Theory and Applications of Categories, Vol. 29, No. 1, 2014, pp. 1–16.

SOME STABILITY PROPERTIES OF EPIMORPHISM CLASSES

DALI ZANGURASHVILI

ABSTRACT. It is proved that in any pointed category with pullbacks, coequalizers and regular epi-mono factorizations, the class of regular epimorphisms is stable under pullback along the so-called balanced effective descent morphisms. Here "balanced" can be omitted if the category is additive. A balanced effective descent morphism is defined as an effective descent morphism $p : E \to B$ such that any subobject of E is a pullback of some morphism along p. It is shown that, in any category with pullbacks and coequalizers, the class of effective descent morphisms is stable under pushout if and only if any regular epimorphism is an effective descent morphism. Moreover, it is shown that the class of descent morphisms is stable under pushout if and only if the class of regular epimorphisms is stable under pullback.

1. Introduction

Throughout the paper the ground category is assumed to have pullbacks and coequalizers.

The main issue considered in this paper consists in finding a possibly large class of morphisms along which the pullback preserves regular epimorphisms. If the category is regular, then obviously all morphisms are of this kind. We focus our consideration on the condition

(*) the class of regular epimorphisms is stable under pullback along effective descent morphisms,

which appeared in the joint paper [15] of the early 1990's by Sobral and Tholen in connection with the problem whether the class of effective descent morphisms is closed under composition (not known yet to be the fact at that time). Namely, they showed that under condition (*), the closedness of this class under composition is a quick consequence of the sufficient condition for the composition of monadic functors to be monadic given by Pfender in [11]. Later, in [12] Sobral, Tholen and Reiterman showed that condition (*) holds not always (namely, it fails for the category of topological spaces). However, in the same paper they proved the invariance theorem for descent data, which implies that the class of effective descent morphisms is always closed under composition, regardless

Financial support from Volkswagen Foundation (Ref.: 85 989) and Shota Rustaveli National Science Foundation (Ref.: ST09 730 3-105) is gratefully acknowledged

Received by the editors 2013-04-12 and, in revised form, 2014-01-12.

Transmitted by Walter Tholen. Published on 2014-01-19.

2010 Mathematics Subject Classification: 18A20, 18A32, 18A30, 18C20.

Key words and phrases: (effective) descent morphism, balanced morphism, factorization system, stability under pullback/pushout.

whether (*) is satisfied. Thereafter the question when condition (*) is valid was not investigated, to the best of our knowledge. We study this issue in this paper and, moreover, answer the question by what kind of effective descent morphisms we must replace all effective descent morphisms in condition (*) so that the resulting condition be valid under some natural restrictions on the category.

We show that if a category has regular epi-mono factorizations and is pointed (and hence have kernels), then

(**) the class of regular epimorphisms is stable under pullback along balanced effective descent morphisms.

A balanced morphism is defined here as a morphism $p : E \to B$ such that, for any subobject $\gamma : C \rightarrowtail E$ of E that contains the kernel of p, there exists a morphism $\xi : E \times_B C \to C$ such that the composition $\gamma\xi$ is equal to the first projection π_1 in the pullback

$$\begin{tikzcd} E \times_B C \arrow[r, "\pi_2"] \arrow[dd, "\pi_1"] & C \arrow[d, "\gamma"] \\ & E \arrow[d, "p"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

If p is an effective descent morphism, then the latter condition is equivalent to the following one: any subobject $C \rightarrowtail E$ of E that contains the kernel of p is the pullback of some morphism along p. The prototype of a balanced morphism is a homomorphism of groups — if a subgroup contains the kernel of p, then it either contains a coset wholly or does not intersect it, and therefore the subgroup is the inverse image of some subgroup of B. As different from the category of groups, only very few effective descent morphisms are balanced in the category of pointed sets.

Further, it is proved that all morphisms are balanced in any additive category. This implies that condition (*) is always satisfied in such a category if it has regular epi-mono factorizations.

We derive the above result on the validity of condition (**) from a more general statement given in this paper, which provides yet another stability property for the class $\mathbb{E}$ in any factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subset Mono$. Namely, it asserts that the class $\mathbb{E}$ is stable under pullback along $\mathbb{M}$-balanced $\mathbb{M}$-reflecting effective descent morphisms. An $\mathbb{M}$-balanced morphism is defined similarly to a balanced morphism (replacing "subobject" by "$\mathbb{M}$-subobject" in the definition), while an $\mathbb{M}$-reflecting morphism is defined as a morphism such that the pullback along it reflects $\mathbb{M}$-morphisms. For instance, any normal epimorphism is $NormMono$-balanced. Moreover, every effective descent morphism is $Mono$-reflecting; here $NormMono$ (resp. $Mono$) denotes the class of all normal monomorphisms (resp. all monomorphisms). Applying these facts and the above general result to the factorization system $(Epi, NormMono)$ in the relevant situations, we obtain the sufficient conditions for the class of epimorphisms to be stable under pullback along normal effective descent morphisms.

One more issue considered in the paper is whether the class of effective descent morphisms and the class of descent morphisms are stable under pushout. Some other stability properties of these classes are well-known: they are stable under pullback [14], [15], have the right cancellation property [12], and, as has already been mentioned here, are closed under composition [12]. We show that the class of effective descent morphisms is stable under pushout if and only if every regular epimorphism is an effective descent morphism. Moreover, the class of descent morphisms is stable under pushout if and only if the category is regular. Under the above-mentioned requirements on a category to have pullbacks and coequalizers, we call a category regular if the class of regular epimorphisms is stable under pullback.

2. Preliminaries

We begin with the needed definitions (see, for example, the paper [8] by Janelidze and Tholen). Throughout the paper $\mathcal{C}$ denotes, unless specified otherwise, an arbitrary category with pullbacks and coequalizers. Moreover, $\mathbb{M}$ denotes a morphism class which is closed under composition with isomorphisms and stable under pullback.

Let $p : E \to B$ be a morphism in $\mathcal{C}$, and let $\mathcal{D}es_{\mathbb{M}}(p)$ be the category of $\mathbb{M}$-descent data with respect to p [8]. Recall that such data is defined as a triple (C, γ, ξ) with $C \in \operatorname{Ob}\mathcal{C}$, where γ and ξ are morphisms $C \to E$ and $E \times_B C \to C$, respectively, such that $\gamma \in \mathbb{M}$, and the following equalities are valid (see the diagrams (2.4) and (2.5) below):

$$\gamma\xi = \pi_1, \tag{2.1}$$

$$\xi(\gamma, 1_C) = 1_C, \tag{2.2}$$

$$\xi(1_E \times_B \pi_2) = \xi(1_E \times_B \xi), \tag{2.3}$$

$$\begin{tikzcd}
C \arrow[dr, "{(\gamma,1_C)}"] \arrow[drr, "1_C"] \arrow[dddr, "\gamma"'] & & \\
& E \times_B C \arrow[r, "\xi"] \arrow[dd, "\pi_1"] & C \arrow[d, "\gamma"] \\
& & E \arrow[d, "p"] \\
& E \arrow[r, "p"] & B
\end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd}
E \times_B (E \times_B C) \arrow[rr] \arrow[dr, shift left, "1_E \times_B \pi_2"] \arrow[dr, shift right, "1_E \times_B \xi"'] \arrow[dddr] & & E \times_B C \arrow[d, shift right, "\xi"'] \arrow[d, shift left, "\pi_2"] \\
& E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] \arrow[dd, "\pi_1"] & C \arrow[d, "\gamma"] \\
& & E \arrow[d, "p"] \\
& E \arrow[r, "p"] & B
\end{tikzcd} \tag{2.5}$$

A morphism $(C, \gamma, \xi) \to (C', \gamma', \xi')$ in $\mathcal{D}es_{\mathbb{M}}(p)$ is defined as a $\mathcal{C}$-morphism $h : C \to C'$ such that $\gamma' h = \gamma$ and $\xi'(1_E \times_B h) = h\xi$.

We have the comparison functor

$$\Phi^p : \mathbb{M}/B \to \mathcal{D}es_{\mathbb{M}}(p),$$

where $\mathbb{M}/B$ is the full subcategory of the slice-category $\mathcal{C}/B$ with objects being morphisms from $\mathbb{M}$. The functor Φ^p sends $f : D \to B$ to the descent data

$$(E \times_B D, \pi_1', 1_E \times_B \pi_2'),$$

where π_1' and π_2' are the pullbacks of f and p, resp. along each other.

p is called an $\mathbb{M}$-descent (resp. effective $\mathbb{M}$-descent) morphism if Φ^p is full and faithful (resp. the equivalence of categories). If $\mathbb{M}$ is the class of all morphisms, then we omit the prefix $\mathbb{M}$.

If $\mathbb{M}$ is closed under composition with p from the left ($e \in \mathbb{M} \Rightarrow pe \in \mathbb{M}$), then the change-of-base functor

$$p^* : \mathbb{M}/B \to \mathbb{M}/E \tag{2.6}$$

(pulling back along the morphism p) has the left adjoint (composing with p on the left). It is proved in [8] that then the Eilenberg-Moore category of the monad induced by the adjunction

$$p_! \dashv p^* \tag{2.7}$$

is equivalent to the category $\mathcal{D}es_{\mathbb{M}}(p)$. This implies that then p is an $\mathbb{M}$-descent (resp. an effective $\mathbb{M}$-descent) morphism if and only if the change-of-base functor p^* is premonadic (resp. monadic).

2.1. Theorem. [7], [8] *Let $\mathbb{M}$ be closed under composition with a morphism p from the left. p is an $\mathbb{M}$-descent morphism if and only if it is an $\mathbb{M}$-universal regular epimorphism, i.e., a morphism such that any of its pullbacks along any $\mathbb{M}$-morphism is a regular epimorphism.*

A special case of interest is the one, where there exists a morphism class $\mathbb{E}$, such that the pair $(\mathbb{E}, \mathbb{M})$ is a factorization system in the usual sense of Freyd and Kelly [5]. Recall that this means that both classes $\mathbb{E}$ and $\mathbb{M}$ are closed under composition with isomorphisms, every morphism admits an $(\mathbb{E}, \mathbb{M})$-factorization (i.e., there exist morphisms $e \in \mathbb{E}$ and $m \in \mathbb{M}$ with $f = me$), and any $e \in \mathbb{E}$ is orthogonal to any $m \in \mathbb{M}$ (i.e., for any commutative square $\beta e = m\alpha$ there exists a unique morphism δ with $\alpha = \delta e$ and $\beta = m\delta$). Recall also the well-known stability properties of the classes $\mathbb{E}$ and $\mathbb{M}$ in a factorization system $(\mathbb{E}, \mathbb{M})$: both are closed under composition, the class $\mathbb{E}$ (resp. $\mathbb{M}$) is stable under pushout (resp. pullback), and is right cancellable ($ee', e' \in \mathbb{E} \Rightarrow e \in \mathbb{E}$), while the class $\mathbb{M}$ is left cancellable. Moreover, it is well known that the class $\mathbb{E}$ (resp. $\mathbb{M}$) coincides with the class $\mathbb{M}^{\uparrow}$ (resp. $\mathbb{E}^{\downarrow}$) of morphisms orthogonal (resp. co-orthogonal) to all $\mathbb{M}$-morphisms (resp. $\mathbb{E}$-morphisms). This in particular implies that if $\mathbb{M} \subset Mono$, then all regular epimorphisms are contained in $\mathbb{E}$. Moreover, the condition $\mathbb{M} \subset Mono$ is equivalent to the strong version of the right cancellation property for the class $\mathbb{E}$ ($e\alpha \in \mathbb{E} \Rightarrow e \in \mathbb{E}$) [1].

Let us recall some facts from [8]. Let $(\mathbb{E}, \mathbb{M})$ be a factorization system. Then the change-of-base functor (2.6) has a left adjoint. It sends a morphism f to the $\mathbb{M}$-morphism in the $(\mathbb{E}, \mathbb{M})$-factorization of the composition pf. However, in general, the category $\mathcal{Des}_{\mathbb{M}}(p)$ is not equivalent to the Eilenberg-Moore category of the monad induced by the corresponding adjunction. But if $\mathbb{E}$ is stable under pullback, then this is the case. One has the following

2.2. Proposition. [8] *Let $(\mathbb{E}, \mathbb{M})$ be a factorization system with $\mathbb{M} \subset Mono$. Then the following conditions are equivalent:*

(i) p^* is premonadic;

(ii) p^* is monadic;

(iii) p is an $\mathbb{M}$-universal $\mathbb{E}$-morphism.

Proposition 2.2 implies

2.3. Theorem. [8] *Let $(\mathbb{E}, \mathbb{M})$ be a factorization system with $\mathbb{M} \subset Mono$, and let $\mathbb{E}$ be stable under pullback. Then every $\mathbb{M}$-descent morphism is effective, and the class of such morphisms coincides with $\mathbb{E}$.*

3. Balanced morphisms

Let $\mathbb{M} \subset Mono$.

3.1. Lemma. *Let $p : E \to B$ be any morphism, and $\gamma : C \rightarrowtail E$ be an $\mathbb{M}$-morphism. The conditions (i)-(iii) below are equivalent and implied by the condition (iv):*

(i) there exist a morphism ξ such that (C, γ, ξ) is $\mathbb{M}$-descent data with respect to p;

(ii) there exist a morphism ξ *such that* $\gamma\xi = \pi_1$*;*

(iii) the morphism $\gamma \times_B 1_C$ *is an isomorphism;*

$$\begin{tikzcd}
C \times_B C \arrow[r, "\gamma \times_B 1_C"] \arrow[dd, "\pi_1'"] & E \times_B C \arrow[r, "\pi_2"] \arrow[dd, "\pi_1"] & C \arrow[d, "\gamma"] \\
 & & E \arrow[d, "p"] \\
C \arrow[r, "\gamma"] & E \arrow[r, "p"] & B
\end{tikzcd}$$

(iv) there exist morphisms $\delta \in \mathbb{M}$ *and* p' *such that the square*

$$\begin{tikzcd}
C \arrow[r, "p'"] \arrow[d, "\gamma"] & A \arrow[d, "\delta"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

is a pullback.

If p *is an effective* $\mathbb{M}$*-descent morphism, then the conditions (i)-(iv) are equivalent.*
If $(\mathbb{E} = \mathbb{M}^{\uparrow}, \mathbb{M})$ *is a factorization system,* $\mathbb{E}$ *is stable under pullback, and* p *is any morphism from* $\mathbb{E}$*, then the conditions (i)-(iv) are also equivalent to the condition*

(v) the square

$$\begin{tikzcd}
C \arrow[r, "e"] \arrow[d, "\gamma"] & A \arrow[d, "m"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

is a pullback, where me *is the* $(\mathbb{E}, \mathbb{M})$*-factorization of the composition* $p\gamma$*.*

If $\mathcal{C}$ *is a pointed category (and hence have kernels), then any of the conditions (i)-(iv) implies*

(vi) there exists a morphism $\kappa : \mathrm{Ker}\, p \to C$ *with* $\gamma\kappa = i$*, where* i *is the embedding* $\mathrm{Ker}\, p \to E$*.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) and the implication (i) $\Rightarrow$ (iv) (when p is an effective descent morphism) are obvious.

(ii) $\Rightarrow$ (iii): One can verify that the morphism (ξ, π_2) is the inverse for $\gamma \times_B 1_C$.

(iii) $\Rightarrow$ (ii): The morphism $\pi_1'(\gamma \times_B 1_C)^{-1}$ is obviously the sought-morphism.

(iv) $\Rightarrow$ (ii): We can take the canonical morphism ξ in the diagram

$$\begin{tikzcd}
& E\times_B C \arrow[rr, "\pi_2"] \arrow[dl, "\xi"'] \arrow[dd, "\pi_1"] & & C \arrow[dl, "p'"'] \arrow[d, "\gamma"] \\
C \arrow[rr] \arrow[dr, "\gamma"'] & & A \arrow[dr, "\delta"] & E \arrow[d, "p"] \\
& E \arrow[rr, "p"] & & B
\end{tikzcd}$$

(ii) $\Rightarrow$ (vi): The morphism $\kappa = \xi(i, 0)$ is obviously the sought-morphism (see the diagram below)

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[dr, "(i{,}0)"] \arrow[drr, "0"] \arrow[dddr, "i"'] & & \\
& E\times_B C \arrow[r, "\xi"] \arrow[dd, "\pi_1"] & C \arrow[d, "\gamma"] \\
& & E \arrow[d, "p"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

The equivalence (i) $\Leftrightarrow$ (v) immediately follows from Theorem 2.3.

■

In the remainder of this section and Sections 4,5 we impose on $\mathcal{C}$ the additional requirement to be a pointed category.

If, for any $\gamma \in \mathbb{M}$, the condition (vi) is equivalent to (i)-(iii), then p is called $\mathbb{M}$-*balanced*. If p is $Mono$-balanced, we say that it is *balanced*.

3.2. Example. Let $\mathbb{M}$ contain isomorphisms and all morphisms with the zero domain. Let $p : E \to B$ be a morphism such that, for any $\mathbb{M}$-subobject C of E that contains $\operatorname{Ker} p$, C coincides with either $\operatorname{Ker} p$ or E. Then it is obvious that p is $\mathbb{M}$-balanced.

3.3. Example. Any morphism in a semi-abelian variety is balanced. Recall that a variety of universal algebras is called semi-abelian if it contains a unique constant 0, and there exist an n-ary term θ and binary terms α_1, α_2, ..., α_{n-1} such that the identities

$$\theta(\alpha_1(a, c), \alpha_2(a, c), ..., \alpha_{n-1}(a, c), c) = a, \tag{3.1}$$

$$\alpha_i(a, a) = 0, \tag{3.2}$$

for any i, are satisfied [2]. Assume that C contains $\operatorname{Ker} p$, and that $(a, c) \in E \times_B C$ for some $a \in E$ and $c \in C$. Then, $p(a) = p(c)$, and from (3.2) we obtain that $p(\alpha_i(a, c)) = \alpha_i(p(a), p(c)) = 0$. Hence $\alpha_i(a, c) \in C$ for any i. The identity (3.1) implies that $a \in C$.

3.4. Example. Let $(\mathbb{E}, \mathbb{M})$ be a factorization system. Any $\mathbb{M}$-universal split monomorphism is $\mathbb{M}$-balanced. Indeed, let n be such a monomorphism, let $\gamma \in \mathbb{M}$, and let me be the $(\mathbb{E}, \mathbb{M})$-factorization of $n\gamma$. Consider the canonical morphism h in the commutative diagram

$$\begin{tikzcd}
C \arrow[dr, "h"] \arrow[drr, "e"] \arrow[ddr, "\gamma"'] & & \\
& P \arrow[r, "n'"] \arrow[d, "m'"] & A \arrow[d, "m"] \\
& E \arrow[r, "n"] & B
\end{tikzcd}$$

where the inner square is a pullback. Since n' is a split monomorphism, $h \in \mathbb{E}$, as follows from Proposition 14.9 of [1]. But $h \in \mathbb{M}$. Hence h is an isomorphism. Therefore the outer quadrangle is a pullback.

3.5. Example. Every normal epimorphism is $NormMono$-balanced. To prove this fact, consider a normal monomorphism $\gamma : C \to E$. Let $\gamma\kappa = i$ for some morphism $\kappa : \operatorname{Ker} p \to C$, and let $\pi : E \to \operatorname{Coker}\gamma$ be the cokernel of γ. Obviously, $\pi i = 0$, and since p is normal, we have a morphism $\alpha : B \to \operatorname{Coker}\gamma$ with $\alpha p = \pi$.

$$\begin{tikzcd}
& E \times_B C \arrow[r, "\pi_2"] \arrow[dd, "\pi_1"] & C \arrow[d, "\gamma"] \\
C \arrow[dr, "\gamma"] & & E \arrow[d, "p"] \\
\operatorname{Ker} p \arrow[u, "\kappa"] \arrow[r, "i"] & E \arrow[r, "p"] \arrow[dr, "\pi"] & B \arrow[d, "\alpha"] \\
& & \operatorname{Coker}\gamma
\end{tikzcd}$$

This implies that $\pi\pi_1 = \alpha p \gamma \pi_2 = \pi\gamma\pi_2 = 0$. Since γ is a normal monomorphism, it is the kernel of π. Therefore we have $\xi : E \times_B C \to C$ such that $\gamma\xi = \pi_1$. Hence p is $NormMono$-balanced.

Note that the converse statement is not valid, as follows from Example 3.3.

4. Factorization systems and certain effective descent morphisms

We say that a morphism p is $\mathbb{M}$-*reflecting* if for any pullback

$$\begin{tikzcd}
E' \arrow[r, "p'"] \arrow[d, "g'"] & B' \arrow[d, "g"] \\
E \arrow[r, "p"] & B
\end{tikzcd} \tag{4.1}$$

with $g' \in \mathbb{M}$, we have $g \in \mathbb{M}$.

4.1. Example. Any effective descent morphism p is $Mono$-reflecting. This follows from the fact that, for such p, the change-of-base functor p^* preserves pullbacks and reflects isomorphisms. Hence p^* reflects monomorphisms.

4.2. Example. Let $\mathcal{C}$ be an ideal-determined category in the sense of Janelidze, Marki, Tholen, and Ursini [6]. Recall that this means that $\mathcal{C}$ is pointed finitely complete and finitely cocomplete and the following conditions are satisfied:

(A) $\mathcal{C}$ is a regular category, where all regular epimorphisms are normal;

(B) for any commutative diagram (4.1) with normal epimorphisms p and p', and monomorphisms g and g', if g' is normal, then g is also normal.

Any semi-abelian category in the sense of [2] is ideal-determined [6].

Since the class of effective descent morphisms is stable under pullback, the conditions (A) and (B) obviously imply that one:

(C) for any commutative diagram (4.1) with an effective descent morphism p and a normal monomorphism g', the morphism g is also a normal monomorphism.

Therefore any effective descent morphism (and also any descent morphism) is $NormMono$-reflecting in any ideal-determined category.

The next lemma is obvious.

4.3. Lemma. *Any* $\mathbb{M}$*-reflecting effective descent morphism is an effective* $\mathbb{M}$*-descent morphism.*

We arrive at

4.4. Proposition. *Let* $(\mathbb{E}, \mathbb{M})$ *be a factorization system on* $\mathcal{C}$ *with* $\mathbb{M} \subset Mono$*. Then the class* $\mathbb{E}$ *is stable under pullback along* $\mathbb{M}$*-balanced* $\mathbb{M}$*-reflecting effective descent morphisms. If* $\mathbb{M} = Mono$*, then "*$\mathbb{M}$*-reflecting" can be omitted.*

Proof. Let me' be the $(\mathbb{E}, \mathbb{M})$-factorization of the pullback r of an $\mathbb{E}$-morphism e along an $\mathbb{M}$-balanced $\mathbb{M}$-reflecting effective descent morphism p. Since $pi = 0 = e0$, there exists a morphism δ such that $i = r\delta = m(e'\delta)$. Lemma 4.3 implies that p is an $\mathbb{M}$-balanced effective $\mathbb{M}$-descent morphism. From Lemma 3.1 it follows that there exists a morphism

$n \in \mathbf{M}$ such that its pullback along p is m.

$$
\begin{tikzcd}
\mathrm{Ker}\, p \arrow[d, "\delta"] \arrow[drr, "i"] & & \\
E' \arrow[rr, "r"] \arrow[dr, "e'"] \arrow[ddd, "p'"] & & E \arrow[ddd, "p"] \\
& C \arrow[ur, "m"'] \arrow[d, "p''"] & \\
& C' \arrow[dr, "n"] & \\
B' \arrow[rr, "e"] & & B
\end{tikzcd}
$$

Obviously p' is a regular epimorphism, and hence lies in $\mathbb{E}$. But $np''e' = ep'$, and therefore n also lies in $\mathbb{E}$. This implies that n is an isomorphism. Then m is an isomorphism as well, and hence $r \in \mathbb{E}$. ■

Proposition 4.4 and Example 4.1 immediately imply

4.5. Theorem. *Let every morphism of a pointed category $\mathcal{C}$ with pullbacks and coequalizers have a regular epi-mono factorization. Then the class of regular epimorphisms is stable under pullback along balanced effective descent morphisms.*

4.6. Remark. In general, the class of regular epimorphisms is not stable under pullback along effective descent morphisms, even if there are regular epi-mono factorizations in a given category, as is pointed out by Reiterman, Sobral and Tholen in [12]. The counterexample constructed in that paper deals with the category of topological spaces. It implies that "balanced" can not be omitted in Theorem 4.5. Indeed, consider the category Top_* of pointed topological spaces, and the forgetful functor $F : Top_* \to Top$. It obviously reflects isomorphisms, and, moreover, preserves and reflects both pullbacks and coequalizers. According to a result of the paper [10] by Mesablishvili, F reflects effective descent morphisms. Moreover, it reflects regular epimorphisms too. This easily implies that regular epimorphisms are not stable under pullback along effective descent morphisms in the category Top_* either.

From Theorem 4.5, Example 3.5, and the fact that a category $\mathcal{C}$ has coequalizers, we obtain

4.7. Corollary. *Under the conditions of Theorem 4.5, let any monomorphism be normal, and any effective descent morphism be a normal epimorphism in $\mathcal{C}$. Then the class of regular epimorphisms is stable under pullback along effective descent morphisms. Moreover, if p is an effective descent morphism, then the change-of-base functor p^* preserves regular epimorphisms.*

Proposition 4.4 and Example 3.5 imply

4.8. Corollary. *Let $\mathcal{C}$ satisfy the condition (C), and let the pair $(Epi, NormMono)$ be a factorization system on $\mathcal{C}$. Then the class of epimorphisms is stable under pullback along normal effective descent morphisms.*

4.9. Corollary. *Let $\mathcal{C}$ be an ideal-determined category, and let $(Epi, NormMono)$ be a factorization system on $\mathcal{C}$. Then the class of epimorphisms is stable under pullback along effective descent morphisms. If, in addition, $\mathcal{C}$ is Barr exact (a semi-abelian variety, for instance), then "effective descent morphisms" can be replaced by "normal epimorphisms".*

5. The case of an additive category

Below we will deal with several pullbacks/products simultaneously. For simplicity, let us agree to use one and the same notation π_1, π_2,..., π_n for projections in all pullbacks/products.

Let $\mathbb{M} \subset Mono$, and let $p : E \to B$ be a morphism in a pointed category $\mathcal{C}$. Let there exist a natural number n such that, for any $\mathbb{M}$-subobject (C, γ) of E and any i $(1 \le i \le n-1)$, there are morphisms $\theta_\gamma : C \times C \times ... \times C \to C$ and $\alpha_{(i,\gamma)} : C \times C \to C$, for which the following conditions are satisfied:

(i) the morphism $\theta_\gamma(\alpha_{(1,\gamma)}, \alpha_{(2,\gamma)}, ..., \alpha_{(n-1,\gamma)}, \pi_2) : C \times C \to C$ coincides with the first projection π_1;

(ii) the squares

$$\begin{tikzcd} C \times C \times ... \times C \arrow[r, "\gamma\times\gamma\times...\times\gamma"] \arrow[d, "\theta_\gamma"] & E \times E \times ... \times E \arrow[d, "\theta_{1_E}"] \\ C \arrow[r, "\gamma"] & E \end{tikzcd}$$

and

$$\begin{tikzcd} C \times C \arrow[r, "\gamma\times\gamma"] \arrow[d, "\alpha_{(i,\gamma)}"] & E \times E \arrow[d, "\alpha_{(i,1_E)}"] \\ C \arrow[r, "\gamma"] & E \end{tikzcd}$$

are commutative for all i;

(iii) the morphism $p(\alpha_{(i,1_E)})(\pi_1, \pi_2) : E \times_B E \to B$ coincides with the zero morphism for all i.

Observe that the condition (iii) is satisfied if there are morphisms $\alpha_{(1,B)}$, $\alpha_{(2,B)}$, ..., $\alpha_{(n-1,B)} : B \times B \to B$ such that p preserves all α_i, and, moreover, $\alpha_{(i,B)}(1_B, 1_B) = 0$ for any i. This follows from the commutativity of the following diagram

$$\begin{tikzcd}
E \times_B E \arrow[r] \arrow[d, "{(\pi_1,\pi_2)}"'] & B \arrow[d, "{(1_B,1_B)}"] \\
E \times E \arrow[r, "p\times p"] \arrow[d, "\alpha_{(i,1_E)}"'] & B \times B \arrow[d, "\alpha_{(i,B)}"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

5.1. Lemma. *The morphism p is $\mathbb{M}$-balanced.*

Proof. Let $\gamma : C \rightarrowtail E$ lie in $\mathbb{M}$. Assume that there exists $\kappa : \operatorname{Ker} p \to C$ with $\kappa\gamma = i$. Consider the diagram

$$\begin{tikzcd}
E \times_B C \arrow[rr, "1_E \times_B \gamma"] \arrow[d, "{(\kappa h_1,\kappa h_2,...,\kappa h_{n-1},1_C)}"'] \arrow[dddrr, "h_i"'] & & E \times_B E \arrow[rr] & & E \times E \arrow[d, "{(\alpha_{(1,1_E)},\alpha_{(2,1_E)},...,\alpha_{(n-1,1_E)},\pi_2)}"'] \\
C \times_B C \times_B ... \times_B C \arrow[rr] & & C \times C \times ... \times C \arrow[rr, "\gamma\times\gamma\times...\times\gamma"] \arrow[d, "\theta_\gamma"] & & E \times E \times ... \times E \arrow[d, "\theta_{1_E}"] \\
& & C \arrow[rr, "\gamma"] & & E \arrow[d, "p"] \\
& & \operatorname{Ker} p \arrow[u, "\kappa"] \arrow[urr, "i"] & & B
\end{tikzcd}$$

where h_i is the canonical morphism induced by the one $\alpha_{(i,1_E)}(\pi_1,\pi_2)(1_E \times_B \gamma)$ for any i. It is easy to verify that the upper rectangle is commutative. This, together with the conditions (i) and (ii), implies that for the morphism

$$\xi = \theta_\gamma(\pi_1,\pi_2,...,\pi_n)(\kappa h_1,\kappa h_2,...,\kappa h_{n-1},1_C)$$

we have

$$\begin{aligned}
\gamma\xi &= \gamma\theta_\gamma(\pi_1,\pi_2,...,\pi_n)(\kappa h_1,\kappa h_2,...,\kappa h_{n-1},1_C) \\
&= \theta_{1_E}(\gamma\times\gamma\times...\times\gamma)(\pi_1,\pi_2,...,\pi_n)(\kappa h_1,\kappa h_2,...,\kappa h_{n-1},1_C) \\
&= \theta_{1_E}(\alpha_{(1,1_E)},\alpha_{(2,1_E)},...,\alpha_{(n-1,1_E)},\pi_2)(\pi_1,\pi_2)(1_E,\gamma) = \pi_1.
\end{aligned}$$

■

Let $\mathcal{C}$ be an additive category. Then, as is well-known, any object C of $\mathcal{C}$ can be equipped with the structure of an internal abelian group (the codiagonal $\nabla_C : C \coprod C \approx C \times C \longrightarrow C$ gives the sought-for internal binary operation $+_C$ on C). At that, any morphism becomes a homomorphism of such groups. Hence, for $n = 2$, $\theta_\gamma = +_C$, $\alpha_{(1,\gamma)} = -_C$, the conditions (i)-(iii) are satisfied. Therefore, from Lemma 5.1 we obtain

5.2. Lemma. *Any morphism is balanced in an additive category.*

5.3. Remark. The conditions (i)-(iii) are, in fact, satisfied if the objects E, B and all subobjects of E are equipped with the structure of an internal $\mathcal{V}$-algebra (for the relevant definitions we refer the reader to [4]), while the morphism p and the corresponding embeddings are homomorphisms with respect to these structures, for any finitely complete category $\mathcal{C}$ and any semi-abelian variety $\mathcal{V}$ of universal algebras. However, apart from the case considered above, no other facts are known, as far as we know, concerning a possibility of introducing such structures on any object of $\mathcal{C}$.

From Theorem 4.5 and Lemma 5.2 we obtain

5.4. Theorem. *The class of regular epimorphisms is stable under pullback along effective descent morphisms in any additive category with regular epi-mono factorizations.*

5.5. Remark. The question naturally arises whether there exists a non-regular additive category with regular epi-mono factorizations. According to the paper [13] by Rump, the answer to this question is positive.

6. Are (effective) descent morphisms stable under pushout?

From the trivial fact that any regular epimorphism is the coequalizer of its kernel pair, we immediately obtain

6.1. Lemma. *Every regular epimorphism is a pushout of a split epimorphism along a split epimorphism.*

Let *EffDes* (*Des* resp.) denote the class of effective descent morphisms (descent morphisms resp.).

6.2. Theorem. *The following conditions are equivalent:*

(i) every regular epimorphism is an effective descent morphism;

(ii) the pair of morphism classes $(\mathit{EffDes}, Mono)$ is a factorization system;

(iii) the pair of morphism classes $(\mathit{EffDes}, (\mathit{EffDes})^{\downarrow})$ is a factorization system;

(iv) the class of effective descent morphisms is stable under pushout.

Proof. For (i) $\Rightarrow$ (ii) it is sufficient to observe that Theorem 2.1 implies that the category is regular, and hence it has regular epi-mono factorizations. The implication (iv) $\Rightarrow$ (i) follows from Lemma 6.1 and the fact that every split epimorphism is an effective descent morphism [7], [9]. All other implications are obvious. ∎

Incidentally, Lemma 6.1 immediately implies

6.3. Proposition. *The following conditions are equivalent:*

(i) every split epimorphism is an isomorphism;

(ii) every effective descent morphism is an isomorphism;

(iii) every descent morphism is an isomorphism;

(iv) every regular epimorphism is an isomorphism;

(v) every split monomorphism is an isomorphism.

If $\mathcal{C}$ has pushouts and equalizers, then these conditions are equivalent also to the following ones:

(vi) every effective codescent morphism is an isomorphism;

(vii) every codescent morphism is an isomorphism;

(viii) every regular monomorphism is an isomorphism.

6.4. Example. As is known, every morphism in the category of fields is a monomorphism. From Proposition 6.3 we conclude that there are no non-isomorphic effective codescent morphisms in any subcategory with pushouts and equalizers of the category of fields.

Similarly, all effective descent morphisms of a $\wedge$-semilattice (considered as a category) are isomorphisms.

Let us now recall some notions from the paper [3] by Carboni, Janelidze, Kelly, and Paré. Let $(\mathbb{E}, \mathbb{M})$ be a factorization system on $\mathcal{C}$. Let $\mathbb{E}'$ be the class of universal $\mathbb{E}$-morphisms, i.e., morphisms e from $\mathbb{E}$ whose any pullback lies in $\mathbb{E}$. The class $\mathbb{E}'$ is called the stabilization of $\mathbb{E}$. Let $\mathbb{M}^*$ be the class of morphisms m which lie in $\mathbb{M}$ locally, i.e., are such that there exist effective descent morphisms along which the pullbacks of m lie in $\mathbb{M}$. The class $\mathbb{M}^*$ is called the localization of $\mathbb{M}$. If any effective descent morphism is $\mathbb{M}$-reflecting, then obviously $\mathbb{M}^* = \mathbb{M}$. In particular, from Example 4.1 we obtain that $Mono^* = Mono$.

6.5. Remark. As proved in [3], if $\mathcal{C}$ is finitely complete, one has

$$\mathbb{E}' \subset (\mathbb{M}^*)^{\uparrow}. \tag{6.1}$$

However, it is by no means true that $\mathbb{E}' = (\mathbb{M}^*)^{\uparrow}$, even in the case where $((\mathbb{M}^*)^{\uparrow}, \mathbb{M}^*)$ is a factorization system, as is shown in [3].

6.6. Remark. The equality $\mathbb{E}' = \mathbb{E}$ implies the one $\mathbb{M}^* = \mathbb{M}$ (while the converse is obviously not valid). Indeed, since $\mathbb{M} \subset \mathbb{M}^*$, we have $(\mathbb{M}^*)^{\uparrow} \subset (\mathbb{M})^{\uparrow} = \mathbb{E}$. From (6.1) we obtain $(\mathbb{M}^*)^{\uparrow} = \mathbb{E}$. Hence $\mathbb{M}^* \subset (\mathbb{M}^*)^{\uparrow\downarrow} = \mathbb{E}^{\downarrow} = \mathbb{M}$. Thus $\mathbb{M}^* = \mathbb{M}$.

6.7. Theorem. [3] *Let $\mathcal{C}$ be finitely complete. The pair $(\mathbb{E}', \mathbf{M}^*)$ is a factorization system if and only if for any morphism f there is an effective descent morphism p such that the $\mathbb{E}$-morphism in the $(\mathbb{E}, \mathbb{M})$-factorization of the pullback of f along p lies in $\mathbb{E}'$.*

Note that if $\mathbb{E}$ is the class of regular epimorphisms, then $\mathbb{E}'$ is precisely the class of descent morphisms. We obtain

6.8. Theorem. *The following conditions are equivalent:*

(i) $\mathcal{C}$ is a regular category;

(ii) the pair $(Des, Mono)$ is a factorization system on $\mathcal{C}$;

(iii) the pair $(Des, (Des)^{\downarrow})$ is a factorization system on $\mathcal{C}$;

(iv) the class of descent morphisms is stable under pushout.

If $\mathcal{C}$ is finitely complete, then these conditions are equivalent also to the following condition:

(v) every morphism has a regular epi-mono factorization, and moreover, for any morphism f there is an effective descent morphism p such that the regular epimorphism in the regular epi-mono factorization of the pullback of f along p is a descent morphism.

Proof. The implication (iv) $\Rightarrow$ (i) follows from Lemma 6.1, while the implication (v) $\Rightarrow$ (iii) follows from Theorem 6.7. All other implications are obvious. ∎

References

[1] J. Adámek, H. Herrlich, G. E. Strecker, Abstract and Concrete Categories, *John Wiley and Sons*, New York, 1990,

[2] D. Bourn, G. Janelidze, Characterization of protomodular varieties of universal algebra, *Theory Appl. Categories*, **11**, No. 6, (2003), 143–147,

[3] A. Carboni, G. Janelidze, G. M. Kelly, R. Paré, On localization and stabilization of factorization systems, *Appl. Categ. Structures*, **5**, No. 1, (1997), 1-58.

[4] M. M. Ebrahimi, Algebra in a Grothendieck topos: injectivity in quasi-equational classes, *J. Pure Appl. Algebra*, **26**, No. 3, (1982), 269-280.

[5] P. J. Freyd, G. M. Kelly, Categories of continious functors, I, *J. Pure Appl. Algebra*, **2** (1972), 169-191.

[6] G. Janelidze, L. Marki, W. Tholen, A. Ursini, Ideal determined categories, *Cahiers Topol. Géom. Différ. Catég.* **LI**, No. 2, (2010), 115-125.

[7] G. Janelidze, W. Tholen, How algebraic is the change-of-base functor? in: *Lecture Notes in Math.*, **1488**, Springer, Berlin, 1991, 174-186

[8] G. Janelidze, W. Tholen, Facets of descent, I. *Appl. Categ. Structures*, **2**(1994), 1-37.

[9] G. Janelidze, W. Tholen, Facets of descent, II. *Appl. Categ. Structures*, **5**, No. 3 (1997), 229-248.

[10] B. Mesablishvili, Descent in categories of (co)algebras, *Homology, Homotopy and Applications*, **7**, No.1, (2005), 1-8.

[11] M. Pfender, Kongruenzen, Konstruktion von Limiten und Cokernen und Algebraishe Kategorien, *Ph.D. Thesis (Technical University of Berlin)* (1971).

[12] J. Reiterman, M. Sobral, W. Tholen, Composites of effective descent maps, *Cahiers Topol. Géom. Différ. Catég.* **XXXIV**, No. 3,(1993), 193-207

[13] W. Rump, A counterexample to Raikov's conjecture, *Bull. London Math. Soc.* **40**(2008), 985-994.

[14] M. Sobral, Some aspects of topological descent, *Appl. Categ. Structures*, **4**(1996), 97-106

[15] M. Sobral, W. Tholen, Effective descent morphisms and effective equivalence relations, *Canadian Math. Soc. Conf. Proc.*, **13**(1992), 421-433

Andrea Razmadze Mathematical Institute, Tbilisi
Centre for Mathematical Sciences, 6 Tamarashvili Str., Tbilisi 0177, Georgia
Email: `dalizan@rmi.ge`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/29/1/29-01.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at http://www.tac.mta.ca/tac/ and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at http://www.tac.mta.ca/tac/.