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FREE A_∞-CATEGORIES

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ABSTRACT. For a differential graded $\Bbbk$-quiver $\mathcal{Q}$ we define the free A_∞-category $\mathcal{FQ}$ generated by $\mathcal{Q}$. The main result is that the restriction A_∞-functor $A_\infty(\mathcal{FQ},\mathcal{A}) \to A_1(\mathcal{Q},\mathcal{A})$ is an equivalence, where objects of the last A_∞-category are morphisms of differential graded $\Bbbk$-quivers $\mathcal{Q} \to \mathcal{A}$.

A_∞-categories defined by Fukaya [Fuk93] and Kontsevich [Kon95] are generalizations of differential graded categories for which the binary composition is associative only up to a homotopy. They also generalize A_∞-algebras introduced by Stasheff [Sta63, II]. A_∞-functors are the corresponding generalizations of usual functors, see e.g. [Fuk93, Kel01]. Homomorphisms of A_∞-algebras (e.g. [Kad82]) are particular cases of A_∞-functors. A_∞-transformations are certain coderivations. Examples of such structures are encountered in studies of mirror symmetry (e.g. [Kon95, Fuk02]) and in homological algebra.

For an A_∞-category there is a notion of units up to a homotopy (homotopy identity morphisms) [Lyu03]. Given two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, one can construct a third A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$, whose objects are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$, and morphisms are A_∞-transformations between such functors (Fukaya [Fuk02], Kontsevich and Soibelman [KS02, KS], Lefèvre-Hasegawa [LH03], as well as [Lyu03]). This allows to define a 2-category, whose objects are unital A_∞-categories, 1-morphisms are unital A_∞-functors and 2-morphisms are equivalence classes of natural A_∞-transformations [Lyu03]. We continue to study this 2-category.

The notations and conventions are explained in the first section. We also describe A_N-categories, A_N-functors and A_N-transformations – truncated at $N < \infty$ versions of A_∞-categories. For instance, A_1-categories and A_1-functors are differential graded $\Bbbk$-quivers and their morphisms. However, A_1-transformations bring new 2-categorical features to the theory. In particular, for any differential graded $\Bbbk$-quiver $\mathcal{Q}$ and any A_∞-category $\mathcal{A}$ there is an A_∞-category $A_1(\mathcal{Q},\mathcal{A})$, whose objects are morphisms of differential graded $\Bbbk$-quivers $\mathcal{Q} \to \mathcal{A}$, and morphisms are A_1-transformations. We recall the terminology related to trees in Section 1.7.

In the second section we define the free A_∞-category $\mathcal{FQ}$ generated by a differential graded $\Bbbk$-quiver $\mathcal{Q}$. We classify functors from a free A_∞-category $\mathcal{FQ}$ to an arbitrary A_∞-category $\mathcal{A}$ in Proposition 2.3. In particular, the restriction map gives a bijection

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between the set of strict A_∞-functors $\mathcal{F}\mathcal{Q} \to \mathcal{A}$ and the set of morphisms of differential graded $\Bbbk$-quivers $\mathcal{Q} \to (\mathcal{A}, m_1)$ (Corollary 2.4). We classify chain maps into complexes of transformations whose source is a free A_∞-category in Proposition 2.8. Description of homotopies between such chain maps is given in Corollary 2.10. Assuming in addition that $\mathcal{A}$ is unital, we obtain our main result: the restriction A_∞-functor restr : $A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A})$ is an equivalence (Theorem 2.12).

In the third section we interpret $A_\infty(\mathcal{F}\mathcal{Q}, \_)$ and $A_1(\mathcal{Q}, \_)$ as strict A^u_∞-2-functors $A^u_\infty \to A^u_\infty$. Moreover, we interpret restr : $A_\infty(\mathcal{F}\mathcal{Q}, \_) \to A_1(\mathcal{Q}, \_)$ as an A^u_∞-2-equivalence. In this sense the A_∞-category $\mathcal{F}\mathcal{Q}$ represents the A^u_∞-2-functor $A_1(\mathcal{Q}, \_)$. This is the 2-categorical meaning of freeness of $\mathcal{F}\mathcal{Q}$.

1. Conventions and preliminaries

We keep the notations and conventions of [Lyu03, LO02], sometimes without explicit mentioning. Some of the conventions are recalled here.

We assume as in [Lyu03, LO02] that most quivers, A_∞-categories, etc. are small with respect to some universe $\mathscr{U}$.

The ground ring $\Bbbk \in \mathscr{U}$ is a unital associative commutative ring. A $\Bbbk$-module means a $\mathscr{U}$-small $\Bbbk$-module.

We use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted $fg : X \to Z$; a map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.

$\mathbb{Z}$-graded $\Bbbk$-modules are functions $X : \mathbb{Z} \ni d \mapsto X^d \in \Bbbk\text{-mod}$. A simple computation shows that the product $X = \prod_{\iota \in I} X_\iota$ in the category of $\mathbb{Z}$-graded $\Bbbk$-modules of a family $(X_\iota)_{\iota \in I}$ of objects $X_\iota : d \mapsto X^d_\iota$ is given by $X : \mathbb{Z} \ni d \mapsto X^d = \prod_{\iota \in I} X^d_\iota$. Everywhere in this article the product of graded $\Bbbk$-modules means the above product.

If P is a $\mathbb{Z}$-graded $\Bbbk$-module, then $sP = P[1]$ denotes the same $\Bbbk$-module with the grading $(sP)^d = P^{d+1}$. The "identity" map $P \to sP$ of degree -1 is also denoted s. The map s commutes with the components of the differential in an A_∞-category (A_∞-algebra) in the following sense: $s^{\otimes n} b_n = m_n s$.

Let $\mathsf{C} = \mathsf{C}(\Bbbk\text{-mod})$ denote the differential graded category of complexes of $\Bbbk$-modules. Actually, it is a symmetric closed monoidal category.

The cone of a chain of a chain map $\alpha : P \to Q$ of complexes of $\Bbbk$-modules is the graded $\Bbbk$-module $\operatorname{Cone}(\alpha) = Q \oplus P[1]$ with the differential $(q, ps)d = (qd^Q + p\alpha, psd^{P[1]}) = (qd^Q + p\alpha, -pd^P s)$.

1.1. A_N-categories. For a positive integer N we define some A_N-notions similarly to the case $N = \infty$. We may say that all data, equations and constructions for A_N-case are the same as in A_∞-case (e.g. [Lyu03]), however, taken only up to level N.

A *differential graded* $\Bbbk$-*quiver* $\mathcal{Q}$ is the following data: a $\mathscr{U}$-small set of objects $\operatorname{Ob}\mathcal{Q}$; a chain complex of $\Bbbk$-modules $\mathcal{Q}(X, Y)$ for each pair of objects X, Y. A morphism of differential graded $\Bbbk$-quivers $f : \mathcal{Q} \to \mathcal{A}$ is given by a map $f : \operatorname{Ob}\mathcal{Q} \to \operatorname{Ob}\mathcal{A}$, $X \mapsto Xf$

and by a chain map $\mathcal{Q}(X,Y) \to \mathcal{A}(Xf, Yf)$ for each pair of objects X, Y of $\mathcal{Q}$. The category of differential graded $\Bbbk$-quivers is denoted A_1.

The category of $\mathscr{U}$-small graded $\Bbbk$-linear quivers, whose set of objects is S, admits a monoidal structure with the tensor product $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A} \otimes \mathcal{B}$, $(\mathcal{A} \otimes \mathcal{B})(X,Y) = \oplus_{Z \in S} \mathcal{A}(X,Z) \otimes_\Bbbk \mathcal{B}(Z,Y)$. In particular, we have tensor powers $T^n\mathcal{A} = \mathcal{A}^{\otimes n}$ of a given graded $\Bbbk$-quiver $\mathcal{A}$, such that $\operatorname{Ob} T^n\mathcal{A} = \operatorname{Ob}\mathcal{A}$. Explicitly,

$$T^n\mathcal{A}(X,Y) = \bigoplus_{X=X_0,X_1,\dots,X_n=Y \in \operatorname{Ob}\mathcal{A}} \mathcal{A}(X_0,X_1) \otimes_\Bbbk \mathcal{A}(X_1,X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk \mathcal{A}(X_{n-1},X_n).$$

In particular, $T^0\mathcal{A}(X,Y) = \Bbbk$ if $X = Y$ and vanishes otherwise. The graded $\Bbbk$-quiver $T^{\leqslant N}\mathcal{A} = \oplus_{n\geqslant 0}^N T^n\mathcal{A}$ is called the restricted tensor coalgebra of $\mathcal{A}$. It is equipped with the cut comultiplication

$$\Delta : T^{\leqslant N}\mathcal{A}(X,Y) \to \bigoplus_{Z\in \operatorname{Ob}\mathcal{A}} T^{\leqslant N}\mathcal{A}(X,Z) \bigotimes_\Bbbk T^{\leqslant N}\mathcal{A}(Z,Y),$$
$$h_1 \otimes h_2 \otimes \cdots \otimes h_n \mapsto \sum_{k=0}^n h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n,$$

and the counit $\varepsilon = \bigl(T^{\leqslant N}\mathcal{A}(X,Y) \xrightarrow{\mathrm{pr}_0} T^0\mathcal{A}(X,Y) \to \Bbbk\bigr)$, where the last map is $\mathrm{id}_\Bbbk$ if $X = Y$, or 0 if $X \neq Y$ (and $T^0\mathcal{A}(X,Y) = 0$). We write $T\mathcal{A}$ instead of $T^{\leqslant\infty}\mathcal{A}$. If $g : T\mathcal{A} \to T\mathcal{B}$ is a map of $\Bbbk$-quivers, then g_{ac} denotes its matrix coefficient $T^a\mathcal{A} \xrightarrow{\mathrm{in}_a} T\mathcal{A} \xrightarrow{g} T\mathcal{B} \xrightarrow{\mathrm{pr}_c} T^c\mathcal{B}$. The matrix coefficient g_{a1} is abbreviated to g_a.

1.2. Definition. *An* A_N-category $\mathcal{A}$ *consists of the following data: a graded* $\Bbbk$*-quiver* $\mathcal{A}$*; a system of* $\Bbbk$*-linear maps of degree 1*

$$b_n : s\mathcal{A}(X_0,X_1) \otimes s\mathcal{A}(X_1,X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1},X_n) \to s\mathcal{A}(X_0,X_n), \qquad 1 \leqslant n \leqslant N,$$

such that for all $1 \leqslant k \leqslant N$

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \to s\mathcal{A}. \tag{1}$$

The system b_n is interpreted as a (1,1)-coderivation $b : T^{\leqslant N}s\mathcal{A} \to T^{\leqslant N}s\mathcal{A}$ of degree 1 determined by

$$b_{kl} = (b\big|_{T^k s\mathcal{A}})\,\mathrm{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{A}, \qquad b_{kl} = \sum_{\substack{r+n+t=k\\ r+1+t=l}} 1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}, \qquad k,l \leqslant N,$$

which is a differential.

1.3. Definition. *A* pointed cocategory homomorphism *consists of the following data: A_N-categories $\mathcal{A}$ and $\mathcal{B}$, a map $f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$ and a system of $\Bbbk$-linear maps of degree 0*

$$f_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{B}(X_0 f, X_n f), \qquad 1 \leqslant n \leqslant N.$$

The above data are equivalent to a cocategory homomorphism $f : T^{\leqslant N}s\mathcal{A} \to T^{\leqslant N}s\mathcal{B}$ of degree 0 such that

$$f_{01} = (f\big|_{T^0 s\mathcal{A}})\operatorname{pr}_1 = 0 : T^0 s\mathcal{A} \to T^1 s\mathcal{B}, \tag{2}$$

(this condition was implicitly assumed in [Lyu03, Definition 2.4]). The components of f are

$$f_{kl} = (f\big|_{T^k s\mathcal{A}})\operatorname{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{B}, \qquad f_{kl} = \sum_{i_1+\cdots+i_l=k} f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}, \tag{3}$$

where $k, l \leqslant N$. Indeed, the claim follows from the following diagram, commutative for all $l \geqslant 0$:

$$\begin{tikzcd}
T^{\leqslant N}s\mathcal{A} \arrow[r, "f"] \arrow[d, "\Delta^{(l)}"'] \arrow[dr, phantom, "="] & T^{\leqslant N}s\mathcal{B} \arrow[r, "\operatorname{pr}_l"] \arrow[d, "\Delta^{(l)}"'] \arrow[dr, phantom, "="] & T^l s\mathcal{B} \arrow[d, equal] \\
(T^{\leqslant N}s\mathcal{A})^{\otimes l} \arrow[r, "f^{\otimes l}"] & (T^{\leqslant N}s\mathcal{B})^{\otimes l} \arrow[r, "\operatorname{pr}_1^{\otimes l}"] & (s\mathcal{B})^{\otimes l}
\end{tikzcd}$$

where $\Delta^{(0)} = \varepsilon$, $\Delta^{(1)} = \mathrm{id}$, $\Delta^{(2)} = \Delta$ and $\Delta^{(l)}$ means the cut comultiplication, iterated $l-1$ times. Notice that condition (2) can be written as $f_0 = 0$.

1.4. Definition. *An A_N-*functor *$f : \mathcal{A} \to \mathcal{B}$ is a pointed cocategory homomorphism, which commutes with the differential b, that is, for all $1 \leqslant k \leqslant N$*

$$\sum_{l>0;i_1+\cdots+i_l=k} (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}) b_l = \sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k s\mathcal{A} \to s\mathcal{B}.$$

We are interested mostly in the case $N = 1$. Clearly, A_1-categories are differential graded quivers and A_1-functors are their morphisms. In the case of one object these reduce to chain complexes and chain maps. The following notion seems interesting even in this case.

1.5. Definition. *An A_N-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ of degree d consists of the following data: A_N-categories $\mathcal{A}$ and $\mathcal{B}$; pointed cocategory homomorphisms $f, g : T^{\leqslant N}s\mathcal{A} \to T^{\leqslant N}s\mathcal{B}$ (or A_N-functors $f, g : \mathcal{A} \to \mathcal{B}$); a system of $\Bbbk$-linear maps of degree d*

$$r_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{B}(X_0 f, X_n g), \qquad 0 \leqslant n \leqslant N.$$

To give a system r_n is equivalent to specifying an (f, g)-coderivation $r : T^{\leqslant N}s\mathcal{A} \to T^{\leqslant N+1}s\mathcal{B}$ of degree d

$$r_{kl} = (r\big|_{T^k s\mathcal{A}})\,\mathrm{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{B}, \qquad k \leqslant N,\ l \leqslant N+1$$

$$r_{kl} = \sum_{\substack{q+1+t=l \\ i_1+\dots+i_q+n+j_1+\dots+j_t=k}} f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_n \otimes g_{j_1} \otimes \dots \otimes g_{j_t}, \tag{4}$$

that is, a $\Bbbk$-quiver morphism r, satisfying $r\Delta = \Delta(f \otimes r + r \otimes g)$. This follows from the commutative diagram

$$\begin{tikzcd}
T^{\leqslant N}s\mathcal{A} \arrow[r, "r"] \arrow[d, "\Delta^{(l)}"'] \arrow[dr, phantom, "="] & T^{\leqslant N+1}s\mathcal{B} \arrow[r, "\mathrm{pr}_l"] \arrow[d, "\Delta^{(l)}"'] \arrow[dr, phantom, "="] & T^l s\mathcal{B} \arrow[d, equal] \\
(T^{\leqslant N}s\mathcal{A})^{\otimes l} \arrow[r, "\sum_{q+1+t=l} f^{\otimes q}\otimes r\otimes g^{\otimes t}"] & (T^{\leqslant N+1}s\mathcal{B})^{\otimes l} \arrow[r, "\mathrm{pr}_1^{\otimes l}"] & (s\mathcal{B})^{\otimes l}
\end{tikzcd}$$

Let $\mathcal{A}$, $\mathcal{B}$ be A_N-categories, and let $f^0, f^1, \dots, f^n : T^{\leqslant N}s\mathcal{A} \to T^{\leqslant N}s\mathcal{B}$ be pointed cocategory homomorphisms. Consider coderivations $r_1, \dots, r_n$ as in

$$f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : T^{\leqslant N}s\mathcal{A} \to T^{\leqslant N}s\mathcal{B}.$$

We construct the following system of $\Bbbk$-linear maps $\theta_{kl} : T^k s\mathcal{A} \to T^l s\mathcal{B}$, $k \leqslant N$, $l \leqslant N+n$ of degree $\deg r^1 + \dots + \deg r^n$ from these data:

$$\theta_{kl} = \sum f^0_{i^0_1} \otimes \dots \otimes f^0_{i^0_{m_0}} \otimes r^1_{j_1} \otimes f^1_{i^1_1} \otimes \dots \otimes f^1_{i^1_{m_1}} \otimes \dots \otimes r^n_{j_n} \otimes f^n_{i^n_1} \otimes \dots \otimes f^n_{i^n_{m_n}}, \tag{5}$$

where summation is taken over all terms with

$$m_0 + m_1 + \dots + m_n + n = l, \quad i^0_1 + \dots + i^0_{m_0} + j_1 + i^1_1 + \dots + i^1_{m_1} + \dots + j_n + i^n_1 + \dots + i^n_{m_n} = k.$$

Equivalently, we write

$$\theta_{kl} = \sum_{\substack{m_0+m_1+\dots+m_n+n=l \\ p_0+j_1+p_1+\dots+j_n+p_n=k}} f^0_{p_0 m_0} \otimes r^1_{j_1} \otimes f^1_{p_1 m_1} \otimes \dots \otimes r^n_{j_n} \otimes f^n_{p_n m_n}.$$

The component θ_{kl} vanishes unless $n \leqslant l \leqslant k+n$. If $n = 0$, then θ_{kl} is expansion (3) of f^0. If $n = 1$, then θ_{kl} is expansion (4) of r^1.

Given an A_K-category $\mathcal{A}$ and an A_{K+N}-category $\mathcal{B}$, $1 \leqslant K, N \leqslant \infty$, we construct an A_N-category $A_K(\mathcal{A}, \mathcal{B})$ out of these. The objects of $A_K(\mathcal{A}, \mathcal{B})$ are A_K-functors $f : \mathcal{A} \to \mathcal{B}$. Given two such functors $f, g : \mathcal{A} \to \mathcal{B}$ we define the graded $\Bbbk$-module $A_K(\mathcal{A}, \mathcal{B})(f, g)$ as the space of all A_K-transformations $r : f \to g$, namely,

$$[A_K(\mathcal{A}, \mathcal{B})(f, g)]^{d+1} = \{r : f \to g \mid A_K\text{-transformation } r : T^{\leqslant K}s\mathcal{A} \to T^{\leqslant K+1}s\mathcal{B} \text{ has degree } d\}.$$

The system of differentials B_n, $n \leqslant N$, is defined as follows:

$$
\begin{gathered}
B_1 : A_K(\mathcal{A},\mathcal{B})(f,g) \to A_K(\mathcal{A},\mathcal{B})(f,g), \quad r \mapsto (r)B_1 = [r,b] = rb - (-)^r br, \\
[(r)B_1]_k = \sum_{i_1+\dots+i_q+n+j_1+\dots+j_t=k} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_n \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) b_{q+1+t} \\
-(-)^r \sum_{\alpha+n+\beta=k} (1^{\otimes\alpha} \otimes b_n \otimes 1^{\otimes\beta}) r_{\alpha+1+\beta}, \qquad k \leqslant K, \\
B_n : A_K(\mathcal{A},\mathcal{B})(f^0,f^1) \otimes \dots \otimes A_K(\mathcal{A},\mathcal{B})(f^{n-1},f^n) \to A_K(\mathcal{A},\mathcal{B})(f^0,f^n), \\
r^1 \otimes \dots \otimes r^n \mapsto (r^1 \otimes \dots \otimes r^n)B_n, \text{ for } 1 < n \leqslant N,
\end{gathered}
$$

where the last A_K-transformation is defined by its components:

$$
[(r^1 \otimes \dots \otimes r^n)B_n]_k = \sum_{l=n}^{n+k} (r^1 \otimes \dots \otimes r^n)\theta_{kl} b_l, \qquad k \leqslant K.
$$

The category of graded $\Bbbk$-linear quivers admits a symmetric monoidal structure with the tensor product $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A} \boxtimes \mathcal{B}$, where $\operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B} = \operatorname{Ob} \mathcal{A} \times \operatorname{Ob} \mathcal{B}$ and $(\mathcal{A} \boxtimes \mathcal{B})\big((X,U),(Y,V)\big) = \mathcal{A}(X,Y) \otimes_\Bbbk \mathcal{B}(U,V)$. The same tensor product was denoted $\otimes$ in [Lyu03], but we will keep notation $\mathcal{A} \otimes \mathcal{B}$ only for tensor product from Section 1.1, defined when $\operatorname{Ob} \mathcal{A} = \operatorname{Ob} \mathcal{B}$. The two tensor products obey

Distributivity law. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be graded $\Bbbk$-linear quivers, such that $\operatorname{Ob} \mathcal{A} = \operatorname{Ob} \mathcal{B}$ and $\operatorname{Ob} \mathcal{C} = \operatorname{Ob} \mathcal{D}$. Then the middle four interchange map $1 \otimes c \otimes 1$ is an isomorphism of quivers

$$
(\mathcal{A} \otimes \mathcal{B}) \boxtimes (\mathcal{C} \otimes \mathcal{D}) \xrightarrow{\sim} (\mathcal{A} \boxtimes \mathcal{C}) \otimes (\mathcal{B} \boxtimes \mathcal{D}), \tag{6}
$$

identity on objects.

Indeed, the both quivers in (6) have the same set of objects $R \times S$, where $R = \operatorname{Ob} \mathcal{A} = \operatorname{Ob} \mathcal{B}$ and $S = \operatorname{Ob} \mathcal{C} = \operatorname{Ob} \mathcal{D}$. Let $X, Z \in R$ and $U, W \in S$. The sets of morphisms from (X,U) to (Z,W) are isomorphic via

$$
\begin{gathered}
\big((\mathcal{A} \otimes \mathcal{B}) \boxtimes (\mathcal{C} \otimes \mathcal{D})\big)\big((X,U),(Z,W)\big) = \\
\big(\oplus_{Y \in R} \mathcal{A}(X,Y) \otimes_\Bbbk \mathcal{B}(Y,Z)\big) \otimes_\Bbbk \big(\oplus_{V \in S} \mathcal{C}(U,V) \otimes_\Bbbk \mathcal{D}(V,W)\big) \\
\Big\downarrow \wr \\
\oplus_{(Y,V) \in R \times S} \mathcal{A}(X,Y) \otimes_\Bbbk \mathcal{B}(Y,Z) \otimes_\Bbbk \mathcal{C}(U,V) \otimes_\Bbbk \mathcal{D}(V,W) \\
\Big\downarrow 1 \otimes c \otimes 1 \\
\oplus_{(Y,V) \in R \times S} \mathcal{A}(X,Y) \otimes_\Bbbk \mathcal{C}(U,V) \otimes_\Bbbk \mathcal{B}(Y,Z) \otimes_\Bbbk \mathcal{D}(V,W) \\
= \big((\mathcal{A} \boxtimes \mathcal{C}) \otimes (\mathcal{B} \boxtimes \mathcal{D})\big)\big((X,U),(Z,W)\big).
\end{gathered}
$$

The notion of a pointed cocategory homomorphism extends to the case of several arguments, that is, to degree 0 cocategory homomorphisms $\psi : T^{\leqslant L^1} s\mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leqslant L^q} s\mathcal{C}^q \to$

$T^{\leqslant N}s\mathcal{B}$, where $N \geqslant L^1+\dots+L^q$. We always assume that $\psi_{00\dots0} : T^0 s\mathcal{C}^1 \boxtimes\dots\boxtimes T^0 s\mathcal{C}^q \to s\mathcal{B}$ vanishes. We call ψ an A-functor if it commutes with the differential, that is,

$$(b \boxtimes 1 \boxtimes \dots \boxtimes 1 + 1 \boxtimes b \boxtimes \dots \boxtimes 1 + \dots + 1 \boxtimes 1 \boxtimes \dots \boxtimes b)\psi = \psi b.$$

For example, the map $\alpha : T^{\leqslant K}s\mathcal{A} \boxtimes T^{\leqslant N}sA_K(\mathcal{A},\mathcal{B}) \to T^{\leqslant K+N}s\mathcal{B}$, $a \boxtimes r^1 \otimes \dots \otimes r^n \mapsto a.[(r^1 \otimes \dots \otimes r^n)\theta]$, is an A-functor.

1.6. Proposition. [cf. Proposition 5.5 of [Lyu03]] *Let $\mathcal{A}$ be an A_K-category, let $\mathcal{C}^t$ be an A_{L^t}-category for $1 \leqslant t \leqslant q$, and let $\mathcal{B}$ be an A_N-category, where $N \geqslant K + L^1 + \dots + L^q$. For any A-functor $\phi : T^{\leqslant K}s\mathcal{A} \boxtimes T^{\leqslant L^1}s\mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leqslant L^q}s\mathcal{C}^q \to T^{\leqslant N}s\mathcal{B}$ there is a unique A-functor $\psi : T^{\leqslant L^1}s\mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leqslant L^q}s\mathcal{C}^q \to T^{\leqslant N-K}sA_K(\mathcal{A},\mathcal{B})$, such that*

$$\phi = \bigl(T^{\leqslant K}s\mathcal{A} \boxtimes T^{\leqslant L^1}s\mathcal{C}^1 \boxtimes\dots\boxtimes T^{\leqslant L^q}s\mathcal{C}^q \xrightarrow{1\ \ \psi} T^{\leqslant K}s\mathcal{A} \boxtimes T^{\leqslant N-K}sA_K(\mathcal{A},\mathcal{B}) \xrightarrow{\alpha} T^{\leqslant N}s\mathcal{B}\bigr).$$

Let $\mathcal{A}$ be an A_N-category, let $\mathcal{B}$ be an A_{N+K}-category, and let $\mathcal{C}$ be an A_{N+K+L}-category. The above proposition implies the existence of an A-functor (cf. [Lyu03, Proposition 4.1])

$$M : T^{\leqslant K}sA_N(\mathcal{A},\mathcal{B}) \boxtimes T^{\leqslant L}sA_{N+K}(\mathcal{B},\mathcal{C}) \to T^{\leqslant K+L}sA_N(\mathcal{A},\mathcal{C}),$$

in particular, $(1 \boxtimes B + B \boxtimes 1)M = MB$. It has the components

$$M_{nm} = M\big|_{T^n\ \ T^m} \mathrm{pr}_1 : T^n sA_N(\mathcal{A},\mathcal{B}) \boxtimes T^m sA_{N+K}(\mathcal{B},\mathcal{C}) \to sA_N(\mathcal{A},\mathcal{C}),$$

$n \leqslant K$, $m \leqslant L$. We have $M_{00} = 0$ and $M_{nm} = 0$ for $m > 1$. If $m = 0$ and n is positive, M_{n0} is given by the formula:

$$\begin{aligned} M_{n0} : sA_N(\mathcal{A},\mathcal{B})(f^0,f^1) \otimes \dots \otimes sA_N(\mathcal{A},\mathcal{B})(f^{n-1},f^n) \boxtimes \Bbbk_{g^0} \to sA_N(\mathcal{A},\mathcal{C})(f^0g^0,f^ng^0),\\ r^1 \otimes \dots \otimes r^n \boxtimes 1 \mapsto (r^1 \otimes \dots \otimes r^n \mid g^0)M_{n0}, \end{aligned}$$

$$[(r^1 \otimes \dots \otimes r^n \mid g^0)M_{n0}]_k = \sum_{l=n}^{n+k} (r^1 \otimes \dots \otimes r^n)\theta_{kl} g^0_l, \qquad k \leqslant N,$$

where $|$ separates the arguments in place of $\boxtimes$. If $m = 1$, then M_{n1} is given by the formula:

$$\begin{aligned} M_{n1} : sA_N(\mathcal{A},\mathcal{B})(f^0,f^1) \otimes \dots \otimes sA_N(\mathcal{A},\mathcal{B})(f^{n-1},f^n) \boxtimes sA_{N+K}(\mathcal{B},\mathcal{C})(g^0,g^1)\\ \to sA_N(\mathcal{A},\mathcal{C})(f^0g^0,f^ng^1), \qquad r^1 \otimes \dots \otimes r^n \boxtimes t^1 \mapsto (r^1 \otimes \dots \otimes r^n \boxtimes t^1)M_{n1}, \end{aligned}$$

$$[(r^1 \otimes \dots \otimes r^n \boxtimes t^1)M_{n1}]_k = \sum_{l=n}^{n+k} (r^1 \otimes \dots \otimes r^n)\theta_{kl} t^1_l, \qquad k \leqslant N.$$

Note that equations

$$[(r^1 \otimes \dots \otimes r^n)B_n]_k = [(r^1 \otimes \dots \otimes r^n \boxtimes b)M_{n1}]_k - (-)^{r^1+\dots+r^n}[(b \boxtimes r^1 \otimes \dots \otimes r^n)M_{1n}]_k$$

imply that

$$\begin{aligned}(r^1 \otimes \dots \otimes r^n)B_n &= (r^1 \otimes \dots \otimes r^n \boxtimes b)M_{n1} - (-)^{r^1+\dots+r^n}(b \boxtimes r^1 \otimes \dots \otimes r^n)M_{1n}, \\ B &= (1 \boxtimes b)M - (b \boxtimes 1)M : \mathrm{id} \to \mathrm{id} : A_N(\mathcal{A}, \mathcal{B}) \to A_N(\mathcal{A}, \mathcal{B}).\end{aligned}$$

Proposition 1.6 implies the existence of a unique A_L-functor

$$A_N(\mathcal{A}, \_) : A_{N+K}(\mathcal{B}, \mathcal{C}) \to A_K(A_N(\mathcal{A}, \mathcal{B}), A_N(\mathcal{A}, \mathcal{C})),$$

such that

$$\begin{aligned}M = \big[&T^{\leqslant K} sA_N(\mathcal{A}, \mathcal{B}) \boxtimes T^{\leqslant L} sA_{N+K}(\mathcal{B}, \mathcal{C}) \xrightarrow{1 \quad A_N(\mathcal{A},\_)} \\ &T^{\leqslant K} sA_N(\mathcal{A}, \mathcal{B}) \boxtimes T^{\leqslant L} sA_K(A_N(\mathcal{A}, \mathcal{B}), A_N(\mathcal{A}, \mathcal{C})) \xrightarrow{\alpha} T^{\leqslant K+L} sA_N(\mathcal{A}, \mathcal{C})\big].\end{aligned}$$

The A_L-functor $A_N(\mathcal{A}, \_)$ is strict, cf. [Lyu03, Proposition 6.2].

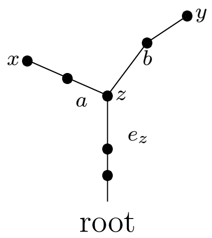
Let $\mathcal{A}$ be an A_N-category, and let $\mathcal{B}$ be a unital A_∞-category with a unit transformation $\mathbf{i}^{\mathcal{B}}$. Then $A_N(\mathcal{A}, \mathcal{B})$ is a unital A_∞-category with the unit transformation $(1 \boxtimes \mathbf{i}^{\mathcal{B}})M$ (cf. [Lyu03, Proposition 7.7]). The unit element for an object $f \in \operatorname{Ob} A_N(\mathcal{A}, \mathcal{B})$ is ${}_f\mathbf{i}_0^{A_N(\mathcal{A},\mathcal{B})} : \Bbbk \to (sA_N)^{-1}(\mathcal{A}, \mathcal{B})$, $1 \mapsto f\mathbf{i}^{\mathcal{B}}$.

When $\mathcal{A}$ is an A_K-category and $N < K$, we may forget part of its structure and view $\mathcal{A}$ as an A_N-category. If furthermore, $\mathcal{B}$ is an A_{K+L}-category, we have the restriction strict A_L-functor $\mathrm{restr}_{K,N} : A_K(\mathcal{A}, \mathcal{B}) \to A_N(\mathcal{A}, \mathcal{B})$. To prove the results mentioned above, we notice that they are restrictions of their A_∞-analogs to finite N. Since the proofs of A_∞-results are obtained in [Lyu03] by induction, an inspection shows that the proofs of the above A_N-statements are obtained as a byproduct.

1.7. Trees. Since the notions related to trees might be interpreted with some variations, we give precise definitions and fix notation. A *tree* is a non-empty connected graph without cycles. A vertex which belongs to only one edge is called *external*, other vertices are *internal*. A *plane tree* is a tree equipped for each internal vertex v with a cyclic ordering of the set E_v of edges, adjacent to v. Plane trees can be drawn on an oriented plane in a unique way (up to an ambient isotopy) so that the cyclic ordering of each E_v agrees with the orientation of the plane. An external vertex distinct from the root is called *input vertex*.

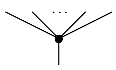
A *rooted tree* is a tree with a distinguished external vertex, called *root*. The set of vertices $V(t)$ of a rooted tree t has a canonical ordering: $x \preccurlyeq y$ iff the minimal path connecting the root with y contains x. A *linearly ordered tree* is a rooted tree t equipped with a linear order $\leqslant$ of the set of internal vertices $IV(t)$, such that for all internal vertices x, y the relation $x \preccurlyeq y$ implies $x \leqslant y$. For each vertex $v \in V(t) - \{\mathrm{root}\}$ of a rooted tree, the set E_v has a distinguished element e_v – the beginning of a minimal path from v to the root. Therefore, for each vertex $v \in V(t) - \{\mathrm{root}\}$ of a rooted plane tree, the set E_v admits a unique linear order $<$, for which e_v is minimal and the induced cyclic order is the given one. An internal vertex v has degree d, if $\mathrm{Card}(E_v) = d + 1$.

For any $y \in V(t)$ let $P_y = \{x \in V(t) \mid x \preccurlyeq y\}$. With each plane rooted tree t is associated a linearly ordered tree $t_< = (t, \leqslant)$ as follows. If $x, y \in IV(t)$ are such that $x \not\preccurlyeq y$ and $y \not\preccurlyeq x$, then $P_x \cap P_y = P_z$ for a unique $z \in IV(t)$, distinct from x and y. Let $a \in E_z - \{e_z\}$ (resp. $b \in E_z - \{e_z\}$) be the beginning of the minimal path connecting z and x (resp. y). If $a < b$, we set $x < y$. Graphically we $<$-order the internal vertices by height. Thus, an internal vertex x on the left is depicted lower than a $\preccurlyeq$-incomparable internal vertex y on the right:



$$, \qquad a < b \implies x < y.$$

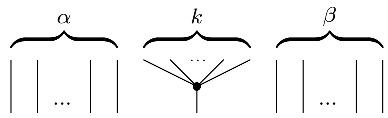
A *forest* is a sequence of plane rooted trees. Concatenation of forests is denoted $\sqcup$. The vertical composition $F_1 \cdot F_2$ of forests F_1, F_2 is well-defined if the sum of lengths of sequences F_1 and F_2 equals the number of external vertices of F_2. These operations allow to construct any tree from elementary ones

$$1 = \Big| \quad , \qquad \text{and} \qquad \mathfrak{t}_k =$$  $$(k \text{ input vertices}).$$

Namely, any linearly ordered tree $(t, \leqslant)$ has a unique presentation of the form

$$(t, \leqslant) = (1^{\sqcup\alpha_1} \sqcup \mathfrak{t}_{k_1} \sqcup 1^{\sqcup\beta_1}) \cdot (1^{\sqcup\alpha_2} \sqcup \mathfrak{t}_{k_2} \sqcup 1^{\sqcup\beta_2}) \cdot \dots \cdot \mathfrak{t}_{k_N}, \tag{7}$$

where $N = |t| \overset{\text{def}}{=} \operatorname{Card}(IV(t))$ is the number of internal vertices. Here

$$1^{\sqcup\alpha} \sqcup \mathfrak{t}_k \sqcup 1^{\sqcup\beta} =$$ 

.

In (7) the highest vertex is indexed by 1, the lowest – by N.

2. Properties of free A_∞-categories

2.1. Construction of a free A_∞-category. The category strictA_∞ has A_∞-categories as objects and strict A_∞-functors as morphisms. There is a functor $\mathcal{U}$: strict$A_\infty \to A_1$, $\mathcal{A} \mapsto (\mathcal{A}, m_1)$ which sends an A_∞-category to the underlying differential graded $\Bbbk$-quiver, forgetting all higher multiplications. Following Kontsevich and Soibelman [KS02] we are going to prove that $\mathcal{U}$ has a left adjoint functor $\mathcal{F} : A_1 \to$ strictA_∞, $\mathcal{Q} \mapsto \mathcal{F}\mathcal{Q}$. The A_∞-category $\mathcal{F}\mathcal{Q}$ is called free. Below we describe its structure for an arbitrary differential graded $\Bbbk$-quiver $\mathcal{Q}$. We shall work with its shift $(s\mathcal{Q}, d)$.

Let us define an A_∞-category $\mathcal{F}\mathcal{Q}$ via the following data. The class of objects $\operatorname{Ob}\mathcal{F}\mathcal{Q}$ is $\operatorname{Ob}\mathcal{Q}$. The $\mathbb{Z}$-graded $\Bbbk$-modules of morphisms between $X, Y \in \operatorname{Ob}\mathcal{Q}$ are

$$\begin{aligned} s\mathcal{F}\mathcal{Q}(X,Y) &= \oplus_{n\geqslant 1} \oplus_{t\in\mathcal{T}^n_{\geqslant 2}} s\mathcal{F}_t\mathcal{Q}(X,Y), \\ s\mathcal{F}_t\mathcal{Q}(X,Y) &= \oplus^{X_0=X,\, X_n=Y}_{X_0,\dots,X_n\in\operatorname{Ob}\mathcal{Q}} s\mathcal{Q}(X_0,X_1)\otimes\dots\otimes s\mathcal{Q}(X_{n-1},X_n)\bigl[-|t|\bigr], \end{aligned}$$

where $\mathcal{T}^n_{\geqslant 2}$ is the class of plane rooted trees with $n+1$ external vertices, such that $\operatorname{Card}(E_v) \geqslant 3$ for all $v \in IV(t)$. We use the following convention: if M, N are (differential) graded $\Bbbk$-modules, then[1]

$$\begin{gathered} (M\otimes N)[k] = M\otimes\bigl(N[k]\bigr), \\ \bigl(M\otimes N \xrightarrow{s^k} (M\otimes N)[k]\bigr) = \bigl(M\otimes N \xrightarrow{1\otimes s^k} M\otimes(N[k])\bigr). \end{gathered}$$

The quiver $\mathcal{F}\mathcal{Q}$ is equipped with the following operations. For $k>1$ the operation b_k is a direct sum of maps

$$b_k = s^{|t_1|}\otimes\dots\otimes s^{|t_{k-1}|}\otimes s^{|t_k|-|t|} : s\mathcal{F}_{t_1}\mathcal{Q}(Y_0,Y_1)\otimes\dots\otimes s\mathcal{F}_{t_k}\mathcal{Q}(Y_{k-1},Y_k) \to s\mathcal{F}_t\mathcal{Q}(Y_0,Y_k), \quad (8)$$

where $t = (t_1 \sqcup \dots \sqcup t_k)\cdot \mathfrak{t}_k$. In particular, $|t| = |t_1| + \dots + |t_k| + 1$. The operation b_1 restricted to $s\mathcal{F}_t\mathcal{Q}$ is

$$b_1 = d \oplus (-1)^{\beta(t')}s^{-1} : s\mathcal{F}_t\mathcal{Q}(X,Y) \to s\mathcal{F}_t\mathcal{Q}(X,Y) \oplus \bigoplus_{t'=t+\text{edge}} s\mathcal{F}_{t'}\mathcal{Q}(X,Y), \quad (9)$$

where the sum extends over all trees $t' \in \mathcal{T}^n_{\geqslant 2}$ with a distinguished edge e, such that contracting e we get t from t'. The sign is determined by

$$\beta(t') = \beta(t',e) = 1 + h(\text{highest vertex of } e),$$

where an isomorphism of ordered sets

$$h : IV(t'_<) \xrightarrow{\sim} \bigl[1, |t'|\bigr] \cap \mathbb{Z}$$

is simply the height of a vertex in the linearly ordered tree $t'_<$, canonically associated with t'. In (9) d means $d\otimes 1\otimes\dots\otimes 1 + \dots + 1\otimes\dots\otimes d\otimes 1 + 1\otimes\dots\otimes 1\otimes d$, where the last d is $d_{s\mathcal{Q}[-|t|]} = (-)^{|t|}s^{|t|}\cdot d_{s\mathcal{Q}}\cdot s^{-|t|}$, as usual. According to our conventions, s^{-1} in (9) means $1^{\otimes n-1}\otimes s^{-1}$.

2.2. Proposition. *$\mathcal{F}\mathcal{Q}$ is an A_∞-category.*

[1] Another gauge choice $(M\otimes N)[1] = M[1]\otimes N$, $s = s\otimes 1$ seems less convenient.

Proof. First we prove that $b_1^2 = 0$ on $s\mathcal{F}_t\mathcal{Q}$. Indeed,

$$b_1^2 = d^2 \oplus (-)^{\beta(t',e)}(s^{-1}d + ds^{-1}) \oplus \left[(-1)^{\beta(t'_1,e_1)+\beta(t'',e_2)} + (-1)^{\beta(t'_2,e_2)+\beta(t'',e_1)}\right] s^{-2} : \\ s\mathcal{F}_t\mathcal{Q} \to s\mathcal{F}_t\mathcal{Q} \oplus \bigoplus_{t'=t+e} s\mathcal{F}_{t'}\mathcal{Q} \oplus \bigoplus_{t''=t+e_1+e_2} s\mathcal{F}_{t''}\mathcal{Q},$$

where t'' contains two distinguished edges e_1, e_2, contraction along which gives t; t'_2 is t'' contracted along e_1, and t'_1 is t'' contracted along e_2. We may assume that highest vertex of e_1 is lower than highest vertex of e_2 in $t''_<$. Then $\beta(t'_1, e_1) = \beta(t'', e_1)$ and $\beta(t'', e_2) = \beta(t'_2, e_2) + 1$, hence,

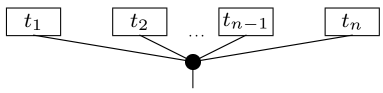
$$(-1)^{\beta(t'_1,e_1)+\beta(t'',e_2)} + (-1)^{\beta(t'_2,e_2)+\beta(t'',e_1)} = 0.$$

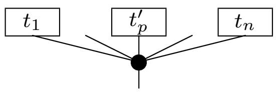
Obviously, $d^2 = 0$ and $s^{-1}d + ds^{-1} = 0$, hence, $b_1^2 = 0$.

Let us prove for each $n > 1$ that

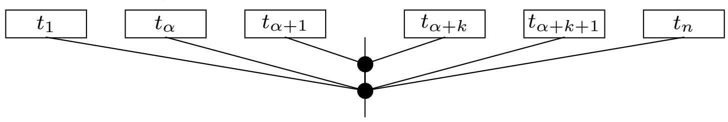
$$b_n b_1 + \sum_{p=1}^{n} (1^{\otimes p-1} \otimes b_1 \otimes 1^{\otimes n-p}) b_n + \sum_{\substack{\alpha+k+\beta=n \\ \alpha+\beta>0}}^{k>1} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta} = 0 : \\ s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \to s\mathcal{F}_t\mathcal{Q} \oplus \bigoplus_{p,t'_p} s\mathcal{F}_{t'}\mathcal{Q} \oplus \bigoplus_{\substack{\alpha+k+\beta=n \\ \alpha+\beta>0}}^{k>1,\, t''} s\mathcal{F}_{t''}\mathcal{Q},$$

where

$t = (t_1 \sqcup \cdots \sqcup t_n) \cdot \mathfrak{t}_n =$  , (10)

$t' = (t_1 \sqcup \cdots \sqcup t'_p \sqcup \cdots \sqcup t_n) \cdot \mathfrak{t}_n =$  ,

$t'' = (t_1 \sqcup \cdots \sqcup t_n) \cdot (1^{\sqcup\alpha} \sqcup \mathfrak{t}_k \sqcup 1^{\sqcup\beta}) \cdot \mathfrak{t}_{\alpha+1+\beta}$

$=$  ,

and contraction of t'_p along distinguished edge e_p gives t_p. According to the three types of summands in the target, the required equation follows from anticommutativity of the following three diagrams:

$$\begin{tikzcd}
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\cdots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|}", "b_n"'] \arrow[d, "1^{\otimes n-1}\otimes d+\cdots+d\otimes 1^{\otimes n-1}"'] \arrow[dr, phantom, "-"] & s\mathcal{F}_t\mathcal{Q} \arrow[d, "d"] \\
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\cdots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|}", "b_n"'] & s\mathcal{F}_t\mathcal{Q}
\end{tikzcd}$$

that is,

$$(1^{\otimes n-1} \otimes d + \dots + d \otimes 1^{\otimes n-1})(s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}) \\ + (s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|})(1^{\otimes n-1} \otimes d + \dots + d \otimes 1^{\otimes n-1}) = 0;$$

$$\begin{tikzcd}
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_p}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\dots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|}"] \arrow[d, "(-)^{\beta(t'_p)}1^{\otimes p-1}\otimes s^{-1}\otimes 1^{\otimes n-p}"] \arrow[dr, phantom, "-"] & s\mathcal{F}_t\mathcal{Q} \arrow[d, "(-)^{1+|t_1|+\dots+|t_{p-1}|+\beta(t'_p)}s^{-1}" '] \\
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t'_p}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\dots\otimes s^{|t_{p-1}|}\otimes s^{|t_p|+1}\otimes s^{|t_{p+1}|}\otimes\dots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|-1}" '] & s\mathcal{F}_{t'}\mathcal{Q}
\end{tikzcd}$$

that is,

$$(-1)^{\beta(t'_p)}(1^{\otimes p-1} \otimes s^{-1} \otimes 1^{\otimes n-p})(s^{|t_1|} \otimes \dots \otimes s^{|t_{p-1}|} \otimes s^{|t_p|+1} \otimes s^{|t_{p+1}|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|-1}) \\ + (s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|})(-1)^{1+|t_1|+\dots+|t_{p-1}|+\beta(t'_p)}(1^{\otimes n-1} \otimes s^{-1}) = 0,$$

in the particular case $p = n$ it holds as well;

$$\begin{tikzcd}
s\mathcal{F}_{t_1}\mathcal{Q}\otimes\dots\otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\dots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|}"] \arrow[d, "1^{\otimes\alpha}\otimes s^{|t_{\alpha+1}|}\otimes\dots\otimes s^{|t_{\alpha+k-1}|}\otimes s^{-|t_{\alpha+1}|-\dots-|t_{\alpha+k-1}|-1}\otimes 1^{\otimes\beta}"] \arrow[dr, phantom, "-"] & s\mathcal{F}_t\mathcal{Q} \arrow[d, "(-)^{1+|t_1|+\dots+|t_\alpha|}s^{-1}" '] \\
s\mathcal{F}_{t_1}\mathcal{Q}\otimes\dots\otimes s\mathcal{F}_{t_\alpha}\mathcal{Q}\otimes s\mathcal{F}_{\hat{t}}\mathcal{Q}\otimes s\mathcal{F}_{t_{\alpha+k+1}}\mathcal{Q}\otimes\dots\otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\dots\otimes s^{|t_\alpha|}\otimes s^{|\hat{t}|}\otimes s^{|t_{\alpha+k+1}|}\otimes\dots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|-1}" '] & s\mathcal{F}_{t''}\mathcal{Q}
\end{tikzcd}$$

where $\hat{t} = (t_{\alpha+1} \sqcup \dots \sqcup t_{\alpha+k}) \cdot \mathfrak{t}_k$, that is,

$$(1^{\otimes\alpha} \otimes s^{|t_{\alpha+1}|} \otimes \dots \otimes s^{|t_{\alpha+k-1}|} \otimes s^{-|t_{\alpha+1}|-\dots-|t_{\alpha+k-1}|-1} \otimes 1^{\otimes\beta}) \cdot \\ (s^{|t_1|} \otimes \dots \otimes s^{|t_\alpha|} \otimes 1^{\otimes k-1} \otimes s^{|t_{\alpha+1}|+\dots+|t_{\alpha+k}|+1} \otimes s^{|t_{\alpha+k+1}|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|-1}) \\ + (s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|})(-1)^{1+|t_1|+\dots+|t_\alpha|}(1^{\otimes n-1} \otimes s^{-1}) = 0,$$

in the particular case $\beta = 0$ it holds as well.

Therefore, $\mathcal{F}\mathcal{Q}$ is an A_∞-category. ■

Let us establish a property of free A_∞-categories, which explains why they are called free.

2.3. PROPOSITION. [A_∞-functors from a free A_∞-category] *Let $\mathcal{Q}$ be a differential graded quiver, and let $\mathcal{A}$ be an A_∞-category. Let $f_1 : s\mathcal{Q} \to (s\mathcal{A}, b_1)$ be a chain morphism of differential graded quivers with the underlying mapping of objects $\operatorname{Ob} f : \operatorname{Ob}\mathcal{Q} \to \operatorname{Ob}\mathcal{A}$. Suppose given $\Bbbk$-quiver morphisms $f_k : T^k s\mathcal{F}\mathcal{Q} \to s\mathcal{A}$ of degree 0 with the same underlying map $\operatorname{Ob} f$ for all $k > 1$. Then there exists a unique extension of f_1 to a quiver morphism $f_1 : s\mathcal{F}\mathcal{Q} \to s\mathcal{A}$ such that $(f_1, f_2, \dots)$ are components of an A_∞-functor $f : \mathcal{F}\mathcal{Q} \to \mathcal{A}$.*

Proof. For each $n > 1$ we have to satisfy the equation

$$b_n f_1 = \sum_{i_1+\dots+i_l=n} (f_{i_1} \otimes \dots \otimes f_{i_l}) b_l - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) f_{\alpha+1+\beta} : T^n s\mathcal{F}\mathcal{Q} \to s\mathcal{A}. \quad (11)$$

It is used to define recursively f_1 on $s\mathcal{F}\mathcal{Q}$. Suppose that $t_1, \dots, t_n$ are trees, $n > 1$, and $f_1 : s\mathcal{F}_{t_i}\mathcal{Q} \to s\mathcal{A}$ is already defined for all $1 \leqslant i \leqslant n$. Since

$$b_n = s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|} : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \to s\mathcal{F}_t\mathcal{Q}$$

is invertible for $t = (t_1 \sqcup \dots \sqcup t_n) \cdot \mathfrak{t}_n$, formula (11) determines $f_1 : s\mathcal{F}_t\mathcal{Q} \to s\mathcal{A}$ uniquely as

$$f_1 = \bigl(s\mathcal{F}_t\mathcal{Q} \xrightarrow{b_n^{-1}} s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \xrightarrow{\sum (f_{i_1}\otimes\dots\otimes f_{i_l}) b_l - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha}\otimes b_k \otimes 1^{\otimes\beta}) f_{\alpha+1+\beta}} s\mathcal{A}\bigr).$$

This proves uniqueness of the extension of f_1.

Let us prove that the cocategory homomorphism f with so defined components $(f_1, f_2, \dots)$ is an A_∞-functor. Equations (11) are satisfied by construction of f_1. So it remains to prove that f_1 is a chain map. Equation $f_1 b_1 = b_1 f_1$ holds on $s\mathcal{F}_|\mathcal{Q}$ by assumption. We are going to prove by induction on $|t|$ that it holds on $s\mathcal{F}_t\mathcal{Q}$. Considering $t = (t_1 \sqcup \dots \sqcup t_n) \cdot \mathfrak{t}_n$, $n > 1$, we assume that $f_1 b_1 = b_1 f_1 : s\mathcal{F}_{t'}\mathcal{Q} \to s\mathcal{A}$ for all trees t' with $|t'| < |t|$. To prove that $f_1 b_1 = b_1 f_1 : s\mathcal{F}_t\mathcal{Q} \to s\mathcal{A}$ it suffices to show that $b_n f_1 b_1 = b_n b_1 f_1$ for all $n > 1$ due to invertibility of b_n. Using (11) and the equation $b^2 \operatorname{pr}_1 = 0$ we find

$$\begin{aligned}
b_n f_1 b_1 - b_n b_1 f_1 &= \sum_{i_1+\dots+i_l=n} (f_{i_1} \otimes \dots \otimes f_{i_l}) b_l b_1 - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) f_{\alpha+1+\beta} b_1 \\
&\quad + \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta} f_1 \\
&= - \sum_{i_1+\dots+i_l=n} (f_{i_1} \otimes \dots \otimes f_{i_l}) \sum_{\gamma+p+\delta=l}^{\gamma+\delta>0} (1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta}) b_{\gamma+1+\delta} \\
&\quad + \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \Biggl[\sum_{j_1+\dots+j_r=\alpha+1+\beta}^{r>1} (f_{j_1} \otimes \dots \otimes f_{j_r}) b_r \\
&\qquad\qquad - \sum_{\gamma+p+\delta=\alpha+1+\beta}^{\gamma+\delta>0} (1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta}) f_{\gamma+1+\delta} \Biggr] \\
&= \sum_{r>1} \Biggl[\sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \sum_{j_1+\dots+j_r=\alpha+1+\beta} f_{j_1} \otimes \dots \otimes f_{j_r} \\
&\qquad - \sum_{i_1+\dots+i_l=n} (f_{i_1} \otimes \dots \otimes f_{i_l}) \sum_{\substack{\gamma+p+\delta=l \\ \gamma+1+\delta=r}} 1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta} \Biggr] b_r
\end{aligned}$$

$$-\sum_{r>1}\Bigl[\sum_{\substack{\alpha+k+\beta=n\\ \alpha+\beta>0}}(1^{\otimes\alpha}\otimes b_k\otimes 1^{\otimes\beta})\sum_{\substack{\gamma+p+\delta=\alpha+1+\beta\\ \gamma+1+\delta=r}}1^{\otimes\gamma}\otimes b_p\otimes 1^{\otimes\delta}\Bigr]f_r.$$

Let us show that the expressions in square brackets vanish. The first one is the matrix coefficient $bf - fb : s\mathcal{F}_{t_1}\mathcal{Q}\otimes\cdots\otimes s\mathcal{F}_{t_n}\mathcal{Q}\to T^r s\mathcal{A}$. Indeed, for $r>1$ the inequality $r\leqslant j_1+\cdots+j_r=\alpha+1+\beta$ automatically implies that $\alpha+\beta>0$, so this condition can be omitted. Using the induction hypothesis one can transform the left hand side of equation

$$\sum_{\alpha+k+\beta=n}(1^{\otimes\alpha}\otimes b_k\otimes 1^{\otimes\beta})\sum_{j_1+\cdots+j_r=\alpha+1+\beta}f_{j_1}\otimes\cdots\otimes f_{j_r}$$
$$=\sum_{i_1+\cdots+i_l=n}(f_{i_1}\otimes\cdots\otimes f_{i_l})\sum_{\substack{\gamma+p+\delta=l\\ \gamma+1+\delta=r}}1^{\otimes\gamma}\otimes b_p\otimes 1^{\otimes\delta} : s\mathcal{F}_{t_1}\mathcal{Q}\otimes\cdots\otimes s\mathcal{F}_{t_n}\mathcal{Q}\to T^r s\mathcal{A}$$

into the right hand side for all $n,r\geqslant 1$.

The second expression

$$\sum_{\substack{\alpha+k+\beta=n\\ \alpha+\beta>0}}(1^{\otimes\alpha}\otimes b_k\otimes 1^{\otimes\beta})\sum_{\substack{\gamma+p+\delta=\alpha+1+\beta\\ \gamma+1+\delta=r}}1^{\otimes\gamma}\otimes b_p\otimes 1^{\otimes\delta} \tag{12}$$

is the matrix coefficient

$$(b-b\,\mathrm{pr}_1)b\,\mathrm{pr}_r : T^n s\mathcal{F}\mathcal{Q}\to T^r s\mathcal{F}\mathcal{Q}$$

of the endomorphism $(b-b\,\mathrm{pr}_1)b : Ts\mathcal{F}\mathcal{Q}\to Ts\mathcal{F}\mathcal{Q}$. However,

$$(b-b\,\mathrm{pr}_1)b\,\mathrm{pr}_r = b^2\,\mathrm{pr}_r - b\,\mathrm{pr}_1\,b\,\mathrm{pr}_r = -b\,\mathrm{pr}_1\,b\,\mathrm{pr}_1\,\mathrm{pr}_r = 0$$

for $r>1$, because $\mathrm{pr}_1\,b=\mathrm{pr}_1\,b\,\mathrm{pr}_1$. Therefore, (12) vanishes and equation $b_nf_1b_1=b_nb_1f_1$ is proven. ∎

Let $\mathrm{strict}A_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})\subset A_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})$ be a full A_∞-subcategory, whose objects are strict A_∞-functors. Recall that $\mathrm{Ob}\,A_1(\mathcal{Q},\mathcal{A})$ is the set of chain morphisms $\mathcal{Q}\to\mathcal{A}$ of differential graded quivers.

2.4. Corollary. *A chain morphism $f:\mathcal{Q}\to\mathcal{A}$ admits a unique extension to a strict A_∞-functor $\widehat{f}:\mathcal{F}\mathcal{Q}\to\mathcal{A}$. The maps $f\mapsto\widehat{f}$ and*

$$\mathrm{restr}:\mathrm{Ob}\,\mathrm{strict}A_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})\to\mathrm{Ob}\,A_1(\mathcal{Q},\mathcal{A}),\qquad g\mapsto(\mathrm{Ob}\,g, g_1\big|_{s\mathcal{Q}})$$

are inverse to each other.

Indeed, strict A_∞-functors g are distinguished by conditions $g_k=0$ for $k>1$.

We may view $\mathrm{strict}A_\infty$ as a category, whose objects are A_∞-categories and morphisms are strict A_∞-functors. We may also view A_1 as a category consisting of differential graded quivers and their morphisms. There is a functor $\mathcal{U} : \mathrm{strict}A_\infty \to A_1$, $\mathcal{A} \mapsto (\mathcal{A}, m_1)$, which sends an A_∞-category to the underlying differential graded $\Bbbk$-quiver, forgetting all higher multiplications. The restriction map

$$\mathrm{restr} : \mathrm{strict}A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{U}\mathcal{A}), \qquad g \mapsto (\mathrm{Ob}\, g, g_1\big|_{s\mathcal{Q}}) \tag{13}$$

is functorial in $\mathcal{A}$.

2.5. Corollary. *There is a functor $\mathcal{F} : A_1 \to \mathrm{strict}A_\infty$, $\mathcal{Q} \mapsto \mathcal{F}\mathcal{Q}$, left adjoint to $\mathcal{U}$.*

2.6. Explicit formula for the constructed strict A_∞-functor. Let us obtain a more explicit formula for $\widehat{f}_1\big|_{s\mathcal{F}_t\mathcal{Q}}$. We define $\widehat{f}_1$ for $t = (t_1 \sqcup \dots \sqcup t_n) \cdot \mathfrak{t}_n$ recursively by a commutative diagram

$$\begin{tikzcd}
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \arrow[r, "s^{|t_1|}\otimes\dots\otimes s^{|t_{n-1}|}\otimes s^{|t_n|-|t|}", "b_n"'] \arrow[d, "\widehat{f}_1^{\otimes n}"'] \arrow[dr, phantom, "\overset{\mathrm{def}}{=:}"] & s\mathcal{F}_t\mathcal{Q} \arrow[d, "\widehat{f}_1"] \\
s\mathcal{A} \otimes \dots \otimes s\mathcal{A} \arrow[r, "b_n"] & s\mathcal{A}
\end{tikzcd}$$

Notice that the top map is invertible. Here n, t_1, …, t_n are uniquely determined by decomposition (10) of t.

Let $(t, \leqslant)$ be a linearly ordered tree with the underlying given plane rooted tree t. Decompose $(t, \leqslant)$ into a vertical composition of forests as in (7). Then the following diagram commutes

$$\begin{tikzcd}
s\mathcal{Q}^{\otimes n} \arrow[r, "1^{\otimes\alpha_1}\otimes b_{k_1}\otimes 1^{\otimes\beta_1}"] \arrow[d, "f_1^{\otimes n}"'] & s\mathcal{Q}^{\otimes\alpha_1} \otimes s\mathcal{F}_{\mathfrak{t}_{k_1}}\mathcal{Q} \otimes s\mathcal{Q}^{\otimes\beta_1} \arrow[r, "1^{\otimes\alpha_2}\otimes b_{k_2}\otimes 1^{\otimes\beta_2}"] \arrow[d, "\widehat{f}_1^{\otimes\alpha_1+1+\beta_1}"] & \dots \arrow[d, "\widehat{f}_1^{\otimes\alpha_2+1+\beta_2}"'] & \dots \arrow[r, "b_{k_N}"] \arrow[d, "\widehat{f}_1^{\otimes k_N}"] & s\mathcal{F}_t\mathcal{Q} \arrow[d, "\widehat{f}_1"] \\
s\mathcal{A}^{\otimes n} \arrow[r, "1^{\otimes\alpha_1}\otimes b_{k_1}\otimes 1^{\otimes\beta_1}"] & s\mathcal{A}^{\otimes\alpha_1} \otimes s\mathcal{A} \otimes s\mathcal{A}^{\otimes\beta_1} \arrow[r, "1^{\otimes\alpha_2}\otimes b_{k_2}\otimes 1^{\otimes\beta_2}"] & \dots & \dots \arrow[r, "b_{k_N}"] & s\mathcal{A}
\end{tikzcd}$$

The upper row consists of invertible maps. One can prove by induction that the composition of maps in the upper row equals $\pm s^{-|t|}$. When $(t, \leqslant) = t_<$ is the linearly ordered tree, canonically associated with t, then the composition of maps in the upper row equals $s^{-|t|}$. This is also proved by induction: if t is presented as $t = (t_1 \sqcup \dots \sqcup t_k) \cdot \mathfrak{t}_k$, then the composition of maps in the upper row is

$$\begin{aligned}(1^{\otimes k-1} \otimes s^{-|t_k|})(1^{\otimes k-2} \otimes s^{-|t_{k-1}|} \otimes 1) \dots (s^{-|t_1|} \otimes 1^{\otimes k-1})(s^{|t_1|} \otimes \dots \otimes s^{|t_{k-1}|} \otimes s^{|t_k|-|t|}) \\ = 1^{\otimes k-1} \otimes s^{-|t|} = s^{-|t|}.\end{aligned}$$

Therefore, for an arbitrary tree $t \in \mathcal{T}^n_{\geqslant 2}$ the map $\widehat{f}_1\big|_{s\mathcal{F}_t\mathcal{Q}}$ is

$$\widehat{f}_1 = \bigl(s\mathcal{F}_t\mathcal{Q} \xrightarrow{s^{|t|}} s\mathcal{Q}^{\otimes n} \xrightarrow{f_1^{\otimes n}} s\mathcal{A}^{\otimes n} \xrightarrow{1^{\otimes\alpha_1}\otimes b_{k_1}\otimes 1^{\otimes\beta_1}} s\mathcal{A}^{\otimes\alpha_1+1+\beta_1} \xrightarrow{1^{\otimes\alpha_2}\otimes b_{k_2}\otimes 1^{\otimes\beta_2}} \dots \xrightarrow{b_{k_N}} s\mathcal{A}\bigr),$$

where the factors correspond to decomposition (7) of $t_<$.

2.7. Transformations between functors from a free A_∞-category. Let $\mathcal{Q}$ be a differential graded quiver, and let $\mathcal{A}$ be an A_∞-category. Then $A_1(\mathcal{Q},\mathcal{A})$ is an A_∞-category as well. The differential graded quiver $(sA_1(\mathcal{Q},\mathcal{A}),B_1)$ is described as follows. Objects are chain quiver maps $\phi:(s\mathcal{Q},b_1)\to(s\mathcal{A},b_1)$, the graded $\Bbbk$-module of morphisms $\phi\to\psi$ is the product of graded $\Bbbk$-modules

$$sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)=\prod_{X\in\operatorname{Ob}\mathcal{Q}}s\mathcal{A}(X\phi,X\psi)\times\prod_{X,Y\in\operatorname{Ob}\mathcal{Q}}\mathsf{C}\bigl(s\mathcal{Q}(X,Y),s\mathcal{A}(X\phi,Y\psi)\bigr),\quad r=(r_0,r_1).$$

The differential B_1 is given by

$$\begin{aligned}(rB_1)_0&=r_0b_1,\\ (rB_1)_1&=r_1b_1+(\phi_1\otimes r_0)b_2+(r_0\otimes\psi_1)b_2-(-)^rb_1r_1.\end{aligned}\tag{14}$$

Restrictions $\phi,\psi:\mathcal{Q}\to\mathcal{A}$ of arbitrary A_∞-functors $\phi,\psi:\mathcal{F}\mathcal{Q}\to\mathcal{A}$ to $\mathcal{Q}$ are A_1-functors (chain quiver maps).

2.8. Proposition. *Let $\phi,\psi:\mathcal{F}\mathcal{Q}\to\mathcal{A}$ be A_∞-functors. For an arbitrary complex P of $\Bbbk$-modules chain maps $u:P\to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)$ are in bijection with the following data: $(u',u_k)_{k>1}$*

1. *a chain map $u':P\to sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)$,*
2. *$\Bbbk$-linear maps*

$$u_k:P\to\prod_{X,Y\in\operatorname{Ob}\mathcal{Q}}\mathsf{C}\bigl((s\mathcal{F}\mathcal{Q})^{\otimes k}(X,Y),s\mathcal{A}(X\phi,Y\psi)\bigr)$$

of degree 0 for all $k>1$.

The bijection maps u to $(u',u_k)_{k>1}$, where $u_k=u\cdot\mathrm{pr}_k$ and

$$u'=\bigl(P\xrightarrow{u}sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)\xrightarrow{\text{restr}}sA_1(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)\xrightarrow{\text{restr}}sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)\bigr).\tag{15}$$

The inverse bijection can be recovered from the recurrent formula

$$\begin{aligned}(-)^pb_k^{\mathcal{F}\mathcal{Q}}(pu_1)=-(pd)u_k+\sum_{a+q+c=k}^{\alpha,\beta}(\phi_{a\alpha}\otimes pu_q\otimes\psi_{c\beta})b^{\mathcal{A}}_{\alpha+1+\beta}\\ -(-)^p\sum_{\alpha+q+\beta=k}^{\alpha+\beta>0}(1^{\otimes\alpha}\otimes b_q^{\mathcal{F}\mathcal{Q}}\otimes1^{\otimes\beta})(pu_{\alpha+1+\beta}):(s\mathcal{F}\mathcal{Q})^{\otimes k}\to s\mathcal{A},\end{aligned}$$

where $k>1$, $p\in P$, and $\phi_{a\alpha}$, $\psi_{c\beta}$ are matrix elements of ϕ, ψ.

Proof. Since the $\Bbbk$-module of (ϕ,ψ)-coderivations $sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$ is a product, $\Bbbk$-linear maps $u: P \to sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$ of degree 0 are in bijection with sequences of $\Bbbk$-linear maps $(u_k)_{k\geqslant 0}$ of degree 0:

$$
\begin{aligned}
u_0 &: P \to \prod_{X\in\operatorname{Ob}\mathcal{Q}} s\mathcal{A}(X\phi, X\psi), && p \mapsto pu_0,\\
u_k &: P \to \prod_{X,Y\in\operatorname{Ob}\mathcal{Q}} \mathsf{C}\big((s\mathcal{FQ})^{\otimes k}(X,Y), s\mathcal{A}(X\phi, Y\psi)\big), && p \mapsto pu_k,
\end{aligned}
$$

for $k\geqslant 1$. The complex $\Phi_0 = (sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi), B_1)$ admits a filtration by subcomplexes

$$\Phi_n = 0\times\dots\times 0\times\prod_{k=n}^{\infty}\prod_{X,Y\in\operatorname{Ob}\mathcal{Q}} \mathsf{C}\big((s\mathcal{FQ})^{\otimes k}(X,Y), s\mathcal{A}(X\phi, Y\psi)\big).$$

In particular, Φ_2 is a subcomplex, and

$$\Phi_0/\Phi_2 = \prod_{X\in\operatorname{Ob}\mathcal{Q}} s\mathcal{A}(X\phi, X\psi) \times \prod_{X,Y\in\operatorname{Ob}\mathcal{Q}} \mathsf{C}\big(s\mathcal{FQ}(X,Y), s\mathcal{A}(X\phi, Y\psi)\big)$$

is the quotient complex with differential (14). Since $(s\mathcal{FQ}, b_1)$ splits into a direct sum of two subcomplexes $s\mathcal{Q}\oplus\big(\oplus_{|t|>0}s\mathcal{F}_t\mathcal{Q}\big)$, the complex Φ_0/Φ_2 has a subcomplex

$$\Big(0\times\prod_{X,Y\in\operatorname{Ob}\mathcal{Q}} \mathsf{C}\big(\oplus_{|t|>0}s\mathcal{F}_t\mathcal{Q}(X,Y), s\mathcal{A}(X\phi, Y\psi)\big), [\_, b_1]\Big).$$

The corresponding quotient complex is $sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)$. The resulting quotient map $\mathrm{restr}_1 : sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi) \to sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)$ is the restriction map. Denoting $u' = u\cdot\mathrm{restr}_1$, we get the discussed assignment $u\mapsto (u', u_n)_{n>1}$. The claim is that if u is a chain map, then the missing part

$$u_1'': P \to \prod_{X,Y\in\operatorname{Ob}\mathcal{Q}} \mathsf{C}\big(\oplus_{|t|>0}s\mathcal{F}_t\mathcal{Q}(X,Y), s\mathcal{A}(X\phi, Y\psi)\big),$$

of $u_1 = u_1'\times u_1''$ is recovered in a unique way.

Let us prove that the map $u\mapsto(u',u_n)_{n>1}$ is injective. The chain map u satisfies $pdu = puB_1$ for all $p\in P$. That is, $pdu_k = (puB_1)_k$ for all $k\geqslant 0$. Since $puB_1 = (pu)b^{\mathcal{A}} - (-)^p b^{\mathcal{FQ}}(pu)$, these conditions can be rewritten as

$$pdu_k = \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha}\otimes pu_q\otimes\psi_{c\beta})b^{\mathcal{A}}_{\alpha+1+\beta} - (-)^p\sum_{\alpha+q+\beta=k}(1^{\otimes\alpha}\otimes b_q\otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}), \quad (16)$$

where $\phi_{a\alpha}: T^a s\mathcal{FQ}(X,Y)\to T^\alpha s\mathcal{A}(X\phi,Y\phi)$ are matrix elements of ϕ, and $\psi_{c\beta}$ are matrix elements of ψ. The same formula can be rewritten as

$$(-)^p b_k^{\mathcal{FQ}}(pu_1) = -(pd)u_k + \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta}$$
$$- (-)^p \sum_{\alpha+q+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_q^{\mathcal{FQ}} \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}) : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \to s\mathcal{A}. \quad (17)$$

When $k > 1$, the map $b_k^{\mathcal{FQ}} : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \to s\mathcal{F}_t\mathcal{Q}$, $t = (t_1 \sqcup \cdots \sqcup t_k)\mathfrak{t}_k$ is invertible, thus, $pu_1 : s\mathcal{F}_t\mathcal{Q} \to s\mathcal{A}$ in the left hand side is determined in a unique way by u_0, u_n for $n > 1$ and by $pu_1 : s\mathcal{F}_{t_i}\mathcal{Q} \to s\mathcal{A}$, $1 \leqslant i \leqslant k$, occurring in the right hand side. Since the restriction u_1' of u_1 to $s\mathcal{F}_{|}\mathcal{Q} = s\mathcal{Q}$ is known by 1), the map u_1'' is recursively recovered from $(u_0, u_1', u_n)_{n>1}$.

Let us prove that the map $u \mapsto (u', u_n)_{n>1}$ is surjective. Given $(u_0, u_1', u_n)_{n>1}$ we define maps u_1'' of degree 0 recursively by (17). This implies equation (16) for $k > 1$. For $k = 0$ this equation in the form $pdu_0 = pu_0 b_1$ holds due to condition 1). It remains to prove equation (16) for $k = 1$:

$$(pd)u_1 = (pu_1)b_1^{\mathcal{A}} + (\phi_1 \otimes pu_0)b_2^{\mathcal{A}} + (pu_0 \otimes \psi_1)b_2^{\mathcal{A}} - (-)^p b_1(pu_1) :$$
$$s\mathcal{F}_t\mathcal{Q}(X,Y) \to s\mathcal{A}(X\phi, Y\psi) \quad (18)$$

for all trees $t \in \mathcal{T}_{\geqslant 2}$. For $t = |$ it holds due to assumption 1). Let $N > 1$ be an integer. Assume that equation (18) holds for all trees $t \in \mathcal{T}_{\geqslant 2}$ with the number of input leaves $\operatorname{in}(t) < N$. Let $t \in \mathcal{T}^N_{\geqslant 2}$ be a tree (with $\operatorname{in}(t) = N$). Then $t = (t_1 \sqcup \cdots \sqcup t_k)\mathfrak{t}_k$ for some $k > 1$ and some trees $t_i \in \mathcal{T}_{\geqslant 2}$, $\operatorname{in}(t_i) < N$. For such t equation (18) is equivalent to

$$(-)^p b_k(pd)u_1 = (-)^p b_k(pu_1)b_1^{\mathcal{A}} + (-)^p b_k(\phi_1 \otimes pu_0)b_2^{\mathcal{A}} + (-)^p b_k(pu_0 \otimes \psi_1)b_2^{\mathcal{A}}$$
$$+ \sum_{\gamma+j+\delta=k}^{\gamma+\delta>0} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta}) b_{\gamma+1+\delta}(pu_1) : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \to s\mathcal{A}.$$

Substituting definition (17) of u_1 we turn the above equation into an identity

$$- \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pdu_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} \quad (19)$$
$$- (-)^p \sum_{\alpha+q+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pdu_{\alpha+1+\beta}) \quad (20)$$
$$= -(pdu_k)b_1 \quad (21)$$
$$+ \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta} b_1$$

$$- (-)^p \sum_{\alpha+q+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta})b_1 \tag{22}$$

$$+ (-)^p b_k(\phi_1 \otimes pu_0)b_2 + (-)^p b_k(pu_0 \otimes \psi_1)b_2 \tag{23}$$

$$+ (-)^p \sum_{\gamma+j+\delta=k}^{\gamma+\delta>0} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta}) \Big[-pdu_{\gamma+1+\delta} \tag{24}$$

$$+ \sum_{a+q+c=\gamma+1+\delta}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta} \tag{25}$$

$$- (-)^p \sum_{\alpha+q+\beta=\gamma+1+\delta}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}) \Big], \tag{26}$$

whose validity we are going to prove now. First of all, terms (20) and (24) cancel each other. Term (26) vanishes because for an arbitrary integer g the sum

$$\sum_{\substack{\gamma+j+\delta=k \\ \alpha+q+\beta=\gamma+1+\delta}}^{\alpha+1+\beta=g} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta}) \tag{27}$$

is the matrix coefficient $b^2 = 0 : T^k s\mathcal{FQ} \to T^g s\mathcal{FQ}$, thus, it vanishes. Notice that condition $\alpha + \beta > 0$ in (26) automatically implies $\gamma + \delta > 0$. Furthermore, term (21) cancels one of the terms of sum (19). In the remaining terms of (19) we may use the induction assumptions and replace pdu_q with the right hand side of (16). We also absorb terms (23) into sum (25), allowing $\gamma = \delta = 0$ in it and allowing simultaneously $\alpha = \beta = 0$ in (22) to compensate for the missing term $b_k(pu_1)b_1$:

$$- \sum_{a+q+c=k}^{\alpha+\beta>0} \sum_{e+j+f=q}^{\gamma,\delta} \big[\phi_{a\alpha} \otimes (\phi_{e\gamma} \otimes pu_j \otimes \psi_{f\delta}) b^{\mathcal{A}}_{\gamma+1+\delta} \otimes \psi_{c\beta}\big] b^{\mathcal{A}}_{\alpha+1+\beta} \tag{28}$$

$$+ (-)^p \sum_{a+q+c=k}^{\alpha+\beta>0} \sum_{\gamma+j+\delta=q} \big[\phi_{a\alpha} \otimes (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(pu_{\gamma+1+\delta}) \otimes \psi_{c\beta}\big] b^{\mathcal{A}}_{\alpha+1+\beta} \tag{29}$$

$$= \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} b^{\mathcal{A}}_1 \tag{30}$$

$$- (-)^p \sum_{\alpha+q+\beta=k} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}) b^{\mathcal{A}}_1 \tag{31}$$

$$+ (-)^p \sum_{\gamma+j+\delta=k} \sum_{a+q+c=\gamma+1+\delta}^{\alpha,\beta} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta}.$$

Recall that ϕ_{a0} vanish for all a except $a = 0$. Therefore, we may absorb term (30) into sum (28) and term (31) into sum (29), allowing terms with $\alpha = \beta = 0$ in them. Denote $r = pu \in sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$. The proposition follows immediately form the following

2.9. Lemma. *For all $r \in sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$ and all $k \geqslant 0$ we have*

$$\begin{aligned}
&- \sum_{a+q+c=k}^{\alpha,\beta} \sum_{e+j+f=q}^{\gamma,\delta} \big[\phi_{a\alpha} \otimes (\phi_{e\gamma} \otimes r_j \otimes \psi_{f\delta}) b^{\mathcal{A}}_{\gamma+1+\delta} \otimes \psi_{c\beta}\big] b^{\mathcal{A}}_{\alpha+1+\beta} \\
&+ (-)^r \sum_{a+q+c=k}^{\alpha,\beta} \sum_{\gamma+j+\delta=q} \big[\phi_{a\alpha} \otimes (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta}) r_{\gamma+1+\delta} \otimes \psi_{c\beta}\big] b^{\mathcal{A}}_{\alpha+1+\beta} \\
&= (-)^r \sum_{\gamma+j+\delta=k} \sum_{a+q+c=\gamma+1+\delta}^{\alpha,\beta} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(\phi_{a\alpha} \otimes r_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta}. \qquad (32)
\end{aligned}$$

Proof. Sum (32) is split into three sums accordingly to output of b_j being an input of $\phi_{a\alpha}$ or r_q or $\psi_{c\beta}$:

$$\begin{aligned}
&- \sum_{a+e+j+f+c=k}^{\alpha,\beta,\gamma,\delta} (\phi_{a\alpha} \otimes \phi_{e\gamma} \otimes r_j \otimes \psi_{f\delta} \otimes \psi_{c\beta})(1^{\otimes\alpha} \otimes b^{\mathcal{A}}_{\gamma+1+\delta} \otimes 1^{\otimes\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} \qquad (33) \\
&+ (-)^r \sum_{a+\gamma+j+\delta+c=k}^{\alpha,\beta} (1^{\otimes a+\gamma} \otimes b_j \otimes 1^{\otimes\delta+c})(\phi_{a\alpha} \otimes r_{\gamma+1+\delta} \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} \qquad (34) \\
&= (-)^r \sum_{x+q+c=k}^{a,\alpha,\beta} (b_{xa}\phi_{a\alpha} \otimes r_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} \qquad (35) \\
&+ (-)^r \sum_{a+y+c=k}^{\alpha,q,\beta} (\phi_{a\alpha} \otimes b_{yq} r_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} \qquad (36) \\
&+ \sum_{a+q+z=k}^{\alpha,\beta,c} (\phi_{a\alpha} \otimes r_q \otimes b_{zc}\psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta}. \qquad (37)
\end{aligned}$$

Here $b_{xa} : T^x s\mathcal{FQ} \to T^a s\mathcal{FQ}$ is a matrix element of $b^{\mathcal{FQ}}$. Terms (34) and (36) cancel each other. We shall use A_∞-functor identities $b\phi = \phi b$, $b\psi = \psi b$ for terms (35) and (37). Being a cocategory homomorphism, ϕ satisfies the identity

$$\sum_{a+e=h} \phi_{a\alpha} \otimes \phi_{e\gamma} = \big[\Delta(\phi \otimes \phi)\big]_{h;\alpha,\gamma} = \phi_{h,\alpha+\gamma} \Delta_{\alpha+\gamma;\alpha,\gamma}$$

for all non-negative integers h, where Δ is the cut comultiplication. Similarly for ψ. Using this identity in (33) we get the equation to verify:

$$\begin{aligned} &-\sum_{x+q+z=k}^{v,w}(\phi_{xv}\otimes r_q\otimes\psi_{zw})\sum_{\alpha+y+\beta=v+1+w}^{\alpha\leqslant v,\beta\leqslant w}(1^{\otimes\alpha}\otimes b^{\mathcal{A}}_y\otimes 1^{\otimes\beta})b^{\mathcal{A}}_{\alpha+1+\beta}\\ &=\sum_{x+q+z=k}^{v,w,\alpha}(\phi_{xv}\otimes r_q\otimes\psi_{zw})(b^{\mathcal{A}}_{v\alpha}\otimes 1^{\otimes 1+w})b^{\mathcal{A}}_{\alpha+1+w}\\ &+\sum_{x+q+z=k}^{v,w,\beta}(\phi_{xv}\otimes r_q\otimes\psi_{zw})(1^{\otimes v+1}\otimes b^{\mathcal{A}}_{w\beta})b^{\mathcal{A}}_{v+1+\beta}. \end{aligned}$$

It follows from the identity $b^2\,\mathrm{pr}_1=0:T^{v+1+w}s\mathcal{A}\to s\mathcal{A}$ valid for arbitrary non-negative integers v, w, which we may rewrite like this:

$$\sum_{\alpha+y+\beta=v+1+w}^{\alpha\leqslant v,\beta\leqslant w}(1^{\otimes\alpha}\otimes b^{\mathcal{A}}_y\otimes 1^{\otimes\beta})b^{\mathcal{A}}_{\alpha+1+\beta}+\sum_{\alpha}(b^{\mathcal{A}}_{v\alpha}\otimes 1^{\otimes 1+w})b^{\mathcal{A}}_{\alpha+1+w}+\sum_{\beta}(1^{\otimes v+1}\otimes b^{\mathcal{A}}_{w\beta})b^{\mathcal{A}}_{v+1+\beta}=0.$$

So the lemma is proved. ■

The proposition follows. ■

Let us consider now the question, when the discussed chain map is null-homotopic.

2.10. Corollary. *Let $\phi,\psi:\mathcal{FQ}\to\mathcal{A}$ be A_∞-functors. Let P be a complex of $\Bbbk$-modules. Let $u:P\to sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$ be a chain map. The set (possibly empty) of homotopies $h:P\to sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)$, $\deg h=-1$, such that $u=dh+hB_1$ is in bijection with the set of data $(h',h_k)_{k>1}$, consisting of*

1. *a homotopy $h':P\to sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)$, $\deg h'=-1$, such that $dh'+h'B_1=u'$, where u' is given by (15);*
2. *$\Bbbk$-linear maps*
$$h_k:P\to\prod_{X,Y\in\mathrm{Ob}\,\mathcal{Q}}\mathsf{C}\big((s\mathcal{FQ})^{\otimes k}(X,Y),s\mathcal{A}(X\phi,Y\psi)\big)$$
of degree -1 for all $k>1$.

The bijection maps h to $(h',h_k)_{k>1}$, where $h_k=h\cdot\mathrm{pr}_k$ and

$$h'=\big(P\xrightarrow{h}sA_\infty(\mathcal{FQ},\mathcal{A})(\phi,\psi)\xrightarrow{\mathrm{restr}}sA_1(\mathcal{FQ},\mathcal{A})(\phi,\psi)\xrightarrow{\mathrm{restr}}sA_1(\mathcal{Q},\mathcal{A})(\phi,\psi)\big).$$

The inverse bijection can be recovered from the recurrent formula

$$\begin{aligned}(-)^pb_k(ph_1)=pu_k-(pd)h_k-\sum_{a+q+c=k}^{\alpha,\beta}(\phi_{a\alpha}\otimes ph_q\otimes\psi_{c\beta})b_{\alpha+1+\beta}\\ -(-)^p\sum_{a+q+c=k}^{a+c>0}(1^{\otimes a}\otimes b_q\otimes 1^{\otimes c})(ph_{a+1+c}):(s\mathcal{FQ})^{\otimes k}\to s\mathcal{A},\end{aligned}$$

where $k > 1$, $p \in P$, and $\phi_{a\alpha}$, $\psi_{c\beta}$ are matrix elements of ϕ, ψ.

Proof. We shall apply Proposition 2.8 to the complex $\operatorname{Cone}(\mathrm{id} : P \to P)$ instead of P. The graded $\Bbbk$-module $\operatorname{Cone}(\mathrm{id}_P) = P \oplus P[1]$ is equipped with the differential $(q, ps)d = (qd + p, -pds)$, $p, q \in P$. The chain maps $\overline{u} : \operatorname{Cone}(\mathrm{id}_P) \to C$ to an arbitrary complex C are in bijection with pairs $(u : P \to C, h : P \to C)$, where $u = dh + hd$ and $\deg h = -1$. The pair $(u, h) = (\mathrm{in}_1 \overline{u}, s\,\mathrm{in}_2 \overline{u})$ is assigned to $\overline{u}$, and the map $\overline{u} : P \oplus P[1] \to C$, $(q, ps) \mapsto qu + ph$ is assigned to a pair (u, h). Indeed, $\overline{u}$ being chain map is equivalent to

$$(q, ps)d\overline{u} = qdu + pu - pdh = qud + phd = (q, ps)\overline{u}d,$$

that is, to conditions $du = ud$, $u = dh + hd$.

Thus, for a fixed chain map $u : P \to C$ the set of homotopies $h : P \to C$, such that $u = dh + hd$, is in bijection with the set of chain maps $\overline{u} : \operatorname{Cone}(\mathrm{id}_P) \to C$ such that $\mathrm{in}_1 \overline{u} = u : P \to C$. Applying this statement to $u : P \to C = sA_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A})(\phi, \psi)$ we find by Proposition 2.8 that the set of homotopies $h : P \to sA_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A})(\phi, \psi)$ such that $u = dh + hB_1$ is in bijection with the set of data $(\overline{u}', \overline{u}_k)_{k>1}$, such that

$$\begin{aligned} &\overline{u}' : \operatorname{Cone}(\mathrm{id}_P) \to sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi) \qquad \text{is a chain map,} && \mathrm{in}_1 \overline{u}' = u', \\ &\overline{u}_k : \operatorname{Cone}(\mathrm{id}_P) \to \prod_{X,Y \in \operatorname{Ob}\mathcal{Q}} \mathsf{C}\big((s\mathcal{F}\mathcal{Q})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)\big), \quad \deg \overline{u}_k = 0, && \mathrm{in}_1 \overline{u}_k = u_k, \end{aligned}$$

therefore, in bijection with the set of data $(h', h_k)_{k>1} = (s\,\mathrm{in}_2 \overline{u}', s\,\mathrm{in}_2 \overline{u}_k)_{k>1}$, as stated in corollary. ∎

2.11. Restriction as an A_∞-functor. Let $\mathcal{Q}$ be a ($\mathscr{U}$-small) differential graded $\Bbbk$-quiver. Denote by $\mathcal{F}\mathcal{Q}$ the free A_∞-category generated by $\mathcal{Q}$. Let $\mathcal{A}$ be a ($\mathscr{U}$-small) unital A_∞-category. There is the restriction strict A_∞-functor

$$\operatorname{restr} : A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A}), \qquad (f : \mathcal{F}\mathcal{Q} \to \mathcal{A}) \mapsto (\overline{f} = (f_1\big|_{\mathcal{Q}}) : \mathcal{Q} \to \mathcal{A}).$$

In fact, it is the composition of two strict A_∞-functors: $A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \xrightarrow{\operatorname{restr}_{\infty,1}} A_1(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A})$, where the second comes from the full embedding $\mathcal{Q} \hookrightarrow \mathcal{F}\mathcal{Q}$. Its first component is

$$\begin{aligned} \operatorname{restr}_1 : sA_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A})(f, g) &\to sA_1(\mathcal{Q}, \mathcal{A})(\overline{f}, \overline{g}), \\ r = (r_0, r_1, \dots, r_n, \dots) &\mapsto (r_0, r_1|_{\mathcal{Q}}) = \overline{r}. \end{aligned} \tag{38}$$

2.12. Theorem. *The A_∞-functor* $\operatorname{restr} : A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A})$ *is an equivalence.*

Proof. Let us prove that restriction map (38) is homotopy invertible. We construct a chain map going in the opposite direction

$$u : sA_1(\mathcal{Q}, \mathcal{A})(\overline{f}, \overline{g}) \to sA_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A})(f, g)$$

via Proposition 2.8 taking $P = sA_1(\mathcal{Q},\mathcal{A})(\overline{f},\overline{g})$. We choose

$$u' : sA_1(\mathcal{Q},\mathcal{A})(\overline{f},\overline{g}) \to sA_1(\mathcal{Q},\mathcal{A})(\overline{f},\overline{g})$$

to be the identity map and $u_k = 0$ for $k > 1$. Therefore,

$$u \cdot \mathrm{restr}_1 = u' = \mathrm{id}_{sA_1(\mathcal{Q},\mathcal{A})(\overline{f},\overline{g})} .$$

Denote

$$v = \mathrm{id}_{sA_\infty(\mathcal{FQ},\mathcal{A})(f,g)} - \bigl[sA_\infty(\mathcal{FQ},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_1} sA_1(\mathcal{Q},\mathcal{A})(\overline{f},\overline{g}) \xrightarrow{u} sA_\infty(\mathcal{FQ},\mathcal{A})(f,g)\bigr].$$

Let us prove that v is null-homotopic via Corollary 2.10, taking $P = sA_\infty(\mathcal{FQ},\mathcal{A})(f,g)$. A homotopy $h : sA_\infty(\mathcal{FQ},\mathcal{A})(f,g) \to sA_\infty(\mathcal{FQ},\mathcal{A})(f,g)$, $\deg h = -1$, such that $v = B_1h + hB_1$ is specified by $h' = 0 : sA_\infty(\mathcal{FQ},\mathcal{A})(f,g) \to sA_1(\mathcal{Q},\mathcal{A})(\overline{f},\overline{g})$ and $h_k = 0$ for $k > 1$. Indeed,

$$v' = v \cdot \mathrm{restr}_1 = \mathrm{restr}_1 - \mathrm{restr}_1 \cdot u \cdot \mathrm{restr}_1 = \mathrm{restr}_1 - \mathrm{restr}_1 = 0,$$

so $v' = B_1h' + h'B_1$ and condition 1 of Corollary 2.10 is satisfied[2]. Therefore, u is homotopy inverse to restr_1.

Let $\mathbf{i}^{\mathcal{A}}$ be a unit transformation of the unital A_∞-category $\mathcal{A}$. Then $A_1(\mathcal{Q},\mathcal{A})$ is a unital A_∞-category with the unit transformation $(1\otimes\mathbf{i}^{\mathcal{A}})M$ (cf. [Lyu03, Proposition 7.7]). The unit element for an object $\phi \in \operatorname{Ob} A_1(\mathcal{Q},\mathcal{A})$ is ${}_\phi\mathbf{i}_0^{A_1(\mathcal{Q},\mathcal{A})} : \Bbbk \to sA_1(\mathcal{Q},\mathcal{A})$, $1 \mapsto \phi\mathbf{i}^{\mathcal{A}}$. The A_∞-category $A_\infty(\mathcal{FQ},\mathcal{A})$ is also unital. To establish equivalence of these two A_∞-categories via $\mathrm{restr} : A_\infty(\mathcal{FQ},\mathcal{A}) \to A_1(\mathcal{Q},\mathcal{A})$ we verify the conditions of Theorem 8.8 from [Lyu03].

Consider the mapping $\operatorname{Ob} A_1(\mathcal{Q},\mathcal{A}) \to \operatorname{Ob} A_\infty(\mathcal{FQ},\mathcal{A})$, $\phi \mapsto \widehat{\phi}$, which extends a given chain map to a strict A_∞-functor, constructed in Corollary 2.4. Clearly, $\overline{\widehat{\phi}} = \phi$. It remains to give two mutually inverse cycles, which we choose as follows:

$$\begin{aligned} {}_\phi r_0 &: \Bbbk \to sA_1(\mathcal{Q},\mathcal{A})(\phi,\overline{\widehat{\phi}}), &\qquad 1 &\mapsto \phi\mathbf{i}^{\mathcal{A}}, \\ {}_\phi p_0 &: \Bbbk \to sA_1(\mathcal{Q},\mathcal{A})(\overline{\widehat{\phi}},\phi), &\qquad 1 &\mapsto \phi\mathbf{i}^{\mathcal{A}}. \end{aligned}$$

Clearly, ${}_\phi r_0B_1 = 0$, ${}_\phi p_0B_1 = 0$,

$$\begin{aligned} ({}_\phi r_0 \otimes {}_\phi p_0)B_2 - {}_\phi\mathbf{i}_0^{A_1(\mathcal{Q},\mathcal{A})} &: 1 \mapsto (\phi\mathbf{i}^{\mathcal{A}} \otimes \phi\mathbf{i}^{\mathcal{A}})B_2 - \phi\mathbf{i}^{\mathcal{A}} \in \operatorname{Im} B_1, \\ ({}_\phi p_0 \otimes {}_\phi r_0)B_2 - {}_\phi\mathbf{i}_0^{A_1(\mathcal{Q},\mathcal{A})} &: 1 \mapsto (\phi\mathbf{i}^{\mathcal{A}} \otimes \phi\mathbf{i}^{\mathcal{A}})B_2 - \phi\mathbf{i}^{\mathcal{A}} \in \operatorname{Im} B_1. \end{aligned}$$

Therefore, all assumptions of Theorem 8.8 [Lyu03] are satisfied. Thus, $\mathrm{restr} : A_\infty(\mathcal{FQ},\mathcal{A}) \to A_1(\mathcal{Q},\mathcal{A})$ is an A_∞-equivalence. ■

[2]By the way, the only non-vanishing component of h is h_1.

2.13. Corollary. *Every A_∞-functor $f : \mathcal{F}\mathcal{Q} \to \mathcal{A}$ is isomorphic to the strict A_∞-functor $\widehat{\overline{f}} : \mathcal{F}\mathcal{Q} \to \mathcal{A}$.*

Proof. Note that $\overline{f} = \overline{\widehat{\overline{f}}}$. The A_1-transformation $\overline{f}\mathbf{i}^{\mathcal{A}} : \overline{f} \to \overline{\widehat{\overline{f}}} : \mathcal{Q} \to \mathcal{A}$ with the components $({}_{X f}\mathbf{i}_0^{\mathcal{A}}, \overline{f}_1\mathbf{i}_1^{\mathcal{A}})$ is natural. It is mapped by u into a natural A_∞-transformation $(\overline{f}\mathbf{i}^{\mathcal{A}})u : f \to \widehat{\overline{f}} : \mathcal{F}\mathcal{Q} \to \mathcal{A}$. Its zero component ${}_{X f}\mathbf{i}_0^{\mathcal{A}}$ is invertible, therefore $(\overline{f}\mathbf{i}^{\mathcal{A}})u$ is invertible by [Lyu03, Proposition 7.15]. ■

3. Representable 2-functors $A_\infty^u \to A_\infty^u$

Recall that unital A_∞-categories, unital A_∞-functors and equivalence classes of natural A_∞-transformations form a 2-category [Lyu03]. In order to distinguish between the A_∞-category $A_\infty^u(\mathcal{C}, \mathcal{D})$ and the ordinary category, whose morphisms are equivalence classes of natural A_∞-transformations, we denote the latter by

$$\overline{A_\infty^u}(\mathcal{C}, \mathcal{D}) = H^0(A_\infty^u(\mathcal{C}, \mathcal{D}), m_1).$$

The corresponding notation for the 2-category is $\overline{A_\infty^u}$. We will see that arbitrary A_N-categories can be viewed as 2-functors $\overline{A_\infty^u} \to \overline{A_\infty^u}$. Moreover, they come from certain generalizations called A_∞^u-2-functors. There is a notion of representability of such 2-functors, which explains some constructions of A_∞-categories. For instance, a differential graded $\Bbbk$-quiver $\mathcal{Q}$ will be represented by the free A_∞-category $\mathcal{F}\mathcal{Q}$ generated by it.

3.1. Definition. *A (strict) A_∞^u-2-functor $F : A_\infty^u \to A_\infty^u$ consists of*

1. *a map $F : \operatorname{Ob} A_\infty^u \to \operatorname{Ob} A_\infty^u$;*
2. *a unital A_∞-functor $F = F_{\mathcal{C},\mathcal{D}} : A_\infty^u(\mathcal{C}, \mathcal{D}) \to A_\infty^u(F\mathcal{C}, F\mathcal{D})$ for each pair $\mathcal{C}, \mathcal{D}$ of unital A_∞-categories;*

   such that

3. *$\mathrm{id}_{F\mathcal{C}} = F(\mathrm{id}_{\mathcal{C}})$ for any unital A_∞-category $\mathcal{C}$;*
4. *the equation*

$$\begin{tikzcd} TsA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty^u(\mathcal{D}, \mathcal{E}) \arrow[r, "M"] \arrow[d, "F_{\mathcal{C},\mathcal{D}} \boxtimes F_{\mathcal{D},\mathcal{E}}"'] \arrow[dr, phantom, "="] & TsA_\infty^u(\mathcal{C}, \mathcal{E}) \arrow[d, "F_{\mathcal{C},\mathcal{E}}"] \\ TsA_\infty^u(F\mathcal{C}, F\mathcal{D}) \boxtimes TsA_\infty^u(F\mathcal{D}, F\mathcal{E}) \arrow[r, "M"] & TsA_\infty^u(F\mathcal{C}, F\mathcal{E}) \end{tikzcd} \tag{39}$$

   holds strictly for each triple $\mathcal{C}, \mathcal{D}, \mathcal{E}$ of unital A_∞-categories.

The A_∞-functor $F : A^u_\infty(\mathcal{C},\mathcal{D}) \to A^u_\infty(F\mathcal{C},F\mathcal{D})$ consists of the mapping of objects

$$\operatorname{Ob} F : \operatorname{Ob} A^u_\infty(\mathcal{C},\mathcal{D}) \to \operatorname{Ob} A^u_\infty(F\mathcal{C},F\mathcal{D}), \qquad f \mapsto Ff,$$

and the components F_k, $k \geqslant 1$:

$$F_1 : sA^u_\infty(\mathcal{C},\mathcal{D})(f,g) \to sA^u_\infty(F\mathcal{C},F\mathcal{D})(Ff,Fg),$$
$$F_2 : sA^u_\infty(\mathcal{C},\mathcal{D})(f,g) \otimes sA^u_\infty(\mathcal{C},\mathcal{D})(g,h) \to sA^u_\infty(F\mathcal{C},F\mathcal{D})(Ff,Fh),$$

and so on.

Weak versions of A^u_∞-2-functors and 2-transformations between them might be considered elsewhere.

3.2. Definition. *A (strict)* A^u_∞-2-transformation $\lambda : F \to G : A^u_\infty \to A^u_\infty$ *of strict* A^u_∞*-2-functors is*

1. *a family of unital* A_∞*-functors* $\lambda_\mathcal{C} : F\mathcal{C} \to G\mathcal{C}$, $\mathcal{C} \in \operatorname{Ob} A^u_\infty$*;*

   such that

2. *the diagram of* A_∞*-functors*

$$\begin{tikzcd} A^u_\infty(\mathcal{C},\mathcal{D}) \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "="] & A^u_\infty(F\mathcal{C},F\mathcal{D}) \arrow[d, "(1\ \lambda_\mathcal{D})M"] \\ A^u_\infty(G\mathcal{C},G\mathcal{D}) \arrow[r, "(\lambda_\mathcal{C}\ 1)M"] & A^u_\infty(F\mathcal{C},G\mathcal{D}) \end{tikzcd} \tag{40}$$

   strictly commutes.

An A^u_∞*-2-transformation* $\lambda = (\lambda_\mathcal{C})$ *for which* $\lambda_\mathcal{C}$ *are* A_∞*-equivalences is called a* natural A^u_∞-2-equivalence*.*

Let us show now that the above notions induce ordinary strict 2-functors and strict 2-transformations in 0-th cohomology. Recall that the strict 2-category $\overline{A^u_\infty}$ consists of objects – unital A_∞-categories, the category $\overline{A^u_\infty}(\mathcal{C},\mathcal{D})$ for any pair of objects $\mathcal{C}$, $\mathcal{D}$, the identity functor $\mathrm{id}_\mathcal{C}$ for any unital A_∞-category $\mathcal{C}$, and the composition functor [Lyu03]

$$\overline{A^u_\infty}(\mathcal{C},\mathcal{D})(f,g) \times \overline{A^u_\infty}(\mathcal{D},\mathcal{E})(h,k) \xrightarrow{\bullet^2} \overline{A^u_\infty}(\mathcal{C},\mathcal{E})(fh,gk),$$
$$(rs^{-1},ps^{-1}) \longmapsto (rhs^{-1} \otimes gps^{-1})m_2.$$

Given a strict A^u_∞-2-functor F as in Definition 3.1 we construct from it an ordinary strict 2-functor $\overline{F} = F : \operatorname{Ob}\overline{A^u_\infty} \to \operatorname{Ob}\overline{A^u_\infty}$, $\overline{F} = H^0(sF_1s^{-1}) : \overline{A^u_\infty}(\mathcal{C},\mathcal{D}) \to \overline{A^u_\infty}(F\mathcal{C},F\mathcal{D})$ as follows.

Denote

$$\begin{aligned} M_{10} \odot M_{01} = \bigl\{ & sA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes sA^u_\infty(\mathcal{D},\mathcal{E}) \xrightarrow[\sim]{\Delta_{10}\quad\Delta_{01}} \\ & [sA^u_\infty(\mathcal{C},\mathcal{D}) \otimes T^0 sA^u_\infty(\mathcal{C},\mathcal{D})] \boxtimes [T^0 sA^u_\infty(\mathcal{D},\mathcal{E}) \otimes sA^u_\infty(\mathcal{D},\mathcal{E})] \\ & \xrightarrow{\sim} [sA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes T^0 sA^u_\infty(\mathcal{D},\mathcal{E})] \otimes [T^0 sA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes sA^u_\infty(\mathcal{D},\mathcal{E})] \\ & \xrightarrow{M_{10} \otimes M_{01}} sA^u_\infty(\mathcal{C},\mathcal{E}) \otimes sA^u_\infty(\mathcal{C},\mathcal{E}) \bigr\}, \end{aligned} \tag{41}$$

where the obvious isomorphisms Δ_{10} and Δ_{01} are components of the comultiplication Δ, the middle isomorphism is that of distributivity law (6), and the components M_{10} and M_{01} of M are the composition maps.

Property (39) of F implies that

$$(M_{10} \odot M_{01})(F_1 \otimes F_1) = (F_1 \boxtimes F_1)(M_{10} \odot M_{01}). \tag{42}$$

Indeed, the following diagram commutes

$$\begin{tikzcd} sA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes T^0 sA^u_\infty(\mathcal{D},\mathcal{E}) \arrow[r, "M_{10}"] \arrow[d, "F_1\quad \operatorname{Ob} F"'] & sA^u_\infty(\mathcal{C},\mathcal{E}) \arrow[d, "F_1"] \\ sA^u_\infty(F\mathcal{C},F\mathcal{D}) \boxtimes T^0 sA^u_\infty(F\mathcal{D},F\mathcal{E}) \arrow[r, "M_{10}"] & sA^u_\infty(F\mathcal{C},F\mathcal{E}) \end{tikzcd}$$

due to (39). $\otimes$-tensoring it with one more similar diagram we get

$$(M_{10} \otimes M_{01})(F_1 \otimes F_1) = [(F_1 \boxtimes \operatorname{Ob} F) \otimes (\operatorname{Ob} F \boxtimes F_1)](M_{10} \otimes M_{01}).$$

The isomorphisms in (41) commute with F in expected way, so (42) follows.

We claim that the diagram

$$\begin{tikzcd} sA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes sA^u_\infty(\mathcal{D},\mathcal{E}) \arrow[r, "(M_{10}\odot M_{01})B_2"] \arrow[d, "F_1\quad F_1"'] & sA^u_\infty(\mathcal{C},\mathcal{E}) \arrow[d, "F_1"] \\ sA^u_\infty(F\mathcal{C},F\mathcal{D}) \boxtimes sA^u_\infty(F\mathcal{D},F\mathcal{E}) \arrow[r, "(M_{10}\odot M_{01})B_2"] & sA^u_\infty(F\mathcal{C},F\mathcal{E}) \end{tikzcd} \tag{43}$$

homotopically commutes. Indeed, since

$$(1 \otimes B_1 + B_1 \otimes 1)F_2 + B_2 F_1 = (F_1 \otimes F_1)B_2 + F_2 B_1,$$

we get

$$\begin{aligned} & (M_{10} \odot M_{01})B_2 F_1 \\ & \quad = (M_{10} \odot M_{01})(F_1 \otimes F_1)B_2 + (M_{10} \odot M_{01})F_2 B_1 - (M_{10} \odot M_{01})(1 \otimes B_1 + B_1 \otimes 1)F_2 \\ & \quad = (F_1 \boxtimes F_1)(M_{10} \odot M_{01})B_2 + (M_{10} \odot M_{01})F_2 B_1 - (1 \boxtimes B_1 + B_1 \boxtimes 1)(M_{10} \odot M_{01})F_2. \end{aligned}$$

We have used equations

$$
\begin{aligned}
(M_{10} \odot M_{01})(1 \otimes B_1) &= (1 \boxtimes B_1)(M_{10} \odot M_{01}), \\
(M_{10} \odot M_{01})(B_1 \otimes 1) &= (B_1 \boxtimes 1)(M_{10} \odot M_{01}),
\end{aligned}
$$

which can be proved similarly to (42) due to M being an A_∞-functor. Passing to cohomology we get from (43) a strictly commutative diagram of functors

$$
\begin{tikzcd}
H^0(A^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes A^u_\infty(\mathcal{D},\mathcal{E})) \arrow[r, "\bullet^2"] \arrow[d, "H^0(sF_1s^{-1}\quad sF_1s^{-1})"'] \arrow[dr, phantom, "="] & H^0(A^u_\infty(\mathcal{C},\mathcal{E})) \arrow[d, "H^0(sF_1s^{-1})"] \\
H^0(A^u_\infty(F\mathcal{C},F\mathcal{D}) \boxtimes A^u_\infty(F\mathcal{D},F\mathcal{E})) \arrow[r, "\bullet^2"] & H^0(A^u_\infty(F\mathcal{C},F\mathcal{E}))
\end{tikzcd}
$$

since $\bullet^2 = H^0((s \boxtimes s)(M_{10} \odot M_{01})B_2 s^{-1})$. Using the Künneth map we come to strictly commutative diagram of functors

$$
\begin{tikzcd}
\overline{A^u_\infty}(\mathcal{C},\mathcal{D}) \times \overline{A^u_\infty}(\mathcal{D},\mathcal{E}) \arrow[r, "\bullet^2"] \arrow[d, "H^0(sF_1s^{-1})\times H^0(sF_1s^{-1})"'] \arrow[dr, phantom, "="] & \overline{A^u_\infty}(\mathcal{C},\mathcal{E}) \arrow[d, "H^0(sF_1s^{-1})"] \\
\overline{A^u_\infty}(F\mathcal{C},F\mathcal{D}) \times \overline{A^u_\infty}(F\mathcal{D},F\mathcal{E}) \arrow[r, "\bullet^2"] & \overline{A^u_\infty}(F\mathcal{C},F\mathcal{E})
\end{tikzcd}
$$

that is, to a usual strict 2-functor $\overline{F} : \overline{A^u_\infty} \to \overline{A^u_\infty}$.

Let us show that an A^u_∞-2-transformation $\lambda : F \to G : A^u_\infty \to A^u_\infty$ as in Definition 3.2 induces an ordinary strict 2-transformation $\overline{\lambda} : \overline{F} \to \overline{G} : \overline{A^u_\infty} \to \overline{A^u_\infty}$ in cohomology. Indeed, diagram (40) implies commutativity of diagram

$$
\begin{tikzcd}
sA^u_\infty(\mathcal{C},\mathcal{D}) \arrow[r, "F_1"] \arrow[d, "G_1"'] \arrow[dr, phantom, "="] & sA^u_\infty(F\mathcal{C},F\mathcal{D}) \arrow[d, "(1\quad \lambda_{\mathcal{D}})M_{10}"] \\
sA^u_\infty(G\mathcal{C},G\mathcal{D}) \arrow[r, "(\lambda_{\mathcal{C}}\quad 1)M"] & sA^u_\infty(F\mathcal{C},G\mathcal{D})
\end{tikzcd}
$$

$$
r : f \to g : G\mathcal{C} \to G\mathcal{D} \longmapsto \lambda_{\mathcal{C}} r : \lambda_{\mathcal{C}} f \to \lambda_{\mathcal{C}} g : F\mathcal{C} \to G\mathcal{D}.
$$

Passing to cohomology we get

$$
\begin{tikzcd}
\overline{A^u_\infty}(\mathcal{C},\mathcal{D}) \arrow[r, "H^0(sF_1s^{-1})"] \arrow[d, "H^0(sG_1s^{-1})"'] \arrow[dr, phantom, "="] & \overline{A^u_\infty}(F\mathcal{C},F\mathcal{D}) \arrow[d, "=\overline{A^u_\infty}(F\mathcal{C},\lambda_{\mathcal{D}})", "\_\cdot\lambda_{\mathcal{D}}"'] \\
\overline{A^u_\infty}(G\mathcal{C},G\mathcal{D}) \arrow[r, "\lambda_{\mathcal{C}}\cdot\_", "\overline{A^u_\infty}(\lambda_{\mathcal{C}},G\mathcal{D})"'] & \overline{A^u_\infty}(F\mathcal{C},G\mathcal{D})
\end{tikzcd}
$$

Therefore, $\overline{\lambda_{\mathcal{C}}} \in \operatorname{Ob} \overline{A^u_\infty}(F\mathcal{C},G\mathcal{C})$ form a strict 2-transformation $\overline{\lambda} : \overline{F} \to \overline{G} : \overline{A^u_\infty} \to \overline{A^u_\infty}$.

3.3. Examples of A^u_∞-2-functors. Let $\mathcal{A}$ be an A_N-category, $1 \leqslant N \leqslant \infty$. It determines an A^u_∞-2-functor $F = A_N(\mathcal{A}, \_) : A^u_\infty \to A^u_\infty$, given by the following data:

1. the map $F : \operatorname{Ob} A^u_\infty \to \operatorname{Ob} A^u_\infty$, $\mathcal{C} \mapsto A_N(\mathcal{A}, \mathcal{C})$ (the category $A_N(\mathcal{A}, \mathcal{C})$ is unital by [Lyu03, Proposition 7.7]);
2. the unital strict A_∞-functor $F = A_N(\mathcal{A}, \_) : A^u_\infty(\mathcal{C}, \mathcal{D}) \to A^u_\infty(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{D}))$ for each pair $\mathcal{C}, \mathcal{D}$ of unital A_∞-categories (cf. [Lyu03, Propositions 6.2, 8.4]).

Clearly, $\mathrm{id}_{A_N(\mathcal{A},\mathcal{C})} = (1 \boxtimes \mathrm{id}_\mathcal{C})M = A_N(\mathcal{A}, \mathrm{id}_\mathcal{C})$. We want to prove now that the equation

$$\begin{aligned}\big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA^u_\infty(\mathcal{D},\mathcal{E}) \xrightarrow{M} TsA^u_\infty(\mathcal{C},\mathcal{E}) \xrightarrow{A_N(\mathcal{A},\_)} TsA^u_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{E}))\big] \\ = \big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA^u_\infty(\mathcal{D},\mathcal{E}) \xrightarrow{A_N(\mathcal{A},\_)\ \ A_N(\mathcal{A},\_)} \\ TsA^u_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{D})) \boxtimes TsA^u_\infty(A_N(\mathcal{A},\mathcal{D}), A_N(\mathcal{A},\mathcal{E})) \\ \xrightarrow{M} TsA^u_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{E}))\big]\end{aligned} \tag{44}$$

holds strictly for each triple $\mathcal{C}, \mathcal{D}, \mathcal{E}$ of unital A_∞-categories. In fact, this F is a restriction of an A_∞-2-functor $F : \operatorname{Ob} A_\infty \to \operatorname{Ob} A_\infty$, $\mathcal{C} \mapsto A_N(\mathcal{A}, \mathcal{C})$, which is defined just as in Definition 3.1 without mentioning the unitality. Equation (44) follows from a similar equation without the unitality index u. To prove it we consider the compositions

$$\begin{aligned}\big[TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \xrightarrow{1\ M} TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{E}) \\ \xrightarrow{1\ A_N(\mathcal{A},\_)} TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{E})) \xrightarrow{\alpha} TsA_N(\mathcal{A},\mathcal{E})\big] \\ = \big[TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \\ \xrightarrow{1\ M} TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{E}) \xrightarrow{M} TsA_N(\mathcal{A},\mathcal{E})\big] \\ = \big[TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \\ \xrightarrow{M\ 1} TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{E}) \xrightarrow{M} TsA_N(\mathcal{A},\mathcal{E})\big] \\ = \big[TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \xrightarrow{1\ A_N(\mathcal{A},\_)\ 1} \\ TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{D})) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \\ \xrightarrow{\alpha\ 1} TsA_N(\mathcal{A},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \\ \xrightarrow{1\ A_N(\mathcal{A},\_)} TsA_N(\mathcal{A},\mathcal{D}) \boxtimes TsA_\infty(A_N(\mathcal{A},\mathcal{D}), A_N(\mathcal{A},\mathcal{E})) \xrightarrow{\alpha} TsA_N(\mathcal{A},\mathcal{E})\big] \\ = \big[TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{D},\mathcal{E}) \xrightarrow{1\ A_N(\mathcal{A},\_)\ A_N(\mathcal{A},\_)} \\ TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{D})) \boxtimes TsA_\infty(A_N(\mathcal{A},\mathcal{D}), A_N(\mathcal{A},\mathcal{E})) \\ \xrightarrow{1\ M} TsA_N(\mathcal{A},\mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A},\mathcal{C}), A_N(\mathcal{A},\mathcal{E})) \xrightarrow{\alpha} TsA_N(\mathcal{A},\mathcal{E})\big].\end{aligned}$$

By Proposition 1.6 we deduce equation (44) (see also [Lyu03, Proposition 5.5]).

Let now $\mathcal{A}$ be a unital A_∞-category. It determines an A^u_∞-2-functor $G = A^u_\infty(\mathcal{A}, \_) : A^u_\infty \to A^u_\infty$, given by the following data:

1. the map $G : \operatorname{Ob} A^u_\infty \to \operatorname{Ob} A^u_\infty$, $\mathcal{C} \mapsto A^u_\infty(\mathcal{A}, \mathcal{C})$ (the category $A^u_\infty(\mathcal{A}, \mathcal{C})$ is unital by [Lyu03, Proposition 7.7]);

2. the unital strict A_∞-functor $G = A^u_\infty(\mathcal{A}, \_) : A^u_\infty(\mathcal{C}, \mathcal{D}) \to A^u_\infty(A^u_\infty(\mathcal{A}, \mathcal{C}), A^u_\infty(\mathcal{A}, \mathcal{D}))$ for each pair $\mathcal{C}, \mathcal{D}$ of unital A_∞-categories, determined from

$$M = \big[TsA^u_\infty(\mathcal{A}, \mathcal{B}) \boxtimes TsA^u_\infty(\mathcal{B}, \mathcal{C}) \xrightarrow{1 \boxtimes A^u_\infty(\mathcal{A}, \_)} \\ TsA^u_\infty(\mathcal{A}, \mathcal{B}) \boxtimes TsA^u_\infty(A^u_\infty(\mathcal{A}, \mathcal{B}), A^u_\infty(\mathcal{A}, \mathcal{C})) \xrightarrow{\alpha} TsA^u_\infty(\mathcal{A}, \mathcal{C})\big].$$

(cf. [Lyu03, Propositions 6.2, 8.4]).

Clearly, $G\mathcal{C}$ are full A_∞-subcategories of $F\mathcal{C}$ for the A^u_∞-2-functor $F = A_\infty(\mathcal{A}, \_)$. Furthermore, A_∞-functors $G_{\mathcal{C},\mathcal{D}}(f)$ are restrictions of A_∞-functors $F_{\mathcal{C},\mathcal{D}}(f)$, so G is a full A^u_∞-2-subfunctor of F. In particular, G satisfies equation (39). Another way to prove that G is an A^u_∞-2-functor is to repeat the reasoning concerning F.

3.4. Example of an A^u_∞-2-equivalence. Assume that $\mathcal{Q}$ is a differential graded $\Bbbk$-quiver. As usual, $\mathcal{FQ}$ denotes the free A_∞-category generated by it. We claim that $\operatorname{restr} : A_\infty(\mathcal{FQ}, \_) \to A_1(\mathcal{Q}, \_) : A^u_\infty \to A^u_\infty$ is a strict 2-natural A_∞-equivalence. Indeed, it is given by the family of unital A_∞-functors $\operatorname{restr}_\mathcal{C} : A_\infty(\mathcal{FQ}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{C})$, $\mathcal{C} \in \operatorname{Ob} A^u_\infty$, which are equivalences by Theorem 2.12. We have to prove that the diagram of A_∞-functors

$$\begin{tikzcd} A^u_\infty(\mathcal{C}, \mathcal{D}) \arrow[r, "A_\infty(\mathcal{FQ}, \_)"] \arrow[d, "A_1(\mathcal{Q}, \_)"'] \arrow[dr, phantom, "="] & A^u_\infty(A_\infty(\mathcal{FQ}, \mathcal{C}), A_\infty(\mathcal{FQ}, \mathcal{D})) \arrow[d, "(1 \boxtimes \operatorname{restr}_\mathcal{D})M"] \\ A^u_\infty(A_1(\mathcal{Q}, \mathcal{C}), A_1(\mathcal{Q}, \mathcal{D})) \arrow[r, "(\operatorname{restr}_\mathcal{C} \boxtimes 1)M"] & A^u_\infty(A_\infty(\mathcal{FQ}, \mathcal{C}), A_1(\mathcal{Q}, \mathcal{D})) \end{tikzcd} \tag{45}$$

commutes. Notice that all arrows in this diagram are strict A_∞-functors. Indeed, $A_\infty(\mathcal{FQ}, \_)$ and $A_1(\mathcal{Q}, \_)$ are strict by [Lyu03, Proposition 6.2]. For an arbitrary A_∞-functor f the components $[(f \boxtimes 1)M]_n = (f \boxtimes 1)M_{0n}$ vanish for all n except for $n = 1$, thus, $(f \boxtimes 1)M$ is strict. The A_∞-functor $g = \operatorname{restr}_\mathcal{D}$ is strict, hence, the n-th component

$$[(1 \boxtimes g)M]_n : r^1 \otimes \cdots \otimes r^n \mapsto (r^1 \otimes \cdots \otimes r^n \mid g)M_{n0}$$

of the A_∞-functor $(1 \boxtimes g)M$ satisfies the equation

$$[(r^1 \otimes \cdots \otimes r^n \mid g)M_{n0}]_k = (r^1 \otimes \cdots \otimes r^n)\theta_{k1}g_1.$$

If the right hand side does not vanish, then $n \leqslant 1 \leqslant k + n$, so $n = 1$ and $(1 \boxtimes g)M$ is strict.

Given an A_∞-transformation $t : g \to h : \mathcal{C} \to \mathcal{D}$ between unital A_∞-functors we find that

$$
\begin{aligned}
A_\infty(\mathcal{FQ}, \_)(t) &= [(1 \boxtimes t)M : (1 \boxtimes g)M \to (1 \boxtimes h)M : A_\infty(\mathcal{FQ}, \mathcal{C}) \to A_\infty(\mathcal{FQ}, \mathcal{D})], \\
A_1(\mathcal{Q}, \_)(t) &= [(1 \boxtimes t)M : (1 \boxtimes g)M \to (1 \boxtimes h)M : A_1(\mathcal{Q}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{D})], \\
[(1 \boxtimes \mathrm{restr}_{\mathcal{D}})M]A_\infty(\mathcal{FQ}, \_)(t) &= [((1 \boxtimes t)M) \cdot \mathrm{restr}_{\mathcal{D}} : ((1 \boxtimes g)M) \cdot \mathrm{restr}_{\mathcal{D}} \\
&\qquad \to ((1 \boxtimes h)M) \cdot \mathrm{restr}_{\mathcal{D}} : A_\infty(\mathcal{FQ}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{D})], \\
[(\mathrm{restr}_{\mathcal{C}} \boxtimes 1)M]A_1(\mathcal{Q}, \_)(t) &= [\mathrm{restr}_{\mathcal{C}} \cdot ((1 \boxtimes t)M) : \mathrm{restr}_{\mathcal{C}} \cdot ((1 \boxtimes g)M) \\
&\qquad \to \mathrm{restr}_{\mathcal{C}} \cdot ((1 \boxtimes h)M) : A_\infty(\mathcal{FQ}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{D})].
\end{aligned}
$$

We have to verify that the last two A_∞-transformations are equal. First of all, let us show that mappings of objects in (45) commute. Given a unital A_∞-functor $g : \mathcal{C} \to \mathcal{D}$, we are going to check that

$$
[(1 \boxtimes g)M]_n \cdot \mathrm{restr}_1 = \mathrm{restr}_1^{\otimes n} \cdot [(1 \boxtimes g)M]_n \tag{46}
$$

for any $n \geqslant 1$. Indeed, for any n-tuple of composable A_∞-transformations

$$
f^0 \xrightarrow{r^1} f^1 \longrightarrow \dots \xrightarrow{r^n} f^n : \mathcal{FQ} \to \mathcal{C},
$$

we have in both cases

$$
\begin{aligned}
\{(r^1 \otimes \dots \otimes r^n)[(1 \boxtimes g)M]_n\}_0 &= [(r^1 \otimes \dots \otimes r^n | g)M_{n0}]_0 = (r_0^1 \otimes \dots \otimes r_0^n)g_n, \\
\{(r^1 \otimes \dots \otimes r^n)[(1 \boxtimes g)M]_n\}_1 &= [(r^1 \otimes \dots \otimes r^n | g)M_{n0}]_1 \\
&= \sum_{i=1}^n (r_0^1 \otimes \dots \otimes r_0^{i-1} \otimes r_1^i \otimes r_0^{i+1} \otimes \dots \otimes r_0^n)g_n \\
&+ \sum_{i=1}^n (r_0^1 \otimes \dots \otimes r_0^{i-1} \otimes f_1^i \otimes r_0^{i+1} \otimes \dots \otimes r_0^n)g_{n+1}.
\end{aligned}
$$

Note that the right hand sides depend only on 0-th and 1-st components of r^i, f^i. This is precisely what is claimed by equation (46).

The coincidence of A_∞-transformations $((1 \boxtimes t)M) \cdot \mathrm{restr}_{\mathcal{D}} = \mathrm{restr}_{\mathcal{C}} \cdot ((1 \boxtimes t)M)$ follows similarly from the computation:

$$
\begin{aligned}
\{(r^1 \otimes \dots \otimes r^n)[(1 \boxtimes t)M]_n\}_0 &= [(r^1 \otimes \dots \otimes r^n \boxtimes t)M_{n1}]_0 = (r_0^1 \otimes \dots \otimes r_0^n)t_n, \\
\{(r^1 \otimes \dots \otimes r^n)[(1 \boxtimes t)M]_n\}_1 &= [(r^1 \otimes \dots \otimes r^n \boxtimes t)M_{n1}]_1 \\
&= \sum_{i=1}^n (r_0^1 \otimes \dots \otimes r_0^{i-1} \otimes r_1^i \otimes r_0^{i+1} \otimes \dots \otimes r_0^n)t_n \\
&+ \sum_{i=1}^n (r_0^1 \otimes \dots \otimes r_0^{i-1} \otimes f_1^i \otimes r_0^{i+1} \otimes \dots \otimes r_0^n)t_{n+1}.
\end{aligned}
$$

3.5. Representability. An A^u_∞-2-functor $F : A^u_\infty \to A^u_\infty$ is called *representable*, if it is naturally A^u_∞-2-equivalent to the A^u_∞-2-functor $A_\infty(\mathcal{A}, \_) : A^u_\infty \to A^u_\infty$ for some A_∞-category $\mathcal{A}$. The above results imply that the A^u_∞-2-functor $A_1(\mathcal{Q}, \_)$ corresponding to a differential graded $\Bbbk$-quiver $\mathcal{Q}$ is represented by the free A_∞-category $\mathcal{F}\mathcal{Q}$ generated by $\mathcal{Q}$.

This definition of representability has a disadvantage: many different A_∞-categories can represent the same A^u_∞-2-functor. More attractive notion is the following. An A^u_∞-2-functor $F : A^u_\infty \to A^u_\infty$ is called *unitally representable*, if it is naturally A^u_∞-2-equivalent to the A^u_∞-2-functor $A^u_\infty(\mathcal{A}, \_) : A^u_\infty \to A^u_\infty$ for some unital A_∞-category $\mathcal{A}$. Such $\mathcal{A}$ is unique up to an A_∞-equivalence. Indeed, composing a natural 2-equivalence $\overline{\lambda} : \overline{A^u_\infty(\mathcal{A}, \_)} \to \overline{A^u_\infty(\mathcal{B}, \_)} : \overline{A^u_\infty} \to \overline{A^u_\infty}$ with the 0-th cohomology 2-functor $H^0 : \overline{A^u_\infty} \to \mathcal{C}at$, we get a natural 2-equivalence $H^0\overline{\lambda} : H^0\overline{A^u_\infty(\mathcal{A}, \_)} \to H^0\overline{A^u_\infty(\mathcal{B}, \_)} : \overline{A^u_\infty} \to \mathcal{C}at$. However, $H^0\overline{A^u_\infty(\mathcal{A}, \_)} = \overline{A^u_\infty}(\mathcal{A}, \_)$, so using a 2-category version of Yoneda lemma one can deduce that $\mathcal{A}$ and $\mathcal{B}$ are equivalent in $\overline{A^u_\infty}$. We shall present an example of unital representability in subsequent publication [LM04].

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