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# ALGEBRAICALLY CLOSED AND EXISTENTIALLY CLOSED SUBSTRUCTURES IN CATEGORICAL CONTEXT

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Abstract. We investigate categorical versions of algebraically closed (= pure) embeddings, existentially closed embeddings, and the like, in the context of locally presentable categories. The definitions of S. Fakir [Fa, 75], as well as some of his results, are revisited and extended. Related preservation theorems are obtained, and a new proof of the main result of Rosický, Adámek and Borceux ([RAB, 02]), characterizing $\lambda$-injectivity classes in locally $\lambda$-presentable categories, is given.

## Introduction

Algebraically closed embeddings are used in module theory and category theory (where they are called pure morphisms), as well as in model theory. The model theoretic definition allows seeing this concept as one of several related types of morphisms (like elementary embeddings and existentially closed embeddings) defined in terms of preservation of certain families of formulas. Adapting this line of thought, S. Fakir ([Fa, 75]) proposed a categorical version of the concepts of algebraically closed and existentially closed morphisms in the context of locally presentable categories.

In this paper, we first revisit Fakir's definitions, extending and simplifying them using two ideas: the first one is to use $\lambda$-presentable morphisms instead of $\lambda$-presentable (and $\lambda$-generated) objects, and the second one involves what we will call locally presentable factorization systems.

The former idea was already exploited for the algebraically closed case in [$H_1$, 98]. In a category $\mathcal{C}$, a morphism $f\colon A \longrightarrow B$ will be called $\lambda$-presentable if it is $\lambda$-presentable as an object of the comma category $(A \downarrow \mathcal{C})$. In [$H_1$, 98], we characterized the classes closed under algebraically closed subobjects as the ones which are injective with respect to some class of cones formed by $\lambda$-presentable morphisms. As a corollary, we obtained a solution to a problem of L. Fuchs in the context of abelian groups ([Fu, 70]). In this paper, we use this characterization to obtain a different proof of the main result of [RAB, 02], characterizing the $\lambda$-injectivity classes as the ones closed under $\lambda$-algebraically closed subobjects and $\lambda$-reduced products (Theorem 3.5 below).

As for the latter idea, we define a $\lambda$-presentable factorization system as a proper factorization system (E,M) in which the morphisms in E between $\lambda$-presentable objects are sufficient to determine the morphisms in M. In a locally $\lambda$-presentable category $\mathcal{C}$, (Strong

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Epi, Mono) is $\lambda$-presentable, but (Epi, Strong Mono) is not necessarily so. Also, if one sees $\mathcal{C}$ as the concrete category of all the models of a $\lambda$-limit theory, the embeddings (in the usual classical sense) form the right-hand part of a $\lambda$-presentable factorization system. We then propose a definition of existentially closed morphisms which is relative to a given $\lambda$-presentable factorization system. Apart from being somewhat simpler then Fakir's, the definition fits the classical model theoretic one even when the signature contains relation symbols, when one chooses the appropriate factorization system.

As an application, Theorem 2.4 below completes the picture regarding the following set of characterizations. [Fa, 75] gives a categorical proof of the well-known characterization of algebraically closed morphisms as those which can be extended to the canonical diagonal morphism into some ultrapower of their domain. The result also holds when "ultrapower" is replaced by "reduced power", and the proof is given in the infinitary context. Fakir also uses his method to prove that the existentially closed morphisms are those which can be extended *by a monomorphism* to the canonical diagonal morphism into some ultrapower of their domain. However, in this case ultrapowers cannot be replaced by reduced powers, so that in particular there is no obvious extension available in the infinitary context. Our definition allows considering intermediate types of morphisms (between the algebraically closed and the existentially closed ones), and we identify precisely, in these terms, the morphisms which can be extended by a monomorphism to the canonical diagonal morphism into some reduced power of their domain. We call those *weakly $\lambda$-existentially closed* morphisms, and we also characterize them in terms of preservation of a certain type of formulas.

## 1. Preliminary concepts

For basic definitions and results in category theory, we refer the reader to [AR, 94] and [Bo, 94]. For convenience, we recall the following.

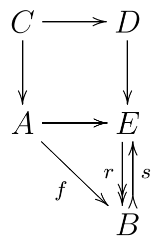
If $\lambda$ is a regular infinite cardinal, an object $A$ of a category $\mathcal{C}$ is *$\lambda$-presentable* if the hom-functor $\mathcal{C}(A,-)\colon\mathcal{C} \longrightarrow$Set preserves $\lambda$-directed colimits. Then, $\mathcal{C}$ is *$\lambda$-accessible* if it has a (small) set $S$ of $\lambda$-presentable objects such that every object is the $\lambda$-directed colimit of a diagram with all its vertices in $S$. Finally, $\mathcal{C}$ is *locally $\lambda$-presentable* if it is $\lambda$-accessible and cocomplete (or, equivalently, complete). We write *finitely presentable* for $\omega$-presentable.

Given an object $A$ in a locally $\lambda$-presentable (resp. $\lambda$-accessible) category $\mathcal{C}$, both the category $\mathcal{C}^2$ (of all morphisms in $\mathcal{C}$) and the comma category $(A\downarrow\mathcal{C})$ (of all morphisms in $\mathcal{C}$ with domain $A$) are also locally $\lambda$-presentable (resp. $\lambda$-accessible). Our definitions and results will be stated for locally $\lambda$-presentable categories, but the interested reader will see easily that all of them apply to $\lambda$-accessible categories with pushouts. Throughout the paper we will adopt the following:

1.1. Convention. A morphism $f\colon A \longrightarrow B$ in a category $\mathcal{C}$ will be called *$\lambda$-presentable* if it is $\lambda$-presentable as an object of the comma category $(A\downarrow\mathcal{C})$.

Note that being a $\lambda$-presentable morphism in our sense is weaker than being a $\lambda$-presentable object in $\mathcal{C}^2$ (which turns out to be the same as a morphism between $\lambda$-presentable objects in $\mathcal{C}$: see [AR, 94], Exercise 2.c). Intuitively (in the context of a variety, say), $f\colon A \longrightarrow B$ being $\lambda$-presentable means that $f$ provides a way to present $B$ by adding less than $\lambda$ generators and relations to some presentation of $A$: see Section 3 for more on that. In categorical terms, this description translates as follows:

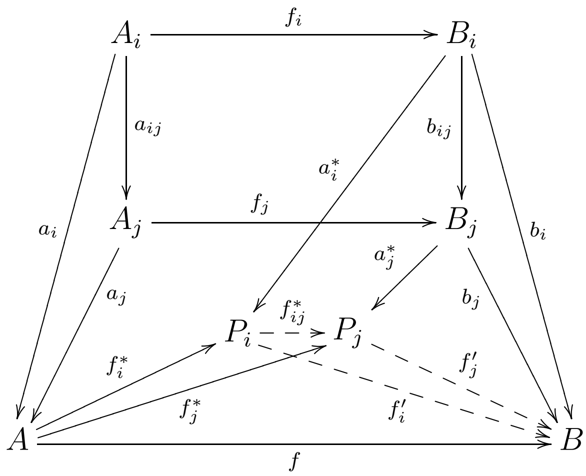
1.2. Proposition. *Let $\mathcal{C}$ be a locally $\lambda$-presentable category. Then $f\colon A \longrightarrow B$ is $\lambda$-presentable if and only if there exists a commutative diagram:*




*where $C$ and $D$ are $\lambda$-presentable, the square is a pushout, and $rs = 1_B$.*

Proof. ($\Rightarrow$) Let $f\colon A \longrightarrow B$ be $\lambda$-presentable. $\mathcal{C}^2$ being locally $\lambda$-presentable, there exists a $\lambda$-directed diagram $((a_{ij}, b_{ij})\colon f_i \longrightarrow f_j)_I$ of $\lambda$-presentable objects of $\mathcal{C}^2$ with colimit $((a_i, b_i)\colon f_i \longrightarrow f)_I$.

Let $(f_i^*, a_i^*)$ be the pushout of $(f_i, a_i)$ $(i \in I)$, and $f_{ij}^*$ and $f_i'$ be the induced morphisms, as in the following diagram:




It is straightforward to verify that in $(A \downarrow \mathcal{C})$, one has $\operatorname*{colim}_I (f_{ij}^*\colon f_i^* \longrightarrow f_j^*)_I =$ $(f_i'\colon f_i^* \longrightarrow f)$. Now, if $f$ is $\lambda$-presentable (in $(A \downarrow \mathcal{C})$), then $1_B\colon f \longrightarrow f$ must factorize through one of the $f_i'\colon f_i^* \longrightarrow f$, i.e., there exist $i \in I$ and $s\colon f \longrightarrow f_i^*$ in $(A \downarrow \mathcal{C})$ with $f_i's = 1_B$. Taking $C = A_i$, $D = B_i$, $P_i = E$ and $r = f_i'$, one gets the required diagram as in the statement of the proposition.

($\Leftarrow$) The converse implication follows from the following facts, which are either known or easily verified: (i) a morphism between $\lambda$-presentable objects in $\mathcal{C}$ is a $\lambda$-presentable morphism, (ii) the pushout (by any morphism) of a $\lambda$-presentable morphism is a $\lambda$-presentable morphism, and (iii) every retraction in $(A \downarrow \mathcal{C})$ of a $\lambda$-presentable morphism is a $\lambda$-presentable morphism. ∎

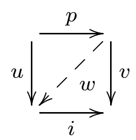
Proposition 1.2 says that $f\colon A \longrightarrow B$ is $\lambda$-presentable if it is, *up to a retraction in* $(A \downarrow \mathcal{C})$, the pushout of a morphism between $\lambda$-presentable objects.

1.3. Open Problem. Is every $\lambda$-presentable morphism (in a locally $\lambda$-presentable category) the pushout of a morphism between $\lambda$-presentable objects?

A positive answer to this rather intriguing question would have interesting consequences for what is to come. See the paragraph following Lemma 2.2.

We quickly recall the definition of a factorization system (see [Bo, 94] or [CHK, 85] for more).

Given morphisms $p$ and $i$, we write $p \perp i$ if for every commutative square $vp = iu$, there exists a unique morphism $w$ making both triangles commute in the following diagram:




If N is a class of morphisms, denote by $E(\mathrm{N})$ and $M(\mathrm{N})$ the classes

$$E(\mathrm{N}) = \{p \mid p \perp i \ \text{ for all } i \in \mathrm{N}\}$$

$$M(\mathrm{N}) = \{i \mid p \perp i \ \text{ for all } p \in \mathrm{N}\}.$$

Then a *factorization system* in $\mathcal{C}$ is a pair (E,M) of classes of morphisms which are closed under composition and contain all isos, and such that: (1) $E(\mathrm{M})=$ E (or, equivalently, $M(\mathrm{E}) = \mathrm{M}$), and (2) every morphism $f$ in $\mathcal{C}$ has a factorization $f = me$ with $m \in \mathrm{M}$ and $e \in \mathrm{E}$. The class of all epimorphisms (resp. strong epimorphisms, monomorphisms, etc.) is denoted by Epi (resp. Strong Epi, Mono, etc.). A factorization system (E,M) is *proper* if E $\subseteq$ Epi and M $\subseteq$ Mono.

In what follows, $\mathrm{E}_\lambda$ denotes the class of all morphisms in E with $\lambda$-presentable domain and codomain.

1.4. Definition. A proper factorization system (E,M) in a category $\mathcal{C}$ is $\lambda$-*presentable* if $M(\mathrm{E}_\lambda) \subseteq \mathrm{M}$.

The idea behind 1.4 is that in order to check that a given morphism is in M, it will be sufficient to verify that it has the diagonal property with respect to all morphisms in E between "small" objects only. This will be used crucially in Theorems 2.4 and 2.5 below.

1.5. Examples.

(a) Let $\Sigma$ be a multisorted $\lambda$-ary signature (with or without relation symbols), and $\mathcal{C} = \mathrm{Mod}(\Sigma)$ be the category of all $\Sigma$-structures and $\Sigma$-homomorphisms. Denoting by Emb the class of all ($\Sigma$-) embeddings in $\mathcal{C}$ (in the usual sense of model theory), and by Sur the class of all surjective homomorphisms, then (Sur, Emb) is a $\lambda$-presentable factorization system in $\mathcal{C}$. This is readily verified directly, but it also follows from Proposition 1.6 (and its proof) below.

(b) It is well-known that in any locally $\lambda$-presentable category, (Strong Epi, Mono) and (Epi, Strong Mono) are (proper) factorization systems. The former is $\lambda$-presentable; again, this will follow from Proposition 1.6 (see Example (c) below). However the latter is not, as we will see from the following counterexample.

Let $\Sigma$ be the signature with a (n+2)-ary relation $R_n$ for each positive integer n. Consider the set

$$T = \{\phi_n |\ n > 0\}$$

where $\phi_n$ is the $\Sigma$-sentence

$$\forall x \exists^{\leq 1}(y_1 \dots y_n)\exists y_{n+1}(R_1(x, y_1, y_2) \wedge R_2(x, y_1, y_2, y_3) \wedge \dots \wedge R_n(x, y_1, \dots, y_{n+1})).$$

Here, "$\exists^{\leq 1}(y_1 \dots y_n)$" means "there exists at most one string $y_1 \dots y_n$". The $\phi_n$'s are equivalent to universal Horn sentences, and consequently the category $\mathrm{Mod}(T)$ of all models of $T$ (with all $\Sigma$-homomorphisms) is locally finitely presentable (see (c) below). Hence (Epi, Strong Mono) is a factorization system in the category. We show it is not finitely presentable.

Let

$$A = \big\langle \{a\}; R_n^A = \emptyset \text{ for all } n \big\rangle$$

be the $\Sigma$-structure with one element $a$ and all relations empty, and

$$B = \langle \{a, b_1, b_2, ...\}\,;\ R_n^B = \{(a, b_1, b_2, ..., b_{n+1})\} \text{ for all } n\rangle.$$

Both are models of $T$, and $f\colon A \longrightarrow B$, defined by $f(a) = a$, is a homomorphism. Then $f$ is easily seen to be an epi in $\mathrm{Mod}(T)$, since every homomorphism from $B$ to another $T$-model is entirely determined by its value on $a$. Hence it is not a strong mono (otherwise it would be an isomorphism). However it is in $M(\mathrm{Epi}_\omega)$. In order to see this, consider a commutative square

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

with $g$ epi and $C$ and $D$ finitely presentable. First note that the finitely presentable objects of $\mathrm{Mod}(T)$ must be finite. Then, the syntactic description of epimorphisms in categories of models due to P.D. Bacsich (see [Ba, 72] or [$H_2$, 98]) says that for every element $d$ in $D$, there exists a finite conjunction $\beta$ of atomic formulas such that

$$T \models \forall \boldsymbol{x} \exists^{\leq 1} y \exists \boldsymbol{z} (\beta(\boldsymbol{x}, y, \boldsymbol{z}))$$

and

$$D \models \exists \boldsymbol{z} (\beta[g(\boldsymbol{c}), d, \boldsymbol{z}])$$

for some string $\boldsymbol{c}$ in $C$. Let us say that such a $d$ is *determined* by $\beta$. We show that $R_n^D = \emptyset$ for all $n$.

Suppose that $D \models R_n[d_0, d_1, \ldots, d_{n+1}]$ holds for some $n$ and some $d_i$'s in $D$. Then all the $d_i$'s must be distinct, since their images in $B$ are distinct. Note also that $d_{n+1}$ cannot be in the image of $g$. Hence, there must be a greatest integer $n$ for which $D \models R_n[d_0, d_1, \ldots, d_{n+1}]$ holds for some $d_i$'s. But then there is no conjunction of atomic formulas $\beta$ which can determine $d_{n+1}$. Hence $R_n^D = \emptyset$. As a consequence, $g$ is surjective, and the required diagonal clearly exists.

(c) Recall that, up to a categorical equivalence, locally $\lambda$-presentable categories are the categories of models of $\lambda$-*limit theories*, i.e., set of sentences of the form

$$\forall \boldsymbol{x} (\varphi(\boldsymbol{x}) \longrightarrow \exists^1 \boldsymbol{y} (\beta(\boldsymbol{x}, \boldsymbol{y}))$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are strings of less than $\lambda$ variables, "$\exists^1 \boldsymbol{y}$" means "there exists exactly one string $\boldsymbol{y}$", and $\varphi$ and $\beta$ are conjunction of less than $\lambda$ atomic formulas in a multisorted $\lambda$-ary signature $\Sigma$ (see [AR, 94] or [H, 01]).

Let $\mathcal{C} = \mathrm{Mod}(T)$ be such a (concrete) category of all the models of a $\lambda$-limit theory $T$ in some signature $\Sigma$. Generalizing Example (a), the following proposition will show that Emb is the right hand part of a $\lambda$-presentable factorization system in $\mathcal{C}$. In addition, note that in the categorical equivalence mentioned above, one can always choose $\Sigma$ to have no relation symbols. Since in this case we have Emb = Mono, the proposition will also imply that (Strong Epi, Mono) is $\lambda$-presentable in every locally $\lambda$-presentable category, as claimed in the first paragraph of Example (b).

1.6. Proposition. *Let $\mathcal{C}$ be the category of all models of a $\lambda$-limit theory. Then (E(Emb), Emb) is a $\lambda$-presentable factorization system in $\mathcal{C}$.*

Proof. It follows immediately from [CHK, 85], Lemma 3.1, that ($E$(Emb), Emb)is a factorization system in $\mathcal{C}$ . From [$H_2$, 98], Theorem 20, one deduces that Strong Mono $\subseteq$ Emb, so that $E$(Strong Mono) = Epi $\supseteq$ $E$(Emb) (since (Epi, Strong Mono) is a factorization system). Hence ($E$(Emb), Emb) is proper. We now show that it is $\lambda$-presentable.

Let $\Sigma$ be the ($\lambda$-ary) signature of the theory. $\mathcal{C}$ is a full reflective subcategory of the category $\mathrm{Mod}(\Sigma)$ of all $\Sigma$-structures and $\Sigma$-homomorphisms. Now, in $\mathrm{Mod}(\Sigma)$, the $\lambda$-presentable objects are precisely the structures presentable in the classical sense by less than $\lambda$ "generators and relations"; this is carefully explained in [AR, 94], 5.5 and 5.28. Let us fix some notations which will be used again later on.

A set of generators and relations can be seen as a set of atomic formulas (identifying the set of generators with a string of variables), so that a typical $\lambda$-presentable object of $\mathrm{Mod}(\Sigma)$ can be denoted by

$$\langle \boldsymbol{x}; \varphi(\boldsymbol{x})\rangle_\Sigma$$

for some set $\varphi$ of less than $\lambda$ atomic formulas. Note that the length of $\boldsymbol{x}$ must then be $< \lambda$. The reflection of $\langle \boldsymbol{x}; \varphi(\boldsymbol{x})\rangle_\Sigma$ in $\mathcal{C}$ is denoted by $\langle \boldsymbol{x}; \varphi(\boldsymbol{x})\rangle_{\mathcal{C}}$, and it is $\lambda$-presentable in $\mathcal{C}$. We show that $M(E(\mathrm{Emb})^*_\lambda) \subseteq \mathrm{Emb}$, where $E(\mathrm{Emb})^*_\lambda$ is the class of morphisms in $E(\mathrm{Emb})$ with $\lambda$-presentable domain and codomain of the form $\langle \boldsymbol{x}; \varphi(\boldsymbol{x})\rangle_{\mathcal{C}}$. This will imply in particular that $M(E(\mathrm{Emb})_\lambda) \subseteq \mathrm{Emb}$, as required.

Given a morphism $f\colon A \longrightarrow B$ in $M(E(\mathrm{Emb})^*_\lambda)$, suppose that $B \models R[f(\boldsymbol{a})]$ for some relation symbol $R$ in $\Sigma \cup \{=\}$ and some string $\boldsymbol{a}$ in $A$. We write $\boldsymbol{x_a}$ for a string of variables corresponding to $\boldsymbol{a}$. Then there exists a commutative diagram

$$\begin{array}{ccc}
\langle \boldsymbol{x_a}; \emptyset\rangle_\Sigma & \xrightarrow[i']{} & \langle \boldsymbol{x_a}; R(\boldsymbol{x_a})\rangle_\Sigma \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\
A & \xrightarrow[f]{} & B
\end{array}$$

in $\mathrm{Mod}(\Sigma)$ with $i(\boldsymbol{x_a}) = \boldsymbol{a}$, $i'$ is the identity function, and $j(\boldsymbol{x_a}) = f(\boldsymbol{a})$. In order to show that $A \models R[\boldsymbol{a}]$, we need to show that there exists a diagonal morphism from $\langle \boldsymbol{x_a}; R(\boldsymbol{x_a})\rangle_\Sigma$ to $A$ making the two triangles commute.

Now, $i$ and $j$ factorize through the reflections:

$$\begin{array}{ccc}
\langle \boldsymbol{x_a}; \emptyset\rangle_\Sigma & \xrightarrow[i']{} & \langle \boldsymbol{x_a}; R(\boldsymbol{x_a})\rangle_\Sigma \\
\Big\downarrow & & \Big\downarrow \\
\langle \boldsymbol{x_a}; \emptyset\rangle_{\mathcal{C}} & \xrightarrow[i'']{} & \langle \boldsymbol{x_a}; R(\boldsymbol{x_a})\rangle_{\mathcal{C}} \\
\Big\downarrow & & \Big\downarrow \\
A & \xrightarrow[f]{} & B
\end{array}$$

It is easily verified that $i'$ is in $E(\mathrm{Emb})$ (in $\mathrm{Mod}(\Sigma)$). From the fact that $i''$ is the reflection of $i'$ in $\mathcal{C}$, it follows that it is in $E(\mathrm{Emb})$ (in $\mathcal{C}$). Hence the bottom square of the above diagram has a diagonal morphism with the required property, and the outer square too, as a consequence. ∎

1.7. Convention. From now on, a locally $\lambda$-presentable category will always be assumed to come equipped with a locally $\lambda$-presentable factorization system (E,M). As is customary, morphisms in E will be denoted by " $\twoheadrightarrow$ ", and morphisms in M by "$\rightarrowtail$ ".

## 2. Algebraically and existentially closed morphisms

In [Fa, 75], Fakir proposed categorical definitions of algebraically closed and existentially closed morphisms in a locally presentable category $\mathcal{C}$. His definitions generalize the familiar model theoretic concepts (see [CK, 90] or [E, 77]). Algebraically closed morphisms generalize the "pure embeddings" in the theory of abelian groups or, more generally, modules. For that reason they are also called "pure morphisms" (see [AR, 94] and [R, 97]). The following reformulates Fakir's definitions. An intermediate concept is also introduced; its purpose will appear clearly in Theorem 2.4 below.

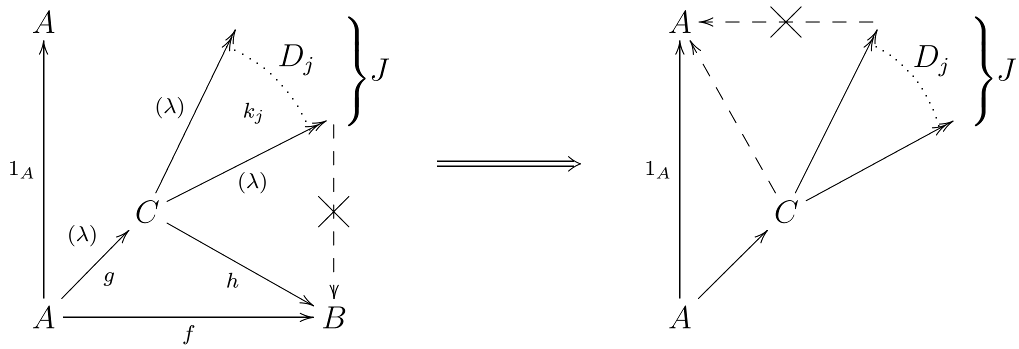
2.1. Definitions. Let $f: A \longrightarrow B$ be a morphism in a locally $\lambda$-presentable category $\mathcal{C}$. In (c) below, a *$\lambda$-presentable cone in E* is a cone $c = (k_j: C \longrightarrow\!\!\!\!\rightarrow D_j\ )_J$ where $|J| < \lambda$ and each $k_j$ is a $\lambda$-presentable morphism in E.

(a) $f$ is *$\lambda$-algebraically closed* if for every factorization $f = hg$ with $g$ $\lambda$-presentable, $g$ has a left inverse.

(b) $f$ is *weakly $\lambda$-existentially closed* if it is $\lambda$-algebraically closed and for every factorization $f = hg$ with $g$ $\lambda$-presentable, and every $\lambda$-presentable morphism $k$ in E from the domain of $h$, and through which $h$ does not factorize, $g$ has a left inverse which does not factorize through $k$.

(c) $f$ is *$\lambda$-existentially closed* if for every factorization $f = hg$ with $g$ $\lambda$-presentable, and every $\lambda$-presentable cone $c$ in E from the domain of $h$, and through which $h$ does not factorize, $g$ has a left inverse which does not factorize through $c$.

The three concepts could be defined in a uniform way using (c), by adjusting the size of the cone accordingly: replacing "$\lambda$-presentable cone" by "cone $c = (k_j)_J$ of $\lambda$-presentable morphisms with $|J| < 1, |J| < 2$ and $|J| < \lambda$" respectively, one gets the definitions (a), (b) and (c). This will be useful in the proofs below. It also allows the same following diagram to be used to illustrate the three definitions:




One reads this as "every diagram of the type on the left can be completed as shown on the right".

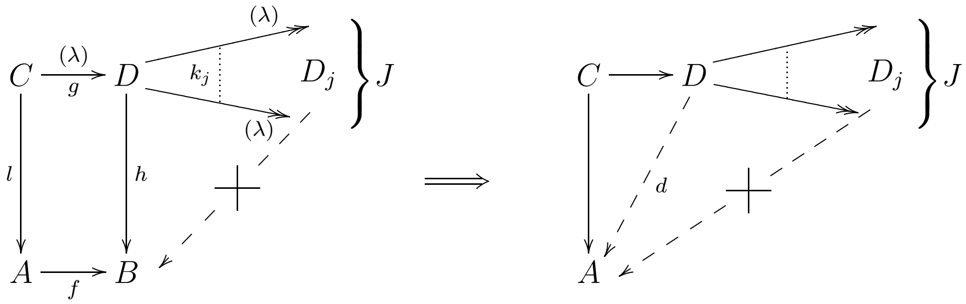
The above definitions are quite different from the formulations in [Fa, 75]. The following lemma shows the connection.

2.2. Lemma. *Let* $f: A \longrightarrow B$ *be a morphism in a locally* $\lambda$*-presentable category* $\mathcal{C}$*. The following are equivalent:*

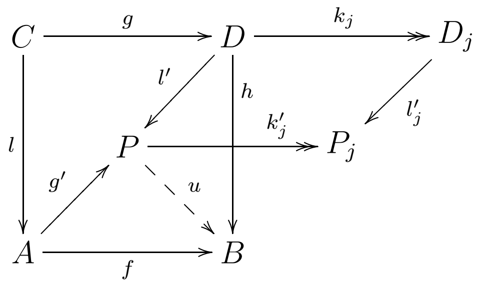
*(i)* $f$ *is* $\lambda$*-algebraically closed (resp. weakly* $\lambda$*-existentially closed,* $\lambda$*-existentially closed);*

*(ii) for every commutative square* $fl = hg$ *with* $g$ $\lambda$*-presentable (resp., and every cone* $c = (k_j: \mathrm{dom}(h) \twoheadrightarrow D_j)_J$ *of less than 2, resp. less than* $\lambda$*,* $\lambda$*-presentable morphisms in E, and through which* $h$ *does not factorize), there exists* $d: \mathrm{dom}(h) \longrightarrow A$ *such that* $dg = l$ *(resp., and such that* $d$ *does not factorize through* $c$*).*

Proof. Condition (ii) can be represented by the diagram:




$(i) \Rightarrow (ii)$: Let $(g', l')$ be the pushout of $(g, l)$, and then $(k'_j, l'_j)$ be the pushout of $(k_j, l')$, $j \in J$:




($u: P \longrightarrow B$ is the morphism induced by the commutativity of the square). Then $g'$ and the $k'_j$ are $\lambda$-presentable. Furthermore, pushouts (by any morphism) of morphisms in E are in E (see [CHK, 85]). Finally, $u$ does not factorize through the cone $c' = (k'_j)_J$ (otherwise $h$ would factorize through the cone $c = (k_j)_J$).

By $(i)$, $g'$ has a left inverse $g''$ which does not factorize through $c'$. It is easily verified that the diagonal morphism $g''l'$ satisfies the conditions required for $d$ in (ii).
$(ii) \Rightarrow (i)$: Take $l: C \longrightarrow A$ in the square to be $1_A$. ∎

Let (ii)′ be the statement obtained from 2.2(ii) by replacing all $\lambda$-presentable morphisms by morphisms with $\lambda$-presentable domain and codomain. Clearly the implication (ii) $\Rightarrow$ (ii)′ holds. It follows easily from 1.2 above that the reverse implication holds for the $\lambda$-algebraically closed case. This shows that our $\lambda$-algebraically closed morphisms are just the $\lambda$-pure morphisms of [AR, 94] (Note however that pushouts are needed here; the two concepts are different in general accessible categories: see [$H_1$, 98]). Whether the equivalence (ii) $\Leftrightarrow$ (ii)′ holds for the other two cases is unknown to us, but one can show that it would be the case if the open problem 1.3 had a positive answer.

Fakir's definitions ([Fa, 75]) are obtained from 2.2(ii) by replacing (1) the $\lambda$-presentable morphism $g$ by a mono from a $\lambda$-generated object to a $\lambda$-presentable object, (2) the cone $c$ by a set $\{(m_j, n_j)\}_J$ of *pairs* of morphisms *from* $\lambda$-presentable objects $D_j$ *to* $D$, and (3) our condition that $h$ (resp. $d$) does not factorize through $c$ by the condition that $hm_j \neq hn_j$ (resp. $dm_j \neq dn_j$). Instead of showing directly that our definitions do generalize the ones of Fakir, we will devote the rest of this section to prove characterization theorems (2.4 and 2.5 below), which will clearly correspond to Fakir's characterizations of his $\lambda$-algebraically closed and $\lambda$-existentially closed morphisms (Theorems 5.7 and 5.10 in [Fa, 75]), when we choose the factorization system (Strong Epi, Mono).

At this point, the reader who wishes to see explicitly the link between the categorical definitions and the classical model theoretic formulations, in terms of preservation of formulas of certain types, may have a look at Proposition 3.1 and its proof.

Recall from [CK, 90] that a $\lambda$-*complete filter* on a set $I$ is a non-empty set $F$ of non-empty subsets of $I$ such that all supersets of elements of $F$ are elements of $F$, and all intersections of less than $\lambda$ elements of $F$ are elements of $F$. An $\omega$-complete filter is just called a *filter*. An *ultrafilter* on $I$ is a filter $F$ in which every subset of $I$ not in $F$ has its complement in $F$.

Given a set $\{A_i\}_{i\in I}$ of objects in a locally $\lambda$-presentable category $\mathcal{C}$, and a $\lambda$-complete filter $F$ on $I$, the *reduced product* $\prod_F A_i$ of $\{A_i\}_{i\in I}$ (with respect to $F$) is the colimit of the ($\lambda$-directed) diagram $(\pi_{K,J}\colon \prod_{i\in K} A_i \longrightarrow \prod_{i\in J} A_i\ ;\ J \subseteq K,\ J, K \in F)$ of the canonical projections between the products.

If $\mathcal{C}$ is the category of all models of a $\lambda$-limit theory in some signature $\Sigma$, as in 1.5 (c) above, one verifies easily that this corresponds to the classical model theoretic concept: $\prod_F A_i$ is the $\Sigma$-structure on the quotient of the product $\prod_{i\in I} A_i$, where for an $\alpha$-ary relation symbol $R$ in $\Sigma \cup \{=\}$, $\prod_F A_i \models R[(f_\beta)_{\beta<\alpha}]$ if and only if $\{i \in I \mid A_i \models R[(f_\beta(i))_{\beta<\alpha}]\} \in F$ (see [CK, 90]). $\prod_F A_i$ is called an *ultraproduct* if $F$ is an ultrafilter.

In [Fa, 75], Fakir shows that his $\omega$-algebraically closed (resp. $\omega$-existentially closed) embeddings coincide with the classical algebraically closed (resp. existentially closed) embeddings. He then gives a categorical proof of the known facts that a morphism $f\colon A \longrightarrow B$ is $\omega$-algebraically closed (resp. $\omega$-existentially closed) if and only if there exists an ultrafilter $F$ and a homomorphism $g\colon B \longrightarrow \prod_F A$ (resp. a monomorphism) to the ultrapower $\prod_F A$, such that $gf$ is the canonical "diagonal" morphism $\Delta_F\colon A \longrightarrow \prod_F A$, induced by the diagonal morphisms to the powers of $A$ (see [E, 77] for the classical model theoretic proof). He actually proves the first part of this result for $\lambda$-algebraically closed morphisms (any $\lambda$), replacing "ultrafilters" by "$\lambda$-complete filters". This does not hold for the existential closed morphisms: for example, even the diagonal morphism $\Delta_F: A \longrightarrow \prod_F A$ into a reduced power is easily seen not to be existentially closed if $F$ is not an ultrafilter. The next theorem fills the gap by showing that the right concept for the extension is the one of weakly $\lambda$-existentially closed. In addition, it covers the concrete cases where relation symbols are present and where one uses embeddings instead of monos. First, a few simpler facts.

2.3. PROPOSITION. *Let $\mathcal{C}$ be a locally $\lambda$-presentable category.*

*(a) Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be morphisms in $\mathcal{C}$. Then:*

*(i) if $gf$ is $\lambda$-algebraically closed, then so is $f$;*

*(ii) if $gf$ is $\lambda$-existentially closed (resp. weakly $\lambda$-existentially closed), then so is $f$, whenever $g$ lies in M.*

*(b) All three concepts in 2.1 are stable under $\lambda$-directed colimits.*

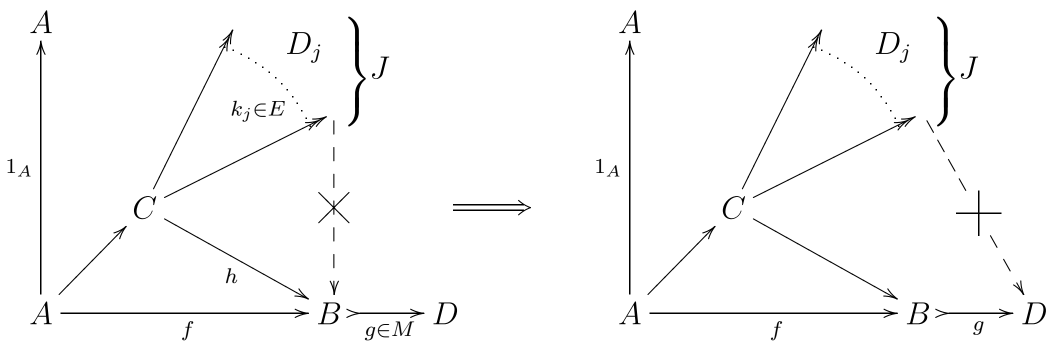
*(c) For any $\lambda$-complete filter $F$, the diagonal morphism $\Delta_F: A \longrightarrow \prod_F A$ to the reduced power of $A$ is weakly $\lambda$-existentially closed.*

*(d) If $\lambda = \omega$, then for any ultrafilter $U$, the diagonal morphism $\Delta_U: A \longrightarrow \prod_U A$ to the ultrapower of $A$ is $\omega$-existentially closed.*

PROOF.

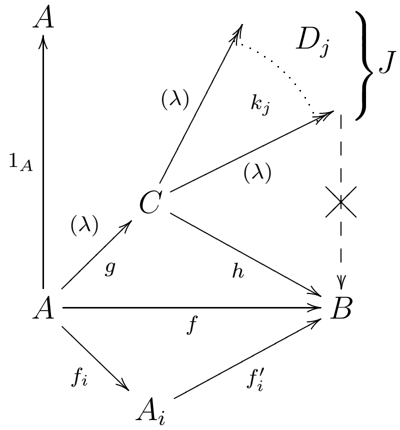
(a) (i) is trivial.

(ii) is an immediate consequence of the implication




(i.e., $h$ does not factorize through the cone $(k_j)_J$ implies that $gh$ does not either), which itself follows from the facts that $k_j \in$E, $g \in$M, and (E,M) is a factorization system.

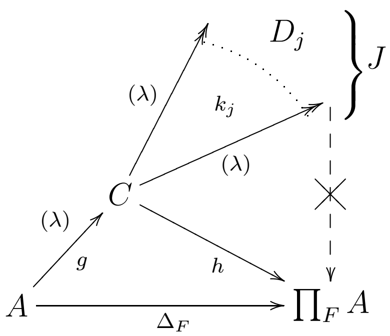
(b) In the diagram



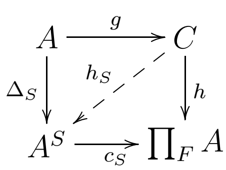

$(f'_i: f_i \longrightarrow f)$ is the colimit of a $\lambda$-directed diagram $(f_{ij}: f_i \longrightarrow f_j)_I$ with each $f_i$ $\lambda$-existentially closed. Then $h: g \longrightarrow f$ must factorize through some $f_i$, and clearly the induced morphism $h_i: C \longrightarrow A_i$ cannot factorize through any of the $k_j$'s. The existence of the required left inverse to $g$ follows since $f_i$ is $\lambda$-existentially closed.

(c)-(d) We prove (c) and (d) in parallel.

Let $F$ be a $\lambda$-complete filter on a set $I$, $\Delta_F: A \longrightarrow \prod_F A$ the canonical diagonal morphism. Consider a diagram




with $g$ $\lambda$-presentable and $|J| < \lambda$, and where $h$ does not factorize through any of the $k_j$'s. $\prod_F A$ is the $\lambda$-directed colimit of the powers $A^S$, $S \in F$, and from this it follows that $\Delta_F$ is the colimit in $(A \downarrow \mathcal{C})$ of the diagonal morphisms $\Delta_S: A \longrightarrow A^S$. $g$ being $\lambda$-presentable, $h: g \longrightarrow \Delta_F$ must factorize $h = c_S \cdot h_S$ through one of the colimit components $c_S: A^S \longrightarrow \prod_F A$ :



Clearly $h_S$ does not factorize through any of the $k_j$'s. Let $s \in S$, and $\pi_s: A^S \longrightarrow A_s = A$ be the canonical projection. Then $\pi_s \cdot h_S \cdot g = \pi_s \cdot \Delta_S = 1_A$, so that $\pi_s h_S$ is a left inverse to $g$. This takes care of the case $J = \emptyset$. If $J = \{j\}$, then $\pi_s h_S$ cannot factorize through $k_j$ for all $s$, otherwise $h_S$ would also factorize through $k_j$. Hence one of the $\pi_s h_S$'s is an adequate left inverse to $g$. These two cases solve (c).

To complete the proof of (d), we now assume that $\lambda = \omega$, and that $F$ is an ultrafilter. For each $j \in J$, let

$$S_j := \{s \in S \mid \pi_s h_S \text{ factorizes through } k_j\}.$$

Then $S_j \notin F$, otherwise the colimit component $c_j: A^{S_j} \longrightarrow \prod_F A$ would provide a factorization of $h$ through $k_j$. Hence its complement $S'_j$ in $I$ is in $F$, and then $S' = S \cap (\bigcap_{j \in J} S'_j)$ is also in $F$ (since $|J| < \omega$). Now $S' = S - (\bigcup_{j \in J} S_j)$, so that one can see that for any $s \in S'$, the morphism $\pi_s h_S: C \to A^S \to A_s = A$ does not factorize through any of the $k_j$'s.

■

An example of a diagonal morphisms $\Delta_I: A \longrightarrow A^I$ which is not $\lambda$-existentially closed, is easily constructed.

The following two theorems should be compared to [Fa, 75]'s Theorems 5.7 and 5.10.

2.4. THEOREM. *Let $\mathcal{C}$ be a locally $\lambda$-presentable category.*

*(a) A morphism $f: A \longrightarrow B$ is $\lambda$-algebraically closed iff there exists a $\lambda$-complete filter $F$ and a morphism $g: B \longrightarrow \prod_F A$ such that $gf = \Delta_F: A \longrightarrow \prod_F A$.*

*(b) A morphism $f: A \longrightarrow B$ is weakly $\lambda$-existentially closed iff there exists a $\lambda$-complete filter $F$ and a morphism $g: B \rightarrowtail \prod_F A$ in $M$ such that $gf = \Delta_F: A \longrightarrow \prod_F A$.*

Before giving the proof, let us state the following theorem, to emphasize that an analogous characterization for the $\lambda$-existentially closed morphisms is known only for $\lambda = \omega$. The proof of both theorems will follow.

2.5. THEOREM. *Let $\mathcal{C}$ be a locally finitely presentable category.*

*A morphism $f: A \longrightarrow B$ is $\omega$-existentially closed iff there exists an ultrafilter $U$ and a morphism $g: B \rightarrowtail \prod_U A$ in $M$ such that $gf = \Delta_U: A \longrightarrow \prod_U A$.*

PROOF.

2.4(a) ($\Leftarrow$) This is immediate from 2.3(a)(i) and (c), since weakly $\lambda$-existentially closed morphisms are $\lambda$-algebraically closed.

($\Rightarrow$) The construction will follow a familiar pattern (see the proof of 2.31 in [AR, 94], for instance), but we recall it since it will also be our starting point for the proofs of (b) and of 2.5 below.

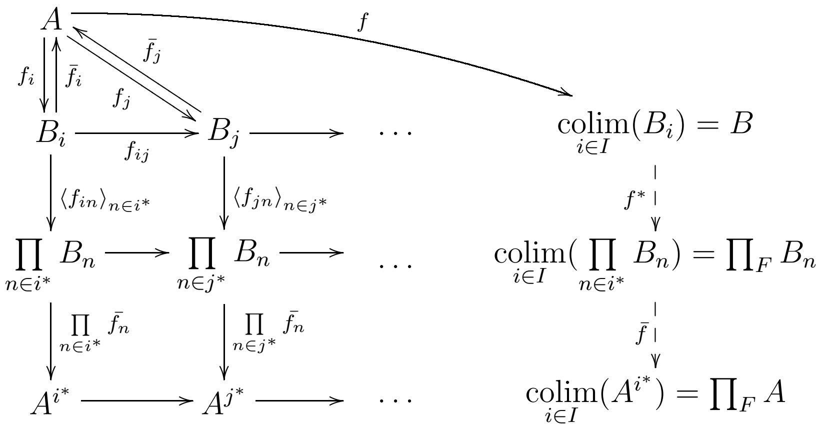
Let $(f_{ij}\colon (f_i\colon A \longrightarrow B_i) \longrightarrow (f_j\colon A \longrightarrow B_j))_I$ be a $\lambda$-directed diagram in $(A \downarrow \mathcal{C})$ with the $f_i$'s $\lambda$-presentable, and with colimit diagram $(f_i'\colon (f_i\colon A \longrightarrow B_i) \longrightarrow (f\colon A \longrightarrow B))_I$.

Let $F$ be the filter defined on (the directed poset) $I$ by

$$F = \{D \subseteq I \mid D \supseteq i^* \text{ for some } i \in I\},$$

where $i^* = \{j \in I \mid j \geq i\}$. From the fact that $I$ is $\lambda$-directed, one sees that $F$ is a $\lambda$-complete filter on $I$. Then the $\lambda$-reduced product $\prod_F B_n$ is the colimit of the diagram $(\pi_{i,j}\colon \prod_{n\in i^*} B_n \longrightarrow \prod_{n\in j^*} B_n)_I$ of the canonical projections $(i \leq j)$ between products (since $(\prod_{n\in i^*} B_n \mid i \in I)$ is final in $(\prod_{n\in D} B_n \mid D \in F)$). Now, for each $j \in I$, there exists a left inverse $\bar{f}_j$ to $f_j$, so that we have the following induced diagram

(1)



It is a straightforward exercise to check that $\bar{f} f^* f\colon A \to B \to \prod_F A$ is the canonical diagonal morphism $\Delta_F$.

2.4(b) $(\Leftarrow)$ This is immediate from 2.3(a)(ii) and (c).

$(\Rightarrow)$ As in the proof of (a), let $(f_i'\colon (f_i \longrightarrow f)_I = \operatorname{colim}_I(f_{ij}\colon f_i \to f_j)$ in $(A \downarrow \mathcal{C})$, with the $f_i$'s $\lambda$-presentable. Let $H$ be the set of all $\lambda$-presentable morphisms in E

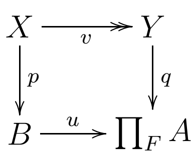
$$h\colon B_i \twoheadrightarrow C\ ,$$

$i \in I$, through which $f_i'$ does not factorize.

First case: $H = \emptyset$. Then consider the same construction than in the proof of (a) $(\Rightarrow)$, see diagram (1). We show that $u = \bar{f} f^*$ is in M.

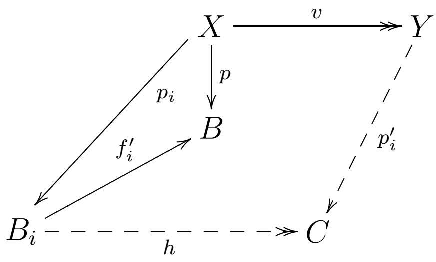
(E,M) being $\lambda$-presentable, this amounts to show that for all commutative square

(2)



with $v \in \mathrm{E}_\lambda$, there exists a unique diagonal $d: Y \longrightarrow B$ such that $dv = p$ and $ud = q$. Since $(f'_i: B_i \longrightarrow B)_I$ is also the colimit *in* $\mathcal{C}$ of $(f_{ij}: B_i \longrightarrow B_j)_I$, there exists a factorization $p = f'_i p_i$ of $p$ for some $i \in I$, $p_i: X \to B_i$. Let $(h, p'_i)$ be the pushout of $(v, p_i)$

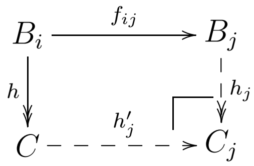
(3)




Note that $h$ is a $\lambda$-presentable morphism in E, since $v$ is. But $H = \emptyset$ means that $f'_i$ factorizes through all $\lambda$-presentable morphisms in E with domain $B_i$. Hence there exists $t: C \to B$ such that $th = f'_i$, and $tp'_i$ is easily checked to be the required diagonal.

Second case: $H \neq \emptyset$. The required reduced power will be the product $\prod\limits_{h \in H} (\prod_{F_h} A)$ of the reduced powers $\prod_{F_h} A$, $h \in H$, where $F_h$ is the filter defined as follows.

Let $I_h$ be the set of all pushouts $h_j: B_j \longrightarrow C_j$ of $h$ by the $f_{ij}$'s, $j \geq i$:

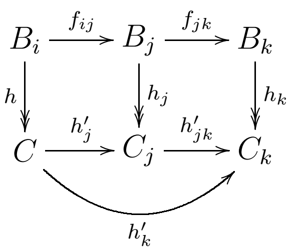



Note that

1) $I_h$ is a set of $\lambda$-presentable morphisms in E,

2) $f'_j: B_j \to B$ does not factorize through $h_j$, and

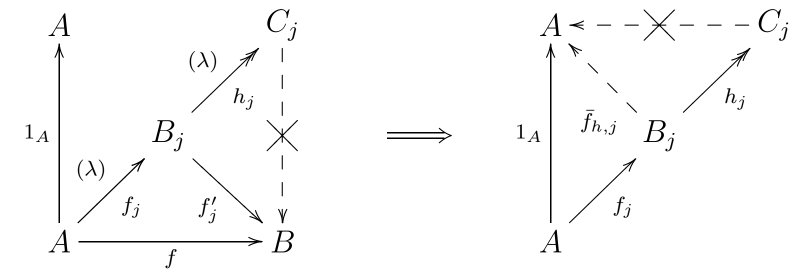
3) if $i \leq j \leq k$ in $I$, then $f_{ij}$ and $h'_k$ induce a morphism $h'_{jk}$ such that the diagram



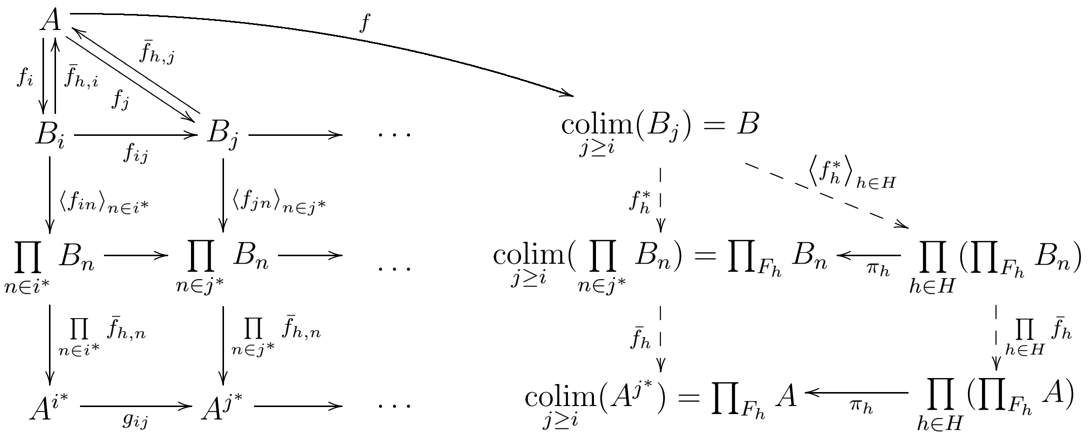

commutes.

For $j \geq i$, define $j^* = \{h_k \mid k \geq j\}$. We then define $F_h = \{D \subseteq I_h \mid D \supseteq j^* \text{ for some } j \geq i\}$. From the fact that $I$ is $\lambda$-directed, one sees that $F_h$ is a $\lambda$-complete filter on $I_h$.

The $\lambda$-reduced product $\prod_{F_h} B_n$ is the colimit of the diagram $(\pi_{j,k}\colon \prod\limits_{n\in j^*} B_n \longrightarrow \prod\limits_{n\in k^*} B_n \mid i \leq j \leq k)$ of the canonical projections, since $(\prod\limits_{n\in j^*} B_n \mid i \leq j)$ is final in $(\prod\limits_{n\in D} B_n \mid D \in F_h)$. Now, for each $j \geq i$ there exists a left inverse $\bar{f}_{h,j}$ to $f_j$, which does not factorize through $h_j$



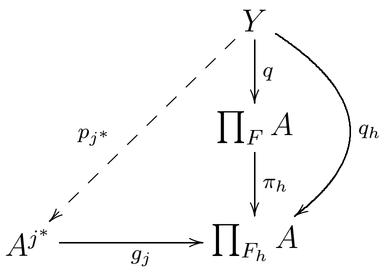
For each $h\colon B_i = B_{i(h)} \longrightarrow\!\!\!\!\!\!\rightarrow C$, we have a diagram similar to (1) in the proof of part (a). Taking the product over all $h \in H$, we get the following diagram



Note that the product $\prod\limits_{h\in H} (\prod_{F_h} A)$ is actually a reduced power $\prod_F A$ for some $\lambda$-complete filter F (see [CK, 90]). The required morphism $u\colon B \longrightarrow \prod_F A$ is the composition $(\prod\limits_{h\in H} \bar{f}_h)\cdot\langle f_h^*\rangle_{h\in H} = \langle \bar{f}_h f_h^*\rangle_{h\in H}$. It is easy to verify that $uf\colon A \longrightarrow \prod_F A$ is the canonical diagonal morphism. What remains to be shown is that $u$ is in M.

As in the case $H = \emptyset$, we need to consider a commutative square $up = qv$, with $v\colon X \to Y$ in $\mathrm{E}_\lambda$, and show that there exists a unique diagonal $d\colon Y \longrightarrow B$ such that $dv = p$ and $ud = q$ (see diagram (2) above). As before, $p$ factorizes $p = f_i' p_i$ through some colimit component $f_i'$, and we consider the pushout $(h, p_i')$ of $(v, p_i)$ (see diagram (3)). Again we'll be looking for a factorization of $f_i'$ through $h$.

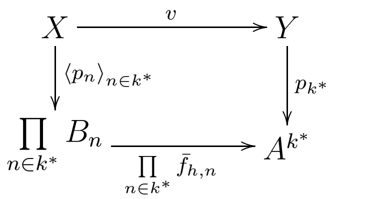
Suppose there is no such factorization of $f_i'$. Then $h \in H$, by definition of $H$. Let $q_h = \pi_h q\colon Y \longrightarrow \prod_F A = \prod\limits_{h\in H} (\prod_{F_h} A) \longrightarrow \prod_{F_h} A$, where $\pi_h$ is the canonical projection. Because $Y$ is $\lambda$-presentable, there exists $j^* \in F_h$ such that $q_h$ factorizes through the colimit component $g_j\colon A^{j^*} \longrightarrow \prod_{F_h} A = \operatorname*{colim}\limits_{j\geq i}(A^{j^*})$:



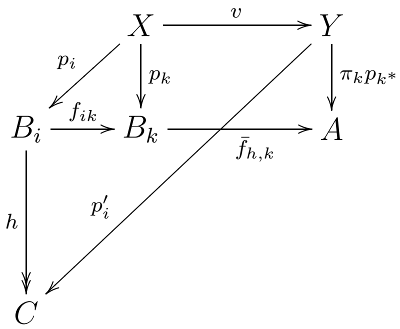

Put $p_n = f_{in}p_i$ and $p_{n^*} = g_{in}p_i$. $X$ being $\lambda$-presentable, the fact that

$$g_j \cdot \prod_{n\in j^*} \bar{f}_{h,n} \cdot \langle p_n \rangle_{n\in j^*} = g_j \cdot p_{j^*} \cdot v$$

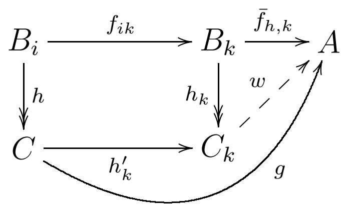
implies that there exists $k \geq j$ such that



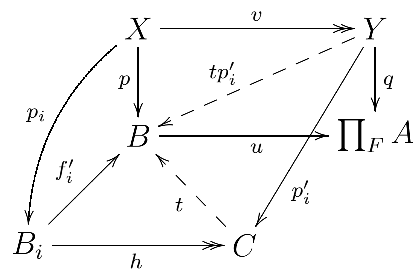

commutes. Composing with the projection $\pi_k\colon A^{k^*} \longrightarrow A_k = A$ , we have a commutative diagram




which induces $g\colon C \longrightarrow A$ such that $gh = \bar{f}_{h,k} f_{ik}$ (since $(h, p_i')$ is the pushout of $(v, p_i)$). This in turn induces $w\colon C_k \longrightarrow A$, which in particular satisfies $wh_k = \bar{f}_{h,k}$:



This contradicts the definition of $\bar{f}_{h,k}$.
Hence there exists $t\colon C \longrightarrow B$ such that $th = f_i'$:




We have $tp_i'v = thp_i = f_i'p_i = p$. Since $v$ is epi, we also have $utp_i' = q$, and the unicity of $tp_i'$ as a diagonal.

2.5 ($\Leftarrow$) This is immediate from 2.3(a)(ii) and (d).

($\Rightarrow$) Let $f\colon A \to B$ be existentially closed, and, as in 2.4 (a) and (b), $(f_i'\colon (f_i \longrightarrow f))_I = \operatorname{colim}_I (f_{ij}\colon f_i \to f_j)$, with the $f_i$'s finitely presentable. Let

$$R\colon = \{(i,c) \mid i \in I,\ c \in \hat{C}_i\},$$

where $\hat{C}_i$ is the set of all finitely presentable cones in E (= finite cones of finitely presentable morphisms in E) with domain $B_i$, and through which $f_i'\colon B_i \to B$ does not factorize.

We define a preorder on $R$ by

$$(i,c) \leq (i',d)$$

if and only if $i \leq i'$ and for all $s\colon B_{i'} \to A$, we have:

$$sf_{ii'} \text{ factorizes through } c \implies s \text{ factorizes through } d.$$

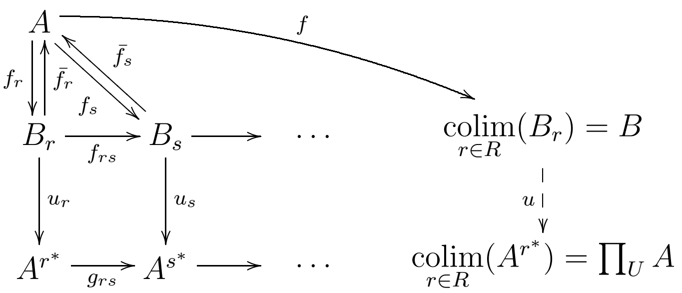
Then $R$ is directed: let $(i,c)$ and $(i',d)$ be in $R$, with $c = (h_{ij} \mid j \in J(c))$ and $d = (h_{i'j} \mid j \in J(d))$. Choose $i'' \geq i, i'$, and let $h_{i''j}$ be the pushout of $h_{ij}$ by $f_{ii''}$ (respectively of $h_{i'j}$ by $f_{i'i''}$). Then, for $e = (h_{i''j} \mid j \in J(c) \cup J(d))$, we clearly have $(i,c), (i',d) \leq (i'',e)$. Note that $e$ is a finitely presentable cone in E.

For $r \in R$, define $r^* = \{r' \mid r \leq r'\}$, and

$$F = \{R' \subseteq R \mid R' \supseteq r^* \text{ for some } r \in R\}.$$

$F$ is a filter on $R$, which can be extended to an ultrafilter $U$. For $r = (i,c) \leq (i',d) = r'$ in $R$, we write $f_{rr'}\colon B_r \to B_{r'}$ for the morphism $f_{ii'}\colon B_i \to B_{i'}$; similarly we put $f_r = f_i$ and $f_r' = f_i'$.

Let $\bar{f}_r\colon B_r \to A$ be a left inverse to $f_r$ which does not factorize through $c$. Then, we get a diagram similar to (1):

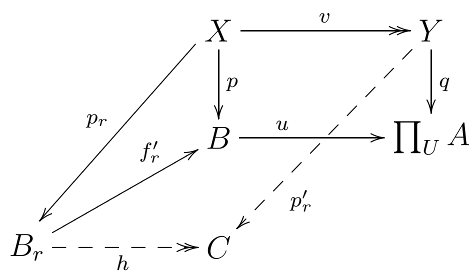


where $u_r = (\prod_{t\in r^*} \bar{f}_t) \cdot \langle f_{rt}\rangle_{t\in r^*} : B_r \longrightarrow \prod_{t\in r^*} B_t \longrightarrow A^{r^*}$.

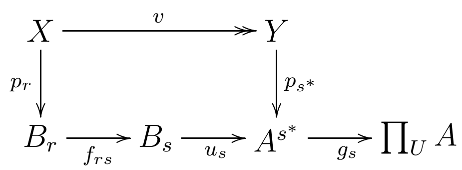
We need to show that $uf = \Delta_U$, and $u \in \mathrm{M}$.

$uf = \Delta_U$ is straightforward. To show that $u \in \mathrm{M}$, we consider, as in 2.4 (b), a commutative square $qv = up$ (see (2) above with $F = U$), with $v \in \mathrm{E}_\omega$. Then again $p$ factorizes $p = f_i'p_i$ for some $i \in I$, and we let $(h, p_i')$ be the pushout of $(v, p_i)$ (see (3) above). $h$ is a finitely presentable morphism in E, and we now show that $f_i'$ factorizes through it.

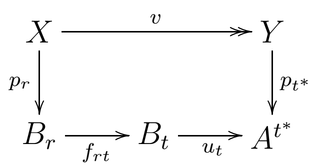
Suppose that there is no such factorization. Then $r = (i, \{h\})$ is in $R$. We rename $B_i$ as $B_r$, $p_i = p_r$ and $f_i' = f_r'$:



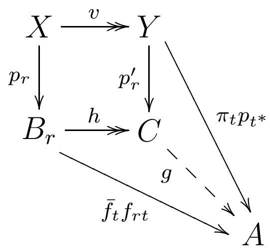
$Y$ being finitely presentable, there exists $s^* \in U$ such that $q$ factorizes $q = g_s p_{s^*}$ through the colimit component $g_s\colon A^{s^*} \longrightarrow \prod_U A$. We can assume $s \geq r$, and we have $g_s \cdot p_{s^*} \cdot v = g_s \cdot u_s \cdot f_{rs} \cdot p_r$



$X$ being finitely presentable, there exists $t \geq s$ and $g_{st}\colon A^{s^*} \to A^{t^*}$ such that $g_{st} \cdot p_{s^*} \cdot v = g_{st} \cdot u_s \cdot f_{rs} \cdot p_r$. Hence the diagram



commutes. Composing with the projection $\pi_t\colon A^{t^*} \to A$, we get $\pi_t u_t f_{rt} = \bar{f}_t f_{rt}$, and the commutative square $\pi_t p_{t^*} v = \bar{f}_t f_{rt} p_r$ induces a morphisms $g\colon C \to A$ making




commute.

Now, from the definition of $r = (i, \{h\}) \le t = (k, c)$, the fact that $\bar{f}_t f_{rt}$ factorizes through $h$ implies that $\bar{f}_t$ factorizes through the cone $c = (c_j\colon B_k \to C_j \mid j \in J(c))$. But this contradicts the definition of $\bar{f}_t$. Hence there must exist $t\colon C \to B$ such that $th = f'_r$, and one checks as in 2.4 (b) that $tp'_r\colon Y \to B$ is the required diagonal in the square $qv = up$. This completes the proof that $u$ lies in M.

■

## 3. Injectivity and syntactic characterizations

In this section, we study the relations between injectivity classes and closure under the three types of subobjects considered so far.

It may be useful first to see explicitly how the categorical Definitions 2.1 specialize in the classical model theoretic context $\mathcal{C} = \mathrm{Mod}(\Sigma)$. We already mentioned that if M is the class Emb of all ($\Sigma$-) embeddings, then the class $E(\mathrm{M})$, in this category, is the family Sur of all the surjective ($\Sigma$-) homomorphisms. Recall also that a *basic* formula is an atomic formula or the negation of an atomic formula. We have:

3.1. Proposition. *Let $\mathcal{C} = \mathrm{Mod}(\Sigma)$, $\Sigma$ a $\lambda$-ary signature, with (E,M)=(Sur, Emb). Below, $\beta$ and $\delta$ (resp. $\gamma$) are arbitrary sets of less than $\lambda$ atomic (resp. basic) formulas. Let $f\colon A \longrightarrow B$ be a morphism in $\mathcal{C}$. Then*

*(a) $f$ is $\lambda$-algebraically closed iff for every $\beta(\boldsymbol{x}, \boldsymbol{y})$ such that*

$$B \models \exists \boldsymbol{y}(\wedge\beta[f(\boldsymbol{a}), \boldsymbol{y}])$$

*for some string $\boldsymbol{a}$ in $A$, we have*

$$A \models \exists \boldsymbol{y}(\wedge\beta[\boldsymbol{a}, \boldsymbol{y}]).$$

*(b) $f$ is weakly $\lambda$-existentially closed iff for every $\beta(\boldsymbol{x}, \boldsymbol{y})$ and $\delta(\boldsymbol{x}, \boldsymbol{y})$ such that*

$$B \models \exists \boldsymbol{y}(\wedge\beta[f(\boldsymbol{a}), \boldsymbol{y}] \wedge (\neg \wedge \delta[f(\boldsymbol{a}), \boldsymbol{y}]))$$

*for some string $\boldsymbol{a}$ in $A$, we have*

$$A \models \exists \boldsymbol{y}(\wedge\beta[\boldsymbol{a}, \boldsymbol{y}] \wedge (\neg \wedge \delta[\boldsymbol{a}, \boldsymbol{y}])).$$

*(c) $f$ is $\lambda$-existentially closed iff for every $\gamma(\boldsymbol{x}, \boldsymbol{y})$ such that*

$$B \models \exists \boldsymbol{y}(\wedge\gamma[f(\boldsymbol{a}), \boldsymbol{y}])$$

*for some string $\boldsymbol{a}$ in $A$, we have*

$$A \models \exists \boldsymbol{y}(\wedge\gamma[\boldsymbol{a}, \boldsymbol{y}]).$$

Proof. For $\lambda = \omega$, (a) and (c) follow from Theorems 2.4 and 2.5, by the well-known model theoretic characterizations of the algebraically closed and existentially closed embeddings ([E, 77]). The general statement in (a) and (b) can be derived from 2.4 and 2.5 using the same standard technique than [E, 77]. However, we will sketch a direct proof, in order to make the connection explicit.

We associate to any given family $\{k_j g: A \longrightarrow C \twoheadrightarrow D_j\}_J$ of $\lambda$-presentable morphisms, $|J| < \lambda$, the following (classical) presentations. First, we present $A$ by its *positive diagram*

$$A = \langle \boldsymbol{X_A}; \Phi(\boldsymbol{X_A})\rangle_\Sigma,$$

i.e., $\Phi(\boldsymbol{X_A}) := \{\varphi(\boldsymbol{x_a}) \mid A \models \varphi[\boldsymbol{a}],\ \varphi \text{ atomic}\}$. Here, as in the proof of 1.6, $\boldsymbol{x_a}$ is a string of variables corresponding to $\boldsymbol{a}$. Capitalized (bold) letters will generally represent strings or sets of unrestricted cardinality; we write $\varphi(\boldsymbol{x_a})$ to emphasize the fact that only a substring $\boldsymbol{x_a}$ of $\boldsymbol{X_A}$ of length $< \lambda$ can actually appear in any given $\varphi$ (since $\Sigma$ is $\lambda$-ary).

Then, applying the characterization of $\lambda$-presentable morphisms (1.2) to $g$, the fact that $\lambda$-presentability in $\mathcal{C} = \mathrm{Mod}(\Sigma)$ has the usual meaning (see the proof of Proposition 1.6) allows one to see that $C$ has a presentation of the form

$$\langle \boldsymbol{X_A} \cup \boldsymbol{x_c}; \Phi(\boldsymbol{X_A}) \cup \beta(\boldsymbol{x_a}, \boldsymbol{x_c})\rangle_\Sigma$$

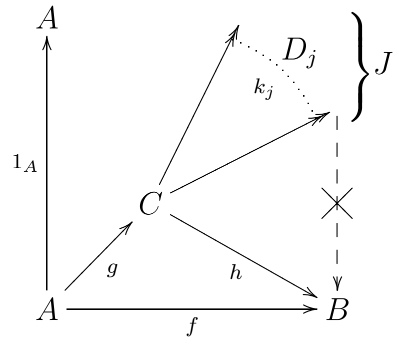
where $\beta$ is some set of less than $\lambda$ atomic formulas, and $\boldsymbol{c}$ is a string of elements of $C$ of length $< \lambda$. In this presentation, $g$ is identified with the function $x_a \mapsto x_a$; if $g(a) = g(a')$, the identification $x_a = x_{a'}$ (in the presentation of $C$) follows from the identities in $\beta$.

Similarly, for each $j$ there exists a presentation of $D_j$ which adds less than $\lambda$ (variables and) atomic formulas to the ones presenting $C$. All $k_j$'s are surjective, so that for each generator $x_d$ of $D_j$, we have $d = k_j(c)$ for some $c \in C$. Because $|J| < \lambda$, we can add one $x_c$ to the generators of $C$ for each $c$ such that $x_{k_j(c)}$ is involved in the presentation of $D_j$ (if it is not already there), and the cardinality of the set of all these extra $x_c$ will still be $< \lambda$. Assuming this has been done in the presentation of $C$ above, we can then present $D_j$ by

$$D_j = \langle \boldsymbol{X_A} \cup \boldsymbol{x_c}; \Phi(\boldsymbol{X_A}) \cup \beta(\boldsymbol{x_a}, \boldsymbol{x_c}) \cup \delta_j(\boldsymbol{x_a}, \boldsymbol{x_c})\rangle_\Sigma,$$

for some set $\delta_j$ of less than $\lambda$ atomic formulas.

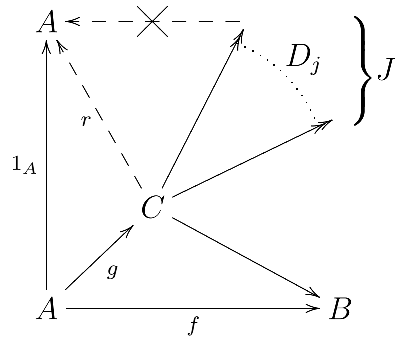
Consider the diagram




where $f$ and $h$ are some $\Sigma$-homomorphisms. Then we have

$$B \models \wedge\beta[f(\boldsymbol{a}), h(\boldsymbol{c})] \wedge (\wedge_{j\in J}(\neg \wedge \delta_j[f(\boldsymbol{a}), h(\boldsymbol{c})])).$$

If the diagram can be completed by some appropriate $r$:




then

$$A \models \wedge\beta[\boldsymbol{a}, r(\boldsymbol{c})] \wedge (\wedge_{j\in J}(\neg \wedge \delta_j[\boldsymbol{a}, r(\boldsymbol{c})])).$$

Using this translation, the details of the proof are now straightforward. ∎

We fix some notations to represent injectivity with respect to specific types of cones and morphisms:

3.2. Definitions. Let $\mathcal{C}$ be a locally $\lambda$-presentable category.

(a) A $\lambda_m$-*cone* is a cone made of $\lambda$-presentable morphisms.

(b) If $d$ is a morphism or a cone with domain $A$, we say that an object $C$ is $d$-*injective*, denoted by

$$C \mid\models d,$$

if every morphism $f\colon A \longrightarrow C$ factorizes through $d$.

(c) If $\mathbf{D}$ is a class of morphisms or of cones, $\mathrm{Mod}_{|\models}(\mathbf{D})$ will denote the class

$$\{C \in \mathcal{C} \mid C \mid\models d \text{ for all } d \in \mathbf{D}\}\,.$$

Then, a class $\mathcal{K}$ of objects is a $\lambda_m$*-injectivity class* (resp. a $\lambda_m$*-cone-injectivity class*) if there exists a class $\mathbf{D}$ of $\lambda$-presentable morphisms (resp. of $\lambda_m$-cones) such that $\mathcal{K} = \mathrm{Mod}_{|\models}(\mathbf{D})$. If $\mathbf{D}$ is a class of morphisms between $\lambda$-presentable objects, we say that $\mathcal{K}$ is a $\lambda$*-injectivity class.*

Consider again the classical context $\mathcal{C} = \mathrm{Mod}(\Sigma)$, with $\Sigma$ $\lambda$-ary, and (E,M) = (Sur, Emb). Proceeding as in the proof of Proposition 3.1, each morphism $g_i$ of a $\lambda_m$-cone $d = (g_i \colon A \longrightarrow B_i)_I$ can be presented by generators and relations

$$\langle \boldsymbol{X_A},\, \Phi(\boldsymbol{X_A})\rangle_\Sigma \xrightarrow{g_i} \langle \boldsymbol{X_A} \cup \boldsymbol{x_{b_i}};\, \Phi(\boldsymbol{X_A}) \cup \beta_i(\boldsymbol{x_{a_i}}, \boldsymbol{x_{b_i}})\rangle_\Sigma$$

where $g_i(x) = x$ for all $x \in \boldsymbol{X_A}$. Then one sees that for $C \in \mathcal{C}$,

$$C \mid\models d$$

if and only if

$$C \models \forall \boldsymbol{X}(\wedge \Phi(\boldsymbol{X}) \longrightarrow \vee_{i\in I}(\exists \boldsymbol{y_i}(\wedge \beta_i(\boldsymbol{x_i}, \boldsymbol{y_i})))),$$

where we wrote $\boldsymbol{X}$ for $\boldsymbol{X_A}$, $\boldsymbol{x_i}$ for $\boldsymbol{x_{a_i}}$ and $\boldsymbol{y_i}$ for $\boldsymbol{x_{b_i}}$. Note that the $\boldsymbol{x_i}$'s are substrings of $\boldsymbol{X}$, and that for each $i \in I$, the formula $\exists \boldsymbol{y_i}(\wedge \beta_i(\boldsymbol{x_i}, \boldsymbol{y_i})))$ is in $L_\lambda(\Sigma)$.

The following was proved in [$\mathrm{H}_1$, 98].

3.3. Theorem. [$\mathrm{H}_1$, 98] *Let $\mathcal{K}$ be a class of objects in a locally $\lambda$-presentable category.*

*(a) The class of all $\lambda$-algebraically closed subobjects of the objects in $\mathcal{K}$ is a $\lambda_m$-cone-injectivity class.*

*(b) $\mathcal{K}$ is a $\lambda_m$-cone-injectivity class iff it is closed under $\lambda$-algebraically closed subobjects.*

*(c) $\mathcal{K}$ is a $\lambda_m$-injectivity class iff it is closed under $\lambda$-algebraically closed subobjects and products.*

The proof uses in particular the characterization in Lemma 2.2.

For $\mathcal{C} = \mathrm{Mod}(\Sigma)$, Theorem 3.3 says that the classes of $\Sigma$-structures closed under $\lambda$-algebraically closed subobjects are the ones axiomatizable by classes of sentences of the appropriate type, as given after Definition 3.2. Adding closure under products is easily seen to correspond to removing the disjunctions:

3.4. Corollary. *A class $\mathcal{K}$ of $\lambda$-ary $\Sigma$-structures is closed in* $\mathrm{Mod}(\Sigma)$ *under $\lambda$-algebraically closed substructures (resp. under $\lambda$-algebraically closed substructures and products) iff it can be axiomatized by a class of sentences of the form*

$$\forall \boldsymbol{X}(\wedge \Phi(\boldsymbol{X}) \longrightarrow \vee_{i\in I}(\exists \boldsymbol{y_i}(\wedge\beta_i(\boldsymbol{x_i}, \boldsymbol{y_i}))))$$

*(resp.*

$$\forall \boldsymbol{X}(\wedge \Phi(\boldsymbol{X}) \longrightarrow \exists \boldsymbol{y}(\wedge\beta(\boldsymbol{x}, \boldsymbol{y})))\ ),$$

*where $\Phi$ is a set (of any cardinality) of atomic formulas, and each $\beta_i$ (resp. $\beta$) is a set of less than $\lambda$ atomic formulas.*

The sentences in Corollary 3.4 are in $L_\infty(\Sigma)$. What prevent them to be in $L_\lambda(\Sigma)$, are the sizes of $\beta$ and of $I$. Adding closure under $\lambda$-directed colimits to closure under products will give the expected preservation theorem in $L_\lambda(\Sigma)$. In terms of injectivity, this means we will pass from injectivity with respect to $\lambda$-presentable morphisms to injectivity with respect to morphisms between $\lambda$-presentable objects (i.e., from $\lambda_m$-injectivity to $\lambda$-injectivity). This will be shown in the context of locally $\lambda$-presentable categories (Theorem 3.5 below), giving us a different proof of the main result of [RAB, 02].

To reduce the class of $\lambda$-presentable morphisms in Theorem 3.3 to a set, the additional condition needed of $\mathcal{C}$ is precisely to be closed under $\alpha$-directed colimits for some $\alpha$. This was shown already in [AR, 93]. We will obtain the refined version of [RAB, 02] by showing that closure under $\lambda$-directed colimits allows replacing $\lambda$-presentable morphisms by morphisms between $\lambda$-presentable objects (of which essentially only a set exists).

It is known from the 60's that an elementary class $\mathcal{K}$ of finitary $\Sigma$-structures is closed under directed colimits and (reduced) products if and only if it can be axiomatized by ($L_\omega(\Sigma)$-) sentences of the form

$$\forall \boldsymbol{x}(\wedge\varphi(\boldsymbol{x})) \longrightarrow \exists \boldsymbol{y}(\wedge\beta(\boldsymbol{x}, \boldsymbol{y}))$$

where $\varphi$ and $\beta$ are finite sets of atomic formulas (see [CK, 90]). More recently, Rothmaler ([R, 97]) noticed that the assumption of elementarity of $\mathcal{K}$ can be replaced by closure under ($\omega$-) algebraically closed substructures. This amounts to replace closure under elementary substructures by closure under algebraically closed substructures. Krause ([K, 98]) also obtained this characterization, in a context which roughly corresponds to the additive locally finitely presentable categories (note that his locally *presented* categories are different from the locally *presentable* categories). Finally, this characterization without the additivity and the finitary assumptions is obtained in [RAB, 02], by very different methods. We reach the same result following a different path altogether, as a consequence of our Theorem 3.3.

3.5. Theorem. [RAB, 02] *Let $\mathcal{K}$ be a class of objects in a locally $\lambda$-presentable category $\mathcal{C}$. Then $\mathcal{K}$ is a $\lambda$-injectivity class iff it is closed under products, $\lambda$-directed colimits and $\lambda$-algebraically closed subobjects (or, equivalently, closed under $\lambda$-reduced products and $\lambda$-algebraically closed subobjects).*

Proof. Clearly, closure under $\lambda$-directed colimits and products implies closure under $\lambda$-reduced products. Hence we need to show that

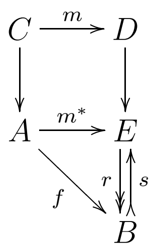
1) if $\mathcal{K}$ is a $\lambda$-injectivity class, then it is closed under $\lambda$-directed colimits, products and $\lambda$-algebraically closed subobjects, and

2) if $\mathcal{K}$ is closed under $\lambda$-reduced products and $\lambda$-algebraically closed subobjects, then it is a $\lambda$-injectivity class.

As for 1), $\lambda$-injectivity classes are special $\lambda_m$-injectivity classes, so by 3.3, only closure under $\lambda$-directed colimits remains to be shown. This is left as a straightforward exercise. We now prove 2).

Products are special $\lambda$-reduced products, so we know from 3.3 that $\mathcal{K} = \mathrm{Mod}_{|\models}(\mathbf{D})$ for some class $\mathbf{D}$ of $\lambda$-presentable morphisms. To be able to replace $\mathbf{D}$ by a class (and then, by a set) of morphisms between $\lambda$-presentable objects of $\mathcal{C}$, we will use Proposition 1.2.

We first note that for a diagram




satisfying the conditions in the statement of 1.2, we have, for any object $X$ in $\mathcal{C}$,

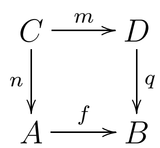
a) $X \mid\models m^* \quad \Leftrightarrow \quad X \mid\models f$, and

b) $X \mid\models m \quad \Rightarrow \quad X \mid\models m^*$.

This follows easily from the facts that the square is a pushout and that $rs = 1_f$ in $(A \downarrow \mathcal{C})$. Then by a), we can assume that the class $\mathbf{D}$ above is made of morphisms of type $m^*$, i.e., pushouts of morphisms between $\lambda$-presentable objects. Using b), we will be done if we prove the following:

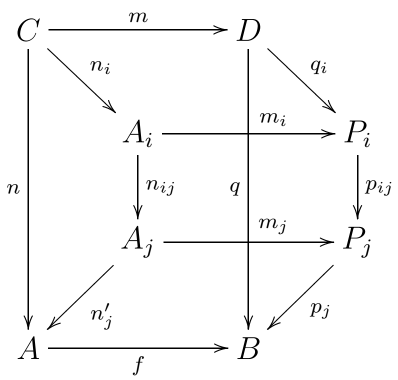
3.6. Lemma. *Let $\mathcal{K}$ be a class of objects in $\mathcal{C}$ closed under $\lambda$-reduced products. Let $\mathcal{K} \mid\models f$ for some $f$ which is the pushout of a morphism between $\lambda$-presentable objects. Then $f$ is the pushout of some morphism $m_f$ between $\lambda$-presentable objects such that $\mathcal{K} \mid\models m_f$.*

Proof. Let then



be a pushout diagram, where $C$ and $D$ are $\lambda$-presentable and $\mathcal{K} \mid\models f$.

Let $(n'_i\colon (n_i\colon C \longrightarrow A_i) \longrightarrow (n\colon C \longrightarrow A))_I$ be the colimit in $(C \downarrow \mathcal{C})$ of the $\lambda$-directed diagram $(n_{ij}\colon n_i \longrightarrow n_j)_I$, where the $n_i$'s are $\lambda$-presentable morphisms. Taking the pushout $(q_i, m_i)$ of $(n_i, m)$ for each $i$, we get the commutative diagram

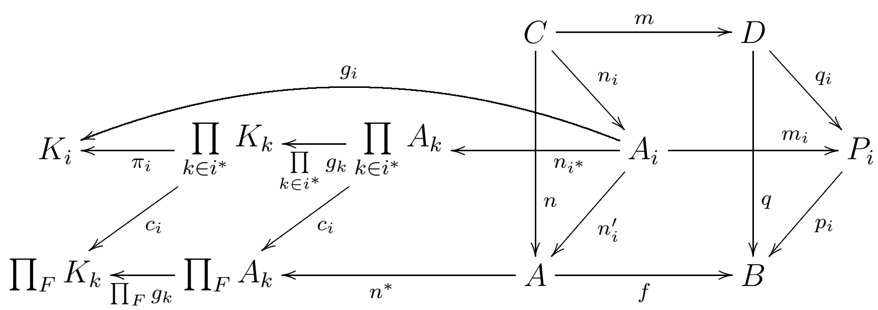



where the $p_i$'s and the $p_{ij}$'s are the induced morphisms. We now assume $\mathcal{K} \mid\models f$, and we suppose $\mathcal{K} \not\mid\models m_i$ for all $i \in I$. We will reach a contradiction.

$\mathcal{K} \not\mid\models m_i$ means that there exists $g_i\colon A_i \longrightarrow K_i$, $K_i \in \mathcal{K}$, which does not factorize through $m_i\colon A_i \longrightarrow P_i$. Choose a $\lambda$-complete filter $F$ on $I$ which contains all $i^* = \{j \in I \mid i \leq j\}$, $i \in I$. Denote by $(c_i\colon \prod_{k\in i^*} A_k \longrightarrow \prod_F A_k)_I$ the colimit diagram (in $\mathcal{C}$) of the $\lambda$-directed diagram of the projections

$$( \pi_{ij}\colon \prod_{k\in i^*} A_k \longrightarrow \prod_{k\in j^*} A_k \mid i \leq j\ )_I.$$

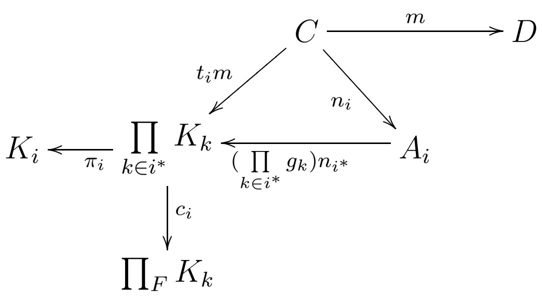
Denoting by $n_{i^*}\colon A_i \longrightarrow \prod_{k\in i^*} A_k$ the morphism $n_{i^*} = \langle n_{ij}\rangle_{j\in i^*}$, the family $\{n_{i^*}\}_{i\in I}$ induces a morphism $n^*\colon A = \operatorname*{colim}_{i\in I}(A_i) \longrightarrow \prod_F A_k = \operatorname*{colim}_{i\in I}(\prod_{k\in i^*} A_k)$ such that $n^* n'_i = c_i n_{i^*}$ for all $i \in I$. The family $\{g_i\colon A_i \longrightarrow K_i\}_I$ induces morphisms $\prod_{k\in i^*} g_k\colon \prod_{k\in i^*} A_k \longrightarrow \prod_{k\in i^*} K_k$ and $\prod_F g_k\colon \prod_F A_k \longrightarrow \prod_F K_k$, such that the diagram



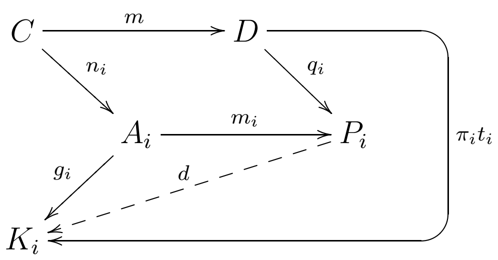

commutes. Now, $\prod_F K_k \in \mathcal{K}$, so that $(\prod_F g_k)n^*\colon A \longrightarrow \prod_F K_k$ must factorize $(\prod_F g_k)n^* = gf$ through $f$, for some $g\colon B \longrightarrow \prod_F K_k$. We have $gq\colon D \longrightarrow \prod_F K_k = \operatorname*{colim}_{i\in I}(\prod_{k\in i^*} K_k)$

with $D$ $\lambda$-presentable, so that there must exist $i \in I$ and $t_i\colon D \longrightarrow \prod_{k\in i^*} K_k$ such that $c_i t_i = gq$.

In the diagram




we have $c_i \cdot (\prod_{k\in i^*} g_k) \cdot (n_{i^*}) = c_i \cdot (t_i m)$ (by chasing the previous diagram). $C$ being $\lambda$-presentable, we can choose $i$ such that $(\prod_{k\in i^*} g_k) n_{i^*} = t_i m$. This equality induces (a unique) $d\colon P_i \longrightarrow K_i$ such that $dm_i = g_i$ (and $dq_i = \pi_i t_i$)




(noting that $g_i = \pi_i \cdot (\prod_{k\in i^*} g_k) \cdot n_{i^*}$). This contradicts the assumption above. ∎

Using 3.6 and the remarks preceding it, it follows that $\mathcal{K} = \mathrm{Mod}_{|\models}(\mathbf{D}')$, where $\mathbf{D}' = \{ m_f \mid f \in \mathbf{D} \}$. This completes the proof of 3.5. ∎

3.7. Notes.

1) Lemma 3.6 can be seen as the categorical version of an infinitary "compactness-like" property: following the translation just after Definition 3.2, its meaning in $\mathcal{C} = \mathrm{Mod}(\Sigma)$, $\Sigma$ $\lambda$-ary, is that if $\mathcal{K}$ is a class of $\Sigma$-structures closed under $\lambda$-reduced products, then for any given sentence of the form

$$\forall \boldsymbol{X}(\wedge \Phi(\boldsymbol{X}) \longrightarrow \exists \boldsymbol{y}(\wedge \beta(\boldsymbol{x}, \boldsymbol{y})))$$

true in $\mathcal{K}$, where $\Phi$ is a set (of any cardinality) of atomic formulas, $\boldsymbol{x} \subseteq \boldsymbol{X}$, and $\beta$ is a set of less than $\lambda$ atomic formulas, there exists a subset $\varphi$ of $\Phi$ of cardinality less than $\lambda$, and $\boldsymbol{x}' \supseteq \boldsymbol{x}$ such that $\mathcal{K}$ satisfies

$$\forall \boldsymbol{x}'(\wedge \varphi(\boldsymbol{x}') \longrightarrow \exists \boldsymbol{y}(\wedge \beta(\boldsymbol{x}, \boldsymbol{y}))).$$

Clearly this and Corollary 3.4 imply that the classes of Σ-structures closed under products, $\lambda$-directed colimits and $\lambda$-algebraically closed substructures are the ones axiomatizable by sets of $L_\lambda(\Sigma)$-sentences as above (called $\lambda$-*regular* sentences). In [RAB, 02], a counterexample shows that the analogous reduction is not possible if the class is not closed under products.

2) What was done in Theorem 3.3 and Corollary 3.4 for algebraically closed embeddings can be done for existentially closed and weakly existentially closed embeddings. First, instead of $\lambda_m$-cones, one considers diagrams of the form $( A \xrightarrow[g_i]{(\lambda)} B_i \xrightarrow[h_{ij}]{(\lambda)} C_{ij} \mid i \in I, j \in J_i )$, with each $|J_i| < \lambda$ (resp., $|J_i| < 2$), and then defines injectivity with respect to these diagrams in an appropriate way. Then, using a very similar technique than the proof in [$H_1$, 98], and using the relevant parts of Lemma 2.2, one shows that the classes of objects closed under $\lambda$-existentially closed (resp., weakly $\lambda$-existentially closed) subobjects are precisely the injectivity classes with respect to the above special types of diagrams. A syntactic characterization (in the context of $\mathcal{C} = \mathrm{Mod}(\Sigma)$) follows easily. However, serious difficulties arise when one adds closure under $\lambda$-directed colimits, in trying to obtain the analogous of the characterization 3.5. This is not surprising, considering the difficulty of the proofs of the classical preservation theorem for reduced products, even for ∀∃-theories (see [CK, 90], Section 3.5). The infinitary case also seems to raise additional problems, see [HS, 81]. Actually, the very fact that the known proof of the classical preservation theorem for reduced products is so intricate may be seen as an incentive to look for a different (more categorical?) line of attack.

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