underset{\sim}{G}$ and $\underset{\sim}{H}$ lie in the same connected component under essential equivalence, then one has a diagram

(3.1.1)
$$\begin{tikzcd}
\underset{\sim}{C} \arrow[r] \arrow[d, "F_0"'] & \underset{\sim}{H} \\
\underset{\sim}{G} &
\end{tikzcd}$$

of fully faithful $\mathbb{E}$-equivariant functors of $\mathbb{E}$-bouquets which are essentially surjective on the underlying set level. From this it immediately follows that $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ is isomorphic to $\underline{\mathrm{Aut}}_{\underset{\sim}{H}}(\beta)$ for any choice of α and β since the assignment
$(g, \xi\colon \alpha \longrightarrow \alpha^g) \mapsto (g, F(\xi)\colon F(\alpha) \longrightarrow F(\alpha)^g = F(\alpha^g))$ is an isomorphism of $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ with $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(F(\alpha))$ for any fully faithful equivariant functor F.

(4.) Neutral elements and split extensions.

(4.0) In spite of the appearance of the composition law (3.0.1) in $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$, this extension is not necessarily split since the identification of the elements of $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ with the set $\mathbb{E} \times \mathbb{N} \equiv \mathbb{E} \times \underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ is possible only through the choice, for each element g of $\mathbb{E}$ of an isomorphism $s(g)\colon \alpha \longrightarrow \alpha^g$ in $\underset{\sim}{G}$. Chosen with $s(e) = \mathrm{id}(\alpha)\colon \alpha \longrightarrow \alpha^e = \alpha$, this identification uses the composition in $\underset{\sim}{G}$ to define $(g, n\colon \alpha \longrightarrow \alpha)$ for a given $(g, \xi\colon \alpha \longrightarrow \alpha^g)$ as that unique arrow of projection g which makes the square

(4.0.0)
$$\begin{tikzcd}
\alpha \arrow[r, "\mathrm{id}"] \arrow[d, "n"'] & \alpha \arrow[d, "\xi"] \\
\alpha \arrow[r, "s(g)"'] & \alpha^g
\end{tikzcd}$$

commutative in $\underset{\sim}{G}$. Under this mapping, the multiplication in $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ becomes on $\mathbb{E} \times \underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$, the familiar product $\overline{(gh, s(gh)^{-1}s(g)s(h)s(h)^{-1}ns(h)m)}$ defined by a normalized section $s\colon \mathbb{E} \longrightarrow \underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ of the extension. The extension is thus split if s could be chosen as a homomorphism, but not in general.

In contrast, if the bouquet $\underset{\sim}{G}$ is an $\mathbb{E}$-group, considered as an $\mathbb{E}$-category with one object $\xrightarrow{\sim} \mathbb{1}$, then it is easy to see that $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\ulcorner \mathbb{1} \urcorner)$ is nothing more than the split extension (= semi-direct product) of the group $\mathbb{E}$ with $\underset{\sim}{G}$ with operators defined by the homomorphism $\varphi\colon \mathbb{E} \longrightarrow \mathrm{Aut}_{\mathrm{Gr}}(\underset{\sim}{G})$ (the group of automorphisms of the underlying group of $\underset{\sim}{G}$) which defines $\underset{\sim}{G}$ as an $\mathbb{E}$-group.

(4.1) Theorem. Let α be an object of an $\mathbb{E}$-bouquet $\underset{\sim}{G}$. In order that the extension $\underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ be split, it is necessary and sufficient that the bouquet $\underset{\sim}{G}$ be neutral (I(4.2)).

In effect, if $A = \underline{\mathrm{Aut}}_{\underset{\sim}{G}}(\alpha)$ is split then $\Gamma(\mathbb{A}/\mathbb{E})$ has an invariant object and is thus split from the left by an $\mathbb{E}$-group. Since $\Gamma(\mathbb{A}/\mathbb{E})$ lies in the same component as $\underset{\sim}{G}$, it follows that $\underset{\sim}{G}$ is neutral. The converse follows from the preceding remarks.

Combining these remarks we have the following

(4.2) Theorem. Let $\mathrm{Ext}[\mathbb{E};\mathbb{N}]$ be the set of isomorphism classes of extensions of $\mathbb{E}$ by a group $\mathbb{N}$ and let $\mathrm{BOUQ}[\mathbb{E}\text{-sets};\mathbb{N}]$ be the set of connected component classes of $\mathbb{E}$-bouquets of which are equivalent, as categories on the underlying set level, to the group $\mathbb{N}$. Then the mapping $\langle\langle \mathbb{F} \mapsto \Gamma(\mathbb{F}/\mathbb{E}) \rangle\rangle$ defines a bijection

$$\boxed{\Gamma\colon \mathrm{Ext}[\mathbb{E};\mathbb{N}] \xrightarrow{\sim} \mathrm{BOUQ}[\mathbb{E}\text{-sets};\mathbb{N}]} \tag{4.2.0}$$

which carries split extensions onto neutral elements.

It follows that the study of group extensions is equivalent to the study of the category of bouquets of $\mathbb{E}$-sets.

(5.0) Non-abelian cocycles and Schreier factor Systems. In the category of $\mathbb{E}$-sets every hypercovering $X_\bullet$ admits, on the underlying set level, a contracting homotopy since the epimorphisms which occur in the definition all have set theoretic sections and a choice of such sections suffices to define a contraction. Since the standard cotriple resolution $\mathbb{G}^+_\bullet(\mathbb{1}) \longrightarrow \mathbb{1}$ of the terminal object $\mathbb{1}$ is universal for such simplicial objects (c.f. DUSKIN (1975) for all details of this process), the contracting homotopy in $\mathbb{E}\mathrm{NS}$ of $X_\bullet$ defines a simplicial map $\varphi\colon \mathbb{G}^+_\bullet(\mathbb{1}) \longrightarrow X_\bullet$ in $\mathbb{E}$-sets. But since $\mathbb{G}^+_\bullet(\mathbb{1}) \xrightarrow{\sim} \mathrm{Cosk}^0(\mathbb{G}^+_\bullet(\mathbb{1}))$, this standard resolution is itself a covering, and we see that the Boolean topos of $\mathbb{E}$-sets has the property that *every hypercovering may be refined by a covering*, in fact here, a standard one. It immediately follows that any 2-cocycle (χ, α, N) on a hypercovering $X_\bullet$ (II.1.1) in $\mathbb{E}$-sets is equivalent, under refinement to a 2-cocycle $\varphi^*(\chi, \alpha, N)$ defined on the standard resolution $\mathbb{G}^+_\bullet(\mathbb{1})$. Thus, it suffices to look at such 2-cocycles: we shall show that they are nothing more than Schreier factor systems (1.1). For this we need the following lemma whose proof we leave to the reader:

For any set X, let $X \times \mathbb{E}$ be the free right $\mathbb{E}$-set on X defined by right multiplication of $\mathbb{E}$ on $X \times \mathbb{E}$.

(5.1) Lemma. For any set X, the functor $\omega_X \colon \mathbb{E}\text{-ENS}/X \times \mathbb{E} \longrightarrow \text{ENS}/X$ defined by pull-back along the set map $x \times e \colon X \longrightarrow X \times \mathbb{E}$,

(5.1.0)
$$\begin{tikzcd}
P_X \arrow[r, hook, "\mathrm{pr}_P"] \arrow[d, "\mathrm{pr}_X", "\omega_X(f)"'] & P \arrow[d, "f"] \\
X \arrow[r, hook, "x\times e"'] & X \times \mathbb{E}
\end{tikzcd}$$

is an equivalence of categories whose quasi-inverse is defined by the assignment

(5.1.1)
$$\begin{tikzcd}
S \arrow[d, "g"'] & & S \times \mathbb{E} \arrow[d, "g\times\mathbb{E}"] \\
X & & X \times \mathbb{E}
\arrow[from=1-1, to=2-3, phantom, "\longmapsto" description]
\end{tikzcd}$$

Remark. This theorem is valid for any group object $\mathbb{E}$ in any topos $\mathbb{T}$. For $x = \mathbb{1}$, this specializes to well known theorem (actually characteristic of the category of $\mathbb{E}$-objects in any topos) which asserts that the category of $\mathbb{E}$-objects above $\mathbb{E}_\delta$ is equivalent to the underlying topos under the functor which assigns to $P \xrightarrow{f} \mathbb{E}_\delta$ its fiber above the unit element of $\mathbb{E}$. This forms the basis for the justification of the statement "for $\mathbb{E}$-objects the underlying object functor is a *functor of localization*" since, under this equivalence it is just the functor pull-back over $\mathbb{E}_\delta \longrightarrow \pi$: ($\mathbb{E}$-objects $\approx$) $\mathbb{E}$-objects/$\mathbb{1} \longrightarrow \mathbb{E}$-objects/$\mathbb{E}_\delta$ ($\approx$ the underlying topos).

(5.2) Using Lemma (5.1) we may transfer the data and axioms for any non-abelian 2-cocycle over the standard resolution of $\mathbb{1}$ in $\mathbb{E}$-sets to an equivalent system in ($\mathbb{E}$NS). The standard resolution of $\mathbb{1}$ in $\mathbb{E}$-sets is the augmented simplicial complex of $\mathbb{E}$-sets and equivariant maps whose structural operators (↓) and face maps (→) are given by

(5.2.0)

$$\begin{tikzcd}[column sep=large]
\mathbb{E}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "\mathbb{E}\times\mathbb{E}\times\mathbb{E}\times\mu"] & \mathbb{E}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\mathbb{E}\times\mathbb{E}\times\mu"] & \mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, "\mathbb{E}\times\mu"] & \mathbb{E}\times\mathbb{E} \arrow[r] \arrow[d, "\mu"] & \mathbb{1}\times\mathbb{E} \arrow[d] & \\
\mathbb{E}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=3, "\mathbb{E}\times\mathbb{E}\times\mu"] \arrow[r, shift left=1, "\mathbb{E}\times\mu\times\mathbb{E}" description] \arrow[r, shift right=1, "\mu\times\mathbb{E}\times\mathbb{E}" description] \arrow[r, shift right=3, "\mathrm{pr}_2\times\mathbb{E}\times\mathbb{E}"'] \arrow[d, equal] & \mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=2, "\mathbb{E}\times\mu"] \arrow[r, "\mu\times\mathbb{E}" description] \arrow[r, shift right=2, "\mathrm{pr}_2\times\mathbb{E}"'] \arrow[d, equal] & \mathbb{E}\times\mathbb{E} \arrow[r, shift left=1, "\mu"] \arrow[r, shift right=1, "\mathrm{pr}_2"'] \arrow[d, equal] & \mathbb{E} \arrow[r] \arrow[d, equal] & \mathbb{1} \arrow[d, equal] & \text{in ENS} \\
\mathbb{E}_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "\varepsilon_i"'] & \mathbb{E}_2 \arrow[r, shift left=2, "\varepsilon_0"] \arrow[r, "\varepsilon_1" description] \arrow[r, shift right=2, "\varepsilon_2"'] & \mathbb{E}_1 \arrow[r, shift left=1, "\varepsilon_0"] \arrow[r, shift right=1, "\varepsilon_1"'] & \mathbb{E}_0 \arrow[r] \arrow[d, equal] & \mathbb{1} \arrow[d, equal] & \text{in } \mathbb{E}\text{-sets} \\
& & & \mathbb{E} & \mathbb{E}_{-1} &
\end{tikzcd}$$

(from which it is easily seen that $\mathbb{G}^\bullet(\mathbb{1}) \xrightarrow{\sim} \mathrm{Cosk}^0(\mathbb{G}^\bullet(\mathbb{1}))$). From Lemma (5.1), we see that category fibers $\mathbb{E}$-sets/$\mathbb{E}_i$ are each equivalent to the corresponding category (ENS)/$\mathbb{E}_{i-1}$ via pull-back along the set map $\mathbb{E}_{i-1} \times s_0 \colon \mathbb{E}_{i-1} \longrightarrow \mathbb{E}_i$, where $s_0 \colon \mathbb{1} \longrightarrow \mathbb{E}$ is just the unit element of $\mathbb{E}$. Using these functors and their quasi-inverses it then is a simple matter to check that the corresponding pull-back functors

$$\varepsilon_i^* \colon \mathbb{E}\text{-sets}/\mathbb{E}_{i+1} \longrightarrow \mathbb{E}\text{-sets}/\mathbb{E}_i \tag{5.2.1}$$

are those induced by pull-back over the structural maps of the nerve of the group $\mathbb{E}$ in (ENS):

(5.2.2)

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathbb{E}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r, shift left=2, "\mathbb{E}\times\mu"] \arrow[r, "\mu\times\mathbb{E}" description] \arrow[r, shift right=2, "\mathrm{pr}_2\times\mathbb{E}"'] & \mathbb{E}\times\mathbb{E} \arrow[r, shift left=1, "\mu"] \arrow[r, shift right=1, "\mathrm{pr}_2"'] & \mathbb{E} \arrow[r] & \mathbb{1} & \mathbb{G}^\bullet(\mathbb{1}) \\
\mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[u, dashed, "\mathbb{E}\times\mathbb{E}\times\mathbb{E}\times s_0"] \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, shift left=3, "\mathbb{E}\times\mathrm{pr}_1"] \arrow[r, shift left=1, "\mathbb{E}\times\mu" description] \arrow[r, shift right=1, "\mu\times\mathbb{E}" description] \arrow[r, shift right=3, "\mathrm{pr}_2\times\mathbb{E}"'] & \mathbb{E}\times\mathbb{E} \arrow[u, dashed, "\mathbb{E}\times\mathbb{E}\times s_0"'] \arrow[ur, dashed, "\mathrm{id}"'] \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "\mu" description] \arrow[r, shift right=2, "\mathrm{pr}_2"'] & \mathbb{E} \arrow[u, dashed, "\mathbb{E}\times s_0"'] \arrow[ur, dashed, "\mathrm{id}"'] \arrow[r] & \mathbb{1} \arrow[u, dashed, "s_0"'] & & \mathrm{Ner}(\mathbb{E}) \arrow[u, phantom, "\wr\wr"]
\end{tikzcd}$$

(5.3) We now translate the data for a 2-cocycle on $\mathbb{G}^\bullet(\mathbb{1})$: *A locally given group* N^1, i.e. a group object in $\mathbb{E}$-sets/$\mathbb{E}_0$, *is equivalent to* a group object in ENS/$\mathbb{1}$, i.e. *an* *ordinary group* N in (ENS). A *gluing* $\alpha' \colon \varepsilon_0^*(N') \xrightarrow{\sim} \varepsilon_1^*(N')$ *is* just an $\mathbb{E}$-isomorphism of groups $\mathbb{E} \times N \xrightarrow{\alpha} \mathbb{E} \times N$, and thus by exponential adjointness, j*ust equivalent to a mapping* $\alpha^* \colon \mathbb{E} \times \mathbb{E} \longrightarrow \underline{\mathrm{Aut}}_{\mathrm{Gr}}(N)$. Finally, *the cocycle* χ just *becomes equivalent to ordinary mapping* $\chi \colon \mathbb{E} \times \mathbb{E} \longrightarrow N$. We leave it to the reader to verify the corresponding translation of the axioms for a 2-cocycle gives those of a factor system on $\mathbb{E}$ (II 2.0). Thus we have the

(5.4) Theorem. In the category of $\mathbb{E}$-sets, the set of isomorphism classes of non-abelian 2-cocycles is bijectively equivalent to the set of isomorphism classes of Schreier factor systems of $\mathbb{E}$.

(5.5) Remark. Similar remaps concern neutrality and “factor free” ($X = e$) Schreier factor systems.

From (I.4.3) in order that a bouquet $\underset{\approx}{G}$ of a topos $\mathbb{T}$ be neutral it is also necessary and sufficient that the groupoid $\underline{\mathrm{TORS}}(\mathbb{T}; \underset{\approx}{G})$ be non-empty, i.e., that there exists a $\mathbb{T}$-torsor under $\underset{\approx}{G}$ or, equivalently, that there exist a simplicial map from the nerve of a covering of $\mathbb{T}$ into the nerve of $\underset{\approx}{G}$. In $\mathbb{E}$-sets we have for neutrality a simplicial map from the nerve of a covering of $\mathbb{T}$ into the nerve of $\underset{\approx}{G}$. In $\mathbb{E}$-sets we thus must have for neutrality a simplicial map from the standard resolution of $\mathbb{1}$ into the nerve of $\underset{\approx}{G}$

$$\begin{tikzcd}
\mathbb{E}\times\mathbb{E}\times\mathbb{E} \arrow[r] \arrow[d, shift right=3, "\mathbb{E}\times\mu"'] \arrow[d, "\mu\times\mathbb{E}"] \arrow[d, shift left=3, "\mathrm{pr}_2\times\mathbb{E}"] & A\times_O A \arrow[r] \arrow[d, shift right=2] \arrow[d] \arrow[d, shift left=2] & K_2 \arrow[dl, shift right=2] \arrow[dl] \arrow[dl, shift left=2] \\
\mathbb{E}\times\mathbb{E} \arrow[u, dashed, bend left=60] \arrow[r] \arrow[d, shift right, "\mu"'] \arrow[d, shift left, "\mathrm{pr}_2"] & A \arrow[d, shift right] \arrow[d, shift left] & \\
\mathbb{E} \arrow[u, dashed, bend left=60, "s_1"] \arrow[ur, dashed] \arrow[r] \arrow[d] & O & \\
\mathbb{1} \arrow[u, dashed, bend left=60, "s_0"] \arrow[ur, dashed] & &
\end{tikzcd}$$

If we use the canonical set maps s_i to translate such a simplicial map of $\mathbb{E}$-sets into set theoretic terms, we see immediately that such a simplicial map is directly equivalent in sets to giving an object $\alpha_0 \in \mathrm{Ob}(\underset{\approx}{G})$ and a function $F \colon \mathbb{E} \longrightarrow A$ such that for each

$h \in \mathbb{E}$ $\quad F(h)\colon \alpha_0 \longrightarrow \alpha_0^h$ and for each $g, h \in \mathbb{E}$, the triangle

(5.5.1)
$$\begin{tikzcd}
\alpha_0 \arrow[r, "F(h)"] \arrow[dr, "F(gh)"'] & \alpha_0^h \arrow[d, "F(g)^h"] \\
& \alpha_0^{gh}
\end{tikzcd}$$

is commutative in $\underset{\approx}{G}$. But in view of the definition of the group extension $\underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha_0)$ this is nothing more than a *homomorphism* $F\colon \mathbb{E} \longrightarrow \underline{\mathrm{Aut}}_{\underset{\approx}{G}}(\alpha_0)$ which splits the projection onto $\mathbb{E}$.

Thus if we started with a non-abelian 2-cocycle on a hypercovering and constructed its associated bouquet $\mathbb{B}(X, \alpha, N)$ there the bouquet $\mathbb{B}(X, \alpha, N)$ is, by construction, the fiber $(X = 0)$ of the canonical 2-cocycle on the hypercovering $\mathrm{Cosk}^1(\mathbb{B}(X, \alpha, N))$ since $K_2 = \mathrm{Cosk}^1(\mathbb{B}(X, \alpha, N))_2$ consists of the triangles of $\mathbb{B}(X, \alpha, N)$ and $A \times_O A \hookrightarrow K_2$ consists of the commutative triangles $f_1 f_0 f_2 = X(f_0, f_1, f_2) = e$. Thus (X, α, N) is neutral if and only if the hypercovering on which it is defined factors through the nerve of $\mathbb{B}(X, \alpha, N)$. In $\mathbb{E}$-sets this is thus just equivalent to saying that the Schreier system defined by the cocycle is (up to equivalence) *factor free* ($c(g, h) = e$ for all $g, h \in \mathbb{E} \times \mathbb{E}$), a condition classically equivalent to saying that the extension which it defines is a semi-direct product which is, of course, equivalent to its being split. (6.0) Ties and abstract kernels.

We now will proceed to show that a tie in $\mathbb{E}$-sets is fully equivalent to a homomorphism $\ell\colon \mathbb{E} \longrightarrow \mathrm{Out}(N)$ whose target is the group of outer automorphisms ($\mathrm{Aut}(N)/\mathrm{Int}(N)$) of some group N, i.e. to an *abstract kernel* in the terminology of EILENBERG-MAC LANE (1954). For this it is convenient to note that a Schreier factor system may succinctly be described as a simplicial map

(6.0.0)
$$S\colon \mathrm{Ner}(\mathbb{E}) \longrightarrow \underline{\mathrm{Aut}}^{(2)}(N)$$

from the nerve of $\mathbb{E}$ into a simplicial complex (actually the nerve of a 2-dimensional groupoid) whose form in low dimensions looks like

(6.0.1)
$$\begin{tikzcd}
N^3 \times \mathrm{Aut}(N)^3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & N \times \mathrm{Aut}(N)^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Aut}(N) \arrow[r] & \mathbb{1}\colon\ \underline{\mathrm{Aut}}^{(2)}(N)
\end{tikzcd}$$

and whose set of 2-simplices may be viewed as triangles of automorphisms of N, coupled with a group element $n \in N$

(6.0.2)
$$\begin{tikzcd}
N \arrow[rr, "\varphi"] \arrow[ddr, "\delta"'] & & e \arrow[ddl, "\rho"] \\
& n & \\
& N &
\end{tikzcd}$$
such that $\mathrm{int}(n)\, \varphi = \delta$.

This complex $\underline{\mathrm{Aut}}^{(2)}(N)$ is supplied canonically with a simplicial map

$$p^{(2)}\colon \underline{\mathrm{Aut}}^{(2)}(N) \longrightarrow \mathrm{Ner}(\mathrm{Out}(N)) \tag{6.0.3}$$

defined by the canonical surjection $p\colon \mathrm{Aut}(N) \longrightarrow \mathrm{Out}(N)$.

In (6.0) of Part II we noted that by refinement any tie of a topos gave rise to a system which was precisely that of 2-cocycle (minus the cocycle condition). This in $\mathbb{E}$-sets any tie gives rise to the same system over the standard resolution $\mathbb{G}^{\bullet}(\mathbb{1})$. From the preceding equivalence (5.1) it thus follows that any tie defines and is equivalent to a truncated map:

$$\bar{\ell}\colon \mathrm{TR}^2(\mathrm{Ner}(\mathbb{E})) \longrightarrow \mathrm{TR}^2(\underline{\mathrm{Aut}}^2(N))\,. \tag{6.0.4}$$

By composition of $\bar{\ell}$ with the truncated map

$$\mathrm{TR}^2(p^{(2)})\colon \mathrm{TR}^2(\underline{\mathrm{Aut}}^{(2)}(N)) \longrightarrow \mathrm{TR}^2(\mathrm{Ner}(\mathrm{Out}(N))),$$

we thus obtain a *homomorphism*: $\ell\colon \mathbb{E} \longrightarrow \mathrm{Out}(N)$, i.e. an abstract kernel as promised. Choosing a section for p, and using the same equivalence, we see immediately that any such abstract kernel defines a lien and the correspondence is established as asserted.

More use of the complex $\underline{\mathrm{Aut}}^{(2)}(N)$ will be made in the sequel.

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John W. Duskin

APPENDIX: THE FORMALISM OF THE THEORY OF DESCENT

INTRODUCTION

If X is a topological space and $(U_i)_{i\in I}$ is an open covering of X, then a familiar problem in topology is that of constructing a fiber space over X by "gluing together" a locally given family $(T_i \longrightarrow U_i)_{i\in I}$ of fiber spaces (of some particular type) using "gluing data" given in the form of system of compatible isomorphisms $\alpha_{ji}\colon T_i|U_i\cap U_j \longrightarrow T_j|U_i\cap U_j$ on the restriction of the members of the given family to the overlaps. In the terminology of the Grothendieck presentation, if such a fiber space over X can be constructed so that its restriction to each of the members of the covering gives back (up to isomorphism) the original family, then the fiber space may be said to have been obtained by "descent" and that the gluing isomorphisms α_{ji} defined an "effective descent datum". Since any fiber space T over X defines by restriction to the U_i (i.e., by "localization") a locally given family which is canonically equipped with an effective descent datum by means of the canonical isomorphisms
$T_i|U_i\cap U_j \xrightarrow{\sim} TU_i\cap U_j \overset{\sim}{\longleftrightarrow} T_jU_i\cap U_j$, the process involved here is, in fact, the study of the properties of the "localization on a cover" functor as it carries fiber spaces of the particular type under study into fiber spaces supplied with descent data defined on the cover.

In the abstract version of this process the notion of a category of fiber spaces over X defined for each space X and stable under restriction is generalized to the abstract notion of a "fibered category", more precisely a (contravariant) pseudofunctor defined on the category, while the corresponding abstract notion of localization is provided through the definition of a (Grothendieck) topology on the category. [ARTIN (1962)].

In this appendix we will first define the abstract notion of a gluing and a descent datum over a family of arrows of some underlying category $\mathbb{E}$ assuming that the category has fiber products. In this form the connection of this abstract notion with the motivating topological problem will be clear. We will then show that an easy modification of this definition will suffice in the absence of fiber products. What will also then be clear, unfortunately, is that this definition (which goes back to Grothendieck's original (1960) formulation) while intuitively clear, is also enormously cumbersome to write down. Fortunately, this state of affairs is not permanent for (following GIRAUD (1962)) we will show that any such descent datum is entirely equivalent to a single cartesian functor from the seive generated by the family into the fibration defined by the pseudofunctor. The resulting formulation of the notion, while extraordinarily elegant, is still not without its drawbacks for in contrast to what can be done for a Grothendieck topology it requires us to go completely outside of the functor category $\mathbb{E}\hat{}$ (hence outside of an extremely well behaved, set-like topos) for its formulation. Fortunately, this difficulty as well can be overcome for we will finally indicate how the whole process can be done in the topos $\mathbb{E}\hat{}$ of presheaves on $\mathbb{E}$.

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What may be then seen is that there are at least four independent but mutually essentially equivalent descriptions of the notions which occur in the original theory which start from the very intuitive notions coming from topology and lead to the rather abstract formulation within the internal category theory of $\mathbb{E}\hat{}$ (or in fact in an elementary topos). Since various writers on the subject use various combinations of portions of these different versions (with their mutual translation not altogether obvious) we have chosen to present them independently and to then describe their mutual equivalence as it occurs regularly in the following table (A) in which, the four columns represent the essentially equivalent descriptions of the corresponding fundamental notions of “the theory of descent” (the rows) as they have evolved from the initial intuitive description (I) of Grothendieck and its modification by Giraud (II and III) finally leading to the form (IV) in which its extension to the elementary topoi of Lawvere-Tierney becomes clear. Of course, most of the material presented here may be found explicitly (or implicitly) in Giraud although, as we have remarked in the text, it is closely related to the work of many authors, among them GRAY (1974), JOYAL (1974), PENON (1978 1979) BOURN (1978), LAWVERE (1974), BUNGE (1979), PARÉ (1979), STREET (1982). It is hoped that by presenting it here in at least a semi-coherent form, the work of these various authors may be made accessible to a wider audience and, at least, its use in the text made clear.

Fundamental Notions $\big($Grothendieck topology on $\mathbb{E}\big)$	Description I ( ) (in presence of fiber products) and without fiber products	Description II ( ) $\big($in 2-CAT$\big)$	Description III ( ) $\big($in Cart$/\mathbb{E}\big)$	Description IV ( ) $\big($in topos $\mathbb{E}\hat{}\,\big)$
Coverings of $\mathbb{E}$	families of maps of $\mathbb{E}$ ( ) $C/X\colon\ (X_\alpha\times_X X_\beta)\rightrightarrows(X_\alpha)\longrightarrow X$	(covering) sieves $\big($as certain subcategories $\mathfrak{R}\hookrightarrow\mathbb{E}/X\big)$	(covering) sieves (as discrete sub-fibrations $\begin{tikzcd} \mathfrak{R} \arrow[rr] \arrow[dr] & & \mathbb{E} \arrow[dl] \\ & \mathbb{E} & \end{tikzcd}$)	(covering) subfunctors $R\hookrightarrow h_X$ of representables.
Presheaves on $\mathbb{E}$	functors $P\colon\mathbb{E}^{\mathrm{op}}\longrightarrow$ENS	functors $P\colon\mathbb{E}^{\mathrm{op}}\longrightarrow$ENS	discrete fibrations $\mathbb{E}/P\longrightarrow\mathbb{E}$	object P of $\mathbb{E}\hat{}$
Sheaf Property	$P(X)\hookrightarrow P(X_\alpha)\rightrightarrows P(X_\alpha\times_X X_\beta)$ is exact for all coverings ( )	$P(X)\longrightarrow\underset{\longleftarrow}{\lim}\,P\vert\mathfrak{R}$ is bijective for each covering seive	$P(X)\longrightarrow\mathrm{Cart}_{\mathbb{E}}(\mathfrak{R},\mathbb{E}/P)$ is an isomorphism for each covering seive	$\mathrm{Hom}_{\mathbb{E}\hat{}}(h_X,P)\longrightarrow\mathrm{Hom}_{\mathbb{E}\hat{}}(R,P)$ $P(X)$ is an iso for each cov. subfunctor
(Theory of Descent) Pseudo-functors on $\mathbb{E}$	$F^c_{(\,)}\colon E^{\mathrm{op}}\rightsquigarrow$CAT ( )	$F^c_{(\,)}\colon E^{\mathrm{op}}\rightsquigarrow$CAT	Grothendieck fibrations $\mathbb{F}\longrightarrow\mathbb{E}$	Category objects $\mathbb{C}\colon\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/-,\mathbb{F})$ of $\mathbb{E}\hat{}$ $\big($complete in canonical topology of $\mathbb{E}\hat{}\,\big)$
Descent Data over a Covering	compatibly glued families of objects in the fibers of $F_{(\,)}$ above a covering ( )	objects of the category $\underset{\longleftarrow}{\lim}\,F^c_{(\,)}\vert\mathfrak{R}$	Cartesian functor $\begin{tikzcd} \mathfrak{R} \arrow[rr] \arrow[dr] & & \mathbb{F} \arrow[dl] \\ & \mathbb{E} & \end{tikzcd}$	Čech-cocycles: $\mathrm{Simpl}_{\mathbb{E}\hat{}}\big(\mathrm{Ner}\,\mathbb{E}/\mathfrak{R},\mathrm{Ner}\,\mathbb{C}\big)$ $=\mathrm{Hom}_{\mathrm{CAT}(\mathbb{E}\hat{}\,)}(\mathbb{E}/\mathfrak{R},\mathbb{C})$
Category of descent data	$\underline{\underline{\mathrm{Desc}}}(C/X;F^c_{(\,)})$	$\underset{\longleftarrow}{\lim}\,F^c_{(\,)}\vert\mathfrak{R}$	$\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathfrak{R},\mathbb{F})$	$\underline{\mathrm{Hom}}_{\mathrm{CAT}(\mathbb{E}\hat{}\,)}(\mathbb{E}/\mathfrak{R},\mathbb{C})$
Stack property	$\mathbb{F}^c_X\xrightarrow{\approx}\underline{\underline{\mathrm{Desc}}}(C/X;F^c_{(\,)})$ an equivalence for each covering of X ( )	$\mathbb{F}^c_X\xrightarrow{\approx}\ \underset{\longleftarrow}{\lim}\,F^c_{(\,)}\vert\mathfrak{R}$ is an equivalence for each covering seive	$\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/X,\mathbb{F})\xrightarrow{\approx}\mathrm{Cart}_{\mathbb{E}}(\mathfrak{R},\mathbb{F})$ is an equivalence for each covering seive	$\mathbb{C}(X)\xrightarrow{\approx}\underline{\mathrm{Hom}}_{\mathbb{E}\hat{}}(R,\mathbb{C})\ \Leftrightarrow$ $\mathbb{C}(X)\xrightarrow{\approx}\underline{\mathrm{Simpl}}(C/X,\mathbb{C})$ is an equivalence $\big(\mathbb{C}$ is complete in induced top on $\mathbb{E}\hat{}\,\big)$

Table A.

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Description I: (in the presence of fibre products)

(1.0) Let $\mathbb{E}$ be a category. A *Grothendieck* (pre-)*topology on* $\mathbb{E}$ is the specification for each object S in $\mathbb{E}$ of a set $R(S)$ of families of arrows of $\mathbb{E}$ which has the following properties:

(a) each family $C \in R(S)$ has the form $C = (S_i \longrightarrow S)_{i\in I}$ for some (not necessarily unique) index set I;

(b) for each $C = (S_i \longrightarrow S)$ in $R(S)$ and any arrow $f\colon T \longrightarrow S$ in $\mathbb{C}$, the family $f^*(C) = (S_i \times_S T \xrightarrow{\mathrm{pr}} T)$ is defined and is a member of $R(T)$;

(c) if $(S_1 \longrightarrow S) \in R(S)$ and for each i, $(T_{ij} \longrightarrow S_i) \in R(S_i)$, then the family $(T_{ij} \longrightarrow S)$ obtained by composition is a member of $R(S)$;

(d) Any one member family consisting of an isomorphism $T \longrightarrow S$ is a member of $R(S)$.

The elements of $R(S)$ are called the *coverings* or *refinements* of S for the (pre-) topology. A category together with a topology is called a *site*.
(1.1) Let F be a presheaf on the underlying category of a site, that is a functor $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$. F is said to be a *sheaf* for the topology provided that for each covering $(S_i \xrightarrow{s_i} S)_{i\in I} \in R(S)$ the diagrams of sets

(1.1.0) $$F(S) \longrightarrow \prod_{i\in I} F(S_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(S_i \times S_j)$$

is exact, i.e., the mapping $s \longmapsto (F(s_i)(s))_{i\in I}$ defines a bijection onto the set of elements $(x_i)_{i\in I} \in \prod_{i\in I} F(S_i)$ such that $F(\mathrm{pr}_i)(x_i) = F(\mathrm{pr}_j)(x_j)$ for all $(i,j) \in I \times I$. The category of presheaves and natural transformation of presheaves on $\mathbb{E}$ is usually denoted by $\mathbb{E}\hat{}$. We define the category $\mathbb{E}\tilde{}$ of *sheaves on the site* the full subcategory of $\mathbb{E}\hat{}$ consisting of the sheaves for the given topology of the site. A category of the form $\mathbb{E}\tilde{}$ for some topology on $\mathbb{E}$ is called a *Grothendieck topos.*
(2.0) <u>Definition</u>: A (contravariant normalized) *pseudo-functor* $\mathbb{F}^{\mathrm{c}}_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ *on* $\mathbb{E}$ [also called a *fibered* (or *indexed*) *category* over $\mathbb{E}$] is an assignment

(1) to each object X of $\mathbb{E}$ of a category $\mathbb{F}^{\mathrm{c}}_X$ (called the *fiber* at X),

(2) to each arrow $f\colon X \longrightarrow Y$ of $\mathbb{E}$ of a functor $\mathbb{F}^{\mathrm{c}}_f\colon \mathbb{F}^{\mathrm{c}}_Y \longrightarrow \mathbb{F}^{\mathrm{c}}_X$ (called the inverse image functor),

(3) to each composab1e pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of arrows of $\mathbb{E}$ of a natural isomorphism, $c(f,g)\colon \mathbb{F}^c_f\mathbb{F}^c_g \longrightarrow \mathbb{F}^c_{gf}$ (called the *cleavage isomorphism*)

(2.0.0)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] & \longmapsto & \mathbb{F}^c_X & \mathbb{F}^c_Y \arrow[l, "\mathbb{F}^c_f"'] \\
& Z & & & \mathbb{F}^i_Z \arrow[ul, "\mathbb{F}_{gf}", ""{name=A}] \arrow[u, "\mathbb{F}^c_g"', ""{name=B, swap}] \arrow[from=B, to=A, bend right, "c(f,s)"', "\sim"]
\end{tikzcd}$$

subject to the following (*coherence*) *conditions*:

(a) for all $f\colon X \longrightarrow Y$ in $\mathbb{E}$, $c(\mathrm{id}, f) = c(f, \mathrm{id}) = \mathrm{id}(\mathbb{F}_f)$ (*normalization*)

(b) for any composable triplet $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbb{E}$, the square

(2.0.1)
$$\begin{tikzcd}
\mathbb{F}^c_f\mathbb{F}^c_g\mathbb{F}^c_h \arrow[r, two heads, "\mathbb{F}^c_f * c(g,h)"] \arrow[d, "c(f,g) * \mathbb{F}^c_h"'] & \mathbb{F}^c_f\mathbb{F}^c_{hg} \arrow[d, "c(f,hg)"] \\
\mathbb{F}^c_{gf}\mathbb{F}^c_h \arrow[r, "c(gf,h)"'] & \mathbb{F}^c_{hgf}
\end{tikzcd}$$

of natural isomorphisms of functors is commutative (2-*cocycle condition*).

If the pseudo functor under discussion is clear, the notation $f^*(X)$ is often used for the value at X of the functor: $\mathbb{F}^c_f\colon \mathbb{F}^c_Y \longrightarrow \mathbb{F}^c_X$.

(2.0.2) Remark. A pseudo-functor is, in fact, a simplicial map from $Ner(\mathbb{C}^{\mathrm{op}})$ into the nerve of (a subcategory of) the 2-category CAT. A description of this will be given in detail in the sequel to this paper.

(2.1) Examples abound in the presence of fiber products, where the fibers are most often some appropriate subcategory of $\mathbb{E}/X$ which is preserved under pull-backs or "change of base", e.g. $\mathbb{G}r(\mathbb{E}/X)$ the category of group objects in $\mathbb{E}/X$; etale spaces (i.e. local homeomorphisms $E \longrightarrow X$) over the category of topological spaces and continuous maps. Many other examples have appeared in the text.

(3.0) Descent on a family.

(3.1) Definition. (in the presence of fiber products). Let $\mathbb{E}$ be a category and $\mathbb{F}^c_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ a normalized pseudo-functor on $\mathbb{E}$ and $(x_i\colon X_i \longrightarrow X)_{i\in I}$ be a family of arrows of $\mathbb{E}$ which is squareable (i. e. the fiber product of any arrow $T \longrightarrow X$ in $\mathbb{E}$ with any member of the family $(x_i)_{i\in I}$ is assumed to exist). By a gluing of a family $(A_i)_{i\in I}$, $A_i \in \mathrm{ob}(\mathbb{F}_{X_i})$ of objects above the family (X_i) we shall mean a system of isomorphisms

$$\theta_{ij}\colon \mathbb{F}^c_{\mathrm{pr}_i}(A_i) = \mathrm{pr}_i^*(A_i) \xrightarrow{\sim} \mathrm{pr}_j^*(A_j) = \mathbb{F}^c_{\mathrm{pr}_j}(A_j), \quad i,j \in I \times I \quad \theta_{ij} \in \mathrm{Ar}(\mathbb{F}_{X_i \times_X X_j})\ .$$

A gluing θ_{ij} will be called a *descent datum on the family A_i* provided that the following conditions are satisfied:

(a) *normalization*: for all $i \in I$, the isomorphism $\theta_{ii}\colon \mathrm{pr}_1^*(A_i) \xrightarrow{\sim} \mathrm{pr}_2^*(A_i)$ in $\mathrm{Ar}(\mathbb{F}_{X_i\times_X X_i})$ has the property that if $\Delta\colon X_i \hookrightarrow X_i \times_X X_i$ is the diagonal, then

(3.1.0)
$$\begin{tikzcd}
\Delta^*(\mathrm{pr}_1^*(A_i)) \arrow[r, "\Delta^*(\theta_{ii})"] \arrow[d, "c(\Delta{,}\mathrm{pr}_1)"', "\wr"] & \Delta^*(\mathrm{pr}_2^*(A_i)) \arrow[d, "c(\Delta{,}\mathrm{pr}_2)", "\wr"'] \\
(\mathrm{pr}_1\Delta)^*(A_i) \arrow[d, equal] & (\mathrm{pr}_2\Delta)^*(A_i) \arrow[d, equal] \\
(\mathrm{id}_{X_i})^*(A_i) \arrow[r, "\mathrm{id}_{A_i}"] & (\mathrm{id}_{X_i})^*(A_i)
\end{tikzcd}$$

is commutative in $\mathbb{F}_{X_i}$;

(b) 1-*cocycle condition*: for all $(i,j,k) \in I \times I \times I$, if $\mathrm{pr}_{ab}\colon X_i \times_X X_j \times_X X_k \longrightarrow X_a \times_X X_b$, for $(a,b) = (i,j)$, (j,k), (i,k), are the canonical projections,

(3.1.2)
$$\begin{tikzcd}
 & X_i \times_X X_j \arrow[r, "\mathrm{pr}_i"] \arrow[dr, "\mathrm{pr}_j"] & X_i \arrow[dr, "x_i"] & \\
X_i \times_X X_j \times_X X_k \arrow[ur, "\mathrm{pr}_{ij}"] \arrow[r, "\mathrm{pr}_{ik}"] \arrow[dr, "\mathrm{pr}_{jk}"'] & X_i \times_X X_k \arrow[ur, "\mathrm{pr}_i"] \arrow[r, "\mathrm{pr}_j"] \arrow[dr, "\mathrm{pr}_k"'] & X_j \arrow[r, "x_j"] & X \\
 & X_j \times_X X_k \arrow[ur, "\mathrm{pr}_j"] \arrow[r, "\mathrm{pr}_k"'] & X_k \arrow[ur, "x_k"'] &
\end{tikzcd}$$

then the diagram

(3.1.3)

$$\begin{tikzcd}
& \mathrm{pr}^*_{ij}(\mathrm{pr}^*_i(A_i)) \arrow[rr, "\mathrm{pr}^*_{ij}(\theta_{ij})", "\sim"'] \arrow[d, "c(ij,i)(A_i)"', "\wr"] & & \mathrm{pr}^*_{ij}(\mathrm{pr}^*_j(A_j)) \arrow[d, "\wr"', "c(ij,j)(A_j)"] & \\
& (\mathrm{pr}_i\mathrm{pr}_{ij})^*(A_i) \arrow[d, equal] & & (\mathrm{pr}_j\mathrm{pr}_{ij})^*(A_j) \arrow[d, equal] & \\
& (\mathrm{pr}_i\mathrm{pr}_{ik})^*(A_i) \arrow[rr, "\widetilde{\mathrm{pr}}^*_{ij}(\theta_{ij})"] \arrow[ddrr, "\widetilde{\mathrm{pr}}^*_{ik}(\theta_{ik})", "\sim"'] & & (\mathrm{pr}_j\mathrm{pr}_{jk})^*(A_j) \arrow[dd, "\wr"', "\widetilde{\mathrm{pr}}_{jk}(\theta_{jk})"] & \mathrm{pr}^*_{jk}(\mathrm{pr}^*_j(A_j)) \arrow[l, "c(jk,j)"', "\sim"] \arrow[dd, "\wr"', "\mathrm{pr}^*_{jk}(\theta_{jk})"] \\
\mathrm{pr}^*_{ik}(\mathrm{pr}^*_i(A_i)) \arrow[ur, "c(ik,i)", "\sim"'] \arrow[dddrr, "\sim", "\mathrm{pr}^*_{ik}(\theta_{ik})"'] & & & & \\
& & & (\mathrm{pr}_k\mathrm{pr}_{jk})^*(A_k) \arrow[d, equal] & \mathrm{pr}^*_{jk}(\mathrm{pr}^*_k(A_k)) \arrow[l, "\sim"', "c(jk,k)"] \\
& & & (\mathrm{pr}_k\mathrm{pr}_{ik})^*(A_k) & \\
& & \mathrm{pr}^*_{ik}(\mathrm{pr}^*_k(A_k)) \arrow[ur, "\sim", "c(ik,k)"'] & &
\end{tikzcd}$$

commutative in $\mathbb{F}_{X_i\times_X X_j\times_X X_k}$, where $\mathrm{pr}^*_{ab}(\theta_{ab}) = \mathbb{F}^c_{\mathrm{pr}_{ab}}(\theta_{ab})$ and $\widetilde{\mathrm{pr}}_{ab}(\theta_{ab}) = c(ab,b)(\theta_b)\mathrm{pr}^*_{ab}(\theta_{ab})c(ab,a)^{-1}(A_a)$ for $(a,b) = (i,j)$, (j,k), and (i,k), respectively (with $c(ab,a) = c(\mathrm{pr}_{ab},\mathrm{pr}_a)$, the cleavage isomorphisms for the pseudofunctor $\mathbb{F}^c_{(\,)}$, etc.), in brief:

(3.1.4) $$\boxed{\widetilde{\mathrm{pr}}_{jk}(\theta_{jk})^*\widetilde{\mathrm{pr}}^*_{ij}(\theta_{ij}) = \widetilde{\mathrm{pr}}^*_{ik}(\theta_{ik})\,.}$$

(3.1.5)

$$\begin{tikzcd}
(\mathbb{F}_{X_i}(X_i\times X_j\times X_k)) \arrow[d, dash] & (\mathbb{F}_{X_i}(X_i\times_X X_j)) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, dash] & (\mathbb{F}_{X_i}) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, dash] & \mathbb{F}_X \arrow[l] \arrow[d, dash] \\
(X_i\times_X X_j\times_X X_h) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (X_i\times_X X_j) \arrow[r, shift left] \arrow[r, shift right] & (X_i) \arrow[r] & X
\end{tikzcd}$$

(3.2) Definition (in the presence of fiber products). By a *morphism of descent* (or gluing) *data* we will mean a family of arrows

$(f\colon A_i \longrightarrow B_i)\quad i \in I,\ f_i \in \mathrm{Ar}(\mathbb{F}_{X_i})$ such that the diagram

(3.2.0)
$$\begin{tikzcd}
\mathrm{pr}_i^*(A_i) \arrow[r, "\mathrm{pr}_i^*(f_i)"] \arrow[d, "\theta_{ij}"'] & \mathrm{pr}_i^*(B_i) \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_j^*(A_j) \arrow[r, "\mathrm{pr}_j^*(f_j)"'] & \mathrm{pr}_j^*(B_j)
\end{tikzcd}$$

is commutative for all $(i,j) \in I \times I$. We thus have defined the *category*

$$\underline{\underline{\mathrm{Desc}}}_{\mathbb{E}}\big(X_i/X;\,\mathbb{F}^c_{(\,)}\big)$$

of families of objects of $\mathbb{F}^c_{(\,)}$ *supplied with descent data relative to the family of arrows* $(x_i\colon X_i \longrightarrow X)_{i\in I}$ of $\mathbb{E}$. If the family of arrows is the family of open inclusions of some open cover of a space X and the pseudofunctor is that defined by the category of fiber spaces over variable base spaces and “restriction to an open subset”, then the above notion is precisely the usual one of a gluing a locally given family of fiber spaces since in this case the usual intersection of subspaces defines the fiber product of the inclusion mappings.

(3.3) Localization and descent on a family, $(x_i\colon X_i \longrightarrow X),\quad i \in I$. If A is an object in $\mathbb{F}^c_X$, then the “localized” family of objects $(x_i^*(A))_{i\in I}$, $\mathbb{F}^c_{X_i}(A) = x_i^*(A) \in \mathrm{ob}(\mathbb{F}^c_{X_i})$ is supplied canonically with a gluing since for any cartesian square in $\mathbb{E}$

(3.3.0)
$$\begin{tikzcd}
 & X_i \times_X X_j \arrow[dl, "\mathrm{pr}_i"'] \arrow[dr, "\mathrm{pr}_j"] & \\
X_i \arrow[dr, "x_i"'] & & X_j \arrow[dl, "x_j"] \\
 & X &
\end{tikzcd}$$

(with common diagonal v), we may define a gluing isomorphism θ_{ij} as that unique isomorphism which makes the diagram

(3.3.1)
$$\begin{tikzcd}
\mathrm{pr}_1^*(x_i^*(A)) \arrow[r, "\theta_{ii}", "\sim"'] \arrow[d, "c(\mathrm{pr}_i{,}x_i)(A)"', "\wr"] & \mathrm{pr}_j^*(x_j^*(A)) \arrow[d, "c(\mathrm{pr}_j{,}x_j)(A)", "\wr"'] \\
(x_i\mathrm{pr}_i)^*(A) \arrow[d, equal] & (x_j\mathrm{pr}_j)^*(A) \arrow[d, equal] \\
v^*(A) \arrow[r, "\mathrm{id}"] & v^*(A)
\end{tikzcd}$$

commutative in $\mathbb{F}^c_{X_i\times_X X_j}$

This gluing is a descent datum on the family $x_i^*(A),\ i \in I$. In effect, for any commutative cube of the form (3.1.2), the axiom (2.0.1) for the cleavage $c(-,-)$ of $\mathbb{F}^c_{(\,)}$ declares that in the cube

(3.3.2)

$$\begin{tikzcd}
\mathrm{pr}_{ij}^*(\mathrm{pr}_i^*(x_i^*(A))) \arrow[rr, "\mathrm{pr}_{ij}^*(\theta_{ij})"] \arrow[dr, "c(\mathrm{pr}_{ij}{,}\mathrm{pr}_i)^* * x_i^*(A)"] \arrow[dd, "\mathrm{pr}_{ij}^* * c(\mathrm{pr}_i{,}x_i)(A)"'] & & \mathrm{pr}_{ij}^*(\mathrm{pr}_j^*(x_j^*(A))) \arrow[dr, "c(\mathrm{pr}_{ij}{,}\mathrm{pr}_j)^* * x_j^*(A)"] \arrow[dd, "\mathrm{pr}_{ij}^* * c(\mathrm{pr}_j{,}x_j)(A)"' near start] & \\
& (\mathrm{pr}_i\mathrm{pr}_{ij})^*(x_i^*(A)) \arrow[rr, "\widetilde{\mathrm{pr}}_{ij}^*(\theta_{ij})" near start] \arrow[dd, "c(\mathrm{pr}_i\mathrm{pr}_{ij}{,}x_i)(A)" near start] & & (\mathrm{pr}_j\mathrm{pr}_{ij})^*(x_j^*(A)) \arrow[dd, "c(\mathrm{pr}_j\mathrm{pr}_{ij}{,}x_j)(A)"] \\
\mathrm{pr}_j^*(x_i\mathrm{pr}_i)^*(A) \arrow[rr, "\mathrm{id}" near end] \arrow[dr, "c(\mathrm{pr}_j{,}x_i\mathrm{pr}_i)(A)"'] & & \mathrm{pr}_{ij}^*(x_j\mathrm{pr}_j)^*(A) \arrow[dr, "c(\mathrm{pr}_{ij}{,}x_j\mathrm{pr}_j)(A)"'] & \\
& (x_i\mathrm{pr}_i\mathrm{pr}_{ij})^*(A) \arrow[rr, "\mathrm{id}"] & & (x_j\mathrm{pr}_j\mathrm{pr}_{ij})^*(A)
\end{tikzcd}$$

the left and right sides are commutative and hence that the isomorphism $\widetilde{\mathrm{pr}}_{ij}^*(\theta_{ij})$ of (3.1.3) is given by $\widetilde{\mathrm{pr}}_{ij}^*(\theta_{ij}) = c(\mathrm{pr}_j\mathrm{pr}_{ij}, x_j)c(\mathrm{pr}_j\mathrm{pr}_{ij}, x_i)^{-1}$. Similarly,
$\widetilde{\mathrm{pr}}_{jk}^*(\theta_{ij}) = c(\mathrm{pr}_k\mathrm{pr}_{jk}, x_k)c(\mathrm{pr}_j\mathrm{pr}_{jk}, x_j)^{-1}$ and $\widetilde{\mathrm{pr}}_{ik}^*(\theta_{ij}) = c(\mathrm{pr}_k\mathrm{pr}_{ik}, x_k)c(\mathrm{pr}_i\mathrm{pr}_{ik}, x_i)^{-1}$
so that

$$\widetilde{\mathrm{pr}}_{jk}^*(\theta_{ij})\widetilde{\mathrm{pr}}_{ij}^*(\theta_{ij}) = c(\mathrm{pr}_k\mathrm{pr}_{jk}, x_k)c(\mathrm{pr}_j\mathrm{pr}_{jk}, x_j)^{-1}c(\mathrm{pr}_j\mathrm{pr}_{ij}, x_j)c(\mathrm{pr}_j\mathrm{pr}_{ij}, x_i)^{-1} = \widetilde{\mathrm{pr}}_{ik}^*(\theta_{ij})$$

as desired. A similar verification gives normalization and we indeed have a descent datum.

Similarly, it is easy to see that for any arrow $f\colon A \longrightarrow B$ in $\mathbb{F}^c_X$, $x_i^*(f)\colon x_i^*(A) \longrightarrow x_i^*(B)$ gives a morphism of descent data and we have defined a *canonical functor of descent*

$$\mathrm{Desc}_X\colon \mathbb{F}^c_X \longrightarrow \underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)}) \tag{3.3.3}$$

which canonically factorizes the *localization* over the covering *functor*

$$\ell oc_{X_i/X}\colon \mathbb{F}^c_X \longrightarrow \underline{\underline{\underline{\mathrm{Fam}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)}) = \textstyle\prod_{i\in I} \mathbb{F}^c_{X_i} \tag{3.3.4}$$

defined by $A \longmapsto x_i^*(A) = \mathbb{F}^c_X(A)$.

(3.3.5)
$$\begin{tikzcd}
 & \underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)}) \arrow[d, "U \quad \text{(underlying family)}"] \\
\mathbb{F}^c_X \arrow[ur, "\mathrm{Desc}_X"] \arrow[r, "\ell oc_{X_i/X}"'] & \underline{\underline{\underline{\mathrm{Fam}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)})
\end{tikzcd}$$

The study of the properties of this descent functor is the subject of the "theory of descent". We make the following

(3.4) Definitions: A family $(x_i\colon X_i \longrightarrow X)$, $i \in I$ of arrows in $\mathbb{E}$ is said to be a *family of* $\mathbb{F}^c_{(\,)}$ - 0, $\mathbb{F}^c_{(\,)}$ - 1, $\mathbb{F}^c_{(\,)}$ - 2 descent provided that the canonical functor $\mathrm{Desc}_X\colon \mathbb{F}^c_X \longrightarrow \underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)})$ of (3.3.3) is *faithful, fully faithful*, or an *equivalence of categories*, respectively. $\mathbb{F}^c_{(\,)}$ - 1 descent is informally referred to as "*morphisms glue*", while the additional property of essential surjectivity in $\mathbb{F}$ - 2 descent is referred to as "*objects glue*".

If $\mathbb{E}$ is supplied with a topology, then a pseudo functor on $\mathbb{E}$ is called a *pre-stack* (resp. a *stack*) for the topology if every covering family for the topology is a family of $\mathbb{F}^c_{(\,)}$ - 1 (resp. $\mathbb{F}^c_{(\,)}$ - 2) descent, i.e., if both objects and arrows in the fibers over any covering glue.

Description II: (without fibre products)

(4.0) The presence of fiber products in the definition of a pre-topology causes some inconvenience as does the fact that several pre-topologies can have the same set of sheaves. The notion of a topology on a category was first made intrinsic by Giraud (1962) through the notion of a seive.

(4.1) Definition. Let $\mathbb{E}$ be a category. A *seive* of $\mathbb{E}$ is a set $\Re$ of objects of $\mathbb{E}$ which has the property that if $Y \in R$ and $f\colon X \longrightarrow Y$ is in $\mathbb{E}$ then $X \in \Re$. It will be identified with the full subcategory of $\mathbb{E}$ defined by the objects of $\Re$. If X is an object of $\mathbb{E}$, a *seive* of X will be a seive of the category $\mathbb{E}/X$ of objects of $\mathbb{E}$ above X. If $f\colon X \longrightarrow Y$ is an arrow of $\mathbb{E}$ and R is a seive of Y, the inverse image $\Re^f$ of $\Re$ by f will be the seive defined by the set of all $t\colon T \longrightarrow X$ such that $ft\colon T \longrightarrow Y$ is (an object) in $\Re$.

(4.2) Definition. A topology on $\mathbb{E}$ is a mapping which associates to each $X \in \mathrm{Ob}(\mathbb{E})$ a non-empty set $J(X)$ of sieves of X (called the *covering sieves* or the *refinements* of X of the topology) such that

(a) for each $f\colon X \longrightarrow Y$ of $\mathbb{E}$ and each $\Re \in J(Y)$, the seive $\Re^f \in J(X)$;

(b) for each $Y \in \mathrm{Ob}(\mathbb{E})$, each $\Re \in J(Y)$, and each seive $\Re'$ of Y, $\Re' \in J(Y)$ if and only if for any object $f\colon X \longrightarrow Y$ in $\Re$, $\Re'^f \in J(X)$.

From these axioms it follows that the intersection of two covering sieves of X is again a covering seive and that any seive which contains a covering seive is itself covering. (Thus, in particular, the seive $\mathbb{E}/X$ is always covering for any topology and the set of covering sieves is filtering.) As with a pre-topology, a category together with a topology is called a site.

(4.3) The category $\mathbb{E}/X$ is supplied with a canonical functor *source* $\mathcal{S}\colon \mathbb{E}/X \xrightarrow{S} \mathbb{E}$ which sends any $Y \longrightarrow X$ in $\mathbb{E}/X$ to Y. Thus for any presheaf $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$, we have a presheaf $F|X$ on $\mathbb{E}/X$ defined by composition with S^{op} and for any seive $\Re \in \mathbb{E}/X$, a presheaf $F|\Re$ on $\Re$ by composition with the inclusion functor into $\mathbb{E}/X$. Consequently, we have a canonical mapping $\varprojlim(F|X) \longrightarrow \varprojlim(F|\Re)$ which is equivalent to a canonical mapping

$$a\colon F(X) \longrightarrow \varprojlim(F|\Re)\ , \tag{4.4.0}$$

since X is terminal in $\mathbb{E}/X$. We thus can make the following

(4.5) Definition. If $\mathbb{E}$ is a site and $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$ is a presheaf on $\mathbb{E}$, F is said to be a *sheaf* on the site provided that for every covering seive $\Re$ of X the canonical mapping $\Re\colon F(X) \longrightarrow \varprojlim(F|\Re)$ is a bijection.

(4.6) The topologies on a category form a partially ordered set under a refinement. We say that a topology J_1 is *courser than* J_2 (or J_2 refines J_1) if for each X, $J_1(X) \subseteq J_2(X)$

so that if F is a sheaf for J_2 it is *a fortiori* sheaf for J_1. There is a coarsest topology called the *discrete topology* in which the only covering seive of X is $\mathbb{E}/X$ itself. For the discrete topology any presheaf is a sheaf. Similarly there is a finest topology called the *indiscrete topology* in which any seive of X is covering. The only sheaf for the indiscrete topologies is the terminal presheaf $\mathbb{1}$ ($\mathbb{1}(X) = \{\emptyset\} = \mathbb{1}$ for all X). The intersection $\cap_{i\in I} J_i$ of any family (J_i) of topologies (defined by $\cap J_i(X)$ at each X) is again a topology. Consequently, given for each X a set $G(X)$ of sieves of X , there is a coarsest topology for which every seive in $G(X)$ is covering; it is said to be the *topology generated* by G. Any family $(x_i \colon X_i \longrightarrow X)_{i\in I}$ of arrows above X generates a seive whose objects are just those of form $x_i s \colon T \longrightarrow X_i \longrightarrow X$ for some $s \colon T \longrightarrow X_i$ in $\mathbb{E}$. The family is said to be a *covering family* for a given topology provided that the seive generated by the family is a covering seive for the given topology. It thus follows that any *pretopo1ogy generates a topology* in which a seive may be shown to be covering if and only if it contains one generated by some covering family of the pre-topology. Thus, a pre-sheaf F is a sheaf for the topology generated by a pre-topology if and only if it is a sheaf for the pre-topology in the sense of (1.1). The coarsest topology for which the representable presheaves are all sheaves is called the *canonical topology* on $\mathbb{E}$. A covering family (or seive)for the canonical topology is called a *strict universal epimorphic family* (or seive). For any topology courser than the canonical one, every representable functor is a sheaf. For the category of open sets and inclusions of a topological space, it is the canonical Grothendieck topology which is taken to give the sheaves on the space in the classical sense.

(4.7) Just as the notion of a topology on category can be made intrinsic through the notion of a seive, the notion of a gluing and descent datum can be made independent of fiber products through the same concept: If one is given a gluing on a family $X_i \longrightarrow X$, then it is clear that any such gluing extends to give a similar formal definition of gluing isomorphism not just on objects of the fiber $\mathbb{F}_{X_i \times_X X_j}$ but in fact, on the objects of any fiber $\mathbb{F}_S$ provided S has a pair of arrows $s_i \colon S \longrightarrow X_j \qquad s_j \colon S \longrightarrow X_j$ such that $x_i s_i = x_j s_j$. The pair then factors through the fiber product and via inverse image provides an isomorphism of $s_i^*(A_i)$ with $s_j^*(A_j)$. Moreover, any such isomorphism remains one under any further factorization $s \colon V \longrightarrow S$. We are thus led to a definition which is entirely equivalent to (3.3) in the presence of fiber products:

(4.8) <u>Definition</u>: Let $\mathbb{E}$ be a category $F^C_{(\,)} \colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ a normalized pseudo-functor on $\mathbb{E}$, and $(x_i \colon X_i \longrightarrow X)$ a family of arrows of $\mathbb{E}$. By a *gluing of a family* $(A_i)_{i\in I}$ where $A_i \in \mathrm{Ob}(\mathbb{F}_{X_i})$ we shall mean a system of isomorphisms $s_{ij} \colon s_i^*(A_i) \xrightarrow{\sim} s_j^*(A_j)$, $s_{ij} \in \mathrm{Ar}(\mathbb{F}_S)$ (one for each ordered pair $s_i \colon S \longrightarrow X_i$, $s_j \colon S \longrightarrow X_j$ of arrows of $\mathbb{E}$ for which $x_i s_i = x_j s_j$), subject to the following condition

(a_0) for any arrow $s\colon V \longrightarrow S$ in $\mathbb{E}$ the diagram

(4.8.0)
$$\begin{tikzcd}
s^*(s_i^*(A_i)) \arrow[r, "s^*(s_{ij})", "\sim"'] \arrow[d, "c(s{,}s_i)"', "\wr"] & s^*(s_j^*(A_j)) \arrow[d, "c(s{,}s_j)", "\wr"'] \\
(s_i s)^*(A_i) \arrow[r, "v_{ij}", "\wr"'] & (A_j s)^*(A_j)
\end{tikzcd}$$

is commutative in $\mathbb{F}_V$. A gluing will be called a *descent datum on the family* (x_i) provided

(a) for any $(s_i, s_i)\colon S \rightrightarrows X_i$, $s_{ij} = \mathrm{id}\colon s_i^*(A_i) \xrightarrow{\sim} s_i^*(A_i)$; and

(b) for each triplet (i, j, k) in $I \times I \times I$, given any commutative cube in $\mathbb{E}$ of the form

(4.8.1)
$$\begin{tikzcd}
 & S \arrow[r, "s_i"] \arrow[dr, "s_j"] & X_i \arrow[dr, "x_i"] & \\
V \arrow[ur, "s"] \arrow[r, "u"] \arrow[dr, "t"'] & U \arrow[ur, "u_i"] \arrow[dr, "u_k"'] \arrow[r, phantom, "u_k"] & X_j \arrow[r, "x_j"] & X \\
 & T \arrow[r, "t_k"'] \arrow[ur, "t_j"'] & X_k \arrow[ur, "x_k"'] &
\end{tikzcd}$$

the diagram

(4.8.2)

$$\begin{tikzcd}
 & s^*(s_i^*(A_i)) \arrow[rr, "s^*(s_{ij})"] \arrow[d, "c(s,s_i)"'] & & s^*(s_j^*(A_j)) \arrow[d, "c(s,s_i)"] & \\
 & (s_is)^*(A_i) \arrow[d, equal] & & (s_js)^*(A_j) \arrow[d, equal] & \\
 & (u_iu)^*(A_i) \arrow[rr, "\widetilde{s}^*(s_{ij})"] \arrow[ddrr, "\widetilde{u}^*(u_{jk})"] & & (t_jt)^*(A_j) \arrow[dd, "\widetilde{t}^*(t_{jk})"] & t^*(t_j^*(A_j)) \arrow[l] \arrow[dd, "t^*(t_{jk})", "\wr"'] \\
u^*(u_i^*(A_i)) \arrow[ru, "c"] \arrow[rrddd, "u^*(u_{jk})"'] & & & & \\
 & & & (t_kt)^*(A_k) \arrow[d, equal] & t^*(t_k^*(A_k)) \arrow[l, "c"] \\
 & & & (u_ku)^*(A_k) & \\
 & & u^*(u_k^*(A_k)) \arrow[ru, "c"'] & &
\end{tikzcd}$$

is commutative in $\mathbb{F}_V$.

(4.9) Definition: By a *morphism of descent data* (or of a gluing), we shall mean a family of arrows $f_i \in \mathrm{Ar}(\mathbb{F}_{X_i})$, $(f_i\colon A_i \longrightarrow A_i')$ such that the diagram

(4.9.0)

$$\begin{tikzcd}
s_i^*(A_i) \arrow[r, "s_{ij}"] \arrow[d, "s_i^*(f_i)"'] & s_j^*(A_j) \arrow[d, "s_j^*(f_j)"] \\
s_i^*(A_i') \arrow[r, "s_{ij}"] & s_j^*(A_j)
\end{tikzcd}$$

is commutative in $\mathbb{F}_S$ for each i, j. We thus have defined the category

(4.9.1)
$$\boxed{\underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}\big(X_i/X;\, \mathbb{F}^c_{(\,)}\big)}$$

of *descent data for the psuedo functor* $\mathbb{F}^c_{(\,)}$ *relative to the family* $(x_i\colon X_i \longrightarrow X)_{i\in I}$ *of arrows of* $\mathbb{E}$.

If we define the category $\underline{\underline{\underline{\mathrm{Fam}}}}_{\mathbb{E}}\big(X_i, X;\, \mathbb{F}^c_{(\,)}\big)$ of families of objects above, the family $(x_i)_{i\in I}$ as the disjoint union of the categories $\mathbb{F}^c_{X_i}$, $x_i\colon X_i \longrightarrow X$ then $\underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}\big(X_i/X;\, \mathbb{F}^c_{(\,)}\big)$

has a canonical a *faithful underlying glued object functor*

(4.9.2)
$$\boxed{U\colon \underline{\underline{\mathrm{Desc}}}_{\mathbb{E}}\big(X_i/X;\, \mathbb{F}^c_{(\,)}\big) \longrightarrow \underline{\underline{\mathbb{F}\mathrm{am}}}_{\mathbb{E}}\big(X_i/X;\, \mathbb{F}^c_{(\,)}\big)}$$

defined in the obvious manner where we view a family of objects above the X_i as being supplied with the "essentially algebraic" structure of a descent datum by a system of isomorphisms s_{ij} which satisfy the (equational) "cocycle condition" for a descent datum.

(4.10) *Localization and descent on a family,* $x_i\colon X_i \longrightarrow X$, $i \in I$. If A is an object in F^c_X then the "localized" family of objects $(x_i^*(A))_{i\in I}$ $\mathbb{F}^c_{x_i}(A) = x_i^*(A) \in \mathrm{Ob}(\mathbb{F}^c_{X_i})$ is supplied canonically with a gluing since given any commutative square in $\mathbb{E}$,

(4.10.0)
$$\begin{tikzcd}
& X_i \times_X X_j \arrow[dl, "s_i"'] \arrow[dr, "s_j"] \arrow[dd, "v"] & \\
X_i \arrow[dr, "x_i"'] & & X_j \arrow[dl, "x_j"] \\
& X &
\end{tikzcd}$$

(with common diagonal v), we may define a gluing isomorphism θ_{ij} as that unique isomorphism which makes the diagram

(4.10.1)
$$\begin{tikzcd}
\mathrm{pr}_1^*(x_i^*(A)) \arrow[r, "s_{ij}", "\sim"'] \arrow[d, "c(s_i,x_i)(A)"', "\wr"] & s_j^*(x_j^*(A)) \arrow[d, "\wr"', "c(s_j,x_j)(A)"] \\
(x_i s_i)^*(A) \arrow[d, equal] & (x_j s_j)^*(A) \arrow[d, equal] \\
v^*(A) \arrow[r, "\mathrm{id}"] & v^*(A)
\end{tikzcd}$$

commutative in $\mathbb{F}^c_S$

This gluing is a descent datum on the family $x_i^*(A)$, $i \in I$. In effect, for any commutative cube of the form (4.8.1), the axiom (2.0.1) for the cleavage $c(-,-)$ of $\mathbb{F}^c_{(\,)}$ declares that in the cube

(4.10.2)

$$\begin{tikzcd}
s(s_i^*(x_i^*(A))(A) \arrow[rr, "s^*(s_{ij})"] \arrow[dr, "c(s,s_i)^* * x_i^*(A)"] \arrow[dd, "s * c(s_i,x_i)"'] & & s(s_j^*(x_j^*(A))) \arrow[dr, "c(s,s_j) * (x_j^*(A))"] \arrow[dd, "s^* * c(s_j,x_j)"'] & \\
 & (s_i s)^*(x_i^*(A)) \arrow[rr, crossing over, "\widetilde{s}^*(s_{ij})"] & & (s_j s)^*(x_j^*(A)) \arrow[dd, "c(s_j s,x_j)(A)"] \\
s_j^*(x_i s_i)^*(A) \arrow[rr, "\mathrm{id}"] \arrow[dr, "c(s_j,x_i s_i)(A)"'] & & s^*(x_j s_j)^*(A) \arrow[dr, "c(s,x_j s_j)(A)"'] & \\
 & (x_i s_i s)^*(A) \arrow[rr, "\mathrm{id}"] \arrow[from=uu, crossing over, "c(s_i s,x_i)(A)"] & & (x_j s_j s)^*(A)
\end{tikzcd}$$

the left and right sides are commutative and hence that the isomorphism $\widetilde{s}^*(s_{ij})$ of (5.3.3) is given by $\widetilde{s}^*(s_{ij}) = c(s_j s, x_j) \cdot c(s_j s, x_i)^{-1}$. Similarly,
$\widetilde{t}^*(s_{ij}) = c(t_k t, x_k) \cdot c(t_j t, x_j)^{-1}$ and $\widetilde{u}^*(s_{ij}) = c(u_k u, x_k) \cdot c(u_i u, x_i)^{-1}$
so that

$$\widetilde{u}^*(s_{ij}) = c(t_k t, x_k)c(t, tx_j)^{-1}c(s_j s, x_j)c(s_i s, x_i)^{-1} = \widetilde{t}^*(s_{ij})\widetilde{s}^*(s_{ij})$$

as claimed. Similarly, the axiom (2.0.1) for the cleavage gives the identity condition (a) and we have indeed defined a descent datum. Similarly, it is easy to verify for any arrow $f\colon A \longrightarrow B$ in $\mathbb{F}^c_X$, $x_i^*(f)\colon x_i^*(A) \longrightarrow x_i^*(B)$, defines a morphism of the corresponding descent data and we thus have a *canonical functor*

$$\boxed{\mathrm{Desc}_X \colon \mathbb{F}^c_X \longrightarrow \underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)})} \tag{4.10.3}$$

which canonically factorizes the *localization functor*

$$\mathbb{F}^c_X \longrightarrow \underline{\underline{\underline{\mathrm{Fam}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)})$$

defined by $A \longmapsto x_i^*(A) = \ell oc_{X_i/X}$.

(5.0) With the fiber product free definition of the category $\underline{\underline{\underline{\mathrm{Desc}}}}_{\mathbb{E}}(X_i/X;\, \mathbb{F}^c_{(\,)})$ the definitions of a family being of $\mathbb{F}^c_{(\,)}$-descent (or effective descent) may be taken as identical to those given in (3.4). Similarly, the property on a pseudo functor being a stack with respect to a topology may be taken as unchanged (and we can even add the notion of the *topology* $\mathbb{F}^2_1$*, descent (or effective descent)* as that generated by the families of $\mathbb{F}^c_{(\,)}$-descent (or effective descent)). However, it is more interesting to observe that these definitions are from a more sophisticated point of view really “2-dimensional versions” of the notion of sheaf as defined in (4.5). This is done through the definition of a *pseudo-limit of a pseudo-functor*.

(5.1) Let $F^c_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ be a normalized pseudo-functor on $\mathbb{E}$. By a (*normalized pseudo-*) *cone* over $F^c_{(\,)}$ we will mean a category $\mathbb{C}$ together with the following data:

(i) for each object X in $\mathbb{E}$, we are given a functor $P_X\colon \mathbb{C} \longrightarrow F^c_X$,

(ii) for each arrow $f\colon X \longrightarrow Y$, we are given a natural transformation $\alpha_f\colon P_X \longrightarrow \mathbb{F}^c_f \cdot P_Y$,

(5.1.0)

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, "P_Y"'] \arrow[dr, "P_X"] & \\
\mathbb{F}_Y \arrow[rr, "\mathbb{F}_f"'] & \arrow[phantom, "\overset{\alpha_f}{\curvearrowleft}"] & \mathbb{F}_X
\end{tikzcd}$$

subject to the following coherence conditions

(a) $\alpha_{\mathrm{id}(X)} = \mathrm{id}(P_X)$ for all objects X in $\mathbb{E}$ (normalization);

(b) for each commutative triangle

(5.1.1)

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

in $\mathbb{E}$, the tetrahedron

(5.1.2)

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, "P_Z"'] \arrow[dr, "P_Y"] \arrow[dd, "P_X"' near start] & \\
\mathbb{F}_Z \arrow[rr, "(f^{-1}{,}g)"'] \arrow[dr, "\mathbb{F}_{fg}"'] & \overset{\alpha_{gf}}{\curvearrowleft} \quad \overset{\alpha_f}{\curvearrowright} & \mathbb{F}_Y \arrow[dl, "\mathbb{F}_f"] \\
& \mathbb{F}_X &
\end{tikzcd}$$

is "commutative" i.e., the canonically associated square

(5.1.3)
$$\begin{tikzcd}
P_X \arrow[r, "\alpha_f"] \arrow[d, "\alpha_{gf}"'] & \mathbb{F}_F P_Y \arrow[d, "\mathbb{F}_f * \alpha_g"] \\
\mathbb{F}_{gf} P_Z \arrow[r, "c(f{,}g)^{-1} * P_2"'] & \mathbb{F}_f \mathbb{F}_g P_Z
\end{tikzcd}$$

of functors and natural isomorphisms is commutative.

By a psuedo-limit $\varprojlim \mathbb{F}^c_{(\,)}$ of a pseudo-functor $\mathbb{F}^c_{(\,)}$ on $\mathbb{E}$, we shall mean an essentially universal cone over $\mathbb{F}$ in the obvious sense that any cone over $\mathbb{F}_{(\,)}$ factors (in CAT) essentially uniquely through $\varprojlim \mathbb{F}^c_{(\,)}$. It is thus unique up to an essentially unique equivalence. Moreover, as in the case of ordinary limits in sets, $\varprojlim \mathbb{F}^c_{(\,)}$ <u>exists</u> and may be taken to be the following category:

(5.2) The objects of $\varprojlim \mathbb{F}^c_{(\,)}$ are ordered pairs $\big((A_X)_{X \in \mathrm{Ob}(\mathbb{E})}\,,\,(\theta_f)_{f \in \mathrm{Ar}(\mathbb{E})}\big)$ of objects and isomorphisms for which $A_X \in \mathbb{F}^c_X$ for each X, $\theta_f \colon A_X \xrightarrow{\sim} \mathbb{F}_f(A_Y)$ for each $f \colon X \longrightarrow Y$ and which satisfy the two conditions:

(a) $\theta_{\mathrm{id}} = \mathrm{id}$, and

(b) for each composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{E}$, the square

(5.2.0)
$$\begin{tikzcd}
A_X \arrow[r, "\theta_f", "\sim"'] \arrow[d, "\theta_{gf}"', "\wr"] & \mathbb{F}_f(A_Y) \arrow[d, "\mathbb{F}_f(\theta_g)", "\wr"'] & A_Y \arrow[d, "\theta_g"] \\
\mathbb{F}_{gf}(A_Z) \arrow[r, "\sim", "c(f{,}g)^{-1}"'] & \mathbb{F}_f(\mathbb{F}_g(A_Z)) & \mathbb{F}_g(A_Z)
\end{tikzcd}$$

is commutative in the fiber $\mathbb{F}^c_X$.

The arrows of $\varprojlim \mathbb{F}^c_{(\,)}$ are families $(a_X \colon A_X \longrightarrow A'_X)_{X \in \mathrm{Ob}(\mathbb{E})}$ such that the square

(5.2.1)
$$\begin{tikzcd}
A_X \arrow[r, "\theta_f", "\sim"'] \arrow[d, "a_X"'] & \mathbb{F}_f(A_Y) \arrow[d, "\mathbb{F}_f(a_Y)"] \\
A'_X \arrow[r, "\sim", "\theta'_f"'] & \mathbb{F}_f(A'_Y)
\end{tikzcd}$$

is commutative for each $f \colon X \longrightarrow Y$ in $\mathbb{E}$.

The canonical projection functors $\mathbb{P}_X \colon \varprojlim \mathbb{F}^c_{(\,)} \longrightarrow \mathbb{F}_X$ now furnish a universal cone over $\mathbb{F}^c_{(\,)}$ when coupled with the isomorphism $\theta_f \colon A_X \longrightarrow \mathbb{F}_f(A_Y)$ which furnish the natural isomorphisms

(5.2.2)

$$\begin{tikzcd}
 & \varprojlim \mathbb{F}^c_{(\,)} \arrow[dl, "\mathbb{P}_Y"'] \arrow[dr, "\mathbb{P}_X"] & \\
\mathbb{F}^c_Y \arrow[rr, "\mathbb{F}^c_f"'] & \arrow[d, phantom, "\swarrow \theta_f"] & \mathbb{F}^c_X \\
 & {} &
\end{tikzcd}$$

(5.3) Notice with this definition of $\varprojlim \mathbb{F}^c_{(\,)}$, even if $F^c_{(\,)}$ is a *functor* into CAT (so that we have a presheaf of categories), one only has an inclusion $\varprojlim \mathbb{F}_{(\,)} \hookrightarrow \varprojlim \mathbb{F}_{(\,)}$ of the ordinary limit of categories since this corresponds to those families $\theta_f \colon A_X \xrightarrow{\sim} \mathbb{F}_f(A_Y)$ which are *identities* for each $f \colon X \longrightarrow Y$ in $\mathbb{E}$, if $\mathbb{F}_X$ is a discrete category, (so that the pseudo-functor $F_{(\,)} \colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ is just a presheaf of sets), is it the case that $\varprojlim \mathbb{F}^c_{(\,)}$ reduces bijectively to the ordinary $\varprojlim \mathbb{F}$ as usually defined.

If $\mathbb{E}$ has a terminal object $\mathbb{1}$, then $\varprojlim \mathbb{F}^c_{(\,)}$ is equivalent to the category $\mathbb{F}^c_{\mathbb{1}}$, the fiber above the terminal object, since for each $X \in \mathrm{ob}(\mathbb{E})$, the canonical map $t_X \colon X \longrightarrow \mathbb{1}$ gives rises to the functor $P_X = \mathbb{F}^c_{t_X} \colon \mathbb{F}^c_{\mathbb{1}} \longrightarrow \mathbb{F}^c_X$, and for each $f \colon X \longrightarrow Y$ in $\mathbb{E}$, the cleavage isomorphism $c(f, t_Y)^{-1}$ gives rise to the universal cone

(5.3.0)

$$\begin{tikzcd}
 & \mathbb{F}_{\mathbb{1}} \arrow[dl, "P_X"'] \arrow[dr, "P_Y"] & \\
\mathbb{F}_Y \arrow[rr, "\mathbb{F}_f"'] & \arrow[d, phantom, "\swarrow"] & \mathbb{F}_X \\
 & {} &
\end{tikzcd}$$

which makes $\mathbb{F}_{\mathbb{1}} \xrightarrow{\cong} \varprojlim \mathbb{F}^c_{(\,)}$.

(5.4) Using this notion of a pseudo-limit, we now claim that the property of completeness of a fibered category becomes a 2-dimensional generalization of the property of being a sheaf which we recall was that for each covering seive $\mathfrak{R} \hookrightarrow \mathbb{E}/X$, the canonical map

$$F(X) \xrightarrow{\sim} \varprojlim (F|\mathbb{E}/X) \longrightarrow \varprojlim F|\mathfrak{R}$$

was a bijection. Here we start with a pseudo-functor $\mathbb{F}^c_{(\,)} \colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ and a seive $\mathfrak{R} \hookrightarrow \mathbb{E}/X$. Using the canonical source functor $S \colon \mathbb{E}/X \longrightarrow \mathbb{E}$ we may compose with $F_{(\,)}$ and the inclusion $i_{\mathfrak{R}} \colon \mathfrak{R}^{\mathrm{op}} \hookrightarrow \mathbb{E}/X^{\mathrm{op}} \longrightarrow \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ to obtain pseudo-functors $\mathbb{F}^c_{(\,)}|X \colon \mathbb{E}/X^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ and $\mathbb{F}^c_{(\,)}|\mathfrak{R} \colon \mathfrak{R}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$. We then obtain in CAT a canonical restriction *functor*

$$L\colon \varprojlim \mathbb{F}^c_{(\,)}|X \longrightarrow \varprojlim \mathbb{F}^c_{(\,)}|\Re \tag{5.4.0}$$

and an equivalence $\mathbb{F}^c_{(\,)} \xrightarrow{\approx} \varprojlim \mathbb{F}^c_{(\,)}|X$ since X is terminal in $\mathbb{F}/X$. We now claim that *we have a canonical equivalence of categories.*

$$\underline{\underline{\mathrm{Desc}}}_{\mathbb{E}}\big(X_i/X;\mathbb{F}^c_{(\,)}\big) \xrightarrow{\approx} \varprojlim\big(\mathbb{F}^c_{(\,)}|\Re\big)\,, \tag{5.4.1}$$

where the family $(x_i\colon X_i \longrightarrow X)_{i\in I}$ consists of the family of objects of the seive $\Re$ (or for that matter any family which generates the seive $\Re$ since any descent datum on such a generating family clearly extends via inverse images to a descent datum on the entire seive) *such that the square*

(5.4.2)
$$\begin{tikzcd}
\mathbb{F}^c_X \arrow[r, "\ell oc_X"] \arrow[d, "\wr"'] & \underline{\underline{\mathrm{Desc}}}_{\mathbb{E}}\big(X_i/X;\mathbb{F}^c_{(\,)}\big) \arrow[d, "\wr"] \\
\varprojlim \mathbb{F}^c_{(\,)}|X \arrow[r] & \varprojlim \mathbb{F}^c_{(c)}|\Re
\end{tikzcd}$$

is essentially commutative.

This fact is easily seen once we observe that any commutative tetrahedron

(5.4.3)
$$\begin{tikzcd}
X_i \arrow[dr, "x_{ij}"] \arrow[drr, "x_i"] \arrow[dd, "x_{ik}"'] & & \\
& X_j \arrow[r, "x_j"] \arrow[dl, "x_{jk}"'] & X \\
X_k \arrow[urr, "x_k"'] & &
\end{tikzcd}$$

in $\mathbb{E}$ gives rise to, and is in fact equivalent to, a commutative cube of the form

(5.4.4)
$$\begin{tikzcd}
& X_i \arrow[r, "\mathrm{id}"] \arrow[dr, "x_{ij}"] & X_i \arrow[dr, "x_i"] & \\
X_i \arrow[ur, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] \arrow[dr, "x_{ij}"'] & X_i \arrow[ur, "\mathrm{id}"] \arrow[dr, "x_{ik}"] & X_j \arrow[r, "x_j"] & X \\
& X_j \arrow[ur, "\mathrm{id}"] \arrow[r, "x_{jk}"'] & X_k \arrow[ur, "x_k"'] &
\end{tikzcd}$$

Thus any descent datum on a family $x_i\colon X_i \longrightarrow X$, $i \in I$ when applied to the cube gives rise to isomorphisms $a_{ij}\colon A_i = \mathrm{id}^*(A_i) \xrightarrow{\sim} x^*_{ij}(A_j)$, $a_{jk}\colon A_j \xrightarrow{\sim} x^*_{jk}(A_k)$, and $a_{ik}\colon A_i \xrightarrow{\sim} x^*_{ik}(A_k)$, such that the diagram

(5.4.5)
$$\begin{tikzcd}
A_i \arrow[r, "a_{ij}", "\sim"'] \arrow[rd, "a_{ik}"', "\sim"] & x^*_{ij}(A_j) \arrow[d, "\widetilde{x}^*_{ij}(a_{jk})", "\wr"'] \arrow[rd, "x^*_{ij}(a_{jk})", "\sim"'] & \\
& x^*_{ik}(A_k) & x^*_{ij}(x^*_{jk}(A_k)) \arrow[l, "\sim", "c(x_{ij},x_{jk})"']
\end{tikzcd}$$

is commutative in $\mathbb{F}^c_{X_i}$, i.e. to an object of $\varprojlim \mathbb{F}^c_{(\,)}|\Re$.

Similarly any morphism of descent defines an arrow between the corresponding objects of $\varprojlim \mathbb{F}^c_{(\,)}|\Re$. Clearly this assignment is functorial and, in fact, fully faithful. It is also essentially surjective since given any object $((A_X),(\theta_f))$ in $\varprojlim \mathbb{F}^c_{(\,)}|\Re$ and any commutative square

(5.4.6)
$$\begin{tikzcd}
& S \arrow[ld, "s_i"'] \arrow[rd, "s_j"] \arrow[dd, "v"] & \\
X_i \arrow[rd, "x_i"'] & & X_j \arrow[ld, "x_j"] \\
& X &
\end{tikzcd}$$

with common diagonal v, we have isomorphisms

$$\theta_{s_j}\colon A_S \xrightarrow{\sim} \mathbb{F}_{s_j}(A_{x_j}) \quad \text{and} \quad \theta_{s_i}\colon A_S \xrightarrow{\sim} \mathbb{F}_{s_i}(A_{x_i})$$

which have the property that the gluing

(5.4.7)
$$\theta_{ij} = \theta_{s_j} \cdot \theta^{-1}_{s_i}\colon \mathbb{F}_{s_i}(A_{X_i}) \longrightarrow \mathbb{F}_{s_j}(A_{X_j})$$

is a descent datum over the family $(X_i \longrightarrow X)$. We leave the proof of this and the commutativity of (5.4.2) to the reader.

In summary we have now established our assertion:

(5.5) <u>Theorem</u>. Let $\mathbb{E}$ be a site and $\mathbb{F}^c_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$. $\mathbb{F}^c_{(\,)}$ is a pre-stack (i.e. is pre-complete), respectively is a stack (i.e. is complete) if and only if for each covering seive $\Re \hookrightarrow \mathbb{E}/X$, the canonical functor

$$\mathbb{F}^c_X \xrightarrow{\approx} \varprojlim \mathbb{F}^c_{(\,)}|X \longrightarrow \varprojlim \mathbb{F}^c_{(\,)}|\Re$$

is fully faithful, respectively, is an equivalence of categories.

(5.6) <u>Corollary</u>. If $\mathbb{F}^c_{(\ )}$ is a functor so that $\mathbb{F}^c_{(\ )}$ is a category object in $\mathbb{E}\hat{}$ then $\mathbb{F}^c_{(\ )}$ is a pre-stack if and only if the canonical functor $\mathbb{F}^c_{(\ )} \longrightarrow a\mathbb{F}^c_{(\ )}$ to the associated sheaf of categories is fully faithful, while if it is fibered in discrete categories, so that $\mathbb{F}^c_{(\ )}$ is an ordinary presheaf, then $\mathbb{F}^c_{(\ )}$ is a stack if and only if it is a sheaf.

Description III: (in $\mathrm{CAT}/\mathbb{E}$)

(6.0) The preceding description of topologies, sheaves and stacks, although satisfactory from an intuitive point of view is, as we have seen, extremely cumbersome to write down. A more elegant formulation of each of these concepts may be provided through the concept of a "Grothendieck fibration" which we will motivate by first examining it in the "discrete case".

(6.1) If $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$ is a pre-sheaf of sets, the usual construction of the co-limit of F describes it is as the co-equalizer of the diagram of sets

(6.1.0)
$$\begin{tikzcd}
\coprod\limits_{f\in \mathrm{Ar}(\mathbb{E})} F(T(f)) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \coprod\limits_{X\in \mathrm{Ob}(\mathbb{E})} F(X)
\end{tikzcd}$$

in which $d_0(y,f) = (y, T(f))$ and $d_1(y,f) = (F(f)(y), S(f))$. This diagram is actually the set of arrows and objects of a category which (for reasons which will later become clear) we will denote by $\mathbb{E}/F$ and amounts to nothing more than viewing an element (y,f) of $\coprod_{f\in \mathrm{Ar}(\mathbb{E})} F(T(f))$ as defining an arrow $(y,f) : (F(f)(y), S(f)) \longrightarrow (y, T(f))$ and composing such arrows in the obvious fashion using the functor F. Moreover, if we place the canonical map $(y,f) \longmapsto f$ and $(x,X) \longmapsto X$ into the picture

(6.1.1)
$$\begin{tikzcd}
\coprod\limits_{f\in \mathrm{Ar}(\mathbb{E})} F(T(f)) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "f_1"'] & \coprod\limits_{X\in \mathrm{Ob}(\mathbb{E})} F(X) \arrow[d, "f_0"] \\
\mathrm{Ar}(\mathbb{E}) \arrow[r, shift left, "T"] \arrow[r, shift right, "S"'] & \mathrm{Ob}(\mathbb{E})
\end{tikzcd} ,$$

we see that we have defined a canonical functor

(6.1.2)
$$s\colon \mathbb{E}/F \longrightarrow \mathbb{E}$$

which has the property that the square

(6.1.3)
$$\begin{tikzcd}
\mathrm{Ar}(\mathbb{E}/F) \arrow[r, "T"] \arrow[d, "s_1"'] & \mathrm{Ob}(\mathbb{E}/F) \arrow[d, "s_0"] \\
\mathrm{Ar}(\mathbb{E}) \arrow[r, "T"'] & \mathrm{Ob}(\mathbb{E})
\end{tikzcd}$$

is cartesian. We will call any functor $F\colon \mathbb{F} \longrightarrow \mathbb{E}$ which satisfies this latter property (6.1.3) a *discrete functor* and when viewing it as an object in CAT/$\mathbb{E}$, a *discrete fibration.* The discrete fibration $s\colon \mathbb{E}/F \longrightarrow \mathbb{E}$ (6.1.2) will be called the discrete fibration associated with the presheaf $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$. The full subcategory of CAT/$\mathbb{E}$ whose objects are the discrete fibrations will be denoted by <u>Disc Fib</u>/$\mathbb{E}$.

(6.2) <u>Theorem</u>. The following categories are equivalent:

$$\underline{\mathrm{OPER}}_{\mathrm{ENS}}(\mathbb{E}) \xrightarrow{\approx} \underline{\mathrm{Disc\ Fib}}/\mathbb{E} \xrightarrow{\approx} \mathbb{E}\hat{\ } \left(= \underline{\mathrm{Hom}}_{\mathrm{CAT}}(\mathbb{E}^{\mathrm{op}}, \mathrm{ENS}) \;=\; \mathrm{ENS}^{\mathbb{E}^{\mathrm{op}}}\right).$$

The category $\underline{\mathrm{OPER}}_{\mathrm{ENS}}(\mathbb{E})$ is the familiar category of $\mathbb{E}$-sets on which the category $\mathbb{E}$ operates (on the right) which generalizes the notion of a monoid operating on a set. Its objects are functions $S \xrightarrow{F_0} \mathrm{Ob}(\mathbb{E})$ together with an action $\alpha\colon S_{F_0} \times_T \mathrm{Ar}(\mathbb{E}) \longrightarrow S$ $(\alpha(x, f\colon X \to F(x)) = x^f)$ such that the conditions

$$\text{(a)}\ \ F(x^f) = X; \quad \text{(b)}\ \ x^{\mathrm{id}} = x; \quad \text{(c)}\ \ (x^f)^g = x^{fg}$$

all hold. The morphisms of <u>OPER</u>($\mathbb{E}$) are equivariant maps, i.e. maps $N\colon S \longrightarrow S'$ above Ob($\mathbb{E}$) such that $N(x^f) = (N(x))^f$.

The equivalence of the first two categories is clear. In effect, given an operation of $\mathbb{E}$ on $S \xrightarrow{F} \mathrm{Ob}(\mathbb{E})$, the projection of $S_F \times_T \mathrm{Ar}(\mathbb{E})$ onto S and the action α from the, respective target and source mappings of category $((x, f)\colon x^f \longrightarrow x)$ for which the functions F and the projection to Ar($\mathbb{E}$) form a discrete functor.

The equivalence of discrete fibrations and presheaves in one direction is already established with the construction of (6.1). The other direction is equally clear. For given any $S \xrightarrow{F} \mathrm{Ob}(\mathbb{E})$ on which $\mathbb{E}$ operates, the assignment $X \longmapsto S_X = F^{-1}(X)$ and $f\colon X \to Y \longmapsto S_f \;=\; (y \mapsto y^f)\colon S_Y \to S_X$ defines a presheaf on $\mathbb{E}$ for which the canonical map $\coprod_{X \in \mathrm{Ob}(\mathbb{E})} S_X \to S$ is a bijection above Ob($\mathbb{E}$). The remainder of the (easy) details are left to the reader.

(6.3) <u>Remark</u>. The equivalence of <u>OPER</u>($\mathbb{E}$) and <u>Disc Fib</u>/$\mathbb{E}$ obviously holds in any category $\mathbb{C}$ with fiber products for any category object $\mathbb{E}$ in $\mathbb{C}$ since all notions may be defined internally in $\mathbb{C}$. As a result, the category $\underline{\mathrm{OPER}}_{\mathbb{C}}(\mathbb{E})$ plays the role of the "internal presheaves on $\mathbb{E}$" and is often denoted by $\mathbb{C}^{\mathbb{E}^{\mathrm{op}}}$ and is called the *category of internal presheaves on* $\mathbb{E}$. If $\mathbb{C}$ is a topos, then so is $\mathbb{C}^{\mathbb{E}^{\mathrm{op}}}$ and this topos plays the same role as $\mathbb{C}\hat{\ }$ does in the set based theory.

The usefulness of $\mathbb{E}/F \longrightarrow \mathbb{E}$ in the definition of a topology on $\mathbb{E}$ and the corresponding notion of a sheaf is based on the following facts;

(6.4) <u>Lemma</u>.

(a) If $P\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$ is a pre-sheaf and $F\colon \mathbb{E}' \longrightarrow \mathbb{E}$ is a functor then the commutative square

(6.4.0)
$$\begin{tikzcd}
\mathbb{E}'/P \cdot F^{\mathrm{op}} \arrow[r] \arrow[d] & \mathbb{E}/P \arrow[d] \\
\mathbb{E}' \arrow[r, "F"'] & \mathbb{E}
\end{tikzcd}$$

is cartesian, i.e. "restriction of presheaves" ($P \longmapsto P \cdot F^{\mathrm{op}}$) corresponds to the pull-back of that associated discrete fibration;

(b) For any object X in $\mathbb{E}$ the functor $S\colon \mathbb{E}/X \longrightarrow \mathbb{E}$ is discrete, with $\mathbb{E}/X \xrightarrow{\sim} \mathbb{E}/hX$, where h_X is the representable pre-sheaf defined by X; $\mathrm{id}\colon \mathbb{E} \longrightarrow \mathbb{E}$ is discrete with $\mathbb{E} \xrightarrow{\sim} \mathbb{E}/\mathbb{1}$ where $\mathbb{1}$ is the terminal presheaf, $\mathbb{1}(X) = \{\phi\}$ for all X.

(c) A subcategory $\Re \hookrightarrow \mathbb{E}/X$ is a seive on X if and only if the inclusion functor is discrete;

(d) If P is a presheaf on $\mathbb{E}$, then one has a canonical bijection

(6.4.1)
$$\varprojlim P \xrightarrow{\sim} \Gamma_{\mathbb{E}}(\mathbb{E}/P)$$

where $\Gamma_{\mathbb{E}}(\mathbb{E}/P)$ is the set of functorial sections of $\mathbb{E}/P \longrightarrow \mathbb{E}$ of $\mathbb{E}/P$ over $\mathbb{E}$; i.e. the set of functors $\ell\colon \mathbb{E} \longrightarrow \mathbb{E}/P$ such that the diagram

(6.4.2)
$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\ell"] \arrow[dr, "\mathrm{id}"'] & \mathbb{E}/P \arrow[d, "s"] \\
 & \mathbb{E}
\end{tikzcd}$$

is commutative.

(a) through (c) are elementary verifications. The verification of (d) is also immediate if one takes as the definition of $\varprojlim P$ the equalizer

(6.4.3)
$$\varprojlim P \hookrightarrow \prod_{X \in \mathrm{Ob}(\mathbb{E})} F(X) \underset{P_1}{\overset{P_0}{\rightrightarrows}} \prod_{f \in \mathrm{Ar}(\mathbb{E})} F(S(f))$$

where $P_0((x_X))_{X \in \mathrm{Ob}(\mathbb{E})} = (x_{S(f)})_{f \in \mathrm{Ar}(\mathbb{E})}$ and $P_1((x_X))_{X \in \mathrm{Ob}(\mathbb{E})} = (F(f)(x_{T(f)}))_{f \in \mathrm{Ar}(\mathbb{E})}$ and notes that for any family of sets $(X_i)_{i \in I}$ there is a canonical bijection

(6.4.4)
$$\prod_{i \in I} X_i \xrightarrow{\sim} \Gamma_I\Big(\coprod X_i\Big)$$

since given any $s \in \coprod_{i\in I} X_i = \{s\colon I \to \cup_{i\in I} X_i \mid s(i) \in X_i \text{ for all } i \in I\}$, $\tilde{s}\colon I \longrightarrow \coprod_{i\in I} X_i$ defined by $\tilde{s}(t) = (s(t), t)$ is a section of the canonical map $\coprod_{i\in I} X_i \longrightarrow I$ defined by $(x,t) \mapsto t$. Thus here a section $\tilde{s}\colon \mathrm{Ob}(\mathbb{E}) \longrightarrow F(X)_{X\in\mathrm{Ob}(\mathbb{E})}$ of the canonical map $s_0\colon \coprod_{X\in\mathrm{Ob}(\mathbb{E})} F(X) \longrightarrow \mathrm{Ob}(\mathbb{E})$ is equivalent to an element $s \in \prod_{X\in\mathrm{Ob}(\mathbb{E})} F(X)$ and is a functorial section if and only if the map $\tilde{s}^{\#}\colon \mathrm{Ar}(\mathbb{E}) \longrightarrow \coprod_{f\in I} F(T(f))$ defined by $\tilde{s}^{\#}(f) = \tilde{s}(T(f))$ satisfies $d_i\tilde{s}^{\#} = \tilde{s}S$, i.e. if and only if $s \in \varprojlim P$.

(6.5) Using the Lemma (6.4), it is now immediate that a *Grothendieck topology on* $\mathbb{E}$ *may be viewed as a function which assigns to any* $X \in \mathrm{Ob}(\mathbb{E})$ *a non-empty set* $J(X)$ *of discrete subfunctors of* $\mathbb{E}/X$ *such that*

(a) for any $f\colon X \to Y$ and any $\Re \in J(Y)$, the discrete subfunctor $\Re^f$ defined by the cartesian square

$$\begin{tikzcd}
\Re^f \arrow[r] \arrow[d, hook] & \Re \arrow[d, hook] \\
\mathbb{E}/X \arrow[r, "\mathbb{E}/f"'] & \mathbb{E}/Y
\end{tikzcd}$$

(b) for any $Y \in \mathrm{Ob}(\mathbb{E})$, any $\Re \in J(Y)$, a discrete subfunctor $\Re'$ of $\mathbb{E}/X$ is a member of $J(Y)$ if and only if for each object $f\colon X \to Y$ in $\Re$, $\Re'^f \in J(X)$.

The corresponding property of a pre-sheaf being a sheaf may now be translated into the property of "completeness" of the discrete fibration associated with the presheaf:

(6.6) Definition. A discrete fibration $F\colon \mathbb{F} \longrightarrow \mathbb{E}$ will be said to be complete with respect to a topology J on $\mathbb{E}$ provided that for every $X \in \mathrm{Ob}(\mathbb{E})$ and every discrete subfunctor $\Re \hookrightarrow \mathbb{E}/X$ which is a member of $J(X)$, the canonical map

$$\mathrm{Hom}_{\mathrm{CAT}/\mathbb{E}}(\mathbb{E}/X, \mathbb{F}) \longrightarrow \mathrm{Hom}_{\mathrm{CAT}/\mathbb{E}}(\Re, \mathbb{F}) \tag{6.6.0}$$

(defined by restriction along the discrete inclusion $\Re \hookrightarrow \mathbb{E}/X$) is a bijection.

(6.7) Theorem. Let $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow \mathrm{ENS}$ be a pre-sheaf, then F is a sheaf if and only if the associated discrete fibration $\mathbb{E}/F \xrightarrow{s} \mathbb{E}$ is complete. In effect, from the preceding observations, we have bijections and commutativity of

(6.7.0)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{CAT}/\mathbb{E}}(\mathbb{E}/X, \mathbb{E}/F) \arrow[r, "\sim"] \arrow[d] & \Gamma_{\mathbb{E}/X}(\mathbb{E}/X\big/FS^{\mathrm{op}}) \arrow[r, "\sim"] & \varprojlim(FS^{\mathrm{op}}) \arrow[r, "\sim"] & \varprojlim(F|X) \cong F(X) \arrow[d] \\
\mathrm{Hom}_{\mathrm{CAT}/\mathbb{E}}(\Re, \mathbb{E}/F) \arrow[r, "\sim"] & \Gamma_{\Re}(\Re/\mathrm{in} \quad S^{\mathrm{op}}F) \arrow[r, "\sim"] & \varprojlim(FS^{\mathrm{op}} \quad \mathrm{in}_{\Re}^{\mathrm{op}}) \arrow[r, "\sim"] & \varprojlim(F|\Re)
\end{tikzcd}$$

which establishes the theorem.

The preceding translation would be only of passing interest if it were not also possible to find a similar context in which pseudo-functors also live. Fortunately there is a construction (due to Grothendieck) which allows us to assign to each pseudo-functor on $\mathbb{E}$ a category above $\mathbb{E}$ in a fashion which generalizes the discrete case at each level. The properties of the resulting functors above $\mathbb{E}$ are not as easily determined as those of simply being a discrete fibration, however, and must be described with more care.

(7.0) Grothendieck fibrations:

(7.1) Definition: Let $\mathbb{F} \xrightarrow{F} \mathbb{E}$ be a functor and $X \xrightarrow{f} Y$ an arrow in $\mathbb{F}$. f is said to be (hyper)-*cartesian* over the arrow $F(f)\colon F(X) \longrightarrow F(Y)$ provided that for each $T \in \mathrm{Ob}(\mathbb{F})$, the commutative square of sets and mappings

(7.1.0)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{F}}(T, X) \arrow[r, "\mathrm{Hom}(T,f)"] \arrow[d, "F"'] & \mathrm{Hom}_{\mathbb{F}}(T, Y) \arrow[d, "F"] \\
\mathrm{Hom}_{\mathbb{E}}(F(T), F(X)) \arrow[r, "\mathrm{Hom}(F(T),F(f))"] & \mathrm{Hom}_{\mathbb{E}}(F(T), F(Y))
\end{tikzcd}$$

is cartesian. In more visual terms this amounts to saying

(7.1.1)

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \\
\mathbb{F} \arrow[dd, "F"'] & & T \arrow[ul, dashed, "\tilde{x}"] \arrow[ur, "k"'] & \\
 & F(X) \arrow[rr, "F(f)"] & & F(Y) \\
\mathbb{E} & & F(T) \arrow[ul, "x"] \arrow[ur, "F(k)"'] &
\end{tikzcd}$$

that given any k and x as above in (7.1.1), there exists a unique $\tilde{x}$ which makes the above triangle commutative and projects onto $\tilde{x}$ (i.e., $F(\tilde{x}) = x$). Since $f\colon X \longrightarrow Y$ then represents the functor $(\mathbb{F}/Y)^{\text{op}} \longrightarrow (\text{ENS})$ defined by the fiber product (7.1.0) it is unique up to a unique isomorphism in $\mathbb{F}/Y$.

(7.2) For a given $\xi\colon A \longrightarrow F(Y)$ in $\mathbb{E}$ and Y in $\mathbb{F}$ a cartesian arrow $X \xrightarrow{f} Y$ in $\mathbb{E}$ projecting onto g will be called an *inverse image* of Y by ξ and denoted by $Y_\xi\colon \xi^*(Y) \longrightarrow Y$ or $Y_\xi\colon Y^\xi \longrightarrow Y$.

(7.3) Proposition. From Definition (7.1) and the standard "calculus of cartesian squares", we have that given any composable pair of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{F}$

(a) if f and g are cartesian, then $gf\colon X \longrightarrow Z$ is cartesian;

(b) if gf is cartesian and g is cartesian, then f is cartesian;

(c) every isomorphism in $\mathbb{F}$ is cartesian. Any cartesian morphism which projects onto an isomorphism is any isomorphism; in fact

(d) if $F(f)$ is a retraction (*resp.* a monomorphism; *resp.* is an isomorphism) and f is cartesian, then f is a retraction, (resp. a monomorphism, resp. an isomorphism);

(e) if F is fully faithful, then every arrow in $\mathbb{F}$ is cartesian;

(f) e.g. cartesian morphisms are stable under change of base, provided that F preserves fibered products (i.e., pull-backs of cartesian morphisms are then cartesian).

(7.4) Definition. A functor $F\colon \mathbb{F} \longrightarrow \mathbb{E}$ is said to be a (Grothendieck) *fibration* (or a *fibered category*) provided that for any Y in $\mathbb{F}$, and $\xi\colon A \longrightarrow F(Y)$ in $\mathbb{E}$ there exists an inverse image of Y by ξ. A choice c of one inverse image $Y^c_\xi\colon Y^\xi \longrightarrow Y$ for each such pair will be said to define a *cleavage* for the fibration F. The cartesian arrows Y^c_ξ are called the *morphisms of transport for the cleavage.* We will assume that any cleavage is *normalized*, i.e., the morphism of transport above any identity arrow in $\mathbb{E}$ is the corresponding identity arrow of its target in $\mathbb{F}$. If F admits a cleavage such that the composition of morphisms of transport is again a morphism of transport then the F is said to be a *split fibration* and the cleavage is then called a *splitting.*

(7.5) Proposition. Fibrations are stable under change of base (i.e. under "pull-back") and composition, etc. in CAT.

(7.6) Definition. By an $\mathbb{E}$-*category* is here again meant an object of CAT/$\mathbb{E}$; an $\mathbb{E}$ *functor* is a morphism (i. e., commutative triangle) of this same category. $\mathbb{E}$-natural transformations of $\mathbb{E}$-functors will be those such $F_2 * \alpha = \text{id}$.

(7.6.0)
$$\begin{tikzcd}
\mathbb{F}_1 \arrow[rr, shift left=2] \arrow[rr, shift right=2] \arrow[rr, phantom, "\downarrow\alpha"] \arrow[dr, "F_1"'] & & \mathbb{F}_2 \arrow[dl, "F_2"] \\
& \mathbb{E} &
\end{tikzcd}$$

An $\mathbb{E}$-functor will be said to be *cartesian* provided it carries cartesian morphisms into cartesian morphisms. We thus have the 2-category CAT/$\mathbb{E}$ of fibrations *cartesian* $\mathbb{E}$-functors and $\mathbb{E}$-natural transformations of same such functors.

(7.7) <u>Examples of fibered categories</u>:

Clearly any *discrete fibration* is indeed a split fibration in the above sense in which any arrow of $\mathbb{F}$ is cartesian.

(7.7.0) The *arrow category* $\underline{\mathrm{Ar}}(\mathbb{E}) \xrightarrow{\sim} \mathbb{E}^2 = \underline{\underline{\mathrm{Hom}}}_{\mathrm{CAT}}(\Delta_1, \mathbb{E})$ is fibered over $\mathbb{E}$ by its *source* functor $\mathbb{E}^2 \xrightarrow{S} \mathbb{E}$

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & X_2 \arrow[d] & & \\
Y_1 \arrow[r, "f_1"'] & Y_2 & \mapsto & X_1 \rightarrow X_2
\end{tikzcd}$$

with the commutative square

(7.7.1)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\underset{\mathrm{id}}{\Longrightarrow}"] & Y \arrow[d, "g"] \\
Z \arrow[r] & Z
\end{tikzcd}$$

forming the morphism of transport for a *splitting* which makes h the inverse image of g by f.

(7.7.2) *The arrow category* $\mathbb{E}^2$ *is fibered over* $\mathbb{E}$ *by its target functor iff* $\mathbb{E}$ admits fiber products. A cartesian arrow $\mathbb{E}^2 \xrightarrow{\mathbb{T}} \mathbb{E}$ is simply a *cartesian square* and a cleavage is a choice of fiber products: Clearly, this fibration is <u>not</u>, in general, *split.* A *homomorphism of groups* (considered as groupoids with a single object is a fibration if and only if it is *surjective* in which case any set theoretic section defines a cleavage (which is a splitting if and only if the section is a homomorphism). TOPSP $\xrightarrow{v}$ ENS is a split fibration with the cartesian morphisms defined by inverse image.

The justification of this notion of fibration may now be given.

<u>Fibrations and pseudo-functors into</u> CAT.

(7.8) If $\mathbb{F} \xrightarrow{F} \mathbb{E}$ is a functor and $X \in \mathrm{Ob}(\mathbb{E})$, we define the *fiber of* $\mathbb{F}$ *at* X to be the subcategory $\mathbb{F}_X \hookrightarrow \mathbb{F}$ consisting of those arrows of $\mathbb{F}$ which project onto the identity

$\mathrm{id}(X)\colon X \longrightarrow X$ of X in $\mathbb{E}$. For example, the category $\mathbb{E}/X$ of objects of $\mathbb{E}$ above $X \in \mathrm{Ob}(\mathbb{E})$ is the fiber of the target functor $T\colon \mathbb{E}^{\Delta_1} \longrightarrow \mathbb{E}$ at $X \in \mathrm{Ob}(\mathbb{E})$. Now let $\mathbb{F} \xrightarrow{F} \mathbb{E}$ be a fibration and c a selected cleavage. Any arrow $f\colon X \longrightarrow Y$ in $\mathbb{E}$ will now define a *functor* $\mathbb{F}^c_f\colon \mathbb{F}_Y \longrightarrow \mathbb{F}_X$ (by sending any T in $\mathbb{F}_Y$ to its inverse image over f for the cleavage; $\mathbb{F}^c_f(T) = T^f$ and sending each arrow $U \xrightarrow{x} T$ in $\mathbb{F}_Y$ to that *unique* arrow in $\mathbb{F}_X$) which makes the diagram

(7.8.0)
$$\begin{tikzcd}
U^f \arrow[r, "U^c_f"] \arrow[d, dashed] & U \arrow[d, "x"] \\
T^f \arrow[r, "T^c_f"'] & T
\end{tikzcd}$$

commutative in $\mathbb{F}$ for the morphisms of transport of the cleavage c. Since the composition of morphisms of transport is not necessarily a morphism of transport, the above assignment does not define a functor $\mathbb{E} \xrightarrow{\varphi} (\mathrm{CAT})$. However, for any cleavage c and any composable pair $(f, g)\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{E}$ and object T in $\mathbb{F}$ there exists unique isomorphism $c(f, g)(T)$ in $\mathbb{F}_X$ which makes commutative the diagram

(7.8.1)
$$\begin{tikzcd}
(T^g)^f \arrow[r, "(T^g)^g_f"] \arrow[d, dashed, "c(f,g)(T)"'] & T^g \arrow[d, "T^c_g"] \\
T^{gf} \arrow[r, "T^c_{gf}"'] & T
\end{tikzcd}$$

in $\mathbb{F}$. It is not difficult to see that the assignment $T \longmapsto c(f, g)(T)$ defines a natural isomorphism of functors $c(f, g)\colon \mathbb{F}^c_f \circ \mathbb{F}^c_g \longrightarrow \mathbb{F}^c_{gf}$

(7.8.2)
$$\begin{tikzcd}
\mathbb{F}_Z \arrow[r, "\mathbb{F}^2_g"] \arrow[rd, "\mathbb{F}^c_{gf}"'] & \mathbb{F}_Y \arrow[d, "\mathbb{F}^c_g"] \arrow[ld, phantom, "\overset{c(f,g)}{\swarrow}"] \\
& \mathbb{F}_X
\end{tikzcd}$$

which has the property that for any composable triplet $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{f} W$ in $\mathbb{E}$, the square of natural transformations of functors

(7.8.3)
$$\begin{tikzcd}
\mathbb{F}^c_f \circ \mathbb{F}^c_g \circ \mathbb{F}^c_h \arrow[r, "\mathbb{F}^c_f * c(g,h)"] \arrow[d, "c(f,g)*\mathbb{F}^c_h"'] & \mathbb{F}^c_f \circ \mathbb{F}^c_{hg} \arrow[d, "c(f,hg)"] \\
\mathbb{F}^c_{gf} \circ \mathbb{F}^c_h \arrow[r, "c(gf,h)"'] & \mathbb{F}^c_{hgf}
\end{tikzcd}$$

is commutative for all composable triplets (f, g, h) in $\mathbb{E}$. Normalization of the cleavage requires further that

(7.8.4)
$$c(\mathrm{id}, f) = c(f, \mathrm{id}) = \mathrm{id}(\mathbb{F}^c_f)$$

for all f in $\mathbb{E}$. Thus *any fibration* $\mathbb{F} \longrightarrow \mathbb{E}$ *with a cleavage* c *defines a pseudo-functor* $(\mathbb{F}, c) \colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ *which is a functor provided the cleavage is a splitting.*

(7.9) If $H \colon : \mathbb{F} \longrightarrow \mathbb{G}$ is a cartesian $\mathbb{E}$-functor of fibrations then it maps fibers into fibers and maps the morphisms of transport of $\mathbb{F}$ into cartesian morphisms with the same projections as those of $\mathbb{G}$. It consequently gives for any choice of cleavages c and d of $\mathbb{F}$ and $\mathbb{G}$, a system H_f, $f \in \mathrm{Ar}(\mathbb{E})$ of natural isomorphisms

(7.9.0)
$$\begin{tikzcd}
\mathbb{F}_Y \arrow[r, "H_y"] \arrow[d, "\mathbb{F}^c_f"'] & \mathbb{G}_Y \arrow[d, "\mathbb{G}^d_f"] \\
\mathbb{F}_X \arrow[r, "H_x"'] \arrow[ur, phantom, "H_f \nearrow \sim"] & \mathbb{G}_X
\end{tikzcd}
\qquad
\begin{tikzcd}
H_X\, \mathbb{F}^c_f \arrow[d, "H_f"', "\wr"] \\
\mathbb{G}^d_f\, H_Y
\end{tikzcd}$$

which satisfies in addition to $H_{\mathrm{id}_x} = \mathrm{id}(H_x)$, the compatibility condition obtained from the diagram

(7.9.1)

$$\begin{tikzcd}
\mathbb{F}_Z \arrow[rr, "H_Z"] \arrow[d, "\mathbb{F}^c_g"'] \arrow[drr, "\mathbb{F}_{gf}"] & & \mathbb{G}_Z \arrow[d, "\mathbb{G}^d_g"] \arrow[drr, "\mathbb{G}^d_{gf}"] \arrow[dl, phantom, "H_g \curvearrowright"] & \\
\mathbb{F}_Y \arrow[rr, crossing over, "H_Y"] \arrow[dr, "\mathbb{F}^c_f"'] \arrow[r, phantom, "c(f{,}g) \curvearrowright"] & & \mathbb{G}_Y \arrow[dr, "\mathbb{G}^d_f"'] \arrow[r, phantom, "d(f{,}g) \curvearrowright"] & \\
 & \mathbb{F}_X \arrow[rr, "H_X"'] & \arrow[r, phantom, "H_{gf} \curvearrowright \; H_f \curvearrowright"] & \mathbb{G}_X
\end{tikzcd}$$

which asserts that the diagram

(7.9.2)

$$\begin{tikzcd}
H_X\mathbb{F}^c_f\mathbb{F}^c_g \arrow[r, "H_X * c(f{,}g)"] \arrow[d, "H_f * \mathbb{F}^c_g"'] & H_X\mathbb{F}^c_{gf} \arrow[dd, "H_{gf}"] \\
\mathbb{G}^d_f H_Y \mathbb{F}^c_g \arrow[d, "\mathbb{G}^d_f * H_g"'] & \\
\mathbb{G}^d_f\mathbb{G}^d_g H_Z \arrow[r, "d(f{,}g) * H_Z"'] & \mathbb{G}^d_{gf} H_Z
\end{tikzcd}$$

is commutative for all composable pairs $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{E}$.

(7.10) Definition: By a *morphism of pseudofunctors* is meant a system H_f of natural transformations as in (7.9.0) which satisfy the compatability condition (7.9.1).

(7.11) Similarly if $\alpha\colon H \longrightarrow J$ is an $\mathbb{E}$-natural transformation of $\mathbb{E}$-functors H and J,

(7.11.0)

$$\begin{tikzcd}
\mathbb{F} \arrow[rr, bend left, "H"] \arrow[rr, "J"'] \arrow[rr, phantom, bend left=15, "\downarrow\alpha"] \arrow[dr] & & \mathbb{G} \arrow[dl, "G"] \\
 & \mathbb{E} &
\end{tikzcd}$$

then since $G * \alpha = \mathrm{id}$, α induces a system of natural transformations between the corresponding fiber functors for each $f\colon x \longrightarrow y$ in $\mathbb{E}$

(7.11.1)

$$\begin{tikzcd}
\mathbb{F}_Y \arrow[r, "H_Y"] \arrow[r, bend right, "J_Y"'] \arrow[r, phantom, bend right=20, "\downarrow\alpha(Y)"] \arrow[d, "\mathbb{F}_f"'] \arrow[dr, phantom, "H_f \quad J_f"] & \mathbb{G}_Y \arrow[d, "\mathbb{G}_f"] \\
\mathbb{F}_X \arrow[r, "H_X"] \arrow[r, bend right, "J_X"'] \arrow[r, phantom, bend right=20, "\downarrow\alpha(X)"] & \mathbb{G}_X
\end{tikzcd}$$

which makes the diagram

(7.11.2)

$$\begin{tikzcd}
H_X\mathbb{F}_f \arrow[r, "\alpha_X*\mathbb{F}_f"] \arrow[d, "H_f"'] & J_X\mathbb{F}_f \arrow[d, "J_f"] \\
\mathbb{G}_f H_Y \arrow[r, "\mathbb{G}_f*\alpha Y"'] & \mathbb{G}_f J_Y
\end{tikzcd}$$

commutative for all arrows f in $\mathbb{E}$.

(7.12) Definition: A system of natural transformation $\alpha(X)$, $Y \in \mathrm{ob}(\mathbb{E})$ which satisfies the conditions of (7.11) will be called a *modification* of the corresponding morphisms of the pseudofunctor H and J.

It now follows that the preceding construction defines a 2-functor

$$\Phi\colon \mathrm{CART}/\mathbb{E} \longrightarrow \text{Pseudo-Functor } (\mathbb{E})$$

whose target is the 2-category of pseudo-functors, morphisms of pseudofunctors and modifications of morphisms of pseudo-functors.

Conversely we have the

(7.13) Fibration defined by a pseudo-functor. If $\Phi\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ is a pseudofunctor, then we may associate with Φ a fibration $\mathrm{Fib}(\Phi)$ supplied canonically with a cleavage c whose associated pseudofunctor is isomorphic to Φ. It is defined as follows:

$$\mathrm{Ob}(\mathrm{Fib}(\Phi)) = \coprod_{x \in \mathrm{ob}(\mathbb{E})} \mathbb{F}^c_x$$

$\mathrm{Ar}(\mathbb{F}(\Phi)) = \coprod\limits_{f\in \mathrm{Ar}(\mathbb{E})} P_f$, where P_f is the fiber product.

(7.13.0)
$$\begin{tikzcd}
P_f \arrow[r] \arrow[d] & \mathrm{Ar}(\mathbb{F}^c_x) \arrow[d] \\
\mathrm{ob}(\mathbb{F}^c_y) \arrow[r, "\mathrm{ob}(\mathbb{F}_f)"'] & \mathrm{ob}(\mathbb{F}^c_x)
\end{tikzcd}$$

An element $(B, \theta\colon A \longrightarrow \mathbb{F}^c_f(B))$ of P_f, $f\colon X \longrightarrow Y$ is called an *arrow* of projection f; its source is A and its target is B. In visual terms

(7.13.1)
$$\begin{tikzcd}
A \arrow[d, "\theta"'] \arrow[drr, dashed, "\theta"] & & \\
\mathbb{F}^c_f(B) & & B \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Thus $\theta\colon A \longrightarrow B$ in $\mathbb{F}(\Phi)$ iff there exists $f\colon X \longrightarrow Y$ in $\mathbb{E}$ such that $\theta\colon A \longrightarrow \mathbb{F}^c_f(B)$ in $\mathbb{F}_X$. Composition of the arrow $\theta\colon A \longrightarrow \mathbb{F}_f(B)$ and $\xi\colon B \longrightarrow \mathbb{F}_g(C)$ is defined via the composition of the sequence

(7.13.2)
$$A \xrightarrow{\theta} \mathbb{F}^c_f(B) \xrightarrow{\mathbb{F}^c_f(\xi)} \mathbb{F}^c_f(\mathbb{F}^c_g(C)) \xrightarrow{c(f,g)} \mathbb{F}^c_{gf}$$

This composition is associative and unitary thanks to normalization and the coherence condition of (2.0). Note that $\theta\colon A \longrightarrow B$ is *cartesian* if and only if $\theta\colon A \longrightarrow \mathbb{F}^c_f(B)$ is an isomorphism and the identity arrow $\mathrm{id}\colon \mathbb{F}^c_f(B) \longrightarrow \mathbb{F}^c_f(B)$ in $\mathbb{F}^c_X$, viewed as an arrow $\mathbb{F}_f(B) \longrightarrow B$ of projection f in $\mathrm{Fib}(\Phi)$ defines a canonical cleavage so that $\mathrm{Fib}(\Phi) \xrightarrow{\mathrm{pr}} \mathbb{E}$ becomes a fibration with a canonical cleavage.

(7.14) In similar fashion if $H\colon \mathbb{F}^c \longrightarrow \mathbb{F}^d$ is a morphism of pseudo functors, then H is easily seen to define a cartesian $\mathbb{E}$-functor, $\mathrm{Fib}(H)\colon \mathrm{Fib}(\mathbb{F}^c) \longrightarrow \mathrm{Fib}(\mathbb{F}^d)$ via the assignment to any $f\colon X \longrightarrow Y$ arrow $x\colon A \longrightarrow \mathbb{F}^c_f(B)$ from A into B in $\mathrm{Fib}(\mathbb{F}^c)$, $x \in \mathrm{Ar}(\mathbb{F}^c_X)$, $B \in \mathrm{ob}(\mathbb{F}^c_Y)$, of the f_X arrow $\mathrm{Fib}(H)(x)\colon H_X(A) \longrightarrow H_Y(B)$ obtained by composition of the pair

(7.14.0)
$$H_X(A) \xrightarrow{H_X(x)} H_X(\mathbb{F}^c_f(B)) \xrightarrow{H_f(B)} \mathbb{G}^d_f(H_Y(B)).$$

Note that this is indeed a cartesian functor since if x is cartesian in $\mathrm{Fib}(\mathbb{F}^c)$, where $x \in \mathrm{Ar}(\mathbb{F}^c_X)$ is an isomorphism, so that $H_X(x)$, as well as $H_f(B) \circ H_X(x)$ are also, thus making $\mathrm{Fib}(H)(x)$ cartesian.
(7.15) Following the same observations, if $\alpha\colon H \longrightarrow J$ is a modification of morphisms of pseudo functors (7.12) then the arrows $\alpha_X(A)$ and $G^d_f * \alpha(B) = G^d_f(\alpha(B))$ define an $\mathbb{E}$-natural transformation of $\mathrm{Fib}(H) \longrightarrow \mathrm{Fib}(G)$ since the square

(7.15.0)
$$\begin{tikzcd}
H_X(A) \arrow[r, "\alpha_X(A)"] \arrow[d, "H_X(x)"] & J_X(A) \arrow[d, "J_X(x)"] \\
H_X(\mathbb{F}^c_f(B)) \arrow[r, "\alpha(\mathbb{F}_f(B))"] \arrow[d, "H_f(B)"] & J_X(\mathbb{F}^c_f(B)) \arrow[d, "J_f(B)"] \\
\mathbb{G}^d_f(H_Y(B)) \arrow[r, "G^d_f(\alpha_Y(B))"] & \mathbb{G}^d_f(J_Y(B))
\end{tikzcd}$$

is then commutative for any f-arrow $x\colon A \longrightarrow B$ in $\mathrm{Fib}(\mathbb{F}^c)$.
(7.16) We leave it to the reader to establish that these two constructions indeed establish a strong 2-equivalence of the 2-category of fibrations above $\mathbb{E}$ and the 2-category of pseudo functors on $\mathbb{E}^{\mathrm{op}}$ for which discrete fibrations are carried equivalently to pre-sheaves.

(7.16.0)
$$\begin{tikzcd}
\mathrm{CART}/\mathbb{E} \arrow[r, "\approx"] & \text{Pseudo-Fun}(\mathbb{E}) \\
\mathrm{DiscFib}/\mathbb{E} \arrow[r, "\approx"] \arrow[u, phantom, sloped, "\subseteq"] & \mathbb{E}\hat{} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

(8.) Descent and CART/$\mathbb{E}$

(8.0) We are now in a position to place the property of completeness of a pseudo-functor $\mathbb{F}^c_{(\,)}$with respect to a topology on $\mathbb{E}$ (i.e. $\mathbb{F}^c_{(\,)}$ is a *stack*) completely in the context of the 2-category CART/$\mathbb{E}$ of fibrations and cartesian $\mathbb{E}$-functors. It is based on the following
(8.1) Proposition. Let $\mathbb{F}^c_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ be a pseudo-functor on $\mathbb{E}$, $\varprojlim \mathbb{F}^c_{(\,)}$ the category which is its pseudo-limit (5.2), and $\mathrm{Fib}(\mathbb{F}^c_{(\,)})$ the fibration above $\mathbb{E}$ which is defined by $\mathbb{F}^c_{(\,)}$ (7.13). With these definitions one has a canonical *isomorphism* of categories

(8.1.0)
$$\boxed{\varprojlim \mathbb{F}^c_{(\,)} \xrightarrow{\sim} \underline{\mathrm{CART}}_{\mathbb{E}}(\mathbb{E}, \mathrm{Fib}(\mathbb{F}^c_{(\,)})) = \Gamma(\mathrm{Fib}(\mathbb{F}^c_{(\,)})/\mathbb{E})}$$

where $\underline{\mathrm{CART}}_{\mathbb{E}}(\mathbb{E},\mathrm{Fib}(\mathbb{F}^c_{(\,)}))$ is the category of cartesian $\mathbb{E}$-functors of $\mathbb{E}$ into $\mathrm{Fib}(\mathbb{F}^c_{(\,)})$, i.e. the *category of cartesian sections of the fibration associated with the pseudo-functor.*

In effect, by definition, an object of $\varprojlim \mathbb{F}^c_{(\,)}$ is an order pair $((A_X)_{X\in\mathrm{ob}(\mathbb{E})},(\theta_f)_{f\in\mathrm{Ar}(\mathbb{E})})$, where $A_X\in\mathbb{F}^c_X$ for each X and $\theta_f\colon A_X \xrightarrow{\sim} \mathbb{F}_f(A_Y)$, i.e. in $\mathrm{Fib}(\mathbb{F}^c_{(\,)})$ a *cartesian* arrow $A_X \xrightarrow{\theta_f} A_Y$ whose projection is $f\colon X\longrightarrow Y$; thus the family defines a cartesian section of $\mathrm{Fib}(\mathbb{F}^c_{(\,)})$ which is functorial since the compatibly conditions (5.2) just say that in $\mathrm{Fib}(\mathbb{F}^c_{(\,)})$, $\theta_g\theta_f=\theta_{gf}$ $(\Leftrightarrow\ c(f,g)\mathbb{F}_f(\theta_g)\theta_f=\theta_{gf}$ in $\mathbb{F}^c_x$). Similarly, the arrows of $\varprojlim \mathbb{F}^c_{(\,)}$ just correspond to $\mathbb{E}$-natural transformations of such functors.

(8.2) Corollary. If $\mathbb{F}^c_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ is discrete. and thus corresponds to a presheaf $F\colon \mathbb{E}^{\mathrm{op}}\longrightarrow \mathrm{ENS}$ then $\mathrm{Fib}(\mathbb{F}^c_{(\,)})\xrightarrow{\sim}\mathbb{E}/F$ (6.1.2) and thus

$\varprojlim \mathbb{F} \xrightarrow{\sim} \underline{\mathrm{CART}}_{\mathbb{E}}(\mathbb{E},\mathbb{E}/F)$ since $\underline{\mathrm{CART}}_{\mathbb{E}}(\mathbb{E},\mathbb{E}/F)$ is a discrete category (i.e., is a set).

(8.3) Corollary. Let $\mathbb{E}$ be a site and $\mathbb{F}^c_{(\,)}\colon \mathbb{E}^{\mathrm{op}} \rightsquigarrow \mathrm{CAT}$ a pseudofunctor, then $F^c_{(\,)}$ is a pre-stack, resp. stack, if and only if for each discrete covering subfunctor $\Re \hookrightarrow \mathbb{E}/X$, the canonical restriction functor

$$\text{(8.3.0)} \qquad \mathbb{F}^c_X \xrightarrow{\approx} \underline{\mathrm{CART}}_{\mathbb{E}}(\mathbb{E}/X,\mathrm{Fib}(\mathbb{F}^c_{(\,)})) \longrightarrow \underline{\mathrm{CART}}_{\mathbb{E}}(\Re,\mathrm{Fib}(\mathbb{F}^c_{(\,)}))$$

is fully faithful, resp. an equivalence of categories. A fibration $\mathbb{F}\longrightarrow\mathbb{E}$ which enjoys the same respective properties is said to be *precomplete*, resp. *complete.*

(8.4) We have thus, as promised, defined a category, CART/$\mathbb{E}$, in which all of the notions of the theory of descent have a natural home. Moreover, since the categories $\underline{\underline{\mathrm{CART}}}_{\mathbb{E}}(\mathbb{A},\mathbb{B})$ of cartesian functors between $\mathbb{E}$-fibrations and $\mathbb{E}$-natural transformations of such functors may also be described in terms of the corresponding category of *cartesian sections* after a base change, we can for any $\mathbb{E}$-category $\mathbb{F}$, define the *category*

$$\text{(8.4.0)} \qquad \varprojlim(\mathbb{F}/\mathbb{E}) = \underline{\underline{\mathrm{CART}}}_{\mathbb{E}}(\mathbb{E},\mathbb{F})\,,$$

the corresponding *category* of cartesian sections. It is then easily seen that the functor defined by $\mathbb{F}\longrightarrow\mathbb{E}\longmapsto\varprojlim(\mathbb{F}/\mathbb{E})$ provides a strict *right* 2-*adjoint* for the change of base along $\mathbb{F}\longrightarrow\Delta_0$ functor $(\mathbb{A}\longmapsto\mathbb{E}\times\mathbb{A}\xrightarrow[\mathrm{pr}]{}\mathbb{E})$

$$\text{(8.4.1)} \qquad \mathrm{CART}_{\mathbb{E}} \xleftarrow{\ \mathbb{E}\times(\,)\ } \mathrm{CAT}/\Delta_0 \cong \mathrm{CAT}$$

$$\varprojlim(\text{-}/\mathbb{E})$$

(where Δ_0 is the one object category which is terminal in CAT) i.e. for any category $\mathbb{A}$ one has a canonical isomorphism of categories

$$\text{(8.4.2)} \qquad \underline{\underline{\mathrm{CART}}}_{\mathbb{E}}(\mathbb{E}\times\mathbb{A},\mathbb{F}) \xrightarrow{\sim} \underline{\underline{\mathrm{CAT}}}(\mathbb{A},\varprojlim(\mathbb{F}/\mathbb{E})) \quad (=\ \underline{\underline{\mathrm{CART}}}_{\Delta_0}(\mathbb{A},\varprojlim(\mathbb{F}/\mathbb{E})))$$

and the fundamental properties of fibered products and sections and allow us to use the canonical "change of base" isomorphism.

$$(8.4.3) \qquad \underline{\underline{\mathrm{CART}}}_{\mathbb{E}}(A,\mathbb{F}) \xrightarrow{\sim} \underline{\underline{\mathrm{CART}}}_{\mathbb{A}}(A,\mathbb{F}\times_{\mathbb{E}} A) = \underset{\longleftarrow}{\lim}(\mathbb{F}\times_A \mathbb{A}/\mathbb{A})$$

to rewrite $\mathbb{F}$ - k descent as the assertion that the canonical change of base functor ("pull-back along inclusion $\mathfrak{R} \hookrightarrow \mathbb{E}/X$) which makes the diagram

$$\begin{tikzcd}
\mathbb{F}_X \arrow[rr] & & \mathrm{CART}_{\mathbb{E}}(\mathfrak{R},\mathbb{F}^{\mathbb{G}}) \\
\lim(\mathbb{F}|X/\mathbb{E}/X) \arrow[u, "\wr"'] \arrow[rr] & & \underset{\longleftarrow}{\lim}(\mathbb{F}|R/\mathfrak{R}) \arrow[u, "\wr"']
\end{tikzcd}$$

commute, is a k-equivalence, $k = 0,1,2$, where (by definition) the following squares are cartesian in CAT.

$$(8.4.4) \qquad \begin{tikzcd}
\mathbb{F}|\mathfrak{R} \arrow[r, hook] \arrow[d] & \mathbb{F}|X \arrow[r] \arrow[d] & \mathbb{F} \arrow[d] \\
\mathfrak{R} \arrow[r, hook] & \mathbb{E}/X \arrow[r, "S"] & \mathbb{E}
\end{tikzcd}$$

It is in this sense that the theory of descent may be described as the "general theory of the behavior of the functor $\underset{\longleftarrow}{\lim}$ along base changes of the form $\mathfrak{R} \hookrightarrow \mathbb{E}/X$, where $\mathfrak{R}$ is a discrete subfunctor (GIRAUD (1962)).

Description IV: (in $\mathbb{E}^{\wedge}$)

(9.0) The preceding description, while exceedingly elegant, requires the relatively unfamiliar and abstract context of CART/$\mathbb{E}$ for its presentation and thus uses a considerable amount of abstract (all-be-it elementary) category theory. As a consequence of this, the amount of "abstract nonsense" used in the presentation is more than many working mathematicians would prefer.

Fortunately, there is still another presentation of the notions involved which keeps the reader within the relatively familiar category $\mathbb{E}^{\wedge}$ of pre-sheaves on $\mathbb{E}$. Working in this category is a very concrete process very much like working in sets, and as $\mathbb{E}^{\wedge}$ is itself a topos (for $\mathbb{E}$ with the discrete topology) it is from observations made here that a generalization of the theory to elementary topoi can be made.

(9) Properties of the category $\mathbb{E}^{\wedge}$.

(9.0) The category $\mathbb{E}^{\wedge}$ has all set based limits and colimits which may be defined "pointwise": $(\varprojlim \mathbb{F}_c)(X) = \varprojlim \mathbb{F}_c(X)$; $(\varinjlim \mathbb{F}_c)(X) = \varinjlim \mathbb{F}_c(X)$; in particular, $(\prod F_c)(X) = \prod F_c(X)$ and $(\coprod F_c)(X) = \coprod F_c(X)$. The canonical functor $h\colon \mathbb{E} \longrightarrow \mathbb{E}^{\wedge}$ which sends any object X to the representable functor $h_X(T) = \mathrm{Hom}_{\mathbb{E}}(T, X)$ is fully faithful, injective on objects, and has the property that $\mathrm{Hom}_{\mathbb{E}^{\wedge}}(h_X, F) \xrightarrow{\sim} F(X)$ ("Yoneda Lemma"). We thus may *identify* $\mathbb{E}$ with the subcategory of representable functors of $\mathbb{E}^{\wedge}$ and with this identification write $h_X(T) = X(T) = \mathrm{Hom}_{\mathbb{E}}(T, X)$ for any $X \in \mathrm{ob}(\mathbb{E})$.

Under this identification an element $x \in F(X)$ is identified with a natural transformation $x\colon X \longrightarrow F$ in $\mathbb{E}^{\wedge}$ and for any arrow $f\colon X \longrightarrow Y$ in $\mathbb{E}$ the triangle

(9.0.0)
$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "F(f)(y)"] & \\
Y \arrow[r] & F
\end{tikzcd}$$

is commutative, as is

(9.0.1)
$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[dr, "\alpha_X(x)"'] & F \arrow[d, "\alpha"] \\
& G
\end{tikzcd}$$

for any natural transformation $\alpha\colon F \longrightarrow G$ in $\mathbb{E}^{\wedge}$.

(9.1) Lemma. For any presheaf $P \in \mathbb{E}^{\wedge}$, let $\mathbb{E}/P$ be category of objects of $\mathbb{E}^{\wedge}$ above P defined as usual and let $\mathbb{E}/P \longrightarrow \mathbb{E}$ be the discrete fibration above $\mathbb{E}$ defined by

$P(\quad)$, then the commutative square of categories and functors

(9.1.0)
$$\begin{tikzcd}
\mathbb{E}/P \arrow[r, "\bar{h}"] \arrow[d] & \mathbb{E}\hat{}/P \arrow[d, "S"] \\
\mathbb{E} \arrow[r, "h"'] & \mathbb{E}\hat{}
\end{tikzcd}$$

(where $\bar{h}\colon \mathbb{E}/P \longrightarrow \mathbb{E}\hat{}/P$ is defined by the assignment $(x, X) \longmapsto h_X \xrightarrow{x} P$, $(x, X) \in \coprod_{x\in\mathrm{ob}(\mathbb{E})} P(X)$ obtained from the Yoneda map) is cartesian. Moreover, one has in addition an equivalence of categories

(9.1.1)
$$\#_P\colon \mathbb{E}\hat{}/P \xrightarrow{\approx} (\mathbb{E}/\mathbb{P})\hat{}$$

which has the property that given any natural transformation $f\colon F \longrightarrow P$ and the corresponding functors

(9.1.2)
$$f_!\colon \mathbb{E}\hat{}/F \longrightarrow \mathbb{E}\hat{}/P$$

(defined by $G \longrightarrow F \longmapsto f_!(g) = fg\colon G \longrightarrow P)$ and

(9.1.3)
$$f^*\colon \mathbb{E}\hat{}/P \longrightarrow \mathbb{E}\hat{}/F$$

(defined by $H \xrightarrow{h} P \longmapsto F_f \times_h H \xrightarrow[\mathrm{pr}]{} F$ and right adjoint to $f_!$), and

(9.1.4)
$$f\hat{}\colon (\mathbb{E}/\mathbb{P})\hat{} \longmapsto (\mathbb{E}/\mathbb{F})\hat{}$$

(defined by restriction $P^\#\colon \mathbb{E}/P^{\mathrm{op}} \longrightarrow \mathrm{ENS} \longmapsto f_!^{\mathrm{op}}P^\#\colon \mathbb{E}/P^{\mathrm{op}} \longrightarrow \mathrm{ENS}$)
diagram

(9.1.5)
$$\begin{tikzcd}
\mathbb{E}\hat{}/P \arrow[r, "\#_P", "\approx"'] \arrow[d, "f^*"'] & (\mathbb{E}/\mathbb{P})\hat{} \arrow[d, "f_!\hat{}"] \\
\mathbb{E}\hat{}/F \arrow[r, "\#_F", "\approx"'] & (\mathbb{E}/\mathbb{F})\hat{}
\end{tikzcd}$$

of categories and functors is commutative. (That is, “restriction of presheaves corresponds to pullback of the representing objects in $\mathbb{E}\hat{}/P$ ”).

In effect the first assertion (9.1.0) is trivial and justifies the notation $\mathbb{E}/P$ used in (6.1). The functor $\#_P$ is easily constructed: Given any $G \xrightarrow{g} P$ in $\mathbb{E}\hat{}/P$ define

$G^{\#}\colon \mathbb{E}/P^{\mathrm{op}} \longrightarrow \mathrm{ENS}$ at any $X \xrightarrow{x} P$ in $\mathbb{E}/P$ as the set of all "liftings of x to G", i.e. as the set of natural transformations $\ell\colon X \longrightarrow G$ such that $g^{\ell} = x$ $(\Leftrightarrow g(X)^{-1}\{x\} \subseteq G(X))$, $g(X)\colon G(X) \longrightarrow P(X)$. Composition of any lifting with

$$\begin{tikzcd} f\colon Y \arrow[r] \arrow[dr] & X \arrow[d] \\ & P \end{tikzcd}$$

clearly makes this functorial on $\mathbb{E}/P^{\mathrm{op}}$.

A quasi-inverse for $\#_P$ may then be defined for any presheaf $H \in (\mathbb{E}/\mathbb{P})\widehat{}$ via

(9.1.6)
$$\begin{tikzcd} H_{\#}(T) \arrow[r, phantom, "="] \arrow[d, "\alpha_T"] & H(t),\ t \in \mathrm{Hom}_{\widehat{\mathbb{E}}}(T, P) \arrow[d, "\mathrm{pr}\, g"] \\ P(T) \arrow[r, "\sim"] & \mathrm{Hom}_{\widehat{\mathbb{E}}}(T, P) \end{tikzcd}$$

for any $T \in \mathrm{ob}(\mathbb{E})$ and the remainder of the observations are easily verified.

(9.2) The preceding lemma is of course applicable if P is representable, say by an object X of $\mathbb{E}$, here for instance it says that given any presheaf P on $\mathbb{E}$, the restriction of P to $\mathbb{E}/X$ is represented in $\widehat{\mathbb{E}}$ by the pull back

(9.2.0)
$$\begin{tikzcd} X \times P \arrow[r] \arrow[d, "\mathrm{pr}_X"'] & P \arrow[d] \\ X \arrow[r] & \mathbb{1} \end{tikzcd}$$

(9.3) Every presheaf on $\mathbb{E}$ is the canonical colimit of representables which may be described as the colimit of diagram in $\widehat{\mathbb{E}}$ defined by $\mathbb{E}/P$.

(9.3.0)
$$\begin{tikzcd} {} \arrow[r, dotted, no head] \arrow[drr] & X \arrow[r, "f"] \arrow[dr] & & Y \arrow[r] \arrow[dl] & \bullet \arrow[dll] \\ & & \mathrm{P} & & \end{tikzcd}$$
(representables)

thus P represents the co-equalizer of

(9.3.1)
$$\coprod_{f \in \mathrm{Ar}(\mathbb{E}/P)} \mathcal{S}(S(f)) \overset{d_1}{\underset{d_0}{\rightrightarrows}} \coprod_{x \in \mathrm{ob}(\mathbb{E}/P)} \mathcal{S}(x) \twoheadrightarrow P$$

where $d_0(T)\colon (T \xrightarrow{t} \mathcal{S}S(f), f) \longmapsto (ft, T(f))$ and

$d_1(T)\colon (T \xrightarrow{t} \mathcal{S}S(f), f) \longmapsto (t, S(f))$.

This assertion is just an internal translation of the definition of a natural transformation: There is a bijective correspondence between maps $p\colon P \longrightarrow Q$ in $\mathbb{E}^{\frown}$ and assignments $X \xrightarrow{x} P$ (in $\mathbb{E}/X$) $\longmapsto p(x)\colon X \longrightarrow Q$ such that for any

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl] \\ & P & \end{tikzcd}$$

in $\mathbb{E}/P$, $p(g)f = p(x)$.

The diagram (9.3.1) is actually the target (d_0) and source (d_1) maps category object in $\mathbb{E}^{\frown}$ whose significance will be discussed shortly.

(9.4) If P is the terminal presheaf $\mathbb{1}$, then (9.3.1) becomes the coequalizer

(9.4.0)
$$\begin{tikzcd} \coprod\limits_{f\in \mathrm{Ar}(\mathbb{E})} S(f) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \coprod\limits_{X\in \mathrm{ob}(\mathbb{E})} X \arrow[r] & \mathbb{1}. \end{tikzcd}$$

That this is a co-equalizer means that for any pre-sheaf P, the top row of the diagram

(9.4.1)
$$\begin{tikzcd} \Gamma_{\mathbb{E}^{\frown}}(P) = \mathrm{Hom}_{\mathbb{E}^{\frown}}(\mathbb{1}, P) \arrow[r, hook] \arrow[dd, "\wr"] & \mathrm{Hom}_{\mathbb{E}^{\frown}}(\coprod\limits_{X\in \mathrm{ob}(\mathbb{E})} X, P) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Hom}_{\mathbb{E}^{\frown}}(\coprod\limits_{f\in \mathrm{Ar}(\mathbb{E})} S(f), P) \arrow[d, "\wr"] \\ & \prod\limits_{X\in \mathrm{ob}(\mathbb{E})} \mathrm{Hom}_{\mathbb{E}^{\frown}}(X, P) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"] & \prod\limits_{f\in \mathrm{Ar}(\mathbb{E})} \mathrm{Hom}_{\mathbb{E}^{\frown}}(S(f), P) \arrow[d, "\wr"] \\ \varprojlim P \arrow[r, hook] & \prod\limits_{X\in \mathrm{ob}(\mathbb{E})} P(X) \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{f\in \mathrm{Ar}(\mathbb{E})} P(S(f)) \end{tikzcd}$$

is exact (i.e., is an equalizer diagram) and is bijectively equivalent by Yoneda to the bottom row which can be taken as the usual definition of $\varprojlim P$. Thus we have a canonical interpretation in $\mathbb{E}^{\frown}$

(9.4.2)
$$\Gamma_{\mathbb{E}^{\frown}}(P) = \mathrm{Hom}_{\mathbb{E}^{\frown}}(\mathbb{1}, P) \xrightarrow{\sim} \varprojlim P$$

of $\varprojlim P$ as the set of global sections of P in $\mathbb{E}^{\frown}$.

The same results holds in $\mathbb{E}/P$; for any presheaf $F\colon \mathbb{E}/P^{\mathrm{op}} \longrightarrow \mathrm{ENS}$,

(9.4.3)
$$\varprojlim F \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}^{\frown}/P}(P, F_{\#}) = \Gamma_{\mathbb{E}^{\frown}}(F_{\#}/P)$$

where $F_{\#} \longrightarrow P$ is the object in $\mathbb{E}^{\frown}/P$ which is equivalent to F.

(9.5) Combining these results we have for any presheaf P on $\mathbb{E}$ and any $r\colon R \longrightarrow X$ in $\mathbb{E}\hat{}$ with X representable the natural mapping,

(9.5.0)
$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathbb{E}\hat{}}(X,P) \arrow[r, "\mathrm{Hom}_{\mathbb{E}}(r,P)"] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathbb{E}\hat{}}(R,P) \arrow[d, "\wr"] \\
 & \Gamma_X(X \times P) \arrow[r] \arrow[d, "\wr"] & \Gamma_R(R \times P) \arrow[d, "\wr"] \\
P(X) \arrow[r, "\sim"] & \varprojlim P|X \arrow[r, "\varprojlim \ker"'] & \varprojlim P|R
\end{tikzcd}$$

which internally translates in $\mathbb{E}\hat{}$ the canonical map $\varprojlim P|X \longrightarrow \varprojlim P|R$.

(10.0) We now can translate the definition of a topology on $\mathbb{E}$ and the attendant notion of a sheaf into the language of $\mathbb{E}\hat{}$. It is, in effect, immediate since a *seive on* X defined as a subcategory $\Re$ of $\mathbb{E}/X$ whose inclusion functor is discrete in $\Re \hookrightarrow \mathbb{E}/X$ *is equivalent* to a *subfunctor* $R \hookrightarrow X$ *of the representable* functor defined by X (with $\Re$ *isomorphic to* $\mathbb{E}/R$) through the above cited equivalences. Moreover, under this identification, for any $f\colon Y \longrightarrow X$ and any seive $\Re \hookrightarrow \mathbb{E}/X$, the inverse image seive $\Re^f \hookrightarrow \mathbb{E}/Y$ corresponds to the pull-back (inverse image)

(10.0.0)
$$\begin{tikzcd}
f^{-1}(R) \arrow[r] \arrow[d, hook] & R \arrow[d, hook] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

in $\mathbb{E}\hat{}$. A direct translation in the language of $\mathbb{E}\hat{}$ now gives the

(10.1) <u>Definition</u> (in $\mathbb{E}\hat{}$): A topology J on $\mathbb{E}$ is a function which assigns to each $X \in \mathrm{ob}(\mathbb{E})$ a non-empty set $J(X)$ of subfunctors of the representable functor X (called the *covering subfunctors* of X or by abuse of language the *covering sieves* or *refinements* of X) such that

(a) If $R \in J(X)$ and $f\colon Y \longrightarrow X$ in $\mathbb{E}$ then $f^{-1}(R) \in J(Y)$

(b) Let $R \hookrightarrow X$ be a subfunctor of $X \in \mathrm{ob}(\mathbb{E})$ and $C \in J(X)$, then $R \in J(X)$ if (and only if in the light of (a)) for all $y\colon Y \longrightarrow C$ $(Y \in \mathrm{ob}(\mathbb{E}))$, $y^{-1}(C \cap R) \in J(Y)$.

$$\begin{tikzcd}
y^{-1}(C \cap R) \arrow[r] \arrow[d, hook] & C \cap R \arrow[r, hook] \arrow[d, hook] & R \arrow[d, hook] \\
Y \arrow[r] & C \arrow[r, hook] & X
\end{tikzcd}$$

(10.2) From (a) and (b) it follows that $R \in J(X)$, $S \in J(X) \implies R \cap S \in J(X)$ and that if $C \hookrightarrow R \hookrightarrow X$, then $C \in J(X) \implies R \in J(X)$, thus since $J(X) \neq \emptyset$, that $X \in J(X)$ as desired. Consequently, $J(X)$ *is filtering*, a fact which has important consequences in what follows. Using (10.1) we now can re-state the

(10.3) <u>Definition</u> (in $\mathbb{E}\hat{}$). A presheaf F is a *separated pre-sheaf* (resp. is a *sheaf*) for the topology J if and only if for every covering subfunctor $R \hookrightarrow X$, the restriction map

$$\mathrm{Hom}_{\mathbb{E}\hat{}}(X, F) \longrightarrow \mathrm{Hom}_{\mathbb{E}\hat{}}(R, F) \tag{10.3.0}$$

is injective (resp. is a bijection).

Thus viewed in $\mathbb{E}\hat{}$, “$\mathbb{F}$ is a sheaf if and only if for any covering subfunctor $R \hookrightarrow X$ given any $t\colon R \longrightarrow F$, there exists a unique $\tilde{t}\colon X \longrightarrow F$ such that the triangle

$$\begin{tikzcd} R \arrow[r, "t"] \arrow[d, hook] & F \\ X \arrow[ur, dashed] & \end{tikzcd} \tag{10.3.1}$$

is commutative”.

(10.4) The advantages of operating in $\mathbb{E}$ with $\mathbb{E}$ identified with the subcategory of representables are numerous. For example let $(X_\alpha \xrightarrow{\alpha} X)_{\alpha \in I}$ be a family of arrows in $\mathbb{E}$. If the family is identified with its representables in $\mathbb{E}\hat{}$, then viewed as a subfunctor of X, the seive generated by the family is nothing more than the subfunctor R which is the *image* in X of the canonical map in $\mathbb{E}\hat{}$

$$\begin{tikzcd} \alpha\colon \coprod\limits_{\alpha \in I} X_\alpha \arrow[rr] \arrow[dr, two heads] & & X \\ & R \arrow[ur, hook] & \end{tikzcd} \tag{10.4.0}$$

defined by the assignment

$$T \xrightarrow{x} X_\alpha \longmapsto \alpha_x\colon T \longrightarrow X \tag{10.4.1}$$

at any $T \in \mathrm{ob}(\mathbb{E})$, thus $t\colon T \longrightarrow R$ $(\in R(T))$ if and only if there exists an arrow $(T \xrightarrow{z} \coprod_{\alpha \in I} X_\alpha$ $(\Leftrightarrow (z, \alpha)$ with $z \in X_\alpha(T))$ such that $\alpha z = t$. Thus a family is a *covering family iff its image* $R \longrightarrow X$ in $\mathbb{E}\hat{}$ is *a covering subfunctor of* X.

Note also that the image R of the family is the co-equalizer of the equivalence relation on $\coprod X_\alpha$ defined by the canonical map $\coprod \alpha\colon \coprod X_\alpha \longrightarrow X$, which has its graph isomorphic in $\mathbb{E}\hat{}$ to the object $\coprod\limits_{(\alpha,\beta) \in I \times I} X_\alpha \times_X X_\beta$, i.e. the diagram

$$\coprod_{(\alpha,\beta) \in I \times I} X_\alpha \times_X X_\beta \overset{d_1}{\underset{d_0}{\rightrightarrows}} \coprod_{\alpha \in I} X_\alpha \twoheadrightarrow R\,(\hookrightarrow X) \tag{10.4.2}$$

is always co-exact (a co-equalizer) in $\widehat{\mathbb{E}}$. This means that for any presheaf F in $\widehat{\mathbb{E}}$ any map $\coprod X_\alpha \longrightarrow F$ which equalizes the projections d_0 and d_1 factors uniquely through R, i.e.

(10.4.3)
$$\begin{tikzcd}
\mathrm{Hom}_{\widehat{\mathbb{E}}}(R,F) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\widehat{\mathbb{E}}}\big(\coprod X_\alpha, F\big) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"] & \mathrm{Hom}_{\widehat{\mathbb{E}}}\big(\coprod X_\alpha \times_X X_\beta, F\big) \arrow[d, "\wr"] \\
\mathrm{Hom}_{\widehat{\mathbb{E}}}(R,F) \arrow[r] & \prod\limits_{\alpha} F(X_\alpha) \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{\alpha,\beta} \mathrm{Hom}(X_\alpha \times_X X_\beta, F)
\end{tikzcd}$$

is exact. Thus if $R \longrightarrow X$ is a covering subfunctor, F satisfies the sheaf property if and only if

(10.4.4)
$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d, "\wr"'] & \prod F(X_\alpha) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathrm{Hom}_{\widehat{\mathbb{F}}}(X_\alpha \times_X X_\beta, F) \\
\mathrm{Hom}_{\widehat{\mathbb{E}}}(X,F) \arrow[r] & \mathrm{Hom}_{\widehat{\mathbb{E}}}(R,F) \arrow[u, hook]
\end{tikzcd}$$

is exact and consequently, if the fiber products $X_\alpha \times_X X_\beta$ representable, if and only if

(10.4.5)
$$F(X) \longrightarrow \prod_{\alpha\in I} F(X_\alpha) \rightrightarrows \prod_{\alpha,\beta\in I\times I} F(X_\alpha \times_X X_\beta)$$

is exact so that we recover the original definition (1.1) of sheaf when the families are the covering families of a pre-topology (1.0).

(11.0) Universal strict epimorphic families and universal effective epimorphic families.

An interesting case of the forgoing occurs if the topology on $\mathbb{E}$ is courser than the canonical topology so that every representable functor is a sheaf. Let $(X \xrightarrow{x_\alpha} X)_{\alpha\in I}$ be a covering family (in $\mathbb{E}$) for such a topology, then for each $T \in \mathrm{ob}(\mathbb{E})$ a natural transformation $g\colon \coprod X_\alpha \longrightarrow T$ must factor uniquely through the canonical transformation $\coprod X_\alpha \longrightarrow X$ if and only if it equalizes the canonical maps $\coprod\limits_{\alpha,\beta\in I\times I} X_\alpha \times_X X_\beta \overset{d_1}{\underset{d_2}{\rightrightarrows}} \coprod X_\alpha$.

Moreover, since for any $f\colon Y \longrightarrow X$ in $\mathbb{E}$, the inverse image of the covering subfunctor which is the image of the $(x_\alpha)_{\alpha\in I}$ is just the image of the family of transformations $(Y \times_X X_\alpha \xrightarrow{\mathrm{pr}} Y)_{\alpha\in I}$ must also be covering, this same factorization property must be true for the family ($(Y \times_X X_\alpha \longrightarrow Y)_{\alpha\in I}$ for any $f\colon Y \longrightarrow X$ in $\mathbb{E}$. In language expressible entirely within the category $\mathbb{E}$, this means that *a family* $(X_\alpha \xrightarrow{x_\alpha} X)$ *which* *is covering for a topology which is coarser than the canonical topology must satisfy the following properties in* $\mathbb{E}$:

(a) for any $T \in \mathrm{ob}(\mathbb{E})$ and any pair of arrows $g_1, g_2 \colon X \rightrightarrows T$, $g_1 = g_2$ if and only if for all $\alpha \in I$, $g_1 x_\alpha = g_2 x_\alpha$ (i.e. $(X_\alpha \xrightarrow{x_\alpha} X)_{\alpha \in I}$ is an *epimorphic family*);

(b) $(x_\alpha)_{\alpha \in I}$ is an epimorphic family and for any $T \in \mathrm{ob}(\mathbb{E})$, for any family $(g_\alpha \colon X_\alpha \longrightarrow T)_{\alpha \in I}$, there exists an arrow $\tilde{g} \colon X \longrightarrow T$ such that $x_\alpha \tilde{g} = g_\alpha$ for each $\alpha \in I$, if and only if for all pairs $U \xrightarrow{u_\alpha} X_\alpha$, $U \xrightarrow{u_\beta} X_\beta$ in $\mathbb{E}$ such that $x_\alpha u_\alpha = x_\beta u_\beta$ for all $\alpha, \beta \in I \times I$, $g_\alpha u_\alpha = g_\beta u_\beta$ (i.e. $(x_\alpha)_{\alpha \in I}$ is a *strict epimorphic family*);

(c) for any $T \in \mathrm{ob}(\mathbb{E})$, for any $f \colon Y \longrightarrow X$ in $\mathbb{E}$, and any pair of arrows $g_1, g_2 \colon Y \rightrightarrows T$, $g_1 = g_2$ if and only if for all $\alpha \in I$ and all pairs $u_\alpha \colon U \longrightarrow X_\alpha$, $u_Y \colon U \longrightarrow Y$ such that $x_\alpha u_\alpha = f u_Y$, $g_1 u_Y = g_2 u_Y$ (i.e. $(x_\alpha)_{\alpha \in I}$ is a *universal epimorphic family*);

(d) for any $T \in \mathrm{ob}(\mathbb{E})$, for any $f \colon Y \longrightarrow X$ in $\mathbb{E}$ and for any family $\big(g_\alpha \colon U \longrightarrow T\ ,\ \ u_Y \colon U \longrightarrow Y\ ,\ \ u_\alpha \colon U \longrightarrow X_\alpha \big)_{\alpha \in I}$such that $f u_Y = x_\alpha u_\alpha$ for each $\alpha \in I$ and $g g_\alpha = g_\beta$ for any $g \colon V \longrightarrow u$ such that $u_Y g = v_y$ and $u_\alpha g = v_\alpha$, $\alpha, \beta \in I \times I$, there exists an arrow $\tilde{g} \colon Y \longrightarrow T$ such that $\tilde{g} u_Y = g_\alpha$ for all $\alpha \in I$, if and only if $u g_\alpha = v g_\beta$ for all $\big(u \colon W \longrightarrow U\ ,\ \ v \colon W \longrightarrow V \big)$ such that $u_Y u = v_Y v$ (i.e., $(X_\alpha \xrightarrow{x_\alpha} X)_{\alpha \in I}$ *is a universal strict (epimorphic) family*).

(11.1) In the presence of fiber products, these rather tedious to state conditions become the following

(a) $(X_\alpha \xrightarrow{x_\alpha} X)_{\alpha \in I}$ is an epimorphic family and for any $T \in \mathrm{ob}(\mathbb{E})$ any family $(g_\alpha \colon X_\alpha \longrightarrow T)_{\alpha \in I}$, there exists an arrow $\tilde{g} \colon X \longrightarrow T$ such that $\tilde{g} x_\alpha = g_\alpha$ if and only if $g_\beta \mathrm{pr} = g_\beta \mathrm{pr}$ for all cartesian squares

(11.1.0)
$$\begin{tikzcd}
X_\alpha \times_X X_\beta \arrow[r, "\mathrm{pr}_\beta"] \arrow[d, "\mathrm{pr}_\alpha"'] & X_\beta \arrow[d, "\alpha_\beta"] \\
X \arrow[r, "x_\alpha"'] & X
\end{tikzcd}$$

(i.e., $(X_\alpha \xrightarrow{x_\alpha} X)_{\alpha \in I}$ is an *effective epimorphic family*) and

(b) $(X_\alpha \xrightarrow{x_\alpha} X)_{\alpha \in I}$ is an effective epimorphic family and for any $f \colon Y \longrightarrow X$, the family $(Y \times_X X_\alpha \xrightarrow[\mathrm{pr}_Y]{} Y)_{\alpha \in I}$ is an effective epimorphic family, (i.e., $(X_\alpha \longrightarrow X)$ is a *universal effective epimorphic family*).

If the family in question reduced to a single arrow $p\colon C \longrightarrow X$, then the corresponding terminology is that of a *universal strict* or *universal effective epimorphism* and if coproducts as well as fibre products exists in $\mathbb{E}$, we can always reduce the above to the consideration of universal effective epimorphisms. The preceding remarks thus justify the term "universal strict epimorphic seive" for any covering subfunctor (seive) for the canonical topology on $\mathbb{E}$ (4.6).

(12.) Induced Topologies on $\mathbb{E}\hat{\ }$ and $\mathbb{E}\hat{\ }/X$.

(12.0) If $\mathbb{E}$ is supplied with a topology J, we may extend the notion of refinement to $\mathbb{E}\hat{\ }$ as follows:

If $i_R\colon R \hookrightarrow P$ is a subfunctor of P, we define R to be a *refinement* (or *covering subfunctor*) of P provided that given any representable X and any transformation $x\colon X \longrightarrow P$, the subfunctor $x^{-1}(R) \hookrightarrow X$ is a refinement of X for the topology J, i.e. if $x^{-1}(R) \in J(X)$. If $J(P)$ now represents the resulting class of covering subfunctors of P, then the assignment $P \longmapsto J(P)$ has the following properties:

(a) If $R \in J(P)$ and $f\colon Q \longrightarrow P$ is an arrow in $\mathbb{E}\hat{\ }$,then $f^{-1}(R) \in J(Q)$ ("the inverse image of a covering subfunctor is covering");

(b) Given a commutative triangle of subfunctors

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[dr, hook] & C \arrow[d, hook] \\
 & P \, ,
\end{tikzcd}$$

then (i) $R \in J(C)$ and $C \in J(P) \Rightarrow R \in J(P)$ ("the composition of covering subfunctors is covering"), and (ii) $R \in J(P) \Rightarrow C \in J(P)$ ("if the composition of two covering subfunctors is covering then the last one was as well");

(c) for all $F \in \mathbb{E}\hat{\ }$, $\mathbb{F} \in J(F)$ ("the identity subfunctor is always covering");

(12.1) Definition. An assignment $P \longmapsto J(P)$ which satisfies properties (a), (b) and (c) above will be called a *topology on* $\mathbb{E}\hat{\ }$ with the above defined one said to be *induced by the topology on* $\mathbb{E}$.

It is easy to see that sheaves remain unchanged: $F \in \mathbb{E}\hat{\ }$ is a sheaf for the induced topology on $\mathbb{E}\hat{\ }$ (i.e. for every covering subfunctor $R \in J(P)$, the restriction map $\mathrm{Hom}_{\mathbb{C}\hat{\ }}(P, F) \longrightarrow \mathrm{Hom}_{\mathbb{C}\hat{\ }}(R, F)$ is a bijection) if and only if F is a sheaf for the given topology on $\mathbb{E}$.

(12.2) Remark. It is easy to show that there is a one-to-one correspondence between topologies on $\mathbb{E}$ and topologies on $\mathbb{E}\hat{\ }$ in the above sense and, indeed, this is only one of several equivalent ways to define a topology on $\mathbb{E}\hat{}$ or, in fact, in any topos or even on an exact category in the sense of Barr. One such alternative approach is given through the notion of a "universal closure operator" on the class of subobjects of any object or through the same notion defined by an endomorphism $j\colon \Omega \longrightarrow \Omega$ (where Ω is the *subobject classifier*). All of them are given in detail in JOHNSTONE (1977) to whom the reader is referred for details.

(12.3) Using the induced topology on $\mathbb{E}\hat{}$ we can equally well define *the induced* topology on $\mathbb{E}\hat{}/P$ (for any presheaf P in $\mathbb{E}\hat{}$): a commutative triangle of the form

$$\begin{tikzcd} R \arrow[rr, hook] \arrow[dr, "r"'] & & S \arrow[dl, "s"] \\ & P & \end{tikzcd} \tag{12.3.0}$$

in $\mathbb{E}\hat{}$ defines r as a subfunctor of s in $\mathbb{E}/P$ and is defined to be a covering subfunctor if and only if $R \hookrightarrow S$ is a covering subfunctor of S in $\mathbb{E}\hat{}$. By restriction this gives topologies on $\mathbb{E}/P$ as well as on $\mathbb{E}/X$, $X \in \mathrm{ob}(\mathbb{E})$. The sheaves for these topologies will be characterized after we give the construction of

(13) The associated sheaf functor.

(13.0) A left adjoint for the canonical inclusion functor $\mathbb{E}\hat{} \hookrightarrow \mathbb{E}\hat{}$ of sheaves on $\mathbb{E}$ into the presheaves on $\mathbb{E}$ may be constructed by a two step iteration of the endofunctor $L\colon \mathbb{E}\hat{} \longrightarrow \mathbb{E}\hat{}$ defined for any presheaf P at any object X in $\mathbb{E}$ by the direct limit

$$LP(X) = \varinjlim_{R \in J(X)} \mathrm{Hom}_{\mathbb{E}\hat{}}(R, P). \tag{13.0.1}$$

Since the inverse image of a covering subfunctor is covering, this indeed defines a presheaf on $\mathbb{E}$ and an endofunctor $P \longmapsto LP$ on $\mathbb{E}\hat{}$ for which the canonical map

$$\ell_P(x)\colon P(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}\hat{}}(X, P) \longrightarrow \varinjlim_{R \in J(X)} \mathrm{Hom}_{\mathbb{E}\hat{}}(R, P) \tag{13.0.2}$$

which sends each element of $P(X)$ to its equivalence class in the direct limit defines a natural transformation

$$\ell_P\colon P \longrightarrow LP \tag{13.0.3}$$

in $\mathbb{E}\hat{}$.

(13.1) Proposition. This natural transformation has from its very construction the following properties:

(a) given any element covering $i_R\colon R \hookrightarrow X$ and any $t\colon R \longrightarrow P$, the transformation $v(t)\colon X \longrightarrow LP$ which represents the equivalence class of t in the direct limit has the property that the diagram

(13.1.0)
$$\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, hook, "i_R"'] & P \arrow[d, "\ell_P"] \\
X \arrow[r, "v(t)"'] & LP
\end{tikzcd}$$

is commutative in $\mathbb{E}\widehat{\ }$.

(b) given any $x\colon X \longrightarrow LP$, there exists a covering subfunctor $i_R\colon R \hookrightarrow X$ and a transfunctor $t\colon R \longrightarrow P$ such that $v(t) = x$.

From these two properties it follows that

(i) P is a separated presheaf (i.e. for any covering subfunctor $R \hookrightarrow X$, $\mathrm{Hom}(X, P) \longrightarrow \mathrm{Hom}(R, P)$ is injective) if and only if $\ell_P\colon P \longrightarrow LP$ is a monomorphism, and that

(ii) LP is a separated presheaf.

(13.2) Remark. $P \xrightarrow{\ell_P} LP$ is *not* the universal separated pre-sheaf associated to P, the subfunctor of LP *which is the image of* P *under* ℓ_P is, however, as may be seen by looking at the equivalence relation on P defined by ℓ_P.

Using the above facts, it follows that for any presheaf P, $aP = LLP$ is a sheaf and that the canonical map a_P obtained from the diagonal in the *commutative* square

(13.2.0)
$$\begin{tikzcd}
P \arrow[r, "\ell_P"] \arrow[d, "\ell_P"'] \arrow[dr] & LP \arrow[d, "\ell_{LP}"] \\
LP \arrow[r, "L(\ell_P)"'] & LLP = aP
\end{tikzcd}$$

is the *universal sheaf associated* to P (i.e. the unit for the left adjoint to the inclusion $\eta\colon \mathbb{E}\widetilde{\ } \longrightarrow \mathbb{E}\widehat{\ }$). The functor $a\colon \mathbb{E}\widehat{\ } \longrightarrow \mathbb{E}\widetilde{\ }$ is called the *associated sheaf functor* and has the following properties:

(13.3) Theorem.

(a) $a\colon \mathbb{E}\widehat{\ } \longrightarrow \mathbb{E}$ is a left exact left adjoint. It thus preserves all finite limits and all co-limits;

(b) A map $p\colon P \longrightarrow Q$ of presheaves is *covering* (i.e. the image of P in Q is a covering subfunctor for the induced topology on $\mathbb{E}\hat{\ }$) if and only if $a(p)\colon a(P) \longrightarrow a(Q)$ is an epimorphism in $\mathbb{E}\tilde{\ }$. In order that $a(p)\colon a(P) \to a(Q)$ be an isomorphism ($p\colon P \longrightarrow Q$ is then said to be *bicovering*) it is necessary and sufficient that both the arrow $p\colon P \longrightarrow Q$ and the canonical monomorphism $\Delta\colon P \longrightarrow P \times_Q P$ be covering.

(c) In order that $p\colon P \longrightarrow Q$ considered as an object of $\mathbb{E}\hat{\ }/Q$ be a *sheaf above* Q (i.e., that the presheaf on $\mathbb{E}/Q$ which is represented by $P(\quad)$ be a sheaf for the induced topology on $\mathbb{E}\hat{\ }/Q$) it is necessary and sufficient that the commutative square

(13.3.0)
$$\begin{tikzcd}
P \arrow[r, "a_P"] \arrow[d, "p"'] & a(P) \arrow[d, "a(P)"] \\
Q \arrow[r, "a_Q"'] & a(Q)
\end{tikzcd}$$

be cartesian in $\mathbb{E}\hat{\ }$.

(13.4) Corollary:

(a) A pre-sheaf P is a sheaf if and only if the canonical map $a_P\colon P \longrightarrow a(P)$ is an isomorphism;

(b) In order that a subfunctor $i_R\colon R \hookrightarrow Q$ be covering, it is necessary an sufficient that $a(i_R)\colon \alpha(R) \hookrightarrow a(Q)$ be an isomorphism.

(c) Every epimorphism $p\colon P \twoheadrightarrow Q$ of sheaves is covering; thus given any element $x \in Q(X)$, there exists a covering family $(X_a \xrightarrow{x_\alpha} X)$ in $\mathbb{E}$ such that $P(X_\alpha)^{-1}(xx_\alpha) \subseteq P(X_\alpha)$ is non-empty for each α. ("for each $X \in \mathrm{ob}(\mathbb{E})$, $p(X)\colon P(X) \longrightarrow Q(X)$ is locally surjective".)

(d) If Q is a sheaf, then $(\mathbb{E}/Q)\tilde{\ } \xrightarrow{\approx} \mathbb{E}\tilde{\ }/Q$; in particular if the topology on $\mathbb{E}$ is courser than the canonical topology so that every representable functor is a sheaf, then $(\mathbb{E}/X)\tilde{\ } \xrightarrow{\approx} \mathbb{E}\tilde{\ }/X$.

The proofs of each of these facts are easily established and, in any case may be found in DEMAZURE (1970) or SGA 4. We have summarized them here because of their relevance to the point of this section which is to complete the description of stacks *internally* in the category $\mathbb{E}\hat{\ }$. We will first make precise the obvious connection between split fibrations and pre-sheaves of categories and then define and use the split fibration associated with any fibration to place the fibrations over $\mathbb{E}$ within the context of category objects in $\mathbb{E}^{\widehat{}}$. We will then translate the notion of stack into this new context. We give the first part of this notion in some detail, in particular the development of the functor "evaluation" and "substitution", for much of the confusion in the subject can be traced to insufficient attention to the subtleties involved.

(14.) FIBRATIONS AND PRESHEAVES OF CATEGORIES.

(14.0) If $F\colon \mathbb{E}^{\mathrm{op}} \longrightarrow (\mathrm{CAT})$ is a functor whose underlying set values lie in (ENS), then F determines (and is determined by) a category object in
$\mathbb{E} = \mathrm{HOM}_{\mathrm{CAT}}(\mathbb{E}^{\mathrm{op}}, (\mathrm{ENS}))$ which has the well known simplicial from

(14.0.1)
$$\begin{tikzcd} F\colon F_1 \times_{F_0} F_1 \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & F_1 \arrow[l, bend left=30, "s_1"] \arrow[l, bend left=50, "s_0"] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & F_0 \arrow[l, bend left=40, "s_0"] \end{tikzcd}$$

defined through $F_0(X) = b(F(X))$, $F_1(X) = \mathrm{Ar}(F(X))$, with $d_0(X) = T(F(X))$ and $d_1(X) = S(F(X))$ etc. in which we use the standard simplicial notation for which d_i always indicates the face *opposite* the i^{th} vertex. The full simplicial object in $\mathbb{E}^{\widehat{}}$ is called the *nerve of the category object* in $\mathbb{E}^{\widehat{}}$. It is, of course, completely determined by its truncation.

It follows that any *split* fibration over $\mathbb{E}$ whose fibers lie in (CAT) has a splitting which as a presheaf of categories we may *identify* with the corresponding *internal* category object in $\mathbb{E}^{\widehat{}}$. If F is representable, then the system of objects and arrows in $\mathbb{E}$ which arises through any representatives of the functors $\mathrm{Ob}(F)$ and $\mathrm{Ar}(F)$ is a category object is in $\mathbb{E}$ and all of the category $\mathrm{CAT}(\mathbb{E})$ of category objects of $\mathbb{E}$ is determined in this fashion. *We will identify it with its image in* $\mathbb{E}^{\widehat{}}$ via the canonical Yoneda functor $\eta\colon \mathbb{E} \longrightarrow \mathbb{E}^{\widehat{}}$ (given by $\eta(X) = h_X$ where $h_X(T) = \mathrm{HOM}_{\mathbb{E}}(T, X) = X(T)$) as usual.

(14.1) The split fibration determined by a category object.

A presheaf $\mathbb{C}$ of categories on $\mathbb{E}$ is a category object in $\mathbb{E}^{\widehat{}}$. It thus determines canonically a presheaf of categories *on* $\mathbb{E}^{\widehat{}}$ (via the assignment $P \longmapsto \mathrm{HOM}_{\mathbb{E}^{\widehat{}}}(P, \quad)$) whose restriction to $\mathbb{E}$ (via $\eta\colon \mathbb{E} \hookrightarrow \mathbb{E}^{\widehat{}}$) is then isomorphic to $\mathbb{C}$ itself. We thus may consider the (split) fibration $\Phi(\mathbb{C})$ (7.13) determined by $\mathbb{C}$ over $\mathbb{E}$ and also by $P \longmapsto \mathrm{HOM}_{\mathbb{E}^{\widehat{}}}(P, \mathbb{C})$ over $\mathbb{E}^{\widehat{}}$. The latter is the (canonical) extension of $\Phi(\mathbb{C})$ *to* $\mathbb{E}^{\widehat{}}$ and will be denoted by $\mathbb{EX}(\mathbb{C})$ and called the *externalization* of $\mathbb{C}$. As with any internal category, the *split fibration determined by the category object* $\mathbb{C}$ may be described as follows: Any object of $\mathbb{EX}(\mathbb{C})$ with projection $P \in \mathrm{Ob}(\mathbb{E}^{\widehat{}})$ is simply a morphism $x\colon P \longrightarrow \mathrm{Ob}(\mathbb{C})$ in $\mathbb{E}^{\widehat{}}$. If $x\colon P \longrightarrow \mathrm{Ob}(\mathbb{C})$ and $y\colon Q \longrightarrow \mathrm{Ob}(\mathbb{C})$ are objects of $\mathbb{EX}(\mathbb{C})$ and $p\colon P \longrightarrow Q$ is a morphism in $\mathbb{E}^{\widehat{}}$, then an arrow $\theta\colon x \longrightarrow y$ in $\mathbb{EX}(\mathbb{C})$ with projection p is a morphism $\theta\colon P \longrightarrow \mathrm{Ar}(\mathbb{C})$ in $\mathbb{E}^{\widehat{}}$ such that $\mathbb{S}\theta = x$ and $\mathbb{T}\theta = yp$.

(14.1.0)
$$\begin{tikzcd}
P \arrow[r, "\theta"] \arrow[d, "p"'] \arrow[dr, ""{name=D}] & \mathrm{Ar}(\mathbb{C}) \arrow[d, shift right, "T(\mathbb{C})"'] \arrow[d, shift left, "S(\mathbb{C})"] \\
Q \arrow[r, "y"', ""{name=Y}] & \mathrm{Ob}(\mathbb{C})
\arrow[from=D, to=Y, dashed, bend right]
\end{tikzcd}$$

i.e. an arrow from x into $yp = \mathrm{Hom}(p, \mathbb{C})(y)$ in the category $\mathrm{Hom}_{\mathbb{E}\hat{\ }}(P, \mathbb{C})$ where $\mathrm{Hom}(f, \mathbb{C})\colon \mathrm{Hom}_{\mathbb{E}\hat{\ }}(Q, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{E}\hat{\ }}(P, \mathbb{C})$ is the "restriction" *functor* defined by composition with p. In $\mathbb{EX}(\mathbb{C})$ composition of θ and ξ

(14.1.1)
$$\begin{tikzcd}
P \arrow[d] \arrow[dr, ""{name=A}] & \\
Q \arrow[d] \arrow[r, ""{name=B}] & \mathrm{Ob}(\quad) \\
R \arrow[ur, ""{name=C}] &
\arrow[from=A, to=B, dashed, bend right]
\arrow[from=B, to=C, dashed, bend right]
\end{tikzcd}$$

is defined by the composition $\xi_p \circ \theta$ inside the category $\mathrm{Hom}_{\mathbb{E}\hat{\ }}(P, \mathbb{C})$. Note that an arrow of $\mathbb{EX}(\mathbb{C})$ *is cartesian if and only if the* defining morphism $\theta\colon P \longrightarrow \mathrm{Ar}(\mathbb{C})$ is an invertible arrow in the category $\mathrm{Hom}_{\mathbb{E}\hat{\ }}(P, \mathbb{C})$. Thus, in particular, if the category object $\mathbb{C}$ is a groupoid, every arrow in $\mathbb{EX}(\mathbb{C})$ is cartesian. For example, since any presheaf may be considered as a discrete groupoid object (i.e., every arrow is an identity), the corresponding split fibration defined by $P \in \mathrm{Ob}(\mathbb{E}\hat{\ })$ is simply the category $\mathbb{E}\hat{\ }/P \xrightarrow{\mathbb{S}} \mathbb{E}\hat{\ }$ whose restriction to $\mathbb{E}$ may be identified with the category $\mathbb{E}/P \xrightarrow{\mathbb{S}} \mathbb{E}$. If P is representable, with $P \xrightarrow{\sim} \eta(X)$, then the *split fibration over* $\mathbb{E}$ defined by the (contravariant) representable functor $\eta(X) = \mathrm{Hom}_{\mathbb{E}}(-, X)$ is just the category $\mathbb{E}/X \xrightarrow{f} \mathbb{E}$.

We now place all fibrations over $\mathbb{E}$ within $\mathrm{CAT}(\mathbb{E}\hat{\ })$:

(15) The category object $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}, \mathbb{F})$ in $\mathbb{E}$ defined by a fibration $F\colon \mathbb{F} \longrightarrow \mathbb{E}$

(15.0) If $F\colon \mathbb{F} \longrightarrow \mathbb{E}$ is any $\mathbb{E}$-category and $X \in \mathrm{Ob}(\mathbb{E})$, we may consider the category $\underline{\underline{\mathrm{Hom}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$ of $\mathbb{E}$-functors from $\mathbb{E}/X$ into $\mathbb{F}$ and its subcategory $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$ consisting of those $\mathbb{E}$-functors which carry every arrow of $\mathbb{E}/X$ into a cartesian arrow in $\mathbb{F}$ (since all arrows of $\mathbb{E}/X$ are cartesian over $\mathbb{E}$). In both cases natural transformations of $\mathbb{E}$-functors are required to project onto the identity (i.e., $F * \alpha = \mathrm{id}$).

(15.0.0)
$$\begin{tikzcd}
\mathbb{E}/X \arrow[rr, shift left=1.5, "c"] \arrow[rr, shift right=1.5, "d"'] \arrow[rr, phantom, "\downarrow\alpha"] \arrow[dr, "\mathbb{S}"'] & & \mathbb{F} \arrow[dl, "F"] \\
& \mathbb{E} &
\end{tikzcd}$$

We now look at $\underline{\underline{\underline{\mathrm{Hom}}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$ in detail. If $c\colon \mathbb{E}/X \longrightarrow \mathbb{F}$ is an $\mathbb{E}$-functor then the value of c on any object $Y \xrightarrow{f} X$ in $\mathbb{E}/X$ lies in the category $\mathbb{F}_Y$ which is the fiber of $\mathbb{F}$ at $Y = \mathbb{S}(f)$ (7 .8). In particular the value of c at the terminal object $X \xrightarrow{\mathrm{id}} X$ of $\mathbb{E}/X$ lies in the fiber $\mathbb{F}_X$ and any $\mathbb{E}$-natural transformation of such $\mathbb{E}$-functors has its value at X lying in this same fiber, so that we have defined a functor, *evaluation at* X,

(15.0.1)
$$\mathrm{ev}_X\colon \mathrm{Hom}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F}) \longrightarrow \mathbb{F}_X\,.$$

(15.1) Proposition. The functor evaluation at X is fully faithful on the subcategory $\underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$ and is an equivalence provided $\mathbb{F}$ is a fibration above $\mathbb{E}$.

In effect let $\alpha\colon c_1 \longrightarrow c_2$ be in $\underline{\underline{\underline{\mathrm{Cart}}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$, then since

(15.1.0)
$$\begin{tikzcd}
Y \arrow[rr, "y"] \arrow[dr, "y"'] & & X \arrow[dl, "\mathrm{id}"] \\
& X &
\end{tikzcd}$$

defines the canonical arrow from f to the terminal object X $(= \mathrm{id}X)$ in $\mathbb{E}/X$, $c_2(y)\colon c_2(Y) \longrightarrow c_2(X)$ is cartesian in $\mathbb{F}$. Hence for any natural transformations $\alpha, \beta\colon c_1 \rightrightarrows c_2$, the diagram(s)

(15.1.1)
$$\begin{tikzcd}
c_1(Y) \arrow[r, "c_1(y)"] \arrow[d, shift right, "\alpha(Y)"'] \arrow[d, shift left, "\beta(Y)"] & c_1(X) \arrow[d, shift right, "\alpha(X)"'] \arrow[d, shift left, "\beta(X)"] \\
c_2(Y) \arrow[r, "c_2(y)"'] & c_2(X)
\end{tikzcd}$$

are commutative for any object $Y \xrightarrow{y} X$ in $\mathbb{E}/X$. If $\mathrm{ev}_X(\alpha) = \mathrm{ev}_X(\beta)$, then $\alpha(X) = \beta(X)$ so that $c_2(y)\alpha(Y) = c_2(y)\beta(Y)$. Thus since $c_2(y)$ is cartesian, $\alpha(Y) = \beta(Y)$ and ev_X

is faithful. That is full follows immediately from the definition of $\alpha(Y)\colon c_1(Y) \longrightarrow c_2(Y)$ as that unique arrow which makes the diagram

(15.1.2)
$$\begin{tikzcd}
c_1(Y) \arrow[r, "c_1(y)"] \arrow[d, "\alpha(Y)"'] & c_1(X) \arrow[d, "f"] \\
c_2(Y) \arrow[r, "c_2(y)"'] & c_2(X)
\end{tikzcd}$$

commutative for a given $f \in \mathrm{Hom}_{\mathbb{F}_X}(c_1(X), C_2(X))$.

Now suppose that $\mathbb{F}$ is a fibration for which some cleavage $\aleph$ has been chosen. Let $C \in \mathrm{Ob}(\mathbb{F}_X)$. We define a cartesian $\mathbb{E}$-functor $C^*\colon \mathbb{E}/X \longrightarrow \mathbb{F}$ via the assignment

(15.1.3)
$$\begin{tikzcd}
Y_1 \arrow[rr, "\alpha"] \arrow[dr, "y_1"'] & & Y \arrow[dl, "y_2"] & \longmapsto & C^*(y_1) \arrow[r, "C(\alpha)"] \arrow[d, equal] & C^*(y_2) \arrow[d, equal] \\
& X & & & y_1^*(C) \arrow[r, "\alpha(C)"'] & y_2^*(C)
\end{tikzcd}$$

where $y_1^*(C)$ and $y_2^*(C)$ are the chosen inverse images of C by y_1 and y_2, and $\alpha(C)$ is that unique arrow in $\mathbb{F}$ which projects onto α and makes the diagram

(15.1.4)
$$\begin{tikzcd}
y_1^*(C) \arrow[rr, "\alpha(c)"] \arrow[dr, "C_{y_1}"'] & & y_2^*(C) \arrow[dl, "C_{y_2}"] \\
& C &
\end{tikzcd}$$

commutative, with C_{y_1} and C_{y_2} the morphisms of transport for the cleavage $\aleph$. Note that since $C_{y_2}\alpha(c) = C_{y_1}$, the fact that C_{y_1} and C_{y_2} are cartesian focus on $\alpha(C)$ to be cartesian as well. Moreover, if $f\colon C \longrightarrow D$ is an arrow in $\mathbb{F}_X$, the unique arrow $y_1^*(f)$ in $\mathbb{F}$ which projects onto Y_1 and makes the diagram

(15.1.5)
$$\begin{tikzcd}
y_1^*(C) \arrow[r, "C_{y_1}"] \arrow[d, dashed, "y_1^*(f)"'] & C \arrow[d, "f"] \\
y_1^*(D) \arrow[r] & D
\end{tikzcd}$$

commutative, clearly defines a natural transformation $f^*\colon C^* \longrightarrow D^*$. Finally, the assignment $C \longmapsto C^*$ defines a functor, *substitution* at X,

(15.1.6)
$$\mathrm{sub}_X\colon \mathbb{F}_X \longrightarrow \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$$

which is clearly a quasi-inverse for evaluation at X. In fact, if the cleavage used to define the functor $\mathrm{sub}_X(C) = C^*$ is normalized, than this quasi-inverse defines a functorial *section* for the evaluation functor.

(15.2) Definition. For any fibration $\mathbb{F}$, the assignment

(15.2.0)
$$\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[d, "f"'] & \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/Y, \mathbb{F}) \\
X \arrow[r, mapsto] & \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F}) \arrow[u, "\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/f, \mathbb{F})"']
\end{tikzcd}$$

defines a presheaf of categories $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ in $\mathbb{E}^{\wedge}$ (provided the fibers of $\mathbb{F}$ are of the appropriate size). Considered as a category object in $\mathbb{E}^{\wedge}$, it will be called the *cartesian internalization* of the fibration $\mathbb{F}$. The split fibration which this category object defines over $\mathbb{E}^{\wedge}$ will be called the *canonical extension* of $\mathbb{F}$ *to* $\mathbb{E}^{\wedge}$ and denoted by $\mathbb{F}^{\wedge} = \mathbb{EX}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{E}))$. Its restriction to $\mathbb{E}$ is the split fibration defined by the functor $X \longmapsto \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F})$. It will be denoted by $\mathbb{S}(\mathbb{F})$ and called the (right adjoint) *split fibration associated* with $\mathbb{F}$.

(15.3) Proposition. The assignment to each object X in $\mathbb{E}$ of the functor evaluation at X has the natural structure of a morphism of pseudofunctors (7.10) from the (canonical) splitting, $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$, associated with $\mathbb{S}(\mathbb{F})$ to the pseudofunctor $\mathbb{F}^{\chi}$ defined by any cleavage χ associated with the fibration $\mathbb{F}$.

As a morphism of pseudofunctors, it is an equivalence with quasi-inverse defined through substitution at X.

If $\mathbb{F}$ is a split fibration, then the functor substitution at X becomes a natural transformation and defines an internal functor from the category object in $\mathbb{E}^{\wedge}$ defined by any splitting of $\mathbb{F}$ into the cartesian internalization $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ of the fibration $\mathbb{F}$. This internal functor is an essential equivalence (i.e., a fully-faithful and essentially epimorphic functor in $\mathrm{CAT}(\mathbb{E}^{\wedge})$) but does not have evaluation at X as internal quasi-inverse.

In effect let $\aleph$ be a cleavage for the fibration $\mathbb{F}$, and $f\colon Y \longrightarrow X$ an arrow in $\mathbb{E}$.

(15.3.0)
$$\begin{tikzcd}
\mathbb{F}^{\aleph}_X \arrow[r, "\mathrm{sub}_X"] \arrow[d, "\mathbb{F}^{\aleph}_f"] & \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/X, \mathbb{F}) \arrow[r, "\mathrm{ev}_X"] \arrow[d, "\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/f, \mathbb{F})"] & \mathbb{F}^{\aleph}_X \arrow[d, "\mathbb{F}^{\aleph}_f"] \\
\mathbb{F}^{\aleph}_Y \arrow[r] \arrow[ur, phantom, "\beta^{\aleph}_f" description] & \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/Y, \mathbb{F}) \arrow[r, "\mathrm{ev}_Y"'] \arrow[ur, phantom, "\alpha^{\aleph}_f" description] & \mathbb{F}^{\aleph}_Y
\end{tikzcd}$$

We define natural isomorphisms $\alpha_f^{\aleph} \colon \mathrm{ev}_Y \circ \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/f, \mathbb{F}) \xrightarrow{\sim} \mathbb{F}_X^{\aleph} \circ \mathrm{ev}_x$ and $\beta_f^{\aleph} \colon \mathrm{sub}_Y \circ \mathbb{F}_f^{\aleph} \xrightarrow{\sim} \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/f, \mathbb{F}) \circ \mathrm{sub}_X$ as follows: If $\theta \colon \mathbb{E}/X \longrightarrow \mathbb{F}$ is a cartesian $\mathbb{E}$-functor, then since the diagram

(15.3.1)

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "f"'] & & Y \arrow[dl, "\mathrm{id}_X"] \\
& X &
\end{tikzcd}$$

is commutative in $\mathbb{E}/X$, $\theta(f) \colon \theta(f) \longrightarrow \theta(\mathrm{id}_X)$ is a cartesian arrow in $\mathbb{F}$ which projects onto f and has $\mathrm{ev}_X(\theta) = \theta(\mathrm{id}_X \colon X \longrightarrow X)$ as its target and $\mathrm{ev}_y \circ \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/f, \mathbb{F})(\theta) = \mathrm{ev}_y(\theta \circ \mathbb{E}/f) = \theta \circ \mathbb{E}/f\,(\mathrm{id}_y) = \theta(f \circ \mathrm{id}_y) = \theta(f)$ as its source. If $\aleph$ is a chosen cleavage for $\mathbb{F}$, then $\mathbb{F}_f^{\aleph} \circ \mathrm{ev}_X(\theta) = f^{\aleph}(\theta(\mathrm{id}_x))$ is the chosen inverse image of $\theta(\mathrm{id}_x)$ by f which has its morphism of transport $f_{\chi} \colon f^{\chi}(\theta(\mathrm{id}_x)) \longrightarrow \theta(\mathrm{id}_x)$ also projecting on $f \colon Y \longrightarrow X$ in $\mathbb{F}$. Thus there exists a unique Y-arrow (necessarily an isomorphism since θ is cartesian) $\alpha_f^{\aleph} \colon \theta(f) \xrightarrow{\sim} f^{\aleph}(\theta(\mathrm{id}_x))$ which makes the diagram

(15.3.2)

$$\begin{tikzcd}
f^{\aleph}(\theta(\mathrm{id}_x)) \arrow[r] \arrow[d, dashed, "\alpha_f^{\aleph}"'] & \theta(\mathrm{id}_x) \\
\theta(f) \arrow[ur, "\theta(f)"'] &
\end{tikzcd}$$

commutative. The so defined α is clearly natural and defines a morphism of the canonical splitting of $\mathbb{S}(\mathbb{F})$ to the pseudofunctor defined by the cleavage $\aleph$. *Note that even if* $\aleph$ *is a splitting for* $\mathbb{F}$*,* α *is not necessarily the identity unless* $\mathbb{F}$ *happens to be discrete.*

We now turn to the definition of $\beta_f^{\aleph}$. If $A \in \mathrm{Ob}(\mathbb{F}_x^{\aleph})$ then $f_{\aleph}(A) \colon f^{\aleph}(A) \longrightarrow A$, the morphism of transport at A for the cleavage $\aleph$, has as source the value of $\mathbb{F}_f^{\aleph}$ at A. $\mathrm{sub}_Y(\mathbb{F}_f^{\aleph}(A))$ is then the functor which has its value at the object $t \colon T \longrightarrow Y$ of $\mathbb{E}/Y$ the inverse image $t^{\aleph}(f^{\aleph}(A)) \in \mathrm{Ob}(\mathbb{F}_T)$. Now on the other side of the square, $\underline{\underline{\mathrm{Cart}}}_{\mathbb{F}}(\mathbb{E}/f)(\mathrm{sub}_X(A)) = \mathrm{sub}_X(A) \circ \mathbb{E}/f$ is the cartesian functor whose value at the object $t \colon T \longrightarrow Y$ of $\mathbb{E}/Y$ is the object $(ft)^{\aleph}(A)$, again in $\mathbb{F}_T$. Thus we may use the canonical isomorphism $\aleph(t, f) \colon t^{\aleph}(f^{\aleph}(A)) \xrightarrow{\sim} (ft)^{\aleph}(A)$

(15.3.3)

$$\begin{tikzcd}
(ft)^{\aleph}(A) \arrow[drr] & & \\
t^{\aleph}(f^{\aleph}(A)) \arrow[u, "\aleph(f,t)", "\wr"'] \arrow[r] & f^{\aleph}(A) \arrow[r] & A
\end{tikzcd}$$

to define $\beta^{\aleph}(A)\colon \mathrm{sub}_y^{\aleph} \circ \mathbb{F}_f^{\aleph}(A) \longrightarrow \mathrm{Cart}(\mathbb{E}/f) \circ \mathrm{sub}_x(A)$ as the natural isomorphism whose value at $t \in \mathrm{Ob}(\mathbb{E}/Y)$ is just $\aleph(f,t)$. This definition again clearly defines the desired natural isomorphism $\beta_f^{\aleph}$. Note here that if, in contrast to the definition of $\alpha_f^{\aleph}$, *if* $\aleph$ *is a splitting for the fibration* $\mathbb{F}$, $\beta_f^{\aleph}$ *is the identity so that substitution becomes a natural transformation of presheaves of categories and thus defines a functor of the corresponding category objects*

$$\mathrm{sub}\colon \mathbb{F} \longrightarrow \underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{E}\mathbb{X}(F))$$

in $\mathbb{E}\hat{}$. Moreover, since for each X, sub_X is fully faithful with ev_X as a quasi-inverse, sub is in fact an essential equivalence of category objects in $\mathbb{E}\hat{}$ (actually injective on objects provided $\aleph$ is normalized as it is usually required to be). Since ev is not necessarily a natural transformation and thus does not lie in $\mathrm{CAT}(\mathbb{E}\hat{})$, it follows that sub does not in general have a quasi-inverse in $\mathrm{CAT}(\mathbb{E}\hat{})$ and Proposition (15.3) is established.

(16.0) The use of the category object in $\mathbb{E}\hat{}$ $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ and the associated sheaf functor $P \longmapsto a(P)$ now will allow us to internalize the notion of completeness of the fibration $\mathbb{F}$ within the topos $\mathbb{E}\hat{}$ of presheaves on $\mathbb{E}$. With this characterization the principal advantage of $\mathrm{CAT}/\mathbb{E}$ over the more manageable topos $\mathbb{E}\hat{}$ as a “common home” for both presheaves and fibrations disappears since all relevant information about $\mathbb{F}$ can be expressed in terms of $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ using the fact that $\mathrm{Hom}_{\mathbb{E}\hat{}}(P, \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F}) \approx \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/P, \mathbb{F})$ established in (I7.9). The characterization turns out to be surprisingly simple to state and allows a unification of the preceding three descriptions of descent theory. The procedure is given for groupoids in the text (I6 and I7) but with minor modifications of the definitions and the replacement of “torsor under a groupoid” with “locally representable internal presheaf on a category” in the proofs all of the results remain valid as stated in the text. We leave this modification to the reader and will content ourselves here with a few observations about the re-formulation of the basic notions of descent for a pseudo-functor $\mathbb{F}_{(\,)}^{C}$ on $\mathbb{E}^{\mathrm{op}}$ in terms of the category object $\underline{\underline{\mathrm{Cart}}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ in $\mathbb{E}\hat{}$, defined through the fibration $\mathbb{F}$ associated with $\mathbb{F}_{(\,)}^{c}$.

(16.1) First note that if $C = (X_\alpha \xrightarrow{x_\alpha} X)_{\alpha \in I}$ is a covering of X in $\mathbb{E}$, then in $\mathbb{E}\hat{}$ we may define its <u>nerve</u> as the simplicial object $\mathrm{Ner}(C/X)$

$$\begin{tikzcd} \coprod X_\alpha \times_X X_\rho \times_X X_\gamma \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod X_\alpha \times_X X_\alpha \arrow[r] & \coprod X_\alpha \arrow[r] & X \end{tikzcd} \tag{16.1.0}$$

in $\mathbb{E}\hat{}$ which always exists (independently of the existence of fiber products or coproducts in $\mathbb{E}$) with the identification of the X_α with the corresponding representable functors in $\mathbb{E}\hat{}$. Now in $\mathbb{E}\hat{}$ using the properties of $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$, in particular (I 7.9), it is easy to see that *a descent datum on the pseudo functor* $\mathbb{F}_{(\,)}^{c}$ *over the covering* $C = (X_\alpha \longrightarrow X)$

(4.8) corresponds to nothing more than *a simplicial map*

(16.1.1)
$$\begin{tikzcd}
\coprod X_\alpha \times X_\rho \times X_\gamma \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \mathrm{Ar}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \times_{\mathrm{ob}} \mathrm{Ar}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\coprod\limits_{\alpha,\beta\in I\times I} X_\alpha \times_X X_\beta \arrow[r] \arrow[d, shift left=2] \arrow[d, shift right=2] & \mathrm{Ar}(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \arrow[d, shift left=2] \arrow[d, shift right=2] \\
\coprod\limits_{\alpha\in I} X_\alpha \arrow[r] \arrow[u] \arrow[d] & \mathrm{Ob}(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \arrow[u] \\
X &
\end{tikzcd}$$

from the (positive part of) *nerve of C into the nerve of the category object* $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$, *while a morphism of descent data corresponds to a homotopy of the corresponding simplicial maps.* (Or. if one prefers, to internal functors and natural transformations from the groupoid $\coprod X_\alpha \times X_\beta \rightrightarrows \coprod X_\alpha$ into the category $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$.

In effect, the translation is simple, for instance

$$\mathrm{Nat}(\coprod_{\alpha\in I} X_\alpha, \mathrm{Ob}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))) \xrightarrow{\sim} \prod_{\alpha\in I} \mathrm{Nat}(X_\alpha, \mathrm{Ob}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))) \xrightarrow{\sim}$$

$$\prod_{\alpha\in I} \mathrm{Ob}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/X_\alpha,\mathbb{F})) \xrightarrow{\sim} \prod_{\alpha\in I} \mathrm{Ob}(\mathbb{F}^c_{X_\alpha})$$

Thus the map $\coprod X_\alpha \longrightarrow \mathrm{ob}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))$ corresponds to a family of objects in the fibers above the X_α of the covering. Similarly the map

$$\coprod X_\alpha \times X_\beta \longrightarrow \mathrm{Ar}(\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))$$

corresponds to a family of (necessarily) isomorphisms of (A 4.8.0) and the simplicial identities then are equivalent to the cocycle condition (A 4.8.2). Thus using $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$ a descent datum really becomes a "Čech-cocycle on the covering with coefficients in a category" in $\mathbb{E}\hat{}$.

(16.2) Moreover the image of $\coprod X_\alpha \longrightarrow X$ is the covering subfunctor $R \hookrightarrow X$ of the topology on $\mathbb{E}$ and since the category $\mathrm{Nat}(R, \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))$ is equivalent to the category $\underline{\mathrm{Cart}}_{\mathbb{E}}(R,\mathbb{F})$, which is, in turn, equivalent to the category of descent data in $\mathbb{F}^c_{(\,)}$ over the covering $(X_\alpha \longrightarrow X)$ we see that the canonical functor

(16.2.0)
$$\mathrm{Hom}_{\mathbb{E}\hat{}}(R, \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \longrightarrow \mathrm{Simpl}_{\mathbb{E}\hat{}}(\mathrm{Ner}(C/X), \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))$$

defined by restriction along the epimorphism $\coprod X_\alpha \longrightarrow R$ is an equivalence of categories. From this *it follows that the category object* $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}))$ *is always complete* in the canonical topology of $\mathbb{E}\hat{\ }$ and that for $\mathbb{F}^c_{(\,)}$ to be complete in the topology of $\mathbb{E}$, it is necessary and sufficient that the canonical functor

$$\text{(16.2.1)} \qquad \mathrm{Nat}(X, \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})) \longrightarrow \mathrm{Simpl}_{\mathbb{E}\hat{\ }}(\mathrm{Ner}(C^+/X), \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F}))$$

defined by restriction along the covering transformation $\coprod X_\alpha \longrightarrow X$ be an equivalence of categories, i.e. that $\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ be complete in the induced topology on $\mathbb{E}\hat{\ }$. In this fashion external completeness of $\mathbb{F}^c_{(\,)}$ becomes internal completeness of $\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ in the induced topology on the topos $\mathbb{E}\hat{\ }$, and completes the translation of the original Grothendieck formulation of descent into terms of the internal category theory of $\mathbb{E}\hat{\ }$ as done in I6 and I7 of the text.

(16.3) In similar terms the second version also becomes immediate in $\mathbb{E}\hat{\ }$; since for any presheaf $P\colon \mathbb{E}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{ENS}$, $\varprojlim P \xrightarrow{\sim} \Gamma(P) = \mathrm{Nat}(\mathbb{1}, P)$ and the pull back of $\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ over P, admits the equivalence

$$\text{(16.3.0)} \qquad \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F}))\ P \xrightarrow{\approx} \underline{\mathrm{Cart}}_{\mathbb{E}/P}(\mathbb{E}/P/\text{-}\,, \mathbb{E}/P \times_{\mathbb{E}} \mathbb{F})$$

the chain of isomorphisms and equivalences

$$\varprojlim \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F}) \xrightarrow{\ \ \sim\ \ } \Gamma(\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})) = \mathrm{Nat}(\mathbb{1}, \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F}))$$

$$\xrightarrow{\ \ \approx\ \ } \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\mathbb{1}, \mathbb{F}) \xrightarrow{\ \ \sim\ \ } \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}, \mathbb{F}) \xrightarrow{\ \ \approx\ \ } \varprojlim \mathbb{F}^c_{(\,)}\,,$$

the formulation of (A 5.5) becomes an immediate consequence. Note that for a presheaf of categories $\mathbb{C}$, the canonical functor substitution in $\mathbb{E}\hat{\ }$

$$\mathrm{sub}\colon \mathbb{C} \longrightarrow \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{E}\mathbb{X}\mathbb{C})$$

is only an *essential equivalence* in $\mathbb{E}\hat{\ }$ and thus the canonical functor

$$\varprojlim \mathbb{C} \longrightarrow \varprojlim \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{E}\mathbb{X}(\mathbb{C})) \approx \varprojlim \mathbb{C}$$

is not an equivalence unless $\mathbb{C}$ is complete in the canonical topology on $\mathbb{E}\hat{\ }$. The objects of $\varprojlim \mathbb{C}$ are thus simplicial maps from a covering of all of $\mathbb{E}$ rather than compatible families of elements of the $\mathbb{C}(X)$.

(17) The Completion of a Pre-complete Fibration.

(17.0) Our final comments of this appendix will be devoted to the description of the completion of a pre-complete fibration since in the case of the description is particularly nice and will justify our handling in the text of the pre-stack of pre-ties over the original topos.

(17.1) Theorem: Let $\mathbb{F}^c_{(\,)}$ be a pre-stack so that its associated fibration $\mathbb{F}$ is pre-complete. Then the presheaf of categories $L\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$ is complete in the topology on $\mathbb{E}$, with the fully faithful bicovering cartesian functor

$$\begin{tikzcd}
 & \mathbb{LSF} \\
\mathbb{F} \arrow[ur] \arrow[r] & \mathbb{EX}_{\mathbb{E}}(L\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \arrow[u, equal]
\end{tikzcd}$$

induced by $\ell\colon \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}) \longrightarrow L\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$ as a universal map. Moreover, the canonical fully faithful covering monomorphic functor $\mathbb{LSF} \longrightarrow \mathbb{KSF}$ induced by the fully faithful bicovering functor

$$\ell_L\colon L\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}) \longrightarrow LL\underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F}) = a\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$$

is a cartesian equivalence of fibered categories so that the fully faithful bicovering functor $\mathbb{F} \longrightarrow \mathbb{KSF}$ may equally be taken as a sheaf theoretic completion of $\mathbb{F}$ (c. f. GIRAUD (1970)).

In effect let $\mathbb{C} = \underline{\mathrm{Cart}}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})$, then using I7.11(c) and (d) we know that the composable pair of canonical functors

(17.1.0)
$$\mathbb{C} \xrightarrow[\ell_{\mathbb{C}}]{} L\mathbb{C} \xrightarrow[\ell_{L\mathbb{C}}]{} a\mathbb{C}\ (= LL\mathbb{C})$$

are both fully faithful since $\mathbb{F}$ is pre-complete. Thus let

$$\begin{tikzcd}
C/X\colon \coprod X_\alpha \times X_\beta \arrow[r, shift left] \arrow[r, shift right] & \coprod X_\alpha \arrow[r] \arrow[dr] & X \\
 & & R \arrow[u, hook]
\end{tikzcd}$$

be the nerve of a covering of X and $d\colon C/\overset{+}{X} \longrightarrow L\mathbb{C}$ be a simplicial map (i.e. a descent datum over C/X in $\mathbb{EX}(L\mathbb{C})$; we wish to show that the canonical functor (obtained by composition) $\mathrm{triv}(p)\colon \mathrm{Hom}_{\mathbb{E}^{\wedge}}(X, L\mathbb{C}) \longrightarrow \mathrm{Simpl}_{\mathbb{E}^{\wedge}}(C/X, L\mathbb{C})$ is an equivalence of categories.

We first note that $\mathrm{triv}(p)$ is fully faithful: Since the commutative diagram of categories and functors

(17.1.1)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{E}^{\wedge}}(X, L\mathbb{C}) \arrow[r, "\ell_{f\mathbb{C}}"] \arrow[d, "\mathrm{triv}(p)"'] & \mathrm{Hom}_{\mathbb{E}^{\wedge}}(X, a(\mathbb{C})) \arrow[d, "\mathrm{triv}(p)"] \\
\underline{\mathrm{Simpl}}_{\mathbb{E}^{\wedge}}(C^+/X, L\mathbb{C}) \arrow[r, "\ell_{L\mathbb{C}}"'] & \underline{\mathrm{Simpl}}_{\mathbb{E}^{\wedge}}(C^+/X, a\mathbb{C})
\end{tikzcd}$$

has its top and bottom functors fully faithful (since $\ell_{L\mathbb{C}}$ is fully faithful) and its right hand side functor fully faithful (since any sheaf of categories is always pre-complete), it follows that the left hand side

$$p\colon \mathrm{Hom}_{\mathbb{E}^{\wedge}}(X, L\mathbb{C}) \longrightarrow \underline{\mathrm{Simpl}}_{\mathbb{E}^{\wedge}}(C^+/X, L\mathbb{C})$$

is fully faithful, as desired.

We now shall show that p^* is essentially surjective. Thus let $d\colon C/X^+ \longrightarrow L\mathbb{C}$ be a descent datum. We wish to show that there exists a map $d'\colon X \longrightarrow \mathrm{Ob}(L\mathbb{C})$ such that the trivial simplicial map: $d'p\colon C^+/X \longrightarrow L\mathbb{C}$ is isomorphic to d in $\underline{\mathrm{Simpl}}_{\mathbb{E}^{\wedge}}(C^+/X, L\mathbb{C})$: Thus consider $d_0\colon \coprod X_\alpha \longrightarrow \mathrm{Ob}(L\mathbb{C})$; since
$\ell_{\mathrm{Ob}(\mathbb{C})}\colon \mathrm{Ob}(\mathbb{C}) \longrightarrow L\mathrm{Ob}(\mathbb{C})$ $(= \mathrm{Ob}(L\mathbb{C}))$ is bicovering,

$$d_0^{\#}\colon \mathrm{Ob}(\mathbb{C}) \times_{\mathrm{Ob}(L\mathbb{C})} \coprod X_\alpha \longrightarrow \coprod X_\alpha$$

is also bicovering. Thus if we represent its source as a quotient of representables $\coprod Y_\gamma \longrightarrow \mathrm{Ob}(\mathbb{C}) \times_{\mathrm{Ob}(L\mathbb{C})} \coprod X_\alpha$, we obtain on composition the following diagram:

(17.1.2)

$$\begin{tikzcd}
 & \mathrm{Ar}(\mathbb{C}) \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] & & L\mathrm{Ar}(\mathbb{C}) \arrow[dd, shift left] \arrow[dd, shift right] \\
\coprod Y_\gamma \times_X Y_\xi \arrow[rr, "r_1"] \arrow[dd, shift left] \arrow[dd, shift right] & & \coprod X_\alpha \times_X X_\beta \arrow[ur] \arrow[dd, shift left] \arrow[dd, shift right] & \\
 & \mathrm{Ob}(\mathbb{C}) \arrow[rr] & & L\mathrm{Ob}(\mathbb{C}) \\
\coprod Y_\gamma \arrow[ur, "d'_0"'] \arrow[rr, "r_0"] \arrow[dr, two heads, "\bar{q}"'] & & \coprod X_\alpha \arrow[ur, "d_0"'] \arrow[d, two heads, "\bar{p}"] & \\
 & R' \arrow[r, hook] \arrow[dr, hook, "\mathrm{i}_{\bar{R}}"'] & R \arrow[d, hook, "\mathrm{i}_R"] & \\
 & & X &
\end{tikzcd}$$

in which $r_0\colon \coprod Y_\gamma \longrightarrow \coprod X_\alpha$ is covering and has the square

(17.1.3)
$$\begin{tikzcd}
\coprod Y_\gamma \times_X Y_\xi \arrow[r, "r_1"] \arrow[d, hook] & \coprod X_\alpha \times_X X_\beta \arrow[d, hook] \\
\coprod Y_\gamma \times \coprod Y_\gamma \arrow[r, "r_0 \times r_0"] & \coprod X_\alpha \times_X \coprod X_\alpha
\end{tikzcd}$$

cartesian since $pr_0 = q$ (so that the equivalence relation $\Re(\bar{q}) \xrightarrow{\sim} \coprod Y_\gamma \times_X Y_\xi$ is isomorphic to the inverse image under r_0 of the equivalence relation $\Re(p) \xrightarrow{\sim} \coprod X_\alpha \times_X X_\beta$).

Letting C'/X be the nerve of the covering $\coprod Y_\gamma \xrightarrow{q} X$, we see that the r_i define a simplicial map of coverings $r\colon C'^{+}/X \longrightarrow C^{+}/X$ and thus by composition with $d\colon C^{+}/X \longrightarrow L\mathbb{C}$ a simplicial map $dr\colon C'/X \longrightarrow L\mathbb{C}$ in which $\ell_{\Re(\ell)} d'_0 = d_0 r_0$.

We now claim that since $\ell\colon \mathbb{C} \longrightarrow L\mathbb{C}$ is fully faithful, $d'_0\colon \coprod Y_\gamma \longrightarrow \mathrm{Ob}(\mathbb{C})$ can be extended to a full simplicial map (descent datum) $d'\colon C'/X \longrightarrow \mathbb{C}$ such that $\ell\, d' = d\, r$: In effect recall that for any augmented simplicial object $X^{+}_{\bullet} \longrightarrow X$ in any category, a simplicial map $s\colon X^{+}_{\bullet} \longrightarrow \mathrm{Ner}(\mathbb{C})$ into the nerve of a category may be described as the set of (normalized) "l-cocyles" of the co-complex of categories and functors obtained by "homing" the complex term by term into the category $\mathbb{C}$:

$$\mathrm{Hom}(X^{+}_{\bullet}, \mathbb{C})\colon\ \mathrm{Hom}(X_0, \mathbb{C}) \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow}} \mathrm{Hom}(X_1, \mathbb{C}) \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} \mathrm{Hom}(X_2, \mathbb{C})\ ,$$

i.e. an object x of $\mathrm{Hom}(X_0, \mathbb{C})$ together with an arrow $f\colon d_1(x) \longrightarrow d_0(x)$ in $\mathrm{Hom}(X_1, \mathbb{C})$ such that $d_0(f) d_2(f) = d_1(f)$ and $s_0(f) = \mathrm{id}(x)$ while a homotopy of such l-cocycles is an arrow $h\colon x_1 \longrightarrow x_2$ in $\mathrm{Hom}(X_0, \mathbb{C})$ such that $f_2 d_l(h) = d_l(h) f_l$. If $X_{\bullet}$ is the nerve of a covering then any such f is necessarily an isomorphism. Now in the case at hand $\ell_{\mathbb{C}}\colon \mathbb{C} \longrightarrow L\mathbb{C}$ is fully faithful so that we have a map of co-complexes of categories and functors

$$\mathrm{Hom}(X^{+}_{\bullet}, \ell_{\mathbb{C}})\colon\ \mathrm{Hom}(X^{+}_{\bullet}, \mathbb{C}) \longrightarrow \mathrm{Hom}(X^{+}_{\bullet}, L\mathbb{C})$$

which is term by term fully faithful and in which we have an object $\alpha'_0\colon \coprod Y_\alpha \longrightarrow \mathrm{Ob}(\mathbb{C})$ whose image under $\ell_{\mathbb{C}}$ is supplied with a cocycle structure in $\mathrm{Hom}(X^{+}_{\bullet}, L\mathbb{C})$. Since $\mathrm{Hom}(X_i, \mathbb{C}) \longrightarrow \mathrm{Hom}(X_i, L\mathbb{C})$ is fully faithful, d'_0 is supplied with a cocycle structure d' in $\mathrm{Hom}(X^{+}_{\bullet}, \mathbb{C})$ whose image under $\ell_{\mathbb{C}}$ is dr. Thus we have obtained the desired descent datum d'.

We may now use the fact that $\mathbb{C} = \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,, \mathbb{F})$ is complete in the canonical topology to obtain an arrow $x\colon R' \longrightarrow \mathrm{Ob}(\mathbb{C})$ such that the trivial map $x\,\bar{q}\colon C'/X \longrightarrow \mathbb{C}$ is isomorphic to $d'\colon C'/X \longrightarrow \mathbb{C}$. Now $x\colon R' \longrightarrow \mathrm{Ob}(\mathbb{C})$ has as its source a covering subfunctor of X and thus, by definition of L, defines an arrow $v(x)\colon X \longrightarrow L\,\mathrm{Ob}(\mathbb{C})$ such that $\ell_{\mathrm{Ob}(\mathbb{C})}x = v(x)\mathrm{i}_{R'}$. We claim that the trivial map defined by $v(x)p$ is isomorphic to d in $\mathrm{Simpl}_{\mathbb{E}^\wedge}(C^+/X, L\mathbb{C})$. For this note that $v\,\mathrm{triv}(v(x)p) = \ell_{\mathbb{C}}\,\mathrm{triv}(x\bar{q})$ and since $\mathrm{triv}(x\bar{q})$ is isomorphic to d', $\ell_{\mathbb{C}}d'$ is isomorphic to $\ell_{\mathbb{C}}\mathrm{triv}(x\bar{q})$. Thus since $\ell_{\mathbb{C}}d' = dr$, we have that dr is isomorphic to $\mathrm{triv}(v(x)p)r$, that is their restrictions along r are isomorphic. Now to conclude that $\mathrm{triv}(v(x)p)$ and d are isomorphic it will be sufficient to observe that $\ell_{L\mathbb{C}}d$ and $\ell_{L\mathbb{C}}\mathrm{triv}(v(x)p)$ are isomorphic since $\ell_L\colon L\mathbb{C} \longrightarrow LL\mathbb{C} = a\mathbb{C}$ is fully faithful $\big($and what we have at present is that $\ell_{L\mathbb{C}}dr$ and $\ell_{L\mathbb{C}}\mathrm{triv}(v(x)p)r\,\big)$. But $LL\mathbb{C} = a\mathbb{C}$ is a sheaf and thus since

(17.1.4)
$$\begin{tikzcd}
\mathrm{Simpl}_{\mathbb{E}^\wedge}(C^+/X, a(\mathbb{C})) \arrow[r, "r^*"] & \mathrm{Simpl}_{\mathbb{E}^\wedge}(C'/X, a(\mathbb{C})) \\
\mathrm{Simpl}_{\mathbb{E}^\wedge}(a(C^+/X), a(\mathbb{C})) \arrow[u, "\wr"'] \arrow[r, "ar^*"] & \mathrm{Simpl}_{\mathbb{E}^\wedge}(a(C'/X), a(\mathbb{C})) \arrow[u, "\wr"']
\end{tikzcd}$$

is commutative with the vertical arrows isomorphisms of categories we will have the desired conclusion provided that the functor restriction along ar is fully faithful. But by construction the simplicial map $r\colon C'^{+}/X \longrightarrow C/X$ (viewed as a functor between groupoids) is fully faithful and has $r_0\colon \mathrm{Ob}(C'^{+}/X) \longrightarrow \mathrm{Ob}(C/X)$ covering, hence $ar\colon a(C'^{+}/X) \longrightarrow a(C^+/X)$ is an essential equivalence of groupoids (since a is left exact and $a(r_0)$ is an epimorphism). Thus the conclusion follows from the first part of the proof of I7.4 of the text:

In a topos, the functor "restriction along an essential equivalence" $H\colon \mathfrak{J}_1 \longrightarrow \mathfrak{J}_2$,

$$\mathrm{Cat}(H, C)\colon \mathrm{Cat}(\mathfrak{J}_2, \mathbb{C}) \longrightarrow \mathrm{Cat}(\mathfrak{J}_1, \mathbb{C})$$

is always fully faithful for any category $\mathbb{C}$.

This completes the proof of the first part of the Theorem (I7.1). For the second part note that since the diagram of categories and functors

(17.1.5)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{E}^\wedge}(X, L\mathbb{C}) \arrow[d, "\mathrm{i}_R^X"'] \arrow[dr, "\mathrm{triv}_p"] & \\
\mathrm{Hom}_{\mathbb{E}^\wedge}(R, L\mathbb{C}) \arrow[r, hook, "\mathrm{triv}_{\mathrm{i}_R}"'] & \mathrm{Simpl}_{\mathbb{E}^\wedge}(C/X, L\mathbb{C})
\end{tikzcd}$$

is commutative, if triv_p is an equivalence, i_R^* is as well since $\mathrm{triv}_{\mathrm{i}_R}$ is always fully faithful. We may use this to establish that the fully faithful functor $\ell_{LC}\colon L\mathbb{C} \longrightarrow LL\mathbb{C} = a(\mathbb{C})$ is essentially surjective for any representable X (so that the fibrations $\mathbb{LS}(\mathbb{F}) = \mathbb{EXX}(\mathbb{LC})$ and $\mathbb{KSF} = \mathbb{EXX}\,a(\mathbb{C})$ are cartesian equivalent: In effect consider any map $X \xrightarrow{x} LL\mathbb{C}$. Since $\ell_{LC}\colon L\mathrm{Ob}(\mathbb{C}) \hookrightarrow a\mathrm{Ob}(\mathbb{C})$ is a covering monomorphism, the pullback

(17.1.6)
$$\begin{tikzcd}
L\mathbb{C} \arrow[r, hook, "\ell_{LC}"] & LL\mathbb{C} \\
R \arrow[u, "x^{\#}"] \arrow[r, hook, "\mathrm{i}_R"'] & X \arrow[u, "x"']
\end{tikzcd}$$

has i_R a covering monomorphism, thus by the above remark, there exists an arrow $x'\colon X \longrightarrow L\mathbb{C}$ such that $x'\,\mathrm{i}_R \cong x^{\#}$. Consequently, $\ell_{LC}x'\,\mathrm{i}_R \cong x\,\mathrm{i}_R$ and thus $\ell_{LC}x' \cong x$ since $LL\mathbb{C}$ is a sheaf. Thus $a(\mathbb{C})$ is also complete in the topology of $\mathbb{E}$. The universal property of $a\colon \mathbb{F} \xrightarrow{\approx} \mathbb{SF} \xrightarrow{\approx} \mathbb{LSF} \xrightarrow{\approx} \mathbb{KSF}$ is easily seen: In effect, using Lemma (I7.13) of the text, if $\mathbb{G}$ is a complete fibration, then we have a chain of cartesian equivalences $\mathbb{G} \xrightarrow{\approx} \mathbb{SG} \xrightarrow{\approx} \mathbb{LSG} \xrightarrow{\approx} \mathbb{KSG}$. If $\mathbb{F}$ is pre-complete then $\mathbb{LSF} \xrightarrow{\approx} \mathbb{KSF}$ are both complete and since $\mathbb{KSF}$ represents the associated sheaf of $\mathbb{SF}$, we obtain the chain of equivalences

(17.1.7)
$$\mathrm{Cart}_{\mathbb{E}}(\mathbb{F},\mathbb{G}) \overset{\approx}{\longrightarrow} \mathrm{Cart}_{\mathbb{E}}(\mathbb{SF},\mathbb{SG}) \overset{\approx}{\longrightarrow} \mathrm{Cart}_{\mathbb{E}}(\mathbb{SF},\mathbb{KSG}) \overset{\approx}{\longrightarrow} \mathrm{Cart}_{\mathbb{E}}(\mathbb{KSF},\mathbb{KSG})$$
$$\overset{\approx}{\longrightarrow} \mathrm{Cart}_{\mathbb{E}}(\mathbb{KSF},\mathbb{G}) \overset{\approx}{\longrightarrow} \mathrm{Cart}_{\mathbb{E}}(\mathbb{LSF},\mathbb{G}) \ ,$$

as desired.

(17.2) As a particular case note that if $\mathbb{C}$ is a sheaf of categories, then $\mathbb{EX}(\mathbb{C})$ is precomplete and if $\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\mathbb{C})$ represents the fibered category of locally representable functors (above $a(X)$ under $\mathbb{C}$ in $\mathbb{E}^{\tilde{}}$), then the canonical functor

(17.2.0)
$$R\colon \mathbb{EX}(\mathbb{C}) \longrightarrow \underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\mathbb{C})$$

which sends any $X \xrightarrow{x} \mathrm{Ob}(\mathbb{C})$ to the representable internal functor defined by $a(x) \xrightarrow{\tilde{x}} \mathrm{Ob}(\mathbb{C})$ is fully faithful and covering (since by definition any member of $\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(X;\mathbb{C})$ is locally representable). Thus in $\mathbb{E}^{\hat{}}$ we have the commutative square of categories and functors

(17.2.1)
$$\begin{tikzcd}
\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{EX}(\mathbb{C})) \arrow[r, "R^*"] \arrow[d, "a_\ell"'] & \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\mathbb{C})) \arrow[d, "a_T"] \\
a\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{EX}(\mathbb{C})) \arrow[r, "a(R)"] & a\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\mathbb{C})) \ .
\end{tikzcd}$$

Since $\underline{\underline{\underline{\mathrm{TORS}}}}_{\mathbb{E}}(\mathbb{C})$ is complete, a_T is an equivalence in $\mathbb{E}^{\hat{}}$ and $a(R)$ is an essential equivalence of sheaves of categories which is an equivalence because $a\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{EX}(\mathbb{C}))$ is complete. Thus in this case we have the equivalences

$$(17.2.2)\quad \mathbb{L}\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{E}\mathbb{X}(\mathbb{C})) \overset{\approx}{\longrightarrow} a\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{E}\mathbb{X}(\mathbb{C})) \overset{\approx}{\longrightarrow} \mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\underline{\underline{\mathrm{TORS}}}_{\mathbb{E}}(\mathbb{C}))$$

as previously claimed in the text.
(17.3) Finally, note that the just described process works (after passage to a larger universe) even if the fibers of the original pre-complete fibrations were not small (as is the case for the fibered category of pre-ties of a topos). A principal advantage of using $\mathbb{LSF}$ as the completion in this case is the simplicity of the description of the completion in terms of $\mathbb{E}$: Since

$$L\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})(X) \;=\; \varinjlim_{R\in\mathrm{Cov}(X)} \mathrm{Hom}_{\mathbb{E}^{\wedge}}(R,\mathrm{Cart}_{\mathbb{E}}(\mathbb{E}/\text{-}\,,\mathbb{F})) \;=\; \varinjlim_{R\in\mathrm{Cov}(X)} \mathrm{Cart}_{\mathbb{E}}(R,\mathbb{F})\;,$$

we see that the completion of a pre-complete fibration may be described as *equivalence* classes *(under refinement) of descent data on coverings*, each of one of which can be represented concretely in terms of the fibration and a given covering as was done in part II of the text for the resulting fibered category of *ties* of the topos $\mathbb{E}^{\sim}$.

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