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ON THE INFINITY CATEGORY OF HOMOTOPY LEIBNIZ ALGEBRAS

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Abstract. We discuss various concepts of ∞-homotopies, as well as the relations between them (focussing on the Leibniz type). In particular ∞-n-homotopies appear as the n-simplices of the nerve of a complete Lie ∞-algebra. In the nilpotent case, this nerve is known to be a Kan complex [Get09]. We argue that there is a quasi-category of ∞-algebras and show that for truncated ∞-algebras, i.e. categorified algebras, this ∞-categorical structure projects to a strict 2-categorical one. The paper contains a shortcut to $(\infty,1)$-categories, as well as a review of Getzler's proof of the Kan property. We make the latter concrete by applying it to the 2-term ∞-algebra case, thus recovering the concept of homotopy of [BC04], as well as the corresponding composition rule [SS07]. We also answer a question of [Sho08] about composition of ∞-homotopies of ∞-algebras.

1. Introduction

1.1. General background. Homotopy, sh, or infinity algebras [Sta63] are homotopy invariant extensions of differential graded algebras. They are of importance, e.g. in BRST of closed string field theory, in Deformation Quantization of Poisson manifolds ... Another technique to increase the flexibility of algebraic structures is categorification [CF94], [Cra95] – a sharpened viewpoint that leads to astonishing results in TFT, bosonic string theory ... Both methods, homotopification and categorificiation are tightly related: the 2-categories of 2-term Lie (resp., Leibniz) homotopy algebras and of Lie (resp., Leibniz) 2-algebras turned out to be equivalent [BC04], [SL10] (for a comparison of 3-term Lie infinity algebras and Lie 3-algebras, as well as for the categorical definition of the latter, see [KMP11]). However, homotopies of ∞-morphisms and their compositions are far from being fully understood. In [BC04], ∞-homotopies are obtained from categorical homotopies, which are God-given. In [SS07], (higher) ∞-homotopies are (higher) derivation homotopies, a variant of infinitesimal concordances, which seems to be the wrong concept [DP12]. In [Sho08], the author states that ∞-homotopies of sh Lie algebra morphisms can

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be composed, but no proof is given and the result is actually not true in whole generality. The objective of this work is to clarify the concept of (higher) ∞-homotopies, as well as the problem of their compositions.

1.2. Structure and main results. In Section 2, we provide explicit formulae for homotopy Leibniz algebras and their morphisms. Indeed, although a category of homotopy algebras is simplest described as a category of quasi-free DG coalgebras, its original nature is its manifestation in terms of brackets and component maps.

We report, in Section 3, on the notions of homotopy that are relevant for our purposes: concordances, i.e. homotopies for morphisms between quasi-free DG (co)algebras, gauge and Quillen homotopies for Maurer-Cartan (MC for short) elements of pronilpotent Lie infinity algebras, and ∞-homotopies, i.e. gauge or Quillen homotopies for ∞-morphisms viewed as MC elements of a complete convolution Lie infinity algebra.

Section 4 starts with the observation that vertical composition of ∞-homotopies of DG algebras is well-defined. However, this composition is not associative and cannot be extended to the ∞-algebra case – which suggests that ∞-algebras actually form an ∞-category. To allow independent reading of the present paper, we provide a short introduction to ∞-categories, see Subsection 4.2. In Subsection 4.14.1, the concept of ∞-n-homotopy is made precise and the class of ∞-algebras is viewed as an ∞-category. Since we apply the proof of the Kan property of the nerve of a nilpotent Lie infinity algebra to the 2-term Leibniz infinity case, a good understanding of this proof is indispensable: we detail the latter in Subsection 4.16.1.

To be complete, we give an explicit description of the category of 2-term homotopy Leibniz algebras at the beginning of Section 5. We show that composition of ∞-homotopies in the nerve-∞-groupoid, which is defined and associative only up to higher ∞-homotopy, projects to a well-defined and associative vertical composition in the 2-term case – thus obtaining the Leibniz counterpart of the strict 2-category of 2-term Lie infinity algebras [BC04], see Subsection 5.5, Theorem 5.7 and Theorem 5.9.

Eventually, we provide, in Section 6, the definitions of the strict 2-category of Leibniz 2-algebras, which is 2-equivalent to the preceding 2-category.

An ∞-category structure on the class of ∞-algebras over a quadratic Koszul operad is being investigated independently of [Get09] in a separate paper.

2. Category of homotopy Leibniz algebras

Let P be a quadratic Koszul operad. A P_∞-structure on a graded vector space V over a field $\mathbb{K}$ of characteristic zero is essentially a sequence ℓ_n of n-ary brackets on V that satisfy a sequence R_n of defining relations, $n \in \{1, 2, \ldots\}$. Surprisingly, these structures are 1:1 [GK94] with codifferentials

$$D \in \mathrm{CoDer}^1(\mathcal{F}^{\mathrm{gr,c}}_{P^{\text{¡}}}(s^{-1}V)) \quad (|\ell_n| = 2-n) \quad \text{or} \quad D \in \mathrm{CoDer}^{-1}(\mathcal{F}^{\mathrm{gr,c}}_{P^{\text{¡}}}(sV)) \quad (|\ell_n| = n-2)\ , \tag{1}$$

or, also, (if V is finite-dimensional) 1:1 with differentials

$$d \in \mathrm{Der}^1(\mathcal{F}^{\mathrm{gr}}_{P^!}(sV^*)) \quad (|\ell_n| = 2-n) \quad \text{or} \quad d \in \mathrm{Der}^{-1}(\mathcal{F}^{\mathrm{gr}}_{P^!}(s^{-1}V^*)) \quad (|\ell_n| = n-2)\ . \quad (2)$$

Here $\mathrm{Der}^1(\mathcal{F}^{\mathrm{gr}}_{P^!}(sV^*))$ (resp., $\mathrm{CoDer}^1(\mathcal{F}^{\mathrm{gr,c}}_{P^{¡}}(s^{-1}V))$), for instance, denotes the space of endomorphisms of the free graded algebra over the Koszul dual operad $P^!$ of P on the suspended linear dual sV^* of V, which have degree 1 (with respect to the grading of the free algebra that is induced by the grading of V) and are derivations for each binary operation in $P^!$ (resp., the space of endomorphisms of the free graded coalgebra over the Koszul dual cooperad $P^{¡}$ on the desuspended space $s^{-1}V$ that are coderivations) (by differential and codifferential we mean of course a derivation or coderivation that squares to 0).

Although the original nature of homotopified or oidified algebraic objects is their manifestation in terms of brackets [BP12], the preceding coalgebraic and algebraic settings are the most convenient contexts to think about such higher structures.

2.1. Zinbiel (co)algebras. Since we take an interest mainly in the case where P is the operad Lei (resp., the operad Lie) of Leibniz (resp., Lie) algebras, the Koszul dual $P^!$ to consider is the operad Zin (resp., Com) of Zinbiel (resp., commutative) algebras. We now recall the relevant definitions and results.

2.2. Definition. *A* graded Zinbiel algebra (GZA) (*resp.,* graded Zinbiel coalgebra (GZC)) is a $\mathbb{Z}$-graded vector space V endowed with a multiplication, i.e. a degree 0 linear map $m : V \otimes V \to V$ (resp., a comultiplication, i.e. a degree 0 linear map $\Delta : V \to V \otimes V$) that satisfies the relation

$$m(\mathrm{id} \otimes m) = m(m \otimes \mathrm{id}) + m(m \otimes \mathrm{id})(\tau \otimes \mathrm{id}) \text{ (resp. } (\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta + (\tau \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta), \quad (3)$$

where $\tau : V \otimes V \ni u \otimes v \mapsto (-1)^{|u||v|} v \otimes u \in V \otimes V$.

When evaluated on homogeneous vectors $u, v, w \in V$, the Zinbiel relation for the multiplication $m(u, v) =: u \cdot v$ reads,

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w + (-1)^{|u||v|} (v \cdot u) \cdot w\ .$$

2.3. Example. *The multiplication $\cdot$ on the reduced tensor module $\overline{T}(V) := \oplus_{n \geq 1} V^{\otimes n}$ over a $\mathbb{Z}$-graded vector space V, defined, for homogeneous $v_i \in V$, by*

$$\begin{aligned}(v_1 ... v_p) \cdot (v_{p+1} ... v_{p+q}) &= \sum_{\sigma \in \mathrm{Sh}(p,q-1)} (\sigma^{-1} \otimes \mathrm{id})(v_1 ... v_{p+q}) = \\ &= \sum_{\sigma \in \mathrm{Sh}(p,q-1)} \varepsilon(\sigma^{-1}) v_{\sigma^{-1}(1)} v_{\sigma^{-1}(2)} ... v_{\sigma^{-1}(p+q-1)} v_{p+q}\ ,\end{aligned} \quad (4)$$

endows $\overline{T}(V)$ with a GZA *structure. In the above equation we wrote tensor products of vectors by simple juxtaposition, and $\mathrm{Sh}(p, q-1)$ is the set of $(p, q-1)$-shuffles and finally $\varepsilon(\sigma^{-1})$ is the Koszul sign,* *Similarly, the comultiplication Δ on $\overline{T}(V)$, defined, for homogeneous $v_i \in V$, by*

$$\Delta(v_1 ... v_p) = \sum_{k=1}^{p-1} \sum_{\sigma \in \mathrm{Sh}(k,p-k-1)} \varepsilon(\sigma) \left(v_{\sigma(1)} ... v_{\sigma(k)}\right) \bigotimes \left(v_{\sigma(k+1)} ... v_{\sigma(p-k-1)} v_p\right) \ , \tag{5}$$

is a GZC *structure on $\overline{T}(V)$.*

As for the GZA multiplication on $\overline{T}(V)$, we have in particular

$$v_1 \cdot v_2 = v_1 v_2 \ ; \quad (v_1 v_2) \cdot v_3 = v_1 v_2 v_3 \ ;$$

$$v_1 \cdot (v_2 v_3) = v_1 v_2 v_3 + (-1)^{|v_1||v_2|} v_2 v_1 v_3; \quad (((v_1 \cdot v_2) \cdot v_3) ...) \cdot v_k = v_1 v_2 ... v_k \ .$$

2.4. Proposition. *The* GZA $(\overline{T}(V), \cdot)$ (*resp., the* GZC $(\overline{T}(V), \Delta)$) *defined in Example 2.3 is the* free GZA (*resp.,* free GZC) *over V. We will denote it by* $\mathrm{Zin}(V)$ (*resp.,* $\mathrm{Zin}^{\mathrm{c}}(V)$).

2.5. Definition. *A* differential graded Zinbiel algebra (DGZA) (*resp., a* differential graded Zinbiel coalgebra) (DGZC) *is a* GZA (V, m) (*resp.,* GZC (V, Δ)) *together with a degree -1 derivation d (resp., coderivation D) that squares to 0. More precisely, d (resp., D) is a degree -1 linear map $d : V \to V$ (resp., $D : V \to V$), such that*

$$d\, m = m\, (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \quad (\textit{resp.,} \quad \Delta\, D = (D \otimes \mathrm{id} + \mathrm{id} \otimes D)\, \Delta)$$

and $d^2 = 0$ (resp., $D^2 = 0$).

Since the GZA $\mathrm{Zin}(V)$ (resp., GZC $\mathrm{Zin}^{\mathrm{c}}(V)$) is free, any degree 1 linear map $d : V \to \mathrm{Zin}(V)$ (resp., $D : \mathrm{Zin}^{\mathrm{c}}(V) \to V$) uniquely extends to a derivation $d : \mathrm{Zin}(V) \to \mathrm{Zin}(V)$ (resp., coderivation $D : \mathrm{Zin}^{\mathrm{c}}(V) \to \mathrm{Zin}^{\mathrm{c}}(V)$).

2.6. Definition. *A* quasi-free *DGZA (resp., a* quasi-free *DGZC) over V is a DGZA (resp., DGZC) of the type* $(\mathrm{Zin}(V), d)$ (*resp.,* $(\mathrm{Zin}^{\mathrm{c}}(V), D)$).

2.7. Homotopy Leibniz algebras. We recall the definition of homological homotopy Leibniz algebras.

2.8. Definition. *A (homological)* homotopy Leibniz algebra *is a graded vector space V together with a sequence of linear maps $l_i : V^{\otimes i} \to V$ of degree $i-2$, $i \geq 1$, such that for any $n \geq 1$, the following* higher Jacobi identity *holds:*

$$\begin{gathered}\sum_{i+j=n+1} \sum_{j \leqslant k \leqslant n} \sum_{\sigma \in \mathrm{Sh}(k-j,j-1)} (-1)^{(n-k+1)(j-1)} (-1)^{j(v_{\sigma(1)}+ ... + v_{\sigma(k-j)})} \varepsilon(\sigma) \operatorname{sign}(\sigma) \\ l_i(v_{\sigma(1)}, ..., v_{\sigma(k-j)}, l_j(v_{\sigma(k-j+1)}, ..., v_{\sigma(k-1)}, v_k), v_{k+1}, ..., v_n) = 0 \ ,\end{gathered} \tag{6}$$

where $\operatorname{sign}\sigma$ *is the signature of σ and where we denoted the degree of the homogeneous $v_i \in V$ by v_i instead of $|v_i|$.*

2.9. Theorem. *There is a 1:1 correspondence between homotopy Leibniz algebras, in the sense of Definition 2.8, over a graded vector space V and quasi-free* DGZC-*s* $(\mathrm{Zin}^{\mathrm{c}}(sV), D)$ (*resp., in the case of a finite-dimensional graded vector space V, quasi-free* DGZA-*s* $(\mathrm{Zin}(s^{-1}V^*), d)$).

In the abovementioned 1:1 correspondence between homotopy algebras over a quadratic Koszul operad P and quasi-free DG$P^{¡}$C (resp., quasi-free DG$P^{!}$A) (self-explaining notation), a P_∞-algebra structure on a graded vector space V is viewed as a representation on V of the DG operad P_∞ – which is defined as the cobar construction $\Omega P^{¡}$ of the Koszul dual cooperad $P^{¡}$. Theorem 2.9 makes this correspondence concrete in the case $P = \mathsf{Lei}$; a proof can be found in [AP10].

2.10. Homotopy Leibniz morphisms.

2.11. Definition. *A* morphism between homotopy Leibniz algebras (V, l_i) *and* (W, m_i) *is a sequence of linear maps $\varphi_i : V^{\otimes i} \to W$ of degree $i-1$, $i \geq 1$, which satisfy, for any $n \geq 1$, the condition*

$$\sum_{i=1}^{n} \sum_{k_1+\ldots+k_i=n} \sum_{\sigma\in \mathfrak{Sh}(k_1,\ldots,k_i)} (-1)^{\sum_{r=1}^{i-1}(i-r)k_r + \frac{i(i-1)}{2}} (-1)^{\sum_{r=2}^{i}(k_r-1)(v_{\sigma(1)}+\ldots+v_{\sigma(k_1+\ldots+k_{r-1})})} \varepsilon(\sigma)\,\mathrm{sign}(\sigma)$$
$$m_i\left(\varphi_{k_1}(v_{\sigma(1)},\ldots,v_{\sigma(k_1)}), \varphi_{k_2}(v_{\sigma(k_1+1)},\ldots,v_{\sigma(k_1+k_2)}),\ldots,\varphi_{k_i}(v_{\sigma(k_1+\ldots+k_{i-1}+1)},\ldots,v_{\sigma(k_1+\ldots+k_i)})\right)$$
$$=$$
$$\sum_{i+j=n+1} \sum_{j\leqslant k\leqslant n} \sum_{\sigma\in \mathrm{Sh}(k-j,j-1)} (-1)^{k+(n-k+1)j} (-1)^{j(v_{\sigma(1)}+\ldots+v_{\sigma(k-j)})} \varepsilon(\sigma)\,\mathrm{sign}(\sigma)$$
$$\varphi_i(v_{\sigma(1)},\ldots,v_{\sigma(k-j)}, l_j(v_{\sigma(k-j+1)},\ldots,v_{\sigma(k-1)},v_k), v_{k+1},\ldots,v_n)\ , \tag{7}$$

where $\mathfrak{Sh}(k_1,\ldots,k_i)$ denotes the set of shuffles $\sigma \in \mathrm{Sh}(k_1,\ldots,k_i)$, such that $\sigma(k_1) < \sigma(k_1+k_2) < \ldots < \sigma(k_1+k_2+\ldots+k_i)$.

2.12. Theorem. *There is a 1:1 correspondence between homotopy Leibniz algebra morphisms from (V, l_i) to (W, m_i) and* DGC *morphisms* $\mathrm{Zin}^{\mathrm{c}}(sV) \to \mathrm{Zin}^{\mathrm{c}}(sW)$ (*resp., in the finite-dimensional case,* DGA *morphisms* $\mathrm{Zin}(s^{-1}W^*) \to \mathrm{Zin}(s^{-1}V^*)$), *where the quasi-free* DGZC-*s* (*resp., the quasi-free* DGZA-*s*) *are endowed with the codifferentials* (*resp., differentials*) *that encode the structure maps l_i and m_i.*

In literature, infinity morphisms of P_∞-algebras are usually defined as morphisms of quasi-free DG$P^{¡}$C-s. However, no explicit formulae seem to exist for the Leibniz case. A proof of Theorem 2.12 can be found in the first author's thesis [Khu13]. Let us also stress that the concept of infinity morphism of P_∞-algebras does not coincide with the notion of morphism of algebras over the operad P_∞.

2.13. Composition of homotopy Leibniz morphisms. Composition of infinity morphisms between P_∞-algebras corresponds to composition of the corresponding morphisms between quasi-free DG$P^{¡}$C-s: the categories P_∞-`Alg` and `qfDG`$P^{¡}$`CoAlg` (self-explaining notation) are isomorphic. Explicit formulae can easily be computed.

3. Leibniz infinity homotopies

3.1. Concordances and their compositions. Let us first look for a proper concept of homotopy in the category `qfDG`$P^{¡}$`CoAlg`, or, dually, in `qfDG`$P^{!}$`Alg`.

3.1.1. Definition and characterization. The following concept of homotopy – referred to as concordance – first appeared in an unpublished work by Stasheff and Schlessinger, which was based on ideas of Bousfield and Gugenheim. It can also be found in [SSS07], for homotopy algebras over the operad `Lie` (algebraic version), as well as in [DP12], for homotopy algebras over an arbitrary operad P (coalgebraic version).

It is well-known that a C^∞-homotopy $\eta : I\times X \to Y$, $I = [0,1]$, connecting two smooth maps p, q between two smooth manifolds X, Y, induces a cochain homotopy between the pullbacks p^*, q^*. Indeed, in the algebraic category,

$$\eta^* : \Omega(Y) \to \Omega(I) \otimes \Omega(X) \ ,$$

and $\eta^*(\omega)$, $\omega \in \Omega(Y)$, reads

$$\eta^*(\omega)(t) = \varphi(\omega)(t) + dt\,\rho(\omega)(t) \ . \tag{8}$$

It is easily checked (see below for a similar computation) that, since η^* is a cochain map, we have

$$d_t\varphi = d_X\rho(t) + \rho(t)d_Y \ ,$$

where d_X, d_Y are the de Rham differentials. When integrating over I, we thus obtain

$$q^* - p^* = d_X h + h d_Y \ ,$$

where $h = \int_I \rho(t)dt$ has degree $+1$ (recall that we use the homological grading).

Before developing a similar approach to homotopies between morphisms of quasi-free DGZA-s, let us recall that tensoring an 'algebra' (resp., 'coalgebra') with a DGCA (resp., DGCC) does not change the considered type of algebra (resp., coalgebra); let us also introduce the 'evaluation' maps

$$\varepsilon_1^i : \Omega(I) = C^\infty(I) \oplus dt\, C^\infty(I) \ni f(t) + dt\, g(t) \mapsto f(i) \in \mathbb{K}, \quad i \in \{0,1\} \ .$$

In the following – in contrast with our above notation – we omit stars. Moreover – although the 'algebraic' counterpart of a homotopy Leibniz algebra over V is $(\mathrm{Zin}(s^{-1}V^*), d_V)$ – we consider Zinbiel algebras of the type $(\mathrm{Zin}(V), d_V)$.

3.2. Definition. *If* $p, q : \mathrm{Zin}(W) \to \mathrm{Zin}(V)$ *are two* DGA *morphisms, a* homotopy *or* concordance $\eta : p \Rightarrow q$ *from* p *to* q *is a* DGA *morphism* $\eta : \mathrm{Zin}(W) \to \Omega(I) \otimes \mathrm{Zin}(V)$, *such that*

$$\varepsilon_1^0 \eta = p \quad \textit{and} \quad \varepsilon_1^1 \eta = q \; .$$

The following proposition is basic.

3.3. Proposition. *Concordances*

$$\eta : \mathrm{Zin}(W) \to \Omega(I) \otimes \mathrm{Zin}(V)$$

between DGA *morphisms* p, q *can be identified with 1-parameter families*

$$\varphi : I \to \mathrm{Hom}_{\mathrm{DGA}}(\mathrm{Zin}(W), \mathrm{Zin}(V))$$

and

$$\rho : I \to \varphi\mathrm{Der}(\mathrm{Zin}(W), \mathrm{Zin}(V))$$

of (degree 0) DGA *morphisms and of degree* 1 φ*-Leibniz morphisms, respectively, such that*

$$d_t \varphi = [d, \rho(t)] \tag{9}$$

and $\varphi(0) = p$, $\varphi(1) = q$. *The* RHS *of the differential equation (9) is defined by*

$$[d, \rho(t)] := d_V \rho(t) + \rho(t) d_W \; ,$$

where d_V, d_W *are the differentials of the quasi-free* DGZA*-s* $\mathrm{Zin}(V), \mathrm{Zin}(W)$.

The notion of φ-derivation or φ-Leibniz morphism appeared for instance in [BKS04]: for $w, w' \in \mathrm{Zin}(W)$, w homogeneous,

$$\rho(w \cdot w') = \rho(w) \cdot \varphi(w') + (-1)^w \varphi(w) \cdot \rho(w') \; ,$$

where we omitted the dependence of ρ on t.

Proof. As already mentioned in Equation (8), $\eta(w)$, $w \in \mathrm{Zin}(W)$, reads

$$\eta(w)(t) = \varphi(w)(t) + dt\, \rho(w)(t) \; ,$$

where $\varphi(t) : \mathrm{Zin}(W) \to \mathrm{Zin}(V)$ and $\rho(t) : \mathrm{Zin}(W) \to \mathrm{Zin}(V)$ have degrees 0 and 1, respectively (the grading of $\mathrm{Zin}(V)$ is induced by that of V and the grading of $\Omega(I)$ is the homological one). Let us now translate the remaining properties of η into properties of φ and ρ. We denote by $d_I = dt\, d_t$ the de Rham differential of I. Since η is a chain map,

$$dt\, d_t \varphi + d_V \varphi - dt\, d_V \rho = (d_I \otimes \mathrm{id} + \mathrm{id} \otimes d_V)\eta = \eta d_W = \varphi d_W + dt\, \rho d_W \; ,$$

so that

$$d_V \varphi = \varphi d_W \quad \text{and} \quad d_t \varphi = d_V \rho + \rho d_W = [d, \rho] \; .$$

As η is also an algebra morphism, we have, for $w, w' \in \mathrm{Zin}(W)$,

$$\varphi(w \cdot w') + dt\, \rho(w \cdot w') = (\varphi(w) + dt\, \rho(w)) \cdot (\varphi(w') + dt\, \rho(w'))$$

$$= \varphi(w) \cdot \varphi(w') + (-1)^{w} dt\, (\varphi(w) \cdot \rho(w')) + dt\, (\rho(w) \cdot \varphi(w'))\ ,$$

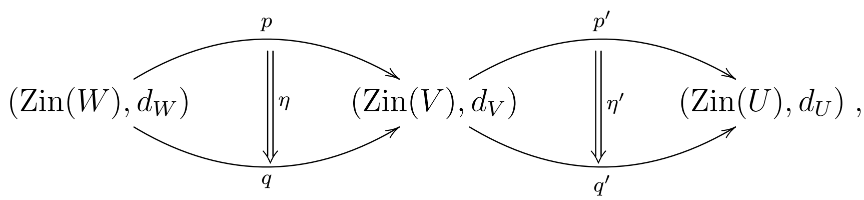
and φ (resp., ρ) is a family of DGA morphisms (resp., of degree 1 φ-Leibniz maps) from $\mathrm{Zin}(W)$ to $\mathrm{Zin}(V)$. Eventually,

$$p = \varepsilon_1^0 \eta = \varphi(0) \quad \text{and} \quad q = \varepsilon_1^1 \eta = \varphi(1)\ .$$

■

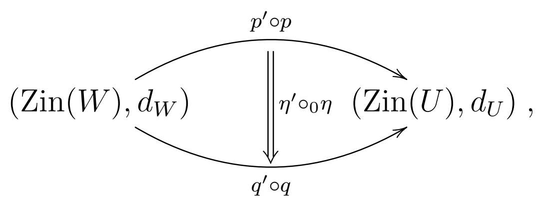
3.3.1. Horizontal and vertical compositions. *In literature, the 'categories' of homotopy Leibniz (resp., Lie) algebras over V (finite-dimensional) and of quasi-free DGZA-s (resp., quasi-free DGCA-s) over $s^{-1}V^*$ are (implicitly or explicitly) considered equivalent.* This conjecture is so far corroborated by the results of this paper. Hence, let us briefly report on compositions of concordances.

Let $\eta : p \Rightarrow q$ and $\eta' : p' \Rightarrow q'$,



(10)

be concordances between DGA morphisms. Their horizontal composite $\eta' \circ_0 \eta : p' \circ p \Rightarrow q' \circ q$,

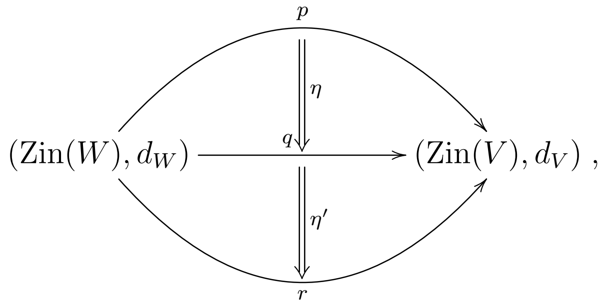


is defined by

$$(\eta' \circ_0 \eta)(t) = (\varphi'(t) \circ \varphi(t)) + dt\, (\varphi'(t) \circ \rho(t) + \rho'(t) \circ \varphi(t))\ , \tag{11}$$

with self-explaining notation. It is easily checked that the first term of the RHS and the coefficient of dt in the second term have the properties needed to make $\eta' \circ_0 \eta$ a concordance between $p' \circ p$ and $q' \circ q$.

As for the vertical composite $\eta' \circ_1 \eta : p \Rightarrow r$ of concordances $\eta : p \Rightarrow q$ and $\eta' : q \Rightarrow r$,



note that the composability condition $\varphi(1) = q = t(\eta) = s(\eta') = q = \varphi'(0)$, where s, t denote the source and target maps, does not encode any information about $\rho(1), \rho'(0)$. Hence, the usual 'half-time' composition cannot be applied.

3.4. Remark. *The preceding observation is actually the shadow of the fact that the 'category' of homotopy Leibniz algebras is not a 2-category (in which compositions of 2-morphisms are well-defined and associative), but an infinity category (in which composition is usually defined and associative only up to higher homotopy).*

3.5. Infinity homotopies. Some authors addressed directly or indirectly the concept of homotopy of Lie infinity algebras (L_∞-algebras). As mentioned above, in the (equivalent) 'category' of quasi-free DGCA-s, the classical picture of homotopy leads to concordances. In the 'category' of L_∞-algebras itself, morphisms can be viewed as Maurer-Cartan (MC) elements of a specific L_∞-algebra [Dol07],[Sho08], which yields the notion of 'gauge homotopy' between L_∞-morphisms. Additional notions of homotopy between MC elements do exist: Quillen and cylinder homotopies. On the other hand, Markl [Mar02] uses colored operads to construct homotopies for ∞-morphisms in a systematic way. The concepts of concordance, operadic homotopy, as well as Quillen, gauge, and cylinder homotopies are studied in detail in [DP12], for homotopy algebras over any Koszul operad, and they are shown to be equivalent, essentially due to homotopy transfer.

In this subsection, we focus on the homotopy Leibniz case and provide a brief account on the relationship between concordances, gauge homotopies, and Quillen homotopies (in the next section, we explain why the latter concept is the bridge to Getzler's [Get09] (and Henriques' [Hen08]) work, as well as to the infinity category structure on the set of homotopy Leibniz algebras).

Let us stress that all series in this section converge under some local finiteness or nilpotency conditions (for instance pronilpotency or completeness).

3.5.1. Gauge homotopic Maurer-Cartan elements. Lie infinity algebras over $\mathfrak{g}$ are in bijective correspondence with quasi-free DGCC-s $(\mathrm{Com}^c(s\mathfrak{g}), D)$, see Equation (1). Depending on the definition of the i-ary brackets ℓ_i, $i \geq 1$, from the corestrictions $D_i : (s\mathfrak{g})^{\odot i} \to s\mathfrak{g}$, where $\odot$ denotes the graded symmetric tensor product, one obtains various sign conventions in the defining relations of a Lie infinity algebra. By setting $\ell_i := D_i$ (resp. $\ell_i := (-1)^{i(i-1)/2} s^{-1} D_i\, s^{\,i}$), we get we a Voronov L_∞-antialgebra [Vor05] (resp. a Getzler L_∞-algebra [Get09]). Our convention is however that of Lada-Stasheff [LS93], namely we set $\ell_i := s^{-1} D_i\, s^i$. The set of graded antisymmetric multilinear maps $\ell_i : \mathfrak{g}^{\times i} \to \mathfrak{g}$ of degree $i-2$ satisfy the conditions

$$\sum_{i+j=r+1} \sum_{\sigma \in \mathrm{Sh}(i,j-1)} (-1)^{i(j-1)} \varepsilon(\sigma)\, \mathrm{sign}(\sigma) \ell_j(\ell_i(v_{\sigma_1}, \ldots, v_{\sigma_i}), v_{\sigma_{i+1}}, \ldots, v_{\sigma_r}) = 0 \;, \tag{12}$$

for all homogeneous $v_k \in \mathfrak{g}$ and all $r \geq 1$.

As the MC equation of a Lie infinity algebra $(\mathfrak{g}, \ell_i)$ must correspond to the MC equation given by the D_i, it depends on the definition of the operations ℓ_i. For a Lada-Stasheff L_∞-algebra, we obtain that the set $\mathrm{MC}(\mathfrak{g})$ of MC elements of $\mathfrak{g}$ is the set of solutions $\alpha \in \mathfrak{g}_{-1}$ of the MC equation

$$\sum_{i=1}^{\infty} \frac{1}{i!} (-1)^{i(i-1)/2} \ell_i(\alpha, \ldots, \alpha) = 0 \;. \tag{13}$$

Hence, we now consider the second MC equation (13). Further, for any $\alpha \in \mathfrak{g}_{-1}$, the twisted brackets

$$\ell_i^{\alpha}(v_1, \ldots, v_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \ell_{k+i}(\alpha^{\otimes k}, v_1, \ldots, v_i) \;,$$

$v_1, \ldots, v_i \in \mathfrak{g}$, are a sequence of graded antisymmetric multilinear maps of degree $i-2$. It is well-known that the ℓ_i^{α} endow $\mathfrak{g}$ with a new Lie infinity structure, if $\alpha \in \mathrm{MC}(\mathfrak{g})$. Finally, any vector $r \in \mathfrak{g}_0$ gives rise to a vector field

$$V_r : \mathfrak{g}_{-1} \ni \alpha \mapsto V_r(\alpha) := -\ell_1^{\alpha}(r) = -\sum_{k=0}^{\infty} \frac{1}{k!} \ell_{k+1}(\alpha^{\otimes k}, r) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \ell_k(r, \alpha^{\otimes (k-1)}) \in \mathfrak{g}_{-1} \;. \tag{14}$$

This field restricts to a vector field of the set $\mathrm{MC}(\mathfrak{g})$ of Maurer-Cartan elements of $\mathfrak{g}$ [DP12]. It follows that the integral curves

$$d_t \alpha = V_r(\alpha(t)) \;, \tag{15}$$

starting from points in $\mathrm{MC}(\mathfrak{g})$, are located inside $\mathrm{MC}(\mathfrak{g})$. Hence, the

3.6. Definition. *([Dol07], [Sho08]) Two MC elements $\alpha, \beta \in \mathrm{MC}(\mathfrak{g})$ of a Lie infinity algebra $\mathfrak{g}$ are* gauge homotopic *if there exists $r \in \mathfrak{g}_0$ and an integral curve $\alpha(t)$ of V_r, such that $\alpha(0) = \alpha$ and $\alpha(1) = \beta$.*

This gauge action is used to define the deformation functor $\mathrm{Def} : \mathtt{L}_\infty \to \mathtt{Set}$ from the category of Lie infinity algebras to the category of sets. Moreover, it will provide a concept of homotopy between homotopy Leibniz morphisms.

Let us first observe that Equation (15) is a 1-variable ordinary differential equation (ODE) and can be solved via an iteration procedure. The integral curve with initial point $\alpha \in \mathrm{MC}(\mathfrak{g})$ is computed in [Get09]. When using our sign convention in the defining relations of a Lie infinity algebra, we get an ODE that contains different signs and the solution of the corresponding Cauchy problem reads

$$\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} e_\alpha^k(r) \ , \tag{16}$$

where the $e_\alpha^k(r)$ admit a nice combinatorial description in terms of rooted trees. Moreover, they can be obtained inductively:

$$\begin{cases} e_\alpha^{i+1}(r) = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n+1)}{2}} \sum_{k_1+\ldots+k_n=i} \frac{i!}{k_1!\ldots k_n!} \ell_{n+1}(e_\alpha^{k_1}(r), \ldots, e_\alpha^{k_n}(r), r) \ , \\ e_\alpha^0(r) = \alpha \ . \end{cases} \tag{17}$$

It follows that $\alpha, \beta \in \mathrm{MC}(\mathfrak{g})$ are gauge homotopic if

$$\beta - \alpha = \sum_{k=1}^{\infty} \frac{1}{k!} e_\alpha^k(r) \ , \tag{18}$$

for some $r \in \mathfrak{g}_0$.

3.6.1. Simplicial de Rham algebra. We first fix the notation.

Let Δ be the *simplicial category* with objects the nonempty finite ordinals $[n] = \{0, \ldots, n\}$, $n \geq 0$, and morphisms the order-respecting functions $f : [m] \to [n]$. Denote by $\delta_n^i : [n-1] \rightarrowtail [n]$ the injection that omits the image i and by $\sigma_n^i : [n+1] \twoheadrightarrow [n]$ the surjection that assigns the same image to i and $i+1$, $i \in \{0, \ldots, n\}$.

A *simplicial object* in a category $\mathtt{C}$ is a functor $X \in [\Delta^{\mathrm{op}}, \mathtt{C}]$. It is completely determined by the simplicial data (X_n, d_i^n, s_i^n), $n \geq 0$, $i \in \{0, \ldots, n\}$, where $X_n = X[n]$ (n-simplices), $d_i^n = X(\delta_n^i)$ (face maps), and $s_i^n = X(\sigma_n^i)$ (degeneracy maps). We denote by $\mathtt{SC}$ the functor category $[\Delta^{\mathrm{op}}, \mathtt{C}]$ of simplicial objects in $\mathtt{C}$.

The simplicial category is embedded in its Yoneda dual category:

$$h_* : \Delta \ni [n] \mapsto \mathrm{Hom}_\Delta(-, [n]) \in [\Delta^{\mathrm{op}}, \mathtt{Set}] = \mathtt{SSet} \ .$$

We refer to the functor of points of $[n]$, i.e. to the simplicial set $\Delta[n] := \mathrm{Hom}_\Delta(-, [n])$, as the *standard simplicial n-simplex*. Moreover, the Yoneda lemma states that

$$\mathrm{Hom}_\Delta([n], [m]) \simeq \mathrm{Hom}(\mathrm{Hom}_\Delta(-, [n]), \mathrm{Hom}_\Delta(-, [m])) = \mathrm{Hom}(\Delta[n], \Delta[m]) \ .$$

This bijection sends $f : [n] \to [m]$ to φ defined by $\varphi_{[k]}(\bullet) = f \circ \bullet$ and φ to $\varphi_{[n]}(\mathrm{id}_{[n]})$. In the following we identify $[n]$ (resp., f) with $\Delta[n]$ (resp., φ).

The set S_n of n-simplices of a simplicial set S is obviously given by

$$S_n \simeq \mathrm{Hom}(\mathrm{Hom}_\Delta(-, [n]), S) = \mathrm{Hom}(\Delta[n], S).$$

Let us also recall the adjunction

$$|-|:\mathtt{SSet}\rightleftarrows\mathtt{Top}:\mathrm{Sing}$$

given by the 'geometric realization functor' $|-|$ and the 'singular complex functor' Sing. To define $|-|$, we first define the realization $|\Delta[n]|$ of the standard simplicial n-simplex to be the *standard topological n-simplex*

$$\Delta^n=\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}:x_i\geq 0,\sum_i x_i=1\}\ .$$

We can view $|-|$ as a functor $|-|\in[\Delta,\mathtt{Top}]$. Indeed, if $f:[n]\to[m]$ is an order-respecting map, we can define a continuous map $|f|:\Delta^n\to\Delta^m$ by

$$|f|[x_0,\ldots,x_n]=[y_0,\ldots,y_m]\ ,$$

where $y_i=\sum_{j\in f^{-1}\{i\}}x_j$.

Let $\wedge^\star_{\mathbb{K}}(x_0,\ldots,x_n,dx_0,\ldots,dx_n)$ be the free graded commutative algebra generated over $\mathbb{K}$ by the degree 0 (resp., degree 1) generators x_i (resp., dx_i). If we divide out the relations $\sum_i x_i=1$ and $\sum_i dx_i=0$ and set $d(x_i)=dx_i$ and $d(dx_i)=0$, we obtain a quotient DGCA

$$\Omega^\star_n=\wedge^\star_{\mathbb{K}}(x_0,\ldots,x_n,dx_0,\ldots,dx_n)/(\textstyle\sum_i x_i-1,\sum_i dx_i)$$

that can be identified, for $\mathbb{K}=\mathbb{R}$, with the algebra of polynomial differential forms $\Omega^\star(\Delta^n)$ of the standard topological n-simplex Δ^n. When defining $\Omega^\star:\Delta^{\mathrm{op}}\to\mathtt{DGCA}$ by $\Omega^\star[n]:=\Omega^\star_n$ and, for $f:[n]\to[m]$, by $\Omega^\star(f):=|f|^*:\Omega^\star_m\to\Omega^\star_n$ (use the standard pullback formula for differential forms given by $y_i=\sum_{j\in f^{-1}\{i\}}x_j$), we obtain a simplicial differential graded commutative algebra $\Omega^\star\in\mathtt{SDGCA}$. Hence, the face maps $d^n_i:\Omega^\star_n\to\Omega^\star_{n-1}$ are the pullbacks by the $|\delta^i_n|:\Delta^{n-1}\to\Delta^n$, and similarly for the degeneracy maps. In particular, $d^2_0=|\delta^0_2|^*:\Omega^\star_2\to\Omega^\star_1$ is induced by $y_0=0,y_1=x_0,y_2=x_1$. Lastly, for $0\leq i\leq n$, let the vertex e_i of Δ^n be defined by the point $x_i=1$, and let the evaluation map $\varepsilon^i_n:\Omega^\star_n\to\mathbb{K}$ at e_i be the restriction $(y_0,\ldots,y_n)=e_i)$.

3.6.2. Quillen homotopic Maurer-Cartan elements. We already mentioned that, if $(\mathfrak{g},\ell_i)$ is an L_∞-algebra and $(A,\cdot,d)$ a DGCA, their tensor product $\mathfrak{g}\otimes A$ has a canonical L_∞-structure $\bar\ell_i$. It is given by

$$\bar\ell_1(v\otimes a)=(\ell_1\otimes\mathrm{id}+\mathrm{id}\otimes d)(v\otimes a)=\ell_1(v)\otimes a+(-1)^v v\otimes d(a)$$

and, for $i\geq 2$, by

$$\bar\ell_i(v_1\otimes a_1,\ldots,v_i\otimes a_i)=\pm\ell_i(v_1,\ldots,v_i)\otimes(a_1\cdot\ldots\cdot a_i)\ ,$$

where $\pm$ is the Koszul sign generated by the commutation of the variables.

The following concept originates in Rational Homotopy Theory.

3.7. Definition. *Two MC elements* $\alpha, \beta \in \mathrm{MC}(\mathfrak{g})$ *of a Lie infinity algebra* $\mathfrak{g}$ *are* Quillen homotopic *if there exists a MC element* $\bar{\gamma} \in \mathrm{MC}(\bar{\mathfrak{g}}_1)$ *of the Lie infinity algebra* $\bar{\mathfrak{g}}_1 := \mathfrak{g} \otimes \Omega_1^\star$*, such that* $\varepsilon_1^0 \bar{\gamma} = \alpha$ *and* $\varepsilon_1^1 \bar{\gamma} = \beta$ *(where the* ε_1^i *are the natural extensions of the evaluation maps).*

From now on, we accept, in the definition of gauge equivalent MC elements, vector fields $V_{r(t)}$ induced by time-dependent $r = r(t) \in \mathfrak{g}_0$. The next result is proved in [Man99] for DGLA-s and essentially proved in [Dol07] for Lie infinity algebras. However, the statement of [Dol07] is actually slightly weaker. A full proof can be found in [DP12]; a sketchy proof will be given below.

3.8. Proposition. *Two MC elements of a Lie infinity algebra are Quillen homotopic if and only if they are gauge homotopic.*

3.8.1. Infinity morphisms as Maurer-Cartan elements and infinity homotopies. The possibility to view morphisms in $\mathrm{Hom}_{\mathrm{DGP^¡C}}(C, \mathcal{F}^{\mathrm{gr,c}}_{P^¡}(sW))$ as MC elements is known from the theory of the bar and cobar constructions of algebras over an operad. In [DP12], the authors showed that the fact that infinity morphisms between P_∞-algebras V and W, i.e. morphisms in

$$\mathrm{Hom}_{\mathrm{DGP^¡C}}(\mathcal{F}^{\mathrm{gr,c}}_{P^¡}(sV), \mathcal{F}^{\mathrm{gr,c}}_{P^¡}(sW)) \;,$$

are 1:1 with Maurer-Cartan elements of an L_∞-structure on

$$\mathrm{Hom}_{\mathbb{K}}(\mathcal{F}^{\mathrm{gr,c}}_{P^¡}(sV), W) \;,$$

is actually a consequence of a more general result based on the encoding of two P_∞-algebras and an infinity morphism between them in a DG colored free operad. In the case $P = \mathsf{Lie}$, one recovers the fact [Sho08] that

$$\mathrm{Hom}_{\mathrm{DGCC}}(C, \mathrm{Com}^c(sW)) \simeq \mathrm{MC}(\mathrm{Hom}_{\mathbb{K}}(C, W)) \;, \tag{19}$$

where C is any locally conilpotent DGCC, where W is an L_∞-algebra, and where the RHS is the set of MC elements of some convolution L_∞-structure on $\mathrm{Hom}_{\mathbb{K}}(C, W)$.

In the sequel we detail the case $P = \mathsf{Lei}$. Indeed, when interpreting infinity morphisms of homotopy Leibniz algebras as MC elements of a Lie infinity algebra, the equivalent notions of gauge and Quillen homotopies provide a concept of homotopy between homotopy Leibniz morphisms.

3.9. Proposition. *Let* (V, ℓ_i) *and* (W, m_i) *be two homotopy Leibniz algebras and let* $(\mathrm{Zin}^c(sV), D)$ *be the quasi-free* DGZC *that corresponds to* (V, ℓ_i)*. The graded vector space*

$$L(V, W) := \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV), W)$$

carries a convolution Lie infinity structure given by

$$\mathcal{L}_1 f = m_1 \circ f + (-1)^f f \circ D \tag{20}$$

and, as for $\mathcal{L}_p(f_1, ..., f_p)$, $p \geq 2$, *by*

$$\mathrm{Zin}^c(sV) \xrightarrow{\Delta^{p-1}} (\mathrm{Zin}^c(sV))^{\otimes p} \xrightarrow{\sum_{\sigma\in S(p)} \varepsilon(\sigma)\ \mathrm{sign}(\sigma)\, f_{\sigma(1)}\otimes\ldots\otimes f_{\sigma(p)}} W^{\otimes p} \xrightarrow{m_p} W\ , \tag{21}$$

where $f, f_1, \ldots, f_p \in L(V, W)$, $\Delta^{p-1} = (\Delta \otimes \mathrm{id}^{\otimes(p-2)})...(\Delta \otimes \mathrm{id})\Delta$ *, where* $S(p)$ *denotes the symmetric group on p symbols, and where the central arrow is the graded antisymmetrization operator.*

Proof. See [Laa02] or [DP12]. A direct verification is possible as well. ∎

3.10. Proposition. *Let* (V, ℓ_i) *and* (W, m_i) *be two homotopy Leibniz algebras. There exists a 1:1 correspondence between the set of infinity morphisms from* (V, ℓ_i) *to* (W, m_i) *and the set of MC elements of the convolution Lie infinity algebra structure* $\mathcal{L}_i$ *on* $L(V, W)$ *defined in Proposition 3.9.*

Observe that the considered MC series converges pointwise. Indeed, the evaluation of $\mathcal{L}_p(f_1, ..., f_p)$ on a tensor in $\mathrm{Zin}^c(sV)$ vanishes for $p \gg$, in view of the local conilpotency of $\mathrm{Zin}^c(sV)$. Moreover, convolution L_∞-algebras are complete, so that their MC equation converges in the topology induced by the filtration (a descending filtration F^iL of the space L of an L_∞-algebra $(L, \mathcal{L}_k)$ is *compatible* with the L_∞-structure $\mathcal{L}_k$, if $\mathcal{L}_k(F^{i_1}L, \ldots, F^{i_k}L) \subset F^{i_1+\ldots+i_k}L$, and it is *complete*, if, in addition, the 'universal' map $L \to \varprojlim L/F^iL$ from L to the (projective) limit of the inverse system L/F^iL is an isomorphism).

Note also that Proposition 3.10 is a specialization, in the case $P = \mathsf{Lei}$, of the above-mentioned 1:1 correspondence between infinity morphisms of P_∞-algebras and MC elements of a convolution L_∞-algebra. To increase the readability of this text, we give nevertheless a sketchy proof.

Proof. An MC element is an $\alpha \in \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV), W)$ of degree -1 that satisfies the MC equation. Hence, $s\alpha : \mathrm{Zin}^c(sV) \to sW$ has degree 0 and, since $\mathrm{Zin}^c(sW)$ is free as GZC, $s\alpha$ coextends uniquely to $\widehat{s\alpha} \in \mathrm{Hom}_{\mathrm{GZC}}(\mathrm{Zin}^c(sV), \mathrm{Zin}^c(sW))$. The fact that α is a solution of the MC equation exactly means that $\widehat{s\alpha}$ is a DGZC-morphism, i.e. an infinity morphism between the homotopy Leibniz algebras V and W. Indeed, when using e.g. the relations $\ell_i = s^{-1}D_i\, s^i$ and $m_i = s^{-1}\mathfrak{D}_i\, s^i$, and the corresponding version of the MC equation, we get

$$\begin{aligned}
&\sum_{p=1}^{\infty} \tfrac{1}{p!}(-1)^{\frac{p(p-1)}{2}} \mathcal{L}_p(\alpha, ..., \alpha) = 0 \Leftrightarrow \\
&\sum_{p=1}^{\infty} (-1)^{\frac{p(p-1)}{2}} m_p(\alpha \otimes ... \otimes \alpha)\Delta^{p-1} + (-1)^{\alpha}\alpha D = 0 \Leftrightarrow \\
&\sum_{p=1}^{\infty} (-1)^{\frac{p(p-1)}{2}} s^{-1}\mathfrak{D}_p\, s^p(\alpha \otimes ... \otimes \alpha)\Delta^{p-1} + (-1)^{\alpha}\alpha D = 0 \Leftrightarrow \\
&\sum_{p=1}^{\infty} s^{-1}\mathfrak{D}_p(s\alpha \otimes ... \otimes s\alpha)\Delta^{p-1} - s^{-1}s\alpha D = 0 \Leftrightarrow \\
&\mathfrak{D}\widehat{(s\alpha)} = \widehat{(s\alpha)}D\ .
\end{aligned}$$

■

Hence, the

3.11. Definition. *Two infinity morphisms f, g between homotopy Leibniz algebras (V, ℓ_i), (W, m_i) are* infinity homotopic*, if the corresponding MC elements $\alpha = \alpha(f)$ and $\beta = \beta(g)$ of the convolution Lie infinity structure $\mathcal{L}_i$ on $L = L(V, W)$ are Quillen (or gauge) homotopic. In other words, f and g are infinity homotopic, if there exists $\bar{\gamma} \in \mathrm{MC}_1(\bar{L})$, i.e. an MC element $\bar{\gamma}$ of the Lie infinity structure $\bar{\mathcal{L}}_i$ on $\bar{L} = L \otimes \Omega_1^\star$ – obtained from the convolution structure $\mathcal{L}_i$ on $L = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV), W)$ via extension of scalars –, such that $\varepsilon_1^0 \bar{\gamma} = \alpha$ and $\varepsilon_1^1 \bar{\gamma} = \beta$.*

3.11.1. Comparison of concordances and infinity homotopies. Since according to the prevalent philosophy, the 'categories' $\texttt{qfDG}P^{¡}\texttt{CoAlg}$ and $P_\infty\texttt{-Alg}$ are 'equivalent', appropriate concepts of homotopy in both categories should be in 1:1 correspondence. It can be shown [DP12] that, for any type of algebras, the concepts of concordance and Quillen homotopy are equivalent (at least if one defines concordances in an appropriate way); and as Quillen homotopies are already known to be equivalent to gauge homotopies, the desired result follows in full generality. We provide now a sketchy explanation of both relationships, defining concordances dually and assuming for simplicity that $\mathbb{K} = \mathbb{R}$.

Remember first that we defined concordances, in conformity with the classical picture, in a contravariant way: two infinity morphisms $f, g : V \to W$ between homotopy Leibniz algebras, i.e. two DGA morphisms $f^*, g^* : \mathrm{Zin}(s^{-1}W^*) \to \mathrm{Zin}(s^{-1}V^*)$, are concordant if there is a morphism

$$\eta \in \mathrm{Hom}_{\mathrm{DGA}}(\mathrm{Zin}(s^{-1}W^*), \mathrm{Zin}(s^{-1}V^*) \otimes \Omega_1^\star) ,$$

whose values at 0 and 1 are equal to f^* and g^*, respectively. Although we will use this definition in the sequel, we temporarily prefer a dual, covariant definition, which has the advantage that the spaces V, W need not be finite-dimensional.

The problem that the linear dual of the infinite-dimensional DGCA $\Omega_1^\star$ is not a coalgebra, has already been addressed in [BM12]. The authors consider a coalgebra Λ, which essentially is the dual $(\Omega_1^\star)^\vee$, except that one needs to fix some completeness issues. We omit the technical details, since this section is purely expository. A concordance can then be defined as a map

$$\eta \in \mathrm{Hom}_{\mathrm{DGC}}(\mathrm{Zin}^c(sV) \otimes \Lambda, \mathrm{Zin}^c(sW))$$

(with the appropriate boundary values). It is easily seen that any Quillen homotopy, i.e. any element in $\mathrm{MC}(L \otimes \Omega_1^\star)$, gives rise to a concordance. Indeed, set $\mathcal{V} := sV$ and note that

$$L \otimes \Omega_1^\star = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(\mathcal{V}), W) \otimes \Omega_1^\star = \mathrm{Hom}_{\mathbb{K}}\left(\bigoplus_{i\geq 1} \mathcal{V}^{\otimes i}, W\right) \otimes \Omega_1^\star = \left(\prod_{i\geq 1} \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W)\right) \otimes \Omega_1^\star$$

$$\longrightarrow \prod_{i\geq 1}\left(\mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W) \otimes \Omega_1^\star\right) \longrightarrow \prod_{i\geq 1} \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i} \otimes \Lambda, W) = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(\mathcal{V}) \otimes \Lambda, W) .$$

Note that the second arrow is not justified without addressing certain finiteness issues, but we again suppress the details and merely give an idea.

The relationship between Quillen and gauge homotopy is (at least on the chosen level of rigor) much clearer. Indeed, an element $\bar{\gamma} \in \mathrm{MC}_1(\bar{L}) = \mathrm{MC}(L \otimes \Omega_1^\star)$ can be decomposed as

$$\bar{\gamma} = \gamma(t) \otimes 1 + r(t) \otimes dt,$$

where $t \in [0,1]$ is the coordinate of Δ^1. When unraveling the MC equation of the $\bar{\mathcal{L}}_i$ according to the powers of dt, one gets

$$\begin{aligned} &\textstyle\sum_{p=1}^{\infty} \frac{1}{p!} \mathcal{L}_p(\gamma(t), ..., \gamma(t)) = 0 \ , \\ &\tfrac{d\gamma}{dt} = -\textstyle\sum_{p=0}^{\infty} \frac{1}{p!} \mathcal{L}_{p+1}(\gamma(t), ..., \gamma(t), r(t)) \ . \end{aligned} \tag{22}$$

A direct computation allows to see that the latter ODE, see Definition 3.6 of gauge homotopies and Equations (15) and (14), is dual (up to dimensional issues) to the ODE (9), see Proposition 3.3 that characterizes concordances.

4. Infinity category of homotopy Leibniz algebras

We already observed that vertical composition of concordances is not well-defined and that homotopy Leibniz algebras should form an infinity category. It is instructive to first briefly look at infinity homotopies between infinity morphisms of DG algebras.

4.1. DG case. Remember that infinity homotopies can be viewed as integral curves of specific vector fields V_r of the MC set (with obvious endpoints). In the DG case, we have, for any $r \in L_0$,

$$V_r : L_{-1} \ni \alpha \mapsto V_r(\alpha) = -\mathcal{L}_1(r) - \mathcal{L}_2(\alpha, r) \in L_{-1} \ .$$

In view of the Campbell-Baker-Hausdorff formula,

$$\exp(tV_r) \circ \exp(tV_s) = \exp(tV_r + tV_s + 1/2 \ t^2[V_r, V_s] + ...) \ .$$

The point is that

$$V : L_0 \to \mathrm{Vect}(L_{-1})$$

is a Lie algebra morphism – also after restriction to the MC set; we will not detail this nonobvious fact. It follows that

$$\exp(tV_r) \circ \exp(tV_s) = \exp(tV_{r+s+1/2 \ t[r,s]+...}) \ .$$

If we accept, as mentioned previously, time-dependent r-s, the problem of the vertical composition of homotopies is solved in the DG situation considered: the integral curve of the composed homotopy of two homotopies $\exp(tV_s)$ (resp., $\exp(tV_r)$) between morphisms f, g (resp., g, h) is given by

$$c(t) = (\exp(tV_r) \circ \exp(tV_s))(f) = \exp(tV_{r+s+1/2\, t[r,s]+\ldots})(f)\ .$$

Note that this vertical composition is not associative. Moreover, the preceding approach does not go through in the homotopy situation (note e.g. that in this case L_0 is no longer a Lie algebra). This again points to the possibility that homotopy algebras form infinity categories.

4.2. Shortcut to infinity categories. This subsection is a short digression that should allow us to grasp the spirit of infinity categories. For additional information, we refer the reader to [Gro10], [Fin11], and [Nog12].

Strict n-categories or *strict ω-categories* (in the sense of strict infinity categories) are well understood. Roughly, they are made up by 0-morphisms (objects), 1-morphisms (morphisms between objects), 2-morphisms (homotopies between morphisms)..., up to n-morphisms, except in the ω-case, where this upper bound does not exist. All these morphisms can be composed in various ways, the compositions being associative, admitting identities, etc. However, in most occurrences of higher categories these defining relations do not hold strictly. A number of concepts of weak infinity category, e.g. infinity categories in which the structural relations hold up to coherent higher homotopy, are developed in literature. Moreover, an (∞, r)-category is roughly an 'infinity category', with the additional requirement that all j-morphisms, $j > r$, be invertible. Hence, an $(\infty, 0)$-category is an infinity category in which all j-morphisms, $j \geq 1$, are invertible, i.e. an $(\infty, 0)$-category is an 'infinity groupoid'. In this subsection, we define *∞-groupoids* and $(\infty, 1)$-categories, which we will simply call *∞-categories*. It is clear that these definitions should be chosen in a way that ∞-groupoids and ordinary categories are specific ∞-categories.

4.2.1. Kan complexes, quasi-categories, nerves of groupoids and of categories. Let us recall that the *nerve functor* $N : \mathtt{Cat} \to \mathtt{SSet}$, provides a fully faithful embedding of the category Cat of all (small) categories into SSet and remembers not only the objects and morphisms, but also the compositions. It associates to any $\mathtt{C} \in \mathtt{Cat}$ the simplicial set

$$(N\mathtt{C})_n = \{C_0 \to C_1 \to \ldots \to C_n\}\ ,$$

where the sequence in the RHS is a sequence of composable C-morphisms between objects $C_i \in \mathtt{C}$; the face (resp., the degeneracy) maps are the compositions and insertions of identities. Let us also recall that the *r-horn* $\Lambda^r[n]$, $0 \leq r \leq n$, of $\Delta[n]$ is 'the part of the boundary of $\Delta[n]$ that is obtained by suppressing the interior of the $(n-1)$-face opposite to r'. More precisely, the r-horn $\Lambda^r[n]$ is the simplicial set, whose nondegenerate k-simplices are the injective order-respecting maps $[k] \to [n]$, except the identity and the map $\delta^r : [n-1] \rightarrowtail [n]$ whose image does not contain r.

We now detail four different situations based on the properties 'Any (inner) horn admits a (unique) filler'.

4.3. Definition. *A simplicial set* $S \in \texttt{SSet}$ *is* fibrant *and called a* Kan complex, *if the map* $S \to \star$, *where* $\star$ *denotes the terminal object, is a Kan fibration, i.e. has the right lifting property with respect to all canonical inclusions* $\Lambda^r[n] \subset \Delta[n]$, $0 \le r \le n$, $n > 0$. *In other words,* S *is a Kan complex, if any horn* $\Lambda^r[n] \to S$ *can be extended to an* n*-simplex* $\Delta[n] \to S$, *i.e. if* any horn in S admits a filler.

The following result is well-known and explains that a simplicial set is a nerve under a quite similar extension condition.

4.4. Proposition. *A simplicial set* S *is the nerve* $S \simeq N\texttt{C}$ *of some category* $\texttt{C}$, *if and only if* any inner horn $\Lambda^r[n] \to S$, $0 < r < n$, has a unique filler $\Delta[n] \to S$.

Indeed, it is quite obvious that for $S = N\texttt{C} \in \texttt{SSet}$, an inner horn $\Lambda^1[2] \to N\texttt{C}$, i.e. two $\texttt{C}$-morphisms $f : C_0 \to C_1$ and $g : C_1 \to C_2$, has a unique filler $\Delta[2] \to N\texttt{C}$, given by the edge $h = g \circ f : C_0 \to C_2$ and the 'homotopy' $\mathrm{id} : h \Rightarrow g \circ f$ (1).

As for Kan complexes $S \in \texttt{SSet}$, the filler property for an outer horn $\Lambda^0[2] \to S$ (resp., $\Lambda^2[2] \to S$) implies for instance that a horn $f : s_0 \to s_1$, $\mathrm{id} : s_0 \to s_2 = s_0$ (resp., $\mathrm{id} : s'_0 \to s'_2 = s'_0$, $g : s'_1 \to s'_2$) has a filler, so that any map has a 'left (resp., right) inverse' (2).

It is clear that simplicial sets $S_0, S_1, S_2, \ldots$ are candidates for ∞-categories. In view of the last remark (2), Kan complexes model ∞-groupoids. Hence, fillers for outer horns should be omitted in the definition of ∞-categories. On the other hand, ∞-categories do contain homotopies $\eta : h \Rightarrow g \circ f$, so that, due to (1), uniqueness of fillers is to be omitted as well. Hence, the

4.5. Definition. *A simplicial set* $S \in \texttt{SSet}$ *is an* ∞-category *if and only if* any inner horn $\Lambda^r[n] \to S$, $0 < r < n$, admits a filler $\Delta[n] \to S$.

We now also understand the

4.6. Proposition. *A simplicial set* S *is the nerve* $S \simeq N\texttt{G}$ *of some groupoid* $\texttt{G}$, *if and only if* any horn $\Lambda^r[n] \to S$, $0 \le r \le n$, $n > 0$, has a unique filler $\Delta[n] \to S$.

It is clear from the above definitions that (nerves of) categories and ∞-groupoids (Kan complexes) are ∞-categories. Note further that what we just defined is a model for ∞-categories called *quasi-categories* or *weak Kan complexes*.

4.6.1. Link with the intuitive picture of an infinity category. In the following, we explain that the preceding model of an ∞-category actually corresponds to the intuitive picture of an $(\infty, 1)$-category, i.e. that in an ∞-category all types of morphisms do exist, that all j-morphisms, $j > 1$, are invertible, and that composition of morphisms is defined and is associative only up to homotopy. This will be illustrated by showing that any ∞-category has a homotopy category, which is an ordinary category.

We denote simplicial sets by $S, S', \ldots$, categories by $\texttt{C}, \texttt{D}, \ldots$, and ∞-categories by $\mathsf{S}, \mathsf{S}', \ldots$

Let $\mathtt{S}$ be an ∞-category. Its *0-morphisms* are the elements of $\mathtt{S}_0$ and its *1-morphisms* are the elements of $\mathtt{S}_1$. The *source* and *target* maps σ, τ are defined, for any 1-morphism $f \in \mathtt{S}_1$, by $\sigma f = d_1 f \in \mathtt{S}_0$, $\tau f = d_0 f \in \mathtt{S}_0$, and the *identity map* is defined, for any 0-morphism $s \in \mathtt{S}_0$, by $\mathrm{id}_s = s_0 s \in \mathtt{S}_1$, with self-explaining notation. In the following, we denote a 1-morphism f with source s and target s' by $f : s \to s'$. In view of the simplicial relations, we have $\sigma\, \mathrm{id}_s = d_1 s_0 s = s$ and $\tau\, \mathrm{id}_s = d_0 s_0 s = s$, so that $\mathrm{id}_s : s \to s$.

Consider now two morphisms $f : s \to s'$ and $g : s' \to s''$. They define an inner horn $\Lambda^1[2] \to \mathtt{S}$, which, as $\mathtt{S}$ is an ∞-category, admits a filler $\phi : \Delta[2] \to \mathtt{S}$, or $\phi \in \mathtt{S}_2$. The face $d_1\phi \in \mathtt{S}_1$ is of course a candidate for the composite $g \circ f$.

4.7. Remark. *Since the face $h := d_1\phi$ of any filler ϕ is a (candidate for the) composite $g \circ f$, composites of morphisms are in ∞-categories not uniquely defined. We will show that they are determined only up to 'homotopy'.*

4.8. Definition. *Let $\mathtt{S}$ be an ∞-category and let $f, g : s \to s'$ be two morphisms. A* 2-morphism *or* homotopy *$\phi : f \Rightarrow g$ between f and g is an element $\phi \in \mathtt{S}_2$ such that $d_0\phi = g, d_1\phi = f, d_2\phi = \mathrm{id}_s$.*

Indeed, if there exists such a 2-simplex ϕ, there are two candidates for the composite $g \circ \mathrm{id}_s$, namely f and, of course, g. If we wish now that all the candidates be homotopic, the existence of ϕ must entail that f and g are homotopic – which is the case in view of Definition 4.8. If f is homotopic to g, we write $f \simeq g$.

4.9. Proposition. *The homotopy relation $\simeq$ is an equivalence in $\mathtt{S}_1$.*

Proof. Let $f : s \to s'$ be a morphism and consider $\mathrm{id}_f := s_0 f \in \mathtt{S}_2$. It follows from the simplicial relations that $d_0\, \mathrm{id}_f = f, d_1\, \mathrm{id}_f = f, d_2\, \mathrm{id}_f = s_0 s = \mathrm{id}_s$, so that id_f is a homotopy between f and f. To prove that $\simeq$ is symmetric, let $f, g : s \to s'$ and assume that ϕ is a homotopy from f to g. We then have an inner horn $\psi : \Lambda^2[3] \to \mathtt{S}$ such that $d_0\psi = \phi$, $d_1\psi = \mathrm{id}_g$, and $d_3\psi = \mathrm{id}_{\mathrm{id}_s} =: \mathrm{id}_s^2$. The face $d_2\Psi$ of a filler $\Psi : \Delta[3] \to \mathtt{S}$ is a homotopy from g to f. Transitivity can be obtained similarly. ■

4.10. Definition. *The* homotopy category $\mathtt{Ho}(\mathtt{S})$ *of an ∞-category $\mathtt{S}$ is the (ordinary) category with objects the objects $s \in \mathtt{S}_0$, with morphisms the homotopy classes $[f]$ of morphisms $f \in \mathtt{S}_1$, with composition $[g] \circ [f] = [g \circ f]$, where $g \circ f$ is any candidate for the composite in $\mathtt{S}$, and with identities $\mathrm{Id}_s = [\mathrm{id}_s]$.*

To check that this definition makes sense, we must in particular show that all composites $g \circ f$ are homotopic, see Remark 4.7. Let thus $\phi_1, \phi_2 \in \mathtt{S}_2$ be two 2-simplices such that $(d_0\phi_1, d_1\phi_1, d_2\phi_1) = (g, h_1, f)$ and $(d_0\phi_2, d_1\phi_2, d_2\phi_2) = (g, h_2, f)$, so that h_1 and h_2 are two candidates. Consider now for instance the inner horn $\psi : \Lambda^2[3] \to \mathtt{S}$ given by $\psi = (\phi_1, \phi_2, \bullet, \mathrm{id}_f)$. The face $d_2\Psi$ of a filler $\Psi : \Delta[3] \to \mathtt{S}$ is then a homotopy from h_2 to h_1. To prove that the composition of morphisms in $\mathtt{Ho}(\mathtt{S})$ is associative, one shows that candidates for $h \circ (g \circ f)$ and for $(h \circ g) \circ f$ are homotopic (we will prove neither this fact, nor the additional requirements for $\mathtt{Ho}(\mathtt{S})$ to be a category).

4.11. Remark. *It follows that in an ∞-category composition of morphisms is defined and associative only up to homotopy.*

We now comment on higher morphisms in ∞-categories, on their composites, as well as on invertibility of j-morphisms, $j > 1$.

4.12. Definition. *Let $\phi_1 : f \Rightarrow g$ and $\phi_2 : f \Rightarrow g$ be 2-morphisms between morphisms $f, g : s \to s'$. A* 3-morphism $\Phi : \phi_1 \Rrightarrow \phi_2$ *is an element $\Phi \in \mathtt{S}_3$ such that $d_0\Phi = \mathrm{id}_g$, $d_1\Phi = \phi_2$, $d_2\Phi = \phi_1$, and $d_3\Phi = \mathrm{id}_s^2$.*

Roughly, a 3-morphism is a 3-simplex with faces given by sources and targets, as well as by identities. Higher morphisms are defined similarly [Gro10].

Concerning the composition and invertibility, let us come back to transitivity of the homotopy relation. There we are given 2-morphisms $\phi_1 : f \Rightarrow g$ and $\phi_2 : g \Rightarrow h$, and must consider the inner horn $\psi = (\phi_2, \bullet, \phi_1, \mathrm{id}_s^2)$. The face $d_1\Psi$ of a filler Ψ is a homotopy between f and h and is a candidate for the composite $\phi_2 \circ \phi_1$ of the 2-morphisms ϕ_1, ϕ_2. If we now look again at the proof of symmetry of the homotopy relation and denote the homotopy from g to f by ψ', we see that $\psi \circ \psi' \simeq \mathrm{id}_g$. We obtain similarly that $\psi' \circ \psi \simeq \mathrm{id}_f$, so that 2-morphisms are 'invertible'.

4.13. Remark. *Eventually, all the requirements of the intuitive picture of an ∞-category are encoded in the existence of fillers of inner horns.*

4.14. Infinity groupoid of infinity morphisms between homotopy Leibniz algebras.

4.14.1. Quasi-category of homotopy Leibniz algebras. Let $\Omega^\star_\bullet$ be the SDGCA introduced in Subsection 3.6.1. The 'Yoneda embedding' of $\Omega^\star_\bullet$ viewed as object of $\mathtt{SSet}$ and $\mathtt{DGCA}$, respectively, gives rise to an adjunction that is well-known in Rational Homotopy Theory:

$$\Omega^\star : \mathtt{SSet} \rightleftarrows \mathtt{DGCA}^{\mathrm{op}} : \mathrm{Spec}_\bullet \ .$$

The functor $\Omega^\star = \mathrm{Hom}_{\mathtt{SSet}}(-, \Omega^\star_\bullet) =: \mathrm{SSet}(-, \Omega^\star_\bullet)$ associates to any $S_\bullet \in \mathtt{SSet}$ its Sullivan DGCA $\Omega^\star(S_\bullet)$ of piecewise polynomial differential forms, whereas the functor $\mathrm{Spec}_\bullet = \mathrm{Hom}_{\mathtt{DGCA}}(-, \Omega^\star_\bullet)$ assigns to any $A \in \mathtt{DGCA}$ its simplicial spectrum $\mathrm{Spec}_\bullet(A)$.

Remember now that an ∞-homotopy between ∞-morphisms between two homotopy Leibniz algebras V, W, is an element in $\mathrm{MC}_1(\bar{L}) = \mathrm{MC}(L \otimes \Omega^\star_1)$, where $L = L(V, W)$.

The latter set is well-known from integration of L_∞-algebras. Indeed, when looking for an integrating topological space or simplicial set of a positively graded L_∞-algebra L of finite type (degree-wise of finite dimension), it is natural to consider the simplicial spectrum of the corresponding quasi-free DGCA $\mathrm{Com}(s^{-1}L^*)$. The dual of Equation (19) yields

$$\mathrm{Spec}_\bullet(\mathrm{Com}(s^{-1}L^*)) = \mathrm{Hom}_{\mathtt{DGCA}}(\mathrm{Com}(s^{-1}L^*), \Omega^\star_\bullet) \simeq \mathrm{MC}(L \otimes \Omega^\star_\bullet) \ .$$

The integrating simplicial set of a nilpotent L_∞-algebra L is actually homotopy equivalent to $\mathrm{MC}_\bullet(\bar{L}) := \mathrm{MC}(L \otimes \Omega^\star_\bullet)$ [Get09]. It is clear that the structure maps of $\mathrm{MC}_\bullet(\bar{L}) \subset L \otimes \Omega^\star_\bullet$ are $\tilde{d}^n_i = \mathrm{id} \otimes d^n_i$ and $\tilde{s}^n_i = \mathrm{id} \otimes s^n_i$, where d^n_i and s^n_i were described in Subsection 3.6.1.

Higher homotopies (n-homotopies) are usually defined along the same lines as standard homotopies (1-homotopies), i.e., e.g., as arrows depending on parameters in $I^{\times n}$ (or Δ^n) instead of I (or Δ^1) [Lei03]. Hence,

4.15. Definition. ∞-n-homotopies (∞-$(n+1)$-morphisms) *between given homotopy Leibniz algebras* V, W *are Maurer-Cartan elements in* $\mathrm{MC}_n(\bar{L}) = \mathrm{MC}(L \otimes \Omega^\star_n)$*, where* $L = L(V, W)$ *and* $n \geq 0$.

Indeed, ∞-1-morphisms are just elements of $\mathrm{MC}(L)$, i.e. standard ∞-morphisms between V and W.

Note that if $\mathtt{S}$ is an ∞-category, the set of n-morphisms, with varying $n \geq 1$, between two fixed objects $s, s' \in \mathtt{S}_0$ can be shown to be a Kan complex [Gro10]. The simplicial set $\mathrm{MC}_\bullet(\bar{L})$, whose $(n-1)$-simplices are the ∞-n-morphisms between the considered homotopy Leibniz algebras V, W, $n \geq 1$, is known to be a Kan complex ($(\infty, 0)$-category) as well [Get09].

4.16. Remark. *We interpret this result by saying that homotopy Leibniz algebras and their infinity higher morphisms form an ∞-category ($(\infty, 1)$-category). Further, as mentioned above and detailed below, composition of homotopies is encrypted in the Kan property.*

Note that $\mathrm{MC}_\bullet(\bar{L})$ actually corresponds to the 'décalage', the 'down-shifting', of the simplicial set $\mathtt{S}$.

Let us also emphasize that Getzler's results are valid only for *nilpotent* L_∞-algebras, hence in principle not for L, which is only complete. Recall that an L_∞-algebra is *pronilpotent*, if it is complete with respect to its lower central series, i.e. the intersection of all its compatible filtrations; and it is *nilpotent*, if its lower central series eventually vanishes. However for our concern, namely the explanation of homotopies and their compositions in the 2-term homotopy Leibniz algebra case, this difficulty is nonexistent. Indeed by interpreting the involved series as formal ones, they become finite simply for degree reasons, and we recover the results on homotopies and their compositions conjectured in [BC04]. A more rigorous approach to these issues is being examined in a separate paper: it is rather technical and requires applying Henriques' method or working over an arbitrary local Artinian algebra.

4.16.1. Kan property. Considering our next purposes, we now review the proof of the Kan property of $\mathrm{MC}_\bullet(\bar{L})$ [Get09]. Since we adopt different sign conventions, certain signs will differ from those of [Get09].

Let us first recall that the lower central filtration of $(L, \mathcal{L}_i)$ is given by $F^1 L = L$ and

$$F^i L = \sum_{i_1 + \cdots + i_k = i} \mathcal{L}_k(F^{i_1} L, \cdots, F^{i_k} L), \quad i > 1 \ .$$

In particular, $F^2L = \mathcal{L}_2(L,L)$, $F^3L = \mathcal{L}_2(L,\mathcal{L}_2(L,L)) + \mathcal{L}_3(L,L,L)$, ..., so that F^kL is spanned by all the nested brackets containing k elements of L. Due to nilpotency, $F^iL = \{0\}$, for $i \gg$.

To simplify notation, let δ be the differential $\mathcal{L}_1$ of L, let d be the de Rham differential $\Omega^\star_n = \Omega^\star(\Delta^n)$ of degree -1, and let $\bar\delta + \bar d$ be the differential $\bar{\mathcal{L}}_1 = \delta \otimes \mathrm{id} + \mathrm{id} \otimes d$ of $L \otimes \Omega^\star(\Delta^n)$. Set now, for any $n \geq 0$ and any $0 \leq i \leq n$,

$$\mathrm{mc}_n(\bar L) := \{(\bar\delta + \bar d)\beta : \beta \in (L \otimes \Omega^\star(\Delta^n))^0\}$$

and

$$\mathrm{mc}^i_n(\bar L) := \{(\bar\delta + \bar d)\beta : \beta \in (L \otimes \Omega^\star(\Delta^n))^0,\ \bar\varepsilon^i_n\beta = 0\}\ ,$$

where $\bar\varepsilon^i_n := \mathrm{id} \otimes \varepsilon^i_n$ is the canonical extension of the evaluation map $\varepsilon^i_n : \Omega^\star(\Delta^n) \to \mathbb{K}$, see 3.6.1.

4.17. Remark. *In the following, we use the extension symbol 'bar' only when needed for clarity.*

- There exist fundamental bijections

$$B^i_n : \mathrm{MC}_n(\bar L) \xrightarrow{\sim} \mathrm{MC}(L) \times \mathrm{mc}^i_n(\bar L) \subset \mathrm{MC}(L) \times \mathrm{mc}_n(\bar L)\ . \tag{23}$$

The proof uses the operators

$$h^i_n :\ \Omega^\star(\Delta^n) \to \Omega^{\star+1}(\Delta^n)$$

defined as follows. Let $\vec t = [t_0, \ldots, t_n]$ be the coordinates of Δ^n (with $\sum_i t_i = 1$) and consider the maps $\phi^i_n :\ I \times \Delta^n \ni (u, \vec t) \mapsto u\vec t + (1-u)\vec e_i \in \Delta^n$. They allow to pull back a polynomial differential form on Δ^n to a polynomial differential form on $I \times \Delta^n$. The operators h^i_n are now given by

$$h^i_n\omega = \int_I (\phi^i_n)^*\omega\ .$$

They satisfy the relations

$$\{d, h^i_n\} = \mathrm{id}_n - \epsilon^i_n, \qquad \{h^i_n, h^j_n\} = 0, \qquad \epsilon^i_n h^i_n = 0\ , \tag{24}$$

where $\{-,-\}$ is the graded commutator (remember that ε^i_n vanishes in nonzero homological degree). The first relation is a higher dimensional analogue of

$$\{d, \int_0^t\}\omega = \{d, \int_0^t\}(f(u)+g(u)du) = d\int_0^t g(u)du + \int_0^t d_u f\, du = g(t)dt + f(t) - f(0) = \omega - \epsilon^0_1\omega\ ,$$

where $\omega \in \Omega^\star(I)$.

The natural extensions of d, h_n^i, and ε_n^i to $L \otimes \Omega^\star(\Delta^n)$ satisfy the same relations, and since we obviously have $\delta h_n^i = -\,h_n^i \delta$, the first relation holds in the extended setting also for d replaced by $\delta + d$.

Define now B_n^i by

$$B_n^i : \mathrm{MC}_n(\bar{L}) \ni \alpha \mapsto B_n^i \alpha := (\epsilon_n^i \alpha, (\delta + d) h_n^i \alpha) \in \mathrm{MC}(L) \times \mathrm{mc}_n^i(\bar{L}) \;. \tag{25}$$

Observe that $\alpha \in (L \otimes \Omega^\star(\Delta^n))^{-1}$ reads $\alpha = \sum_{k=0}^n \alpha^k$, $\alpha^k \in L_{k-1} \otimes \Omega^k(\Delta^n)$, so that $\varepsilon_n^i \alpha = \varepsilon_n^i \alpha^0 \in L_{-1}$. Moreover, it follows from the definition of the extended L_∞-maps $\bar{\mathcal{L}}_i$ that

$$\sum_{i \geq 1} \frac{1}{i!} \mathcal{L}_i(\varepsilon_n^i \alpha, \ldots, \varepsilon_n^i \alpha) = \varepsilon_n^i \sum_{i \geq 1} \frac{1}{i!} \bar{\mathcal{L}}_i(\alpha, \ldots, \alpha) = 0 \;. \tag{26}$$

In view of the last equation (24), the second component of $B_n^i \alpha$ is clearly an element of $\mathrm{mc}_n^i(\bar{L})$.

The construction of the inverse map is based upon a method similar to the iterative approximation procedure that allows us to prove the fundamental theorem of ODE-s. More precisely, consider the Cauchy problem $y'(t) = F(t, y(t))$, $y(0) = Y$, i.e. the integral equation

$$y(s) = Y + \int_0^s F(t, y(t)) dt \;.$$

Choose now the 'Ansatz' $y_0(s) = Y$ and define inductively

$$y_k(s) = Y + \int_0^s F(t, y_{k-1}(t)) dt \;,$$

$k \geq 1$. It is well-known that the y_k converge to a function y, which is the unique solution and depends continuously on the initial value Y.

Note now that, if we are given $\mu \in \mathrm{MC}(L)$ and $\nu = (\delta + d)\beta \in \mathrm{mc}_n^i(\bar{L})$, a solution $\alpha \in \mathrm{MC}_n(\bar{L})$ – i.e. an element $\alpha \in (L \otimes \Omega^\star(\Delta^n))^{-1}$ that satisfies

$$(\delta + d)\alpha + \sum_{i \geq 2} \frac{1}{i!} (-1)^{i(i-1)/2} \bar{\mathcal{L}}_i(\alpha, \ldots, \alpha) =: (\delta + d)\alpha + \bar{R}(\alpha) = 0 \quad -$$

such that $\varepsilon_n^i \alpha = \mu$ and $(\delta + d) h_n^i \alpha = \nu$, also satisfies the integral equation

$$\alpha = \mathrm{id}_n \, \alpha = \{\delta + d, h_n^i\} \alpha + \varepsilon_n^i \alpha = \mu + \nu + h_n^i (\delta + d)\alpha = \mu + \nu - h_n^i \bar{R}(\alpha) \;. \tag{27}$$

We thus choose the 'Ansatz' $\alpha_0 = \mu + \nu$ and set $\alpha_k = \alpha_0 - h_n^i \bar{R}(\alpha_{k-1})$, $k \geq 1$. In view of nilpotency, this iteration stabilizes, i.e. $\alpha_{k-1} = \alpha_k = \ldots =: \alpha$, for $k \gg$, or

$$\alpha = \alpha_0 - h_n^i \bar{R}(\alpha) \;. \tag{28}$$

The limit α is actually a solution in $\mathrm{MC}_n(\bar{L})$. Indeed recall first that, just like the standard curvature, the generalized curvature

$$\bar{\mathcal{F}}(\alpha) = (\delta + d)\alpha + \bar{R}(\alpha) = (\delta + d)\alpha + \sum_{i\geq 2} \frac{1}{i!}(-1)^{i(i-1)/2}\bar{\mathcal{L}}_i(\alpha, \ldots, \alpha) \ ,$$

whose zeros are the MC elements, satisfies the Bianchi identity

$$(\delta + d)\bar{\mathcal{F}}(\alpha) + \sum_{k\geq 1} \frac{1}{k!}(-1)^{k(k+1)/2}\bar{\mathcal{L}}_{k+1}(\alpha, \ldots, \alpha, \bar{\mathcal{F}}(\alpha)) = 0 \ . \tag{29}$$

It follows from (28) and (24) that

$$\bar{\mathcal{F}}(\alpha) = (\delta + d)(\alpha_0 - h_n^i \bar{R}(\alpha)) + \bar{R}(\alpha) = (\delta + d)\mu + h_n^i(\delta + d)\bar{R}(\alpha) + \epsilon_n^i \bar{R}(\alpha) \ .$$

From Equation (26) we know that $\varepsilon_n^i \bar{R}(\alpha) = R(\varepsilon_n^i \alpha) = R(\mu)$, with self-explaining notation. As for $\varepsilon_n^i \alpha = \mu$, note that $\varepsilon_n^i \mu = \mu$ and that

$$\varepsilon_n^i \nu = \varepsilon_n^i (\delta + d)\beta = \varepsilon_n^i (\delta + d) \sum_{k=0}^{n} \beta^k, \ \beta^k \in L_k \otimes \Omega^{-k}(\Delta^n)$$

so that

$$\varepsilon_n^i \nu = \varepsilon_n^i \delta\beta^0 = \delta\varepsilon_n^i \beta^0 = \delta\varepsilon_n^i \beta = 0 \ .$$

Hence,

$$\begin{aligned} \bar{\mathcal{F}}(\alpha) &= (\delta + d)\mu + h_n^i(\delta + d)(\bar{\mathcal{F}}(\alpha) - (\delta + d)\alpha) + R(\mu) \\ &= \mathcal{F}(\mu) + h_n^i(\delta + d)\bar{\mathcal{F}}(\alpha) = -h_n^i \sum_{k\geq 1} \frac{1}{k!}(-1)^{k(k+1)/2}\bar{\mathcal{L}}_{k+1}(\alpha, \ldots, \alpha, \bar{\mathcal{F}}(\alpha)) \ , \end{aligned}$$

in view of (29). Therefore, $\bar{\mathcal{F}}(\alpha) \in F^i\bar{L}$, for arbitrarily large i, and thus $\alpha \in \mathrm{MC}_n(\bar{L})$. This completes the construction of maps

$$\mathcal{B}_n^i : \mathrm{MC}(L) \times \mathrm{mc}_n^i(\bar{L}) \to \mathrm{MC}_n(\bar{L}) \ . \tag{30}$$

We already observed that $\varepsilon_n^i \mathcal{B}_n^i(\mu, \nu) = \varepsilon_n^i \alpha = \mu$. In fact, $B_n^i \mathcal{B}_n^i = \mathrm{id}$, so that B_n^i is surjective. Indeed, Equations (28) and (24) imply that

$$(\delta + d)h_n^i \alpha = -h_n^i(\delta + d)\alpha_0 + \alpha_0 - \varepsilon_n^i \alpha_0 = -h_n^i \delta\mu + \nu = \nu \ .$$

As for injectivity, if $B_n^i \alpha = B_n^i \alpha' =: (\mu, \nu)$, then both α and α' satisfy Equation (27). It is now quite easily seen that nilpotency entails that $\alpha = \alpha'$.

- The bijections

$$B_n^i : \mathrm{MC}_n(\bar{L}) \to \mathrm{MC}(L) \times \mathrm{mc}_n^i(\bar{L})$$

allow proving the Kan property for $\mathrm{MC}_\bullet(\bar{L})$. The extension of a horn in $\mathrm{SSet}(\Lambda^i[n], \mathrm{MC}_\bullet(\bar{L}))$ is sketched in the following diagram:

$$\begin{array}{ccc} \mathrm{SSet}(\Lambda^i[n], \mathrm{MC}_\bullet(\bar{L})) & \dashrightarrow & \mathrm{MC}_n(\bar{L}) \\ \downarrow & & \uparrow \\ \mathrm{SSet}(\Lambda^i[n], \mathrm{MC}(L) \times \mathrm{mc}_\bullet(\bar{L})) & \longrightarrow & \mathrm{MC}(L) \times \mathrm{mc}^i_n(\bar{L}) \end{array} \tag{31}$$

First of all the right arrow is nothing but $\mathcal{B}^i_n$.

$\star$ As for the left arrow, imagine, for simplicity, that $i = 1$ and $n = 2$, and let

$$\alpha \in \mathrm{SSet}(\Lambda^1[2], \mathrm{MC}_\bullet(\bar{L})) \ .$$

The restrictions $\alpha|_{01}$ and $\alpha|_{12}$ to the 1-faces 01, 12 (compositions of the natural injections with α) are elements in $\mathrm{MC}_1(\bar{L})$, so that the map B^1_1 sends $\alpha|_{01}$ to (μ, ν) in $\mathrm{MC}(L) \times \mathrm{mc}_1(\bar{L})$ (and similarly $B^0_1(\alpha|_{12}) = (\mu', \nu') \in \mathrm{MC}(L) \times \mathrm{mc}_1(\bar{L})$). Of course, $\mu = \varepsilon^1_1(\alpha|_{01}) = \varepsilon^0_1(\alpha|_{12}) = \mu'$. Since $\nu = (\delta + d)\beta$ and $\beta(1) = \varepsilon^1_1\beta = 0$, we find $\nu(1) = \varepsilon^1_1\nu = 0$ (and similarly $\nu' = (\delta + d)\beta'$ and $\beta'(1) = \nu'(1) = 0$). Thus,

$$(\mu; \nu, \nu') \in \mathrm{SSet}(\Lambda^1[2], \mathrm{MC}(L) \times \mathrm{mc}_\bullet(\bar{L})) \ ,$$

which explains the left arrow.

$\star$ For the bottom arrow, let again $i = 1, n = 2$. Since μ is constant, it can be extended to the whole simplex. To extend (ν, ν'), it actually suffices to extend (β, β'). Indeed, restriction obviously commutes with δ. As for commutation with d, remember that $\Omega^\star : \Delta^{\mathrm{op}} \to \mathtt{DGCA}$ and that the $\mathtt{DGCA}$-map $d^2_2 = \Omega^\star(\delta^2_2)$ sets the component t_2 to 0. Hence, d^2_2 coincides with restriction to 01 and commutes with d. Let now $\bar{\beta}$ be an extension of (β, β'). Since

$$(d\bar{\beta})|_{01} = d^2_2 d\bar{\beta} = d d^2_2 \bar{\beta} = d\beta$$

and similarly $(d\bar{\beta})|_{12} = d\beta'$.

It now remains to explain that an extension $\bar{\beta}$ does always exist. Consider the slightly more general extension problem of three polynomial differential forms β_0, β_1, and β_2 defined on the 1-faces 12, 02, and 01 of the 2-simplex Δ^2, respectively (it is assumed that they coincide at the vertices). Let $\pi_2 : \Delta^2 \to 01$ be the projection defined, for any $\vec{t} = [t_0, t_1, t_2]$, as the intersection of the line $u\vec{t} + (1-u)\vec{e}_2$ with 01. This projection is of course ill-defined at $\vec{t} = \vec{e}_2$. In coordinates, we get

$$\pi_2 : [t_0, t_1, t_2] \mapsto [t_0/(1 - t_2), t_1/(1 - t_2)] \ .$$

It follows that the pullback $\pi_2^*\beta_2$ is a rational differential form with denominator $(1-t_2)^N$, for some integer N. Hence,

$$\gamma_2 := (1 - t_2)^N \pi_2^* \beta_2$$

is a polynomial differential form on Δ^2 that coincides with β_2 on 01. It now suffices to solve the same extension problem as before, but for the forms $\beta_0 - \gamma_2|_{12}$, $\beta_1 - \gamma_2|_{02}$, and 0. When iterating the procedure – due to Renshaw [Sul77] –, the problem reduces to the extension of $0, 0, 0$ (since the pullback preserves 0). This completes the description of the bottom arrow, as well as the proof of the Kan property of $\mathrm{MC}_\bullet(\bar{L})$.

5. 2-Category of 2-term homotopy Leibniz algebras

Categorification replaces sets (resp., maps, equations) by categories (resp., functors, natural isomorphisms). In particular, rather than considering two maps as equal, one details a way of identifying them. Categorification is thus a sharpened viewpoint that turned out to provide deeper insight. This motivates the interest in e.g. categorified algebras (and in truncated homotopy algebras – see below).

Categorified Lie algebras were introduced under the name of Lie 2-algebras in [BC04] and further studied in [Roy07], [SL10], and [KMP11]. The main result of [BC04] states that Lie 2-algebras and 2-term Lie infinity algebras form equivalent 2-categories. However, **infinity homotopies of 2-term Lie infinity algebras** (resp., **compositions of such homotopies**) are not explained, but appear as some God-given natural transformations read through this EQUIVALENCE (resp., compositions are addressed only in [SS07] and performed in the ALGEBRAIC OR COALGEBRAIC SETTINGS).

This circumstance is not satisfactory, and **the attempt to improve our understanding of infinity homotopies and their compositions is one of the main concerns of the present paper**. Indeed, in [KMP11] (resp., [BP12]), the authors show that the EQUIVALENCE between n-term Lie infinity algebras and Lie n-algebras is, for $n > 2$, not as straightforward as expected – which is essentially due to the largely ignored fact that the category `Vect` n-`Cat` of linear n-categories is symmetric monoidal, but that the corresponding map $\boxtimes : L \times L' \to L \boxtimes L'$ is not an n-functor (resp., that the understanding of a concept in the ALGEBRAIC FRAMEWORK is far from implying its comprehension in the infinity context – a reality that is corroborated e.g. by the comparison of concordances and infinity homotopies).

In this section, we obtain **explicit formulae for infinity homotopies and their compositions**, applying the KAN PROPERTY of $\mathrm{MC}_\bullet(\bar{L})$ to the 2-term case, thus staying inside the INFINITY SETTING.

5.1. CATEGORY OF 2-TERM HOMOTOPY LEIBNIZ ALGEBRAS. For the sake of completeness, we first describe 2-term homotopy Leibniz algebras and their morphisms. Propositions 5.2 and 5.3 are specializations to the 2-term case of Definitions 2.8 and 2.11; see also [SL10]. The informed reader may skip the present subsection.

5.2. PROPOSITION. *A* 2-term homotopy Leibniz algebra *is a graded vector space $V = V_0 \oplus V_1$ concentrated in degrees 0 and 1, together with a linear, a bilinear, and a trilinear* *map l_1, l_2, and l_3 on V, of degree $|l_1| = -1$, $|l_2| = 0$, and $|l_3| = 1$, which satisfy, for any $w, x, y, z \in V_0$ and $h, k \in V_1$,*

(a) $l_1 l_2(x, h) = l_2(x, l_1 h)$,
$l_1 l_2(h, x) = l_2(l_1 h, x)$,

(b) $l_2(l_1 h, k) = l_2(h, l_1 k)$,

(c) $l_1 l_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(y, l_2(x, z)) - l_2(l_2(x, y), z)$,

(d) $l_3(x, y, l_1 h) = l_2(x, l_2(y, h)) - l_2(y, l_2(x, h)) - l_2(l_2(x, y), h)$,
$l_3(x, l_1 h, y) = l_2(x, l_2(h, y)) - l_2(h, l_2(x, y)) - l_2(l_2(x, h), y)$,
$l_3(l_1 h, x, y) = l_2(h, l_2(x, y)) - l_2(x, l_2(h, y)) - l_2(l_2(h, x), y)$,

(e) $l_2(l_3(w, x, y), z) + l_2(w, l_3(x, y, z)) - l_2(x, l_3(w, y, z)) + l_2(y, l_3(w, x, z)) - l_3(l_2(w, x), y, z)$
$+ l_3(w, l_2(x, y), z) - l_3(x, l_2(w, y), z) - l_3(w, x, l_2(y, z)) + l_3(w, y, l_2(x, z)) - l_3(x, y, l_2(w, z))$
$= 0$.

5.3. Proposition. *An* infinity morphism between 2-term homotopy Leibniz algebras *(V, l_1, l_2, l_3) and (W, m_1, m_2, m_3) is made up by a linear and a bilinear map f_1, f_2 from V to W, of degree $|f_1| = 0, |f_2| = 1$, which satisfy, for any $x, y, z \in V_0$ and $h \in V_1$,*

(a) $m_1 f_1 h = f_1 l_1 h$,

(b) $m_2(f_1 x, f_1 y) + m_1 f_2(x, y) = f_1 l_2(x, y)$,

(c) $m_2(f_1 x, f_1 h) = f_1 l_2(x, h) - f_2(x, l_1 h)$,
$m_2(f_1 h, f_1 x) = f_1 l_2(h, x) - f_2(l_1 h, x)$,

(d) $m_3(f_1 x, f_1 y, f_1 z) - m_2(f_2(x, y), f_1 z) + m_2(f_1 x, f_2(y, z)) - m_2(f_1 y, f_2(x, z)) =$
$f_1 l_3(x, y, z) + f_2(l_2(x, y), z) - f_2(x, l_2(y, z)) + f_2(y, l_2(x, z))$.

5.4. Corollary. *The category* `2Lei`$_\infty$ *of 2-term homotopy Leibniz algebras and infinity morphisms is a full subcategory of the category* `Lei`$_\infty$`-Alg` *of homotopy Leibniz algebras and infinity morphisms.*

5.5. From the Kan property to 2-term infinity homotopies and their compositions.

5.6. Definition. *A* 2-term infinity homotopy *between infinity morphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$, which act themselves between 2-term homotopy Leibniz algebras (V, l_1, l_2, l_3) and (W, m_1, m_2, m_3), is a linear map θ_1 from V to W, of degree $|\theta_1| = 1$, which satisfies, for any $x, y \in V_0$ and $h \in V_1$,*

(a) $g_1 x - f_1 x = m_1 \theta_1 x$,

(b) $g_1 h - f_1 h = \theta_1 l_1 h$,

(c) $g_2(x,y) - f_2(x,y) = \theta_1 l_2(x,y) - m_2(f_1 x, \theta_1 y) - m_2(\theta_1 x, g_1 y)$.

The characterizing relations (a) - (c) of infinity Leibniz homotopies are the correct counterpart of the defining relations of infinity Lie homotopies [BC04]. However, rather than choosing the preceding relations as a mere definition, we deduce them here from the Kan property of $\mathrm{MC}_\bullet(\bar{L})$. More precisely,

5.7. Theorem. *There exist surjective maps S_1^i, $i \in \{0,1\}$, from the class $\mathcal{I}$ of ∞-homotopies for 2-term homotopy Leibniz algebras to the class $\mathcal{T}$ of 2-term ∞-homotopies for 2-term homotopy Leibniz algebras.*

5.8. Remark. *The maps S_1^i preserve the source and the target, i.e. they are surjections from the class $\mathcal{I}(f,g)$ of ∞-homotopies from f to g, to the class $\mathcal{T}(f,g)$ of 2-term ∞-homotopies from f to g. In the sequel, we refer to a preimage by S_1^i of an element $\theta_1 \in \mathcal{T}$ as a* lift *of θ_1 by S_1^i .*

Proof. Henceforth, we use again the homological version of infinity algebras (k-ary bracket of degree $k-2$), as well as the Lada-Stasheff sign convention for the higher Jacobi conditions and the MC equation.

Due to the choice of the homological variant of homotopy algebras, $\delta = \mathcal{L}_1$ has degree -1. For consistency, differential forms are then viewed as negatively graded; hence, $d : \Omega^{-k}(\Delta^n) \to \Omega^{-k-1}(\Delta^n)$, $k \in \{0,\ldots,n\}$, and $\bar{\mathcal{L}}_1 = \delta \otimes \mathrm{id} + \mathrm{id} \otimes d$ has degree -1 as well. Similarly, the degree of the operator h_n^i is now $|h_n^i| = 1$. It is moreover easily checked that L cannot contain multilinear maps of nonnegative degree, i.e. that $L = \oplus_{k\ge 0} L_{-k}$. It follows that an element $\bar{\alpha} \in (L \otimes \Omega^\star(\Delta^n))^{-k}$, $k \ge 0$, reads

$$\bar{\alpha} = \sum \alpha_{-k} \otimes \omega^0 + \sum \alpha_{-k+1} \otimes \omega^{-1} + \ldots ,$$

where the RHS is a finite sum. For instance, if $n=2$, an element $\bar{\alpha}$ of degree -1 can be decomposed as

$$\bar{\alpha} = \alpha(s,t) \otimes 1 + \beta(s,t) \otimes ds + \beta'(s,t) \otimes dt ,$$

where (s,t) are coordinates of Δ^2 and where $\alpha(s,t) \in L_{-1}[s,t]$ and $\beta(s,t), \beta'(s,t) \in L_0[s,t]$ are polynomial functions in s,t with coefficients in L_{-1} and L_0, respectively.

In the sequel, we evaluate the L_∞-structure maps $\bar{\mathcal{L}}_i$ of $L \otimes \Omega^\star(\Delta^n)$ mainly on elements of degree -1 and 0, hence we compute the structure maps $\mathcal{L}_i$ of $L = \mathrm{Hom}_{\mathbb{K}}(\mathrm{Zin}^c(sV), W)$ on elements α and β of degree -1 and 0, respectively. Let

$$\alpha = \sum_{p\ge 1} \alpha^p \in L , \quad |\alpha| = -1 ,$$
$$\beta = \sum_{p\ge 1} \beta^p \in L , \quad |\beta| = 0 ,$$

where $\alpha^p, \beta^p : (sV)^{\otimes p} \to W$. The point is that the concentration of V, W in degrees $0,1$ entails that almost all components α^p, β^p vanish and that all series converge (which explains why the formal application of Getzler's method to the present situation leads to the correct counterpart of the findings of [BC04]). Indeed, the only nonzero components of α, β are

$$\begin{aligned} \alpha^1: &\quad sV_0 \to W_0,\quad sV_1 \to W_1\ ,\\ \alpha^2: &\quad (sV_0)^{\otimes 2} \to W_1\ ,\\ \beta^1: &\quad sV_0 \to W_1\ . \end{aligned} \tag{32}$$

Similarly, the nonzero components of the nonzero evaluations of the maps $\mathcal{L}_i$ on α-s and β-s are

$$\begin{aligned} \mathscr{L}_1(\alpha): &\quad sV_1 \to W_0\ ,\quad (sV_0)^{\otimes 2} \to W_0\ ,\quad sV_0 \otimes sV_1 \to W_1\ ,\quad (sV_0)^{\otimes 3} \to W_1\ ,\\ \mathscr{L}_1(\beta): &\quad sV_0 \to W_0\ ,\quad sV_1 \to W_1\ ,\quad (sV_0)^{\otimes 2} \to W_1\ ,\\ \mathscr{L}_2(\alpha_1,\alpha_2): &\quad (sV_0)^{\otimes 2} \to W_0\ ,\quad sV_0 \otimes sV_1 \to W_1\ ,\quad (sV_0)^{\otimes 3} \to W_1\ ,\\ \mathscr{L}_2(\alpha,\beta): &\quad (sV_0)^{\otimes 2} \to W_1\ ,\\ \mathscr{L}_3(\alpha_1,\alpha_2,\alpha_3): &\quad (sV_0)^{\otimes 3} \to W_1\ , \end{aligned} \tag{33}$$

see Proposition 3.9.

We are now ready to make more concrete the iterative construction of $\mathcal{B}_n^i(\mu,\nu) \in \mathrm{MC}_n(\bar{L})$ from $\mu \in \mathrm{MC}(L)$ and $\nu = (\delta + d)\beta$, $\beta \in (L \otimes \Omega^\star(\Delta^n))^0$, $\varepsilon_n^i\beta = 0$ (the explicit forms of $\mathcal{B}_n^i(\mu,\nu)$ for $n = 1$ and $n = 2$ will be the main ingredients of the proofs of Theorems 5.7 and 5.9).

- Let $\alpha \in \mathcal{I}(f,g)$, i.e. let

$$\alpha \in \mathrm{MC}_1(\bar{L}) \xrightarrow{\sim} (\mu, (\delta+d)\beta) \in \mathrm{MC}(L) \times \mathrm{mc}_1^0(\bar{L})\ ,$$

such that $\varepsilon_1^0\alpha = f$ and $\varepsilon_1^1\alpha = g$. To construct

$$\alpha = \mathcal{B}_1^0 B_1^0 \alpha = \mathcal{B}_1^0(\varepsilon_1^0\alpha, (\delta+d)h_1^0\alpha) =: \mathcal{B}_1^0(f, (\delta+d)\beta) =: \mathcal{B}_1^0(\mu,\nu)\ ,$$

we start from

$$\alpha_0 = \mu + (\delta+d)\beta\ .$$

The iteration unfolds as

$$\alpha_k = \alpha_0 - \sum_{j=2}^{\infty} \frac{1}{j!} h_1^0 (-1)^{j(j-1)/2} \bar{\mathscr{L}}_j(\alpha_{k-1}, \cdots, \alpha_{k-1})\ ,\quad k \geq 1\ .$$

Explicitly,

$$\begin{aligned} \alpha_1 &= \mu + (\delta+d)\beta + \frac{1}{2} h_1^0 \bar{\mathscr{L}}_2(\alpha_0,\alpha_0) + \frac{1}{3!} h_1^0 \bar{\mathscr{L}}_3(\alpha_0,\alpha_0,\alpha_0)\\ &= \mu + (\delta+d)\beta + h_1^0 \bar{\mathscr{L}}_2(\mu+\delta\beta, d\beta) + \frac{1}{2} h_1^0 \bar{\mathscr{L}}_3(\mu+\delta\beta, \mu+\delta\beta, d\beta)\\ &= \mu + (\delta+d)\beta + h_1^0 \bar{\mathscr{L}}_2(\mu+\delta\beta, d\beta)\ . \end{aligned}$$

Observe that $\mu+\delta\beta \in L_{-1}[t]$ and $d\beta \in L_0[t]\otimes dt$, that differential forms are concentrated in degrees 0 and -1, that h_1^0 annihilates 0-forms, and that the term in $\bar{\mathcal{L}}_3$ contains a factor of the type $\mathcal{L}_3(\alpha_1,\alpha_2,\beta)$ (notation of (33)), whose components vanish – see above. Analogously,

$$\begin{aligned}\alpha_2 &= \mu+(\delta+d)\beta+h_1^0\bar{\mathscr{L}}_2(\mu+\delta\beta+h_1^0\bar{\mathscr{L}}_2(\mu+\delta\beta,d\beta),d\beta)\\ &\quad+\frac{1}{2}h_1^0\bar{\mathscr{L}}_3(\mu+\delta\beta+h_1^0\bar{\mathscr{L}}_2(\mu+\delta\beta,d\beta),\mu+\delta\beta+h_1^0\bar{\mathscr{L}}_2(\mu+\delta\beta,d\beta),d\beta)\\ &= \mu+(\delta+d)\beta+h_1^0\bar{\mathscr{L}}_2(\mu+\delta\beta,d\beta)\ .\end{aligned}$$

Indeed, the term $h_1^0\bar{\mathcal{L}}_2(h_1^0\bar{\mathcal{L}}_2(\mu+\delta\beta,d\beta),d\beta)$ contains a factor of the type $\mathcal{L}_2(\mathcal{L}_2(\alpha,\beta_1),\beta_2)$ (notation of (33)), and the only nonvanishing component of this factor, as well as of its first internal map $\mathcal{L}_2(\alpha,\beta_1)$, is the component $(sV_0)^{\otimes 2}\to W_1$ – which entails, in view of Proposition 3.9, that the term in consideration vanishes. Hence, the iteration stabilizes already at its second stage and

$$\alpha=\mathscr{B}_1^0(\mu,\nu)=\mu+(\delta+d)\beta+h_1^0\bar{\mathscr{L}}_2(\mu+\delta\beta,d\beta)\in \mathrm{MC}_1(\bar{L})\ . \tag{34}$$

Note first that the integral h_1^0 can be evaluated since $\bar{\mathscr{L}}_2(\mu+\delta\beta,d\beta)$ is a total derivative. Indeed, when setting $\beta=\beta_0\otimes P$ (sum understood), $\beta_0\in L_0$ and $P\in\Omega^0(\Delta^1)$, we see that

$$\bar{\mathcal{L}}_2(\mu,d\beta)=\mathcal{L}_2(\mu,\beta_0)\otimes dP=-d\bar{\mathcal{L}}_2(\mu,\beta)\ .$$

As for the term $\bar{\mathcal{L}}_2(\delta\beta,d\beta)$, we have

$$0=(\delta+d)\bar{\mathcal{L}}_2(\beta,d\beta)=\bar{\mathscr{L}}_2(\delta\beta,d\beta)+\bar{\mathscr{L}}_2(\beta,\delta d\beta)\ ,$$

since $\bar{\mathcal{L}}_1=\delta+d$ is a graded derivation of $\bar{\mathcal{L}}_2$ and as $\bar{\mathcal{L}}_2(\beta,d\beta)=\bar{\mathcal{L}}_2(d\beta,d\beta)=0$. It is now easily checked that

$$\bar{\mathscr{L}}_2(\delta\beta,d\beta)=-\frac{1}{2}d\bar{\mathscr{L}}_2(\delta\beta,\beta)\ .$$

Eventually,

$$\begin{aligned}\alpha &= \mu+(\delta+d)\beta-h_1^0d\bar{\mathscr{L}}_2(\mu,\beta)-\frac{1}{2}h_1^0d\bar{\mathscr{L}}_2(\delta\beta,\beta)\\ &= \mu+(\delta+d)\beta-\bar{\mathscr{L}}_2(\mu,\beta)-\frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta,\beta)\ .\end{aligned}$$

Indeed, it suffices to observe that, for any $\ell_{-1}\otimes P\in L_{-1}\otimes\Omega^0(\Delta^1)$ which vanishes under the action of ε_1^0, we have

$$h_1^0d(\ell_{-1}\otimes P)=-dh_1^0(\ell_{-1}\otimes P)+\ell_{-1}\otimes P-\varepsilon_1^0(\ell_{-1}\otimes P)=\ell_{-1}\otimes P\ .$$

We are now able to write the components of $g = \varepsilon_1^1\alpha \in L_{-1}$ (see (32)) in terms of $f = \mu$ and β:

$$\begin{aligned} g^1 &= \epsilon_1^1(\mu + (\delta + d)\beta - \bar{\mathscr{L}}_2(\mu, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta))^1 = \epsilon_1^1(f + \delta\beta)^1 = f^1 + \epsilon_1^1(\delta\beta)^1 \ , \\ g^2 &= \epsilon_1^1(\mu + (\delta + d)\beta - \bar{\mathscr{L}}_2(\mu, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta))^2 = \epsilon_1^1(f + \delta\beta - \bar{\mathscr{L}}_2(f, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta))^2 \ , \\ g^3 &= 0 \ , \end{aligned} \tag{35}$$

where we recall that the first component of a morphism of the type $\mathcal{L}_2(\alpha, \beta)$ (see (33)) vanishes.

To obtain a 2-term ∞-homotopy $\theta_1 \in \mathcal{T}(f, g)$, it now suffices to further develop the equations (35).

As

$$g_1 := g^1 s, f_1 := f^1 s \in \mathrm{Hom}^0_{\mathbb{K}}(V, W) \ ,$$

we evaluate the first equation on $x \in V_0$ and $h \in V_1$. Therefore, we compute $\varepsilon_1^1(\delta\beta)^1 s = \delta\beta(1)^1 s$ on x and h. Since

$$\delta\beta(1) = \mathcal{L}_1\beta(1) = m_1\beta(1) + \beta(1)D_V \ ,$$

where $D_V \in \mathrm{CoDer}^{-1}(\mathrm{Zin}^c(sV))$, we have $D_V : sV_1 \to sV_0 \ , sV_0 \otimes sV_0 \to sV_0 \ , \ldots$ Hence,

$$\delta\beta(1)^1 sx = m_1\, \beta(1)s\, x = m_1\theta_1 x \ , \tag{36}$$

where we defined the *homotopy parameter* θ_1 by

$$\theta_1 := \beta(1)s = \beta(1)s - \beta(0)s \ . \tag{37}$$

Similarly,

$$\delta\beta(1)^1 sh = \beta(1)D_V sh = \beta(1)s\, s^{-1}D_V s\, h = \theta_1 l_1 h \ . \tag{38}$$

The characterizing equations (a) and (b) follow.

Since

$$g_2 := g^2 s^2, f_2 := f^2 s^2 \in \mathrm{Hom}^1_{\mathbb{K}}(V \otimes V, W) \ ,$$

it suffices to evaluate the second equation on $x, y \in V_0$. When computing e.g.

$$\varepsilon_1^1 \bar{\mathcal{L}}_2(\delta\beta, \beta)^2 s^2(x, y)$$

we get

$$\begin{aligned} \mathcal{L}_2(\delta\beta(1), \beta(1))(sx, sy) = m_2(\delta\beta(1)\, sx, \beta(1)\, sy) + m_2(\beta(1)\, sx, \delta\beta(1)\, sy) = \\ m_2(m_1\theta_1 x, \theta_1 y) + m_2(\theta_1 x, m_1\theta_1 y) = 2m_2(\theta_1 x, m_1\theta_1 y) \ , \end{aligned} \tag{39}$$

in view of Equation (36) and Relation (b) of Proposition 5.2. Similarly,

$$\varepsilon_1^1 \bar{\mathscr{L}}_2(f, \beta)^2 s^2(x, y) = m_2(f_1 x, \theta_1 y) + m_2(\theta_1 x, f_1 y) \ . \tag{40}$$

Further, one easily finds

$$\varepsilon_1^1(\delta\beta)^2 s^2(x,y) = \theta_1 l_2(x,y) \ . \tag{41}$$

When collecting the results (39), (40), and (41), and taking into account Relation (a), we finally obtain the characterizing equation (c).

- Recall that in the preceding step we started from $\alpha \in \mathcal{I}(f,g)$, set $\mu = f$,

$$\beta = h_1^0 \alpha \ ,$$

$\nu = (\delta + d)\beta$, defined

$$\theta_1 = (\beta(1) - \beta(0))s \ ,$$

and deduced the characterizing relations $g = f + \mathcal{E}(f, \beta(1)s) = f + \mathcal{E}(f, \theta_1)$ of $\theta_1 \in \mathcal{T}(f,g)$ by computing

$$\alpha = \mathcal{B}_1^0(\mu,\nu) = \mu + (\delta+d)\beta - \bar{\mathscr{L}}_2(\mu,\beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta,\beta)$$

at 1. Let us mention that instead of defining the map $S_1^0 : \mathcal{I} \ni \alpha \mapsto \theta_1 \in \mathcal{T}$, we can consider the similarly defined map S_1^1.

To prove surjectivity of S_1^i, let $\theta_1 \in \mathcal{T}(f,g)$ and set $\beta(i) = 0$, $i \in \{0,1\}$, and $\beta(1-i) = (-1)^i \theta_1 s^{-1}$. Note that by construction $\theta_1 = (\beta(1) - \beta(0))s$. Use now Renshaw's method [Sul77] to extend $\beta(0)$ and $\beta(1)$ to some $\beta \in L_0 \otimes \Omega^0(\Delta^1)$, set

$$\mu = (1-i)f + ig \quad \text{and} \quad \nu = (\delta+d)\beta \ ,$$

and construct

$$\alpha = \mathcal{B}_1^i(\mu,\nu) \in \mathrm{MC}_1(\bar{L}) \ . \tag{42}$$

If $i = 0$, then

$$\alpha(0) = \varepsilon_1^0\alpha = \mu = f \quad \text{and} \quad \alpha(1) = (\mathcal{B}_1^0(\mu,\nu))(1) = f + \mathcal{E}(f,\theta_1) = g \ ,$$

in view of the characterizing relations (a)-(c) of θ_1. If $i = 1$, one has also $\alpha(1) = g$ and $\alpha(0) = g + \mathcal{E}(g, -\theta_1) = f$, but to obtain the latter result, the characterizing equations (a)-(c), as well as Equation (b) of Proposition 5.2 are needed. To determine the image of $\alpha \in \mathcal{I}(f,g)$ by S_1^i, one first computes $h_1^i\alpha$, which, since h_1^i sends 0-forms to 0, is equal to

$$h_1^i(\delta+d)\beta = -(\delta+d)h_1^i\beta + \beta - \varepsilon_1^i\beta = \beta \ ,$$

then one gets

$$S_1^i\alpha = (\beta(1) - \beta(0))s = \theta_1 \ ,$$

which completes the proof. ∎

5.9. Theorem. [Definition] *If* $\theta_1 : f \Rightarrow g$, $\tau_1 : g \Rightarrow h$ *are 2-term* ∞*-homotopies between infinity morphisms* $f, g, h : V \to W$*, the vertical composite* $\tau_1 \circ_1 \theta_1$ *is given by* $\tau_1 + \theta_1$.

We will actually lift $\theta_1, \tau_1 \in \mathcal{T}$ to $\alpha', \alpha'' \in \mathrm{MC}_1(\bar{L})$ (which involves choices), then compose these lifts in the infinity groupoid $\mathrm{MC}_\bullet(\bar{L})$ (which is not a well-defined operation), and finally project the result back to $\mathcal{T}$ (despite all the intermediate choices, the final result will turn out to be well-defined).

Proof. Let now $n = 2$, take $\mu \in \mathrm{MC}(L)$ and $\nu = (\delta + d)\beta \in \mathrm{mc}_2^1(\bar{L})$, then construct $\alpha = \mathcal{B}_2^1(\mu, \nu)$. The computation is similar to that in the 1-dimensional case and gives the same result:

$$\alpha = \mu + (\delta + d)\beta - \bar{\mathscr{L}}_2(\mu, \beta) - \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta, \beta) . \tag{43}$$

To obtain $\tau_1 \circ_1 \theta_1$, proceed as in (42) and lift θ_1 (resp., τ_1) to

$$\alpha' := \mathcal{B}_1^1(g, (\delta{+}d)\beta') \in \mathcal{I}(f, g) \subset \mathrm{MC}_1(\bar{L}) \text{ (resp., } \alpha'' := \mathcal{B}_1^0(g, (\delta{+}d)\beta'') \in \mathcal{I}(g, h) \subset \mathrm{MC}_1(\bar{L})) ,$$

where

$$\beta'(0) = -\theta_1 s^{-1} \text{ and } \beta'(1) = 0 \quad \text{(resp., } \beta''(0) = 0 \text{ and } \beta''(1) = \tau_1 s^{-1}) .$$

As mentioned above, we have by construction

$$\theta_1 = (\beta'(1) - \beta'(0))s \quad \text{(resp., } \tau_1 = (\beta''(1) - \beta''(0))s) . \tag{44}$$

If we view α' (resp., α'') as defined on the face 01 (resp., 12) of Δ^2, the equation $\varepsilon_1^1\alpha' = \varepsilon_1^0\alpha'' = g$ reads $\varepsilon_2^1\alpha' = \varepsilon_2^1\alpha'' = g =: \mu$. This means that

$$(\alpha', \alpha'') \in \mathrm{SSet}(\Lambda^1[2], \mathrm{MC}_\bullet(\bar{L})) .$$

We now follow the extension square (31). The left arrow leads to

$$(\mu; (\delta + d)\beta', (\delta + d)\beta'') \in \mathrm{SSet}(\Lambda^1[2], \mathrm{MC}(L) \times \mathrm{mc}_\bullet(\bar{L})) ,$$

the bottom arrow to

$$(\mu, (\delta + d)\beta) \in \mathrm{MC}(L) \times \mathrm{mc}_2^1(\bar{L}) ,$$

where β is *any extension* of (β', β'') to Δ^2, and the right arrow provides $\alpha \in \mathrm{MC}_2(\bar{L})$ given by Equation (43). From Subsection 4.6.1, we know that all composites of α', α'' are ∞-2-homotopic and that a possible composite is obtained by restricting α to 02. This restriction $(-)|_{02}$ is given by the `DGCA`-map d_1^2. Hence, we get

$$\alpha|_{02} = \mu + (\delta + d)\beta|_{02} + \bar{\mathscr{L}}_2(\mu, \beta|_{02}) + \frac{1}{2}\bar{\mathscr{L}}_2(\delta\beta|_{02}, \beta|_{02}) \in \mathcal{I}(f, h) \subset \mathrm{MC}_1(\bar{L}) .$$

We now choose the projection $S_1^0\alpha|_{02} \in \mathcal{T}(f, h)$ of the composite-candidate of the chosen lifts of θ_1, τ_1, as composite $\tau_1 \circ_1 \theta_1$. Since

$$h_1^0\alpha|_{02} = -(\delta + d)h_1^0\beta|_{02} + \beta|_{02} - \beta(0) = \beta|_{02} - \beta(0) ,$$

we get

$$S_1^0\alpha|_{02} = (\beta|_{02}(2) - \beta(0) - \beta|_{02}(0) + \beta(0))s = (\beta(2) - \beta(0))s =$$
$$(\beta''(2) - \beta''(1))s + (\beta'(1) - \beta'(0))s = \tau_1 + \theta_1\ ,$$

in view of (44). Hence, by definition, the vertical composite of $\theta_1 \in \mathcal{T}(f,g)$ and $\tau_1 \in \mathcal{T}(g,h)$ is given by

$$\tau_1 \circ_1 \theta_1 = \tau_1 + \theta_1 \in \mathcal{T}(f,h)\ . \tag{45}$$

■

5.10. Remark. *The composition of elements of $\mathcal{I} = \mathrm{MC}_1(\bar{L})$ in the infinity groupoid $\mathrm{MC}_\bullet(\bar{L})$, which is defined and associative only up to higher morphisms, projects to a well-defined and associative vertical composition in $\mathcal{T}$.*

Just as for concordances, there is no problem for the horizontal composition of ∞-homotopies. The horizontal composite of $\theta_1 \in \mathcal{T}(f,g)$ and $\tau_1 \in \mathcal{T}(f',g')$, where $f,g : V \to W$ and $f',g' : W \to X$ act between 2-term homotopy Leibniz algebras, is defined by

$$\tau_1 \circ_0 \theta_1 = g_1'\theta_1 + \tau_1 f_1 = f_1'\theta_1 + \tau_1 g_1\ . \tag{46}$$

The two definitions coincide, since θ_1, τ_1 are chain homotopies between the chain maps f, g and f', g', respectively, see Definition 5.6, Relations (a) and (b). The identity associated to a 2-term ∞-morphism is just the zero-map. As announced in [BC04] (in the Lie case and without information about composition), we have the

5.11. Proposition. *There is a strict 2-category* `2Lei`$_\infty$`-Alg` *of 2-term homotopy Leibniz algebras.*

6. 2-Category of categorified Leibniz algebras

6.1. Category of Leibniz 2-algebras. Leibniz 2-algebras are categorified Leibniz structures on a categorified vector space. More precisely,

6.2. Definition. *A* Leibniz 2-algebra $(L, [-,-], \mathbf{J})$ *is a linear category L equipped with*

1. *a* bracket $[-,-]$, *i.e. a bilinear functor $[-,-] : L \times L \to L$, and*
2. *a* Jacobiator $\mathbf{J}$, *i.e. a trilinear natural transformation*

$$\mathbf{J}_{x,y,z} : [x,[y,z]] \to [[x,y],z] + [y,[x,z]], \quad x,y,z \in L_0,$$

which satisfy, for any $w, x, y, z \in L_0$, the Jacobiator identity

$$
\begin{array}{c}
[w,[x,[y,z]]] \\
\swarrow {\scriptstyle [\mathbf{1}_w, \mathbf{J}_{x,y,z}]} \qquad\qquad {\scriptstyle \mathbf{1}} \searrow \\
\begin{array}{cc}
[w,[[x,y],z]] + [w,[y,[x,z]]] & [w,[x,[y,z]]] \\
\downarrow {\scriptstyle \mathbf{J}_{w,[x,y],z} + \mathbf{J}_{w,y,[x,z]}} & {\scriptstyle \mathbf{J}_{w,x,[y,z]}} \downarrow \\
\begin{array}{c} [[w,[x,y]],z] + [[x,y],[w,z]] \\ +[[w,y],[x,z]] + [y,[w,[x,z]]] \end{array} & [[w,x],[y,z]] + [x,[w,[y,z]]] \\
\downarrow {\scriptstyle \mathbf{1} + [\mathbf{1}_y, \mathbf{J}_{w,x,z}]} & {\scriptstyle \mathbf{1} + [\mathbf{1}_x, \mathbf{J}_{w,y,z}]} \downarrow \\
\begin{array}{c} [[w,[x,y]],z] + [[x,y],[w,z]] \\ +[[w,y],[x,z]] + [y,[[w,x],z]] \\ +[y,[x,[w,z]]] \end{array} & \begin{array}{c} [[w,x],[y,z]] + [x,[[w,y],z]] \\ +[x,[y,[w,z]]] \end{array}
\end{array} \\
{\scriptstyle [\mathbf{J}_{w,x,y}, \mathbf{1}_z]} \searrow \qquad\qquad \swarrow {\scriptstyle \mathbf{J}_{[w,x],y,z} + \mathbf{J}_{x,[w,y],z} + \mathbf{J}_{x,y,[w,z]}} \\
\begin{array}{c} [[[w,x],y],z] + [[x,[w,y]],z] \\ +[[x,y],[w,z]] + [[w,y],[x,z]] \\ +[y,[[w,x],z]] + [y,[x,[w,z]]] \end{array}
\end{array}
\qquad (47)
$$

The Jacobiator identity is a coherence law that should be thought of as a higher Jacobi identity for the Jacobiator.

The preceding hierarchy 'category, functor, natural transformation' together with the coherence law is entirely similar to the known hierarchy 'linear, bilinear, trilinear maps l_1, l_2, l_3' with the L_∞-conditions (a)-(e). More precisely,

6.3. PROPOSITION. *There is a 1-to-1 correspondence between Leibniz 2-algebras and 2-term homotopy Leibniz algebras.*

This proposition was proved in the Lie case in [BC04] and announced for the Leibniz case in [SL10]. A generalization of the latter correspondence to Lie 3-algebras and 3-term Lie infinity algebras can be found in [KMP11]. This paper allows one to understand that the correspondence between higher categorified algebras and truncated infinity algebras is subject to cohomological conditions, and to see how the coherence law corresponds to the last nontrivial L_∞-condition.

The definition of Leibniz 2-algebra morphisms is God-given: such a morphism must be a functor that respects the bracket up to a natural transformation, which in turn respects the Jacobiator. More precisely,

6.4. Definition. *Let* $(L, [-,-], \mathbf{J})$ *and* $(L', [-,-]', \mathbf{J}')$ *be Leibniz 2-algebras* (*in the following, we write* $[-,-], \mathbf{J}$ *instead of* $[-,-]', \mathbf{J}'$). *A* morphism $(F, \mathbf{F})$ of Leibniz 2-algebras *from* L *to* L' *consists of*

1. *a linear functor* $F : L \to L'$, *and*

2. *a bilinear natural transformation*

$$\mathbf{F}_{x,y} : [Fx, Fy] \to F[x,y], \quad x, y \in L_0 \,,$$

which make the following diagram commute

$$\begin{array}{ccc} [Fx,[Fy,Fz]] & \xrightarrow{\mathbf{J}_{Fx,Fy,Fz}} & [[Fx,Fy],Fz] + [Fy,[Fx,Fz]] \\ \downarrow{\scriptstyle [\mathbf{1}_x,\mathbf{F}_{y,z}]} & & \downarrow{\scriptstyle [\mathbf{F}_{x,y},\mathbf{1}_z]+[\mathbf{1}_y,\mathbf{F}_{x,z}]} \\ [Fx,F[y,z]] & & [F[x,y],Fz] + [Fy,F[x,z]] \\ \downarrow{\scriptstyle \mathbf{F}_{x,[y,z]}} & & \downarrow{\scriptstyle \mathbf{F}_{[x,y],z}+\mathbf{F}_{y,[x,z]}} \\ F[x,[y,z]] & \xrightarrow{F\mathbf{J}_{x,y,z}} & F[[x,y],z] + F[y,[x,z]] \end{array} \tag{48}$$

6.5. Proposition. *There is a 1-to-1 correspondence between Leibniz 2-algebra morphisms and 2-term homotopy Leibniz algebra morphisms.*

For a proof, see [BC04] and [SL10].

Composition of Leibniz 2-algebra morphisms $(F, \mathbf{F})$ is naturally given by composition of functors and whiskering of functors and natural transformations.

6.6. Proposition. *There is a category* `Lei2` *of Leibniz 2-algebras and morphisms.*

6.7. 2-morphisms and their compositions. The definition of a 2-morphism is canonical:

6.8. Definition. *Let* $(F, \mathbf{F}), (G, \mathbf{G})$ *be Leibniz 2-algebra morphisms from* L *to* L'. *A* Leibniz 2-algebra 2-morphism $\boldsymbol{\theta}$ *from* F *to* G *is a linear natural transformation* $\boldsymbol{\theta} : F \Rightarrow G$*, such that, for any* $x, y \in L_0$*, the following diagram commutes*

$$\begin{array}{ccc} [Fx, Fy] & \xrightarrow{\mathbf{F}_{x,y}} & F[x,y] \\ \Big\downarrow{\scriptstyle [\boldsymbol{\theta}_x, \boldsymbol{\theta}_y]} & & \Big\downarrow{\scriptstyle \boldsymbol{\theta}_{[x,y]}} \\ [Gx, Gy] & \xrightarrow{\mathbf{G}_{x,y}} & G[x,y] \end{array} \tag{49}$$

6.9. Theorem. *There is a 1:1 correspondence between Leibniz 2-algebra 2-morphisms and 2-term Leibniz* ∞*-homotopies.*

Horizontal and vertical compositions of Leibniz 2-algebra 2-morphisms are those of natural transformations.

6.10. Proposition. *There is a strict 2-category* `Lei2Alg` *of Leibniz 2-algebras.*

6.11. Corollary. *The 2-categories* $\texttt{2Lei}_\infty\texttt{-Alg}$ *and* `Lei2Alg` *are 2-equivalent.*

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