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REALIZABLE HOMOTOPY COLIMITS

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ABSTRACT. We show that the composition of a homotopically meaningful 'geometric realization' (or simple functor) with the simplicial replacement produces all homotopy colimits and Kan extensions in a relative category which is closed under coproducts. Examples (and its duals) include model categories, Δ-closed classes and other concrete examples such as complexes on (AB4) abelian categories, (filtered) commutative dg algebras and mixed Hodge complexes. The resulting homotopy colimits satisfy the expected properties as cofinality and Fubini, and are moreover colimits in a suitable 2-category of relative categories. Conversely, the existence of homotopy colimits satisfying these properties guarantees that $\mathtt{hocolim}_{\Delta^\circ}$ is a simple functor.

Introduction.

In the setting of model categories, the existence, properties and ways of computing homotopy (co)limits are well-known questions which have been widely studied in the literature (see [H], [CS] and [C], among others). However, in the general case of a category $\mathcal{C}$ endowed with a class $\mathcal{W}$ of weak equivalences (where a model structure is not necessarily present), few tools are available to know whether $(\mathcal{C}, \mathcal{W})$ admits homotopy (co)limits or not, as well as to obtain formulae to compute them.

In the present paper we study this question in the more tractable case of a closed-under-coproducts class of weak equivalences. We show how a homotopically meaningful 'geometric realization' $\Delta^\circ\mathcal{C} \to \mathcal{C}$ can be used to obtain homotopy colimits in $(\mathcal{C}, \mathcal{W})$ satisfying the expected properties (and that, in addition, the converse is also true). Dually, a homotopically meaningful totalization $\Delta\mathcal{C} \to \mathcal{C}$ allows one to construct homotopy limits.

To possess a homotopically meaningful (co)totalization is precisely the core of the notion of *(co)simplicial descent category* introduced in [R], which is a simplicial variant of the cubical *(co)homological descent categories* of [GN]. More concretely, $(\mathcal{C}, \mathcal{W})$ is a simplicial descent category if it is equipped with a *simple functor* $\mathbf{s} : \Delta^\circ\mathcal{C} \to \mathcal{C}$ subject to five axioms guaranteeing it has the correct homotopical meaning. Namely, $\mathbf{s}$ must

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be compatible with $\mathcal{W}$, the homotopy equivalences in $\Delta^\circ\mathcal{C}$, coproducts and must satisfy an Eilenberg-Zilber and a normalization property. Examples include other abstract axiomatic theories such as model categories and Voevodsky Δ-closed classes ([V]), as well as further examples. Among these, we treat here the ones of positive complexes on (AB4) abelian categories, (filtered) commutative dg algebras and mixed Hodge complexes. The corresponding simple functors are, respectively, the total complex of a double complex, Navarro's Thom-Whitney simple ([N]) and Deligne's cosimplicial construction ([De]).

Our main result, Theorem 3.1, states that the composition of the simple functor $\mathbf{s}$ with the simplicial replacement of diagrams produces all homotopy colimits in $(\mathcal{C},\mathcal{W})$, which satisfy the expected properties such as cofinality, Fubini and preservation by forming diagram categories. This generalizes the classical construction of homotopy (co)limits for simplicial sets given in [BK], and allows one to obtain explicit formulae to compute homotopy (co)limits in the previous examples of (co)simplicial descent categories.

The homotopy colimits constructed in this way are stronger than the ones in the sense of Grothendieck or derived functors, defined only at the level of localized categories. Instead, they are indeed *colimits* in a suitable 2-category $\mathcal{R}el\mathcal{C}at$ of 'categories with weak equivalences' (or *relative categories*, following [BaK]). We call this kind of homotopy colimit *realizable*. Theorem 3.1 also states that, conversely, if $(\mathcal{C},\mathcal{W})$ admits all realizable homotopy colimits satisfying the cofinality property, then $\mathbf{s} = \mathtt{hocolim}_{\Delta^\circ} : \Delta^\circ\mathcal{C} \to \mathcal{C}$ makes $(\mathcal{C},\mathcal{W})$ a simplicial descent category.
In addition, such a $(\mathcal{C},\mathcal{W})$ admits pointwise homotopy Kan extensions. In the language of Grothendieck derivators, this means that $A \mapsto \mathcal{C}^{A^\circ}[\mathcal{W}^{-1}]$ defines a right derivator (see Theorem 3.16). The bottom line of the results presented here is then that for relative categories closed under coproducts, to posses a simple functor is the same thing than to be homotopically cocomplete.

The paper is organized as follows. The first section is categorical in nature and contains the formal definitions of the 2-category of relative categories and realizable homotopy colimits. The second section deals instead with simplicial homotopy theory and is devoted to the explicit construction of homotopy colimits and Kan extensions for a particular (but not less relevant) case of simplicial descent categories: those induced by a Voevodsky Δ-closed class. Once this is done the general case is treated in Section 3 where Theorems 3.1 and 3.16 are proved. Finally, in the last section we describe the formulae obtained for homotopy (co)limits in the examples previously mentioned.

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2.14's proof.

1. Relative categories and realizable homotopy colimits.

The homotopy (co)limits constructed in the subsequent sections are stronger than those in the sense of Grothendieck derivators and derived functors. They are moreover (co)limits in a suitable 2-category $\mathcal{R}el\mathcal{C}at$ of relative categories. This section contains the 2-categorical formalism needed to define and study them. After introducing the 2-category $\mathcal{R}el\mathcal{C}at$, we study the associated notion of adjunction in this 2-category, and then we focus on the particular case of colimits.

1.1. Relative categories as a 2-category.

1.2. Definition. *A relative category consists of a pair $(\mathcal{C}, \mathcal{W}_\mathcal{C})$ formed by a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}_\mathcal{C}$ of $\mathcal{C}$, whose elements are called weak equivalences. For simplicity, we assume along the paper that the class $\mathcal{W}_\mathcal{C}$, also denoted by $\mathcal{W}$ for brevity, is saturated. That is, $\mathcal{W}$ is the inverse image by the localization functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ of the isomorphisms of $\mathcal{C}[\mathcal{W}^{-1}]$.*

Among all relative categories, we will mainly focus on those in which coproducts exist and preserve the weak equivalences. The reason is that for this kind of relative categories the formulas for homotopy colimits become simplified, as we will see later.

1.3. Definition. *We say that a relative category $(\mathcal{C}, \mathcal{W})$ is closed under (finite) coproducts if $\mathcal{C}$ has an initial object 0 and both $\mathcal{C}$ and $\mathcal{W}$ are closed under (finite) coproducts.*

If $(\mathcal{D}, \mathcal{W})$ is a relative category and C is a category, the category $\mathrm{Fun}(C, \mathcal{D})$ (or just $\mathcal{D}^C$) of functors from C to $\mathcal{D}$ is again a relative category with the class $\mathcal{W}^C$ of *pointwise weak equivalences*. These are, by definition, the natural transformations $\tau : F \to G$ such that $\tau_c : F(c) \to G(c)$ is a weak equivalence of $\mathcal{D}$ for each object c of C. If C is understood, we write $\mathcal{W}^C$ simply as $\mathcal{W}$.

1.4. Remark. The next 2-category structure on relative categories requires to deal with large categories as $\mathrm{Fun}(C, \mathcal{D})$, and even further to invert morphisms there. To be able to do this we assume Grothendieck's axiom of universes. However, when we refer to a small category, we always mean a small category with respect to a fixed universe U_0.

1.5. Definition. *A relative functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ is a weak equivalence preserving functor $F : \mathcal{C} \to \mathcal{D}$. A relative natural transformation between the relative functors $F, G : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ is a morphism $\tau : F \dashrightarrow G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})[\mathcal{W}^{-1}]$. More precisely, τ is represented by a finite zigzag connecting F and G*

$$F \cdots \bullet \to \bullet \leftarrow \bullet \to \bullet \cdots G$$

formed by functors and natural transformation between them, such that those natural transformations going to the left are pointwise weak equivalences.

1.6. Lemma. *Relative categories form a* 2*-category, with relative functors as* 1*-morphisms and relative natural transformations as* 2*-morphisms.*

The proof presents no difficulty and is left to the reader. The composition of relative functors is just the usual composition of functors. The vertical and horizontal compositions of relative functors with relative natural transformations are defined by localizing the corresponding compositions in $\mathcal{C}at$ with respect to the weak equivalences. The resulting 2-category of relative categories is denoted by $\mathcal{R}el\mathcal{C}at$.

1.7. Definition. *A relative natural transformation that is an isomorphism in* $\mathcal{R}el\mathcal{C}at$ *will be called a relative isomorphism. As usual, a relative functor* $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ *is an equivalence of relative categories if there exists a relative functor* $G : (\mathcal{D}, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$ *and relative isomorphisms* $\tau : FG \overset{\sim}{\dashrightarrow} 1_{\mathcal{D}}$, $\rho : GF \overset{\sim}{\dashrightarrow} 1_{\mathcal{C}}$. *That is,* τ *and* ρ *are invertible in* $\mathrm{Fun}(\mathcal{D}, \mathcal{D})[\mathcal{W}^{-1}]$ *and* $\mathrm{Fun}(\mathcal{C}, \mathcal{C})[\mathcal{W}^{-1}]$*, respectively.*

1.8. Remark. Recall from [BaK, 3.3] that a *homotopy equivalence* of relative categories is an equivalence of relative categories such that τ and ρ are in addition zigzags of natural weak equivalences. We remark that an equivalence of relative categories needs not be a homotopy equivalence. However, the two notions agree in case the weak equivalences considered admit a natural calculus of left (or right) fractions, or a natural 3-arrow calculus. This is the case of Brown categories of (co)fibrant objects with functorial cylinders, and of model categories (with functorial factorizations).

The following lemmas easily follow from the definitions.

1.9. Lemma. *Localization by weak equivalences is a* 2*-functor*

$$\mathrm{loc} : \mathcal{R}el\mathcal{C}at \to \mathcal{C}at \quad , \quad (\mathcal{C}, \mathcal{W}) \mapsto \mathcal{C}[\mathcal{W}^{-1}]$$

1.10. Lemma. *Given a small category* I*, exponentiation by* I *is a* 2*-functor*

$$(-)^I : \mathcal{R}el\mathcal{C}at \to \mathcal{R}el\mathcal{C}at \quad , \quad (\mathcal{C}, \mathcal{W}) \mapsto (\mathcal{C}^I, \mathcal{W}^I)$$

1.11. Relative adjunctions.

1.12. Definition. *An adjunction in the 2-category* $\mathcal{R}el\mathcal{C}at$ *will be called a relative adjunction. It consists of:*
1. *Relative functors* $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ *and* $G : (\mathcal{D}, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$.
2. *Relative natural transformations* $\alpha : FG \dashrightarrow 1_{\mathcal{D}}$ *and* $\beta : 1_{\mathcal{C}} \dashrightarrow GF$ *satisfying the triangle identities. That is, the following compositions are the identity in* $\mathcal{R}el\mathcal{C}at$

$$F \overset{F\star\beta}{\dashrightarrow} FGF \overset{\alpha\star F}{\dashrightarrow} F$$
$$G \overset{\beta\star G}{\dashrightarrow} GFG \overset{F\star\alpha}{\dashrightarrow} G$$

A relative adjunction will be denoted either by $F : (\mathcal{C}, \mathcal{W}) \leftrightarrows (\mathcal{D}, \mathcal{W}) : G$, (F, G, α, β) *or just by* (F, G)*. We will say that* F *is a left relative adjoint of (or relative left adjoint to)*

G and that G is a right relative adjoint of (or relative left adjoint to) F.
In case the relative natural transformations α and β are relative isomorphisms, (F, G) is called a relative adjoint equivalence.
Given $\tau : FT' \dashrightarrow T$, the triangle identities allow to construct as usual the (relative) adjoint natural transformation $\tau' : T' \dashrightarrow GT$, and conversely one can get a relative natural transformation τ from such a τ'.

1.13. Example. A relative adjunction $F : (\mathcal{D}, \text{isos}) \rightleftarrows (\mathcal{C}, \text{isos}) : G$ is just an adjunction of categories. More generally, an adjunction $l : \mathcal{D} \rightleftarrows \mathcal{C} : r$ is a relative adjunction $l : (\mathcal{D}, \mathcal{W}) \rightleftarrows (\mathcal{C}, \mathcal{W}) : r$ provided that both l and r preserve weak equivalences.

1.14. Example. Let $l : \mathcal{M} \rightleftarrows \mathcal{N} : r$ be a Quillen adjunction between the model categories $(\mathcal{M}, \mathcal{W})$ and $(\mathcal{N}, \mathcal{W})$. If $Q : \mathcal{M} \to \mathcal{M}$ and $R : \mathcal{N} \to \mathcal{N}$ are, respectively, functorial cofibrant and fibrant replacements, then the following is a relative adjunction

$$lQ : (\mathcal{M}, \mathcal{W}) \rightleftarrows (\mathcal{N}, \mathcal{W}) : rR$$

Next results follow from Lemmas 1.9 and 1.10, and the fact that 2-functors preserve adjunctions.

1.15. Lemma. *The localization of a relative adjunction $F : (\mathcal{C}, \mathcal{W}) \leftrightarrows (\mathcal{D}, \mathcal{W}) : G$ is an adjunction $F : \mathcal{C}[\mathcal{W}^{-1}] \leftrightarrows \mathcal{D}[\mathcal{W}^{-1}] : G$ between the corresponding localized categories.*

1.16. Lemma. *If I is a small category and $F : (\mathcal{C}, \mathcal{W}) \leftrightarrows (\mathcal{D}, \mathcal{W}) : G$ is a relative adjunction then $F^I : (\mathcal{C}^I, \mathcal{W}) \leftrightarrows (\mathcal{D}^I, \mathcal{W}) : G^I$ is again a relative adjunction.*

Then, a relevant feature of a relative adjunction (F, G) is that it does not only provide an adjunction between the corresponding localized categories, but also an adjunction $F^I : \mathcal{C}^I[\mathcal{W}^{-1}] \leftrightarrows \mathcal{D}^I[\mathcal{W}^{-1}] : G^I$ naturally defined for each small category I.

To finish, we introduce the next technical lemma for later use.

1.17. Lemma. *Let $(\mathcal{C}, \mathcal{W})$, $(\mathcal{D}, \mathcal{W})$ be relative categories, $Q : (\mathcal{D}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ a relative functor and $\rho : Q \to 1_{\mathcal{D}}$ a natural transformation which is a weak equivalence. Assume there exists a natural isomorphism*

$$\mathcal{C}(FK, L) \simeq \mathcal{D}(K, GL)$$

of bifunctors $(\mathrm{im}Q)^{\circ} \times \mathcal{C} \to \mathrm{Set}$, where $F : \mathrm{im}Q \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ are functors such that

- ***i.*** *FQ and G are relative functors.*
- ***ii.*** *The relative natural transformations $F\rho Q$ and $FQ\rho$ agree.*

Then (FQ, G) is a relative adjoint pair.

Proof. The isomorphism $\mathcal{C}(FK, L) \simeq \mathcal{D}(K, GL)$ associates a natural transformation $\alpha : FQG \to 1_{\mathcal{C}}$ with $\rho_\star G$, and $\beta' : Q \to GFQ$ with $1_\star FQ$. Then, it is not difficult to check using ii that α and $\beta : 1_{\mathcal{D}} \xleftarrow{\rho} Q \xrightarrow{\beta'} GFQ$ do satisfy the triangle identities. ∎

1.18. Realizable homotopy colimits.

1.19. Definition. *Given a small category I and a relative category $(\mathcal{C},\mathcal{W})$, the constant diagram functor $c_I:\mathcal{C}\to\mathcal{C}^I$ is defined by $(c_I(x))(i)=x$ for all $i\in I$ and $x\in\mathcal{C}$. Clearly, $c_I:(\mathcal{C},\mathcal{W})\to(\mathcal{C}^I,\mathcal{W})$ is a relative functor.*

A realizable homotopy colimit in $(\mathcal{C},\mathcal{W})$ is a relative adjunction $(\mathtt{hocolim}_I,\, c_I,\, \alpha,\, \beta)$ between $(\mathcal{C}^I,\mathcal{W})$ and $(\mathcal{C},\mathcal{W})$. For simplicity, we will often drop the adjunction morphisms and denote a realizable homotopy colimit just by $\mathtt{hocolim}_I:(\mathcal{C}^I,\mathcal{W})\to(\mathcal{C},\mathcal{W})$. When needed, we will write $\mathtt{hocolim}_I^{\mathcal{C}}$ instead of $\mathtt{hocolim}_I$ to emphasize the target category where we are taking homotopy colimits.

A realizable homotopy limit is defined in the dual way. Throughout the paper we focus on realizable homotopy colimits, but all the constructions and results presented here can be dualized to the setting of realizable homotopy limits.

1.20. Example. Note that a usual colimit in $\mathcal{C}$ is the same thing as a realizable homotopy colimit on $(\mathcal{C},\mathcal{W}=\{\text{isomorphisms}\})$.

1.21. Example. If $(\mathcal{M},\mathcal{W})$ is a *combinatorial* model category, the projective model structure on $(\mathcal{M}^I,\mathcal{W})$ is such that $(\mathtt{colim}_I, c_I)$ is a Quillen adjunction. In view of example 1.14, realizable homotopy colimits exist in $(\mathcal{M},\mathcal{W})$ and are the composition of $\mathtt{colim}_I$ with a cofibrant replacement. Realizable homotopy limits are constructed analogously, this time using the injective model structure on $(\mathcal{M}^I,\mathcal{W})$ instead.

1.22. Example. For a general model category $(\mathcal{M},\mathcal{W})$ the above argument is no longer valid because a model structure on $(\mathcal{M}^I,\mathcal{W})$ does not necessarily exist. As a consequence, the proof of the existence of homotopy (co)limits in $(\mathcal{M},\mathcal{W})$ is more involved (see [CS], [C] or [DHKS]). The arguments in these references may be adapted to the setting of relative categories to show that $(\mathcal{M}^I,\mathcal{W})$ has all realizable homotopy (co)limits as well. In addition, these realizable homotopy colimits may be computed using the Bousfield-Kan homotopy colimit construction of [H] (see Section 4).

Further examples of realizable homotopy (co)limits will be given at the end of the paper for chain complexes on (AB4) abelian categories, (filtered) commutative dg algebras and mixed Hodge complexes.

Next we explore some of the properties of realizable homotopy colimits. Firstly we observe that, after localizing by the weak equivalences, they are homotopy colimits in the sense of Grothendieck derivators, hence also left derived functors of usual colimits.

1.23. Proposition. *If $\mathtt{hocolim}_I:(\mathcal{C}^I,\mathcal{W})\to(\mathcal{C},\mathcal{W})$ is a realizable homotopy colimit in $(\mathcal{C},\mathcal{W})$ then $\mathtt{hocolim}_I:\mathcal{C}^I[\mathcal{W}^{-1}]\to\mathcal{C}[\mathcal{W}^{-1}]$ is a homotopy colimit in the sense of Grothendieck derivators. In particular, if the colimit $\mathtt{colim}_I:\mathcal{C}^I\to\mathcal{C}$ exists, then the absolute left derived functor $\mathbb{L}\mathtt{colim}_I$ of $\mathtt{colim}_I$ exists and it agrees with $\mathtt{hocolim}_I:\mathcal{C}^I[\mathcal{W}^{-1}]\to\mathcal{C}[\mathcal{W}^{-1}]$.*

Proof. By Lemma 1.15, the localization of a realizable homotopy colimit produces a left adjoint $\mathtt{hocolim}_I : \mathcal{C}^I[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}]$ of the localized constant diagram functor, i.e., a homotopy colimit in the sense of Grothendieck derivators. The second assertion then follows from [RII, Proposition 4.2]. ∎

An advantage of working with realizable homotopy colimits is that, in contrast to $\mathbb{L}\mathtt{colim}$, they are inherited by diagram categories.

1.24. Lemma. *If* $\mathtt{hocolim}_I : (\mathcal{C}^I, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$ *is a realizable homotopy colimit in* $(\mathcal{C}, \mathcal{W})$ *and* J *is a small category, then the relative functor* $\mathtt{hocolim}_I^J : (\mathcal{C}^{I\times J}, \mathcal{W}) \to (\mathcal{C}^J, \mathcal{W})$ *induced pointwise is a realizable homotopy colimit in* $(\mathcal{C}^J, \mathcal{W})$.

Proof. This follows from Lemma 1.16. ∎

Also, the Fubini property for realizable homotopy colimits is, as happens with colimits, a formal consequence of adjointness.

1.25. Proposition. *Let* $(\mathcal{C}, \mathcal{W})$ *be a relative category. Given small categories* I *and* J, *there is a unique relative isomorphism*

$$\mathtt{hocolim}_{I\times J} \dashrightarrow \mathtt{hocolim}_I \mathtt{hocolim}_J$$

compatible with the adjunction morphisms.

Proof. It follows from Lemma 1.24 and the fact that relative adjunctions are closed under composition that $\mathtt{hocolim}_I\mathtt{hocolim}_J$ is relative left adjoint to $c_J c_I = c_{I\times J} : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}^{I\times J}, \mathcal{W})$. But so is $\mathtt{hocolim}_{I\times J}$, hence the claim follows. ∎

There are other properties of colimits which are formal consequences of adjointness in $\mathcal{C}at$, for instance, they are invariant under right cofinal changes of diagrams. In the homotopical setting the property corresponding to right cofinality is the one of homotopy right cofinality, which we proceed to recall.

1.26. Definition. *A functor* $f : I \to J$ *is homotopy right cofinal if for each* $j \in J$ *the simplicial nerve* $\mathrm{N}(j/f)$ *of the undercategory* (j/f) *is weakly contractible. That is,* $\mathrm{N}(j/f) \to \Delta[0]$ *is a weak homotopy equivalence of simplicial sets.*

1.27. Given a functor $f : I \to J$ between small categories, consider the relative natural transformation

$$\mathtt{hocolim}(f) : \mathtt{hocolim}_I f^* \dashrightarrow \mathtt{hocolim}_J$$

defined as the relative adjunct natural transformation of $f^* \star \beta : f^* \to f^* c_J \mathtt{hocolim}_J = c_I \mathtt{hocolim}_J$, where $\beta : 1_{\mathcal{C}^J} \to c_J \mathtt{hocolim}_J$ is the adjunction morphism of $(\mathtt{hocolim}_J, c_J)$.

1.28. Definition. *We say that realizable homotopy colimits in* $(\mathcal{C}, \mathcal{W})$ *satisfy the cofinality property if for each homotopy right cofinal functor* f, $\mathtt{hocolim}(f)$ *is a relative isomorphism.*

1.29. Remark. There is no known example of (realizable) homotopy colimit which does not satisfy this property. However, as remarked in [M, Remarque 4.23], it does not seem to be a formal consequence of the definition.

To finish, we study the preservation of realizable homotopy colimits by relative functors.

1.30. Definition. *Given a relative functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$, consider the relative natural transformation*

$$\rho_F^I : \mathtt{hocolim}_I^{\mathcal{D}} F \dashrightarrow F\, \mathtt{hocolim}_I^{\mathcal{C}}$$

adjoint to $F\star\beta : F \dashrightarrow F c_I \mathtt{hocolim}_I^{\mathcal{C}} = c_I F\, \mathtt{hocolim}_I^{\mathcal{C}}$, where $\beta : 1_{\mathcal{C}^I} \dashrightarrow c_I \mathtt{hocolim}_I^{\mathcal{C}}$ is the adjunction morphism of $(\mathtt{hocolim}_I^{\mathcal{C}}, c_I)$. Then, the relative functor F commutes with I-homotopy colimits if ρ_F^I is a relative isomorphism. Note that this definition does not depend on the representatives chosen for $\mathtt{hocolim}_I^{\mathcal{C}}$ and $\mathtt{hocolim}_I^{\mathcal{D}}$.
If F commutes with I-homotopy colimits for each small category I we simply say that F commutes with homotopy colimits. In case F commutes with Ω-homotopy colimits for each discrete category Ω, we say that F commutes with homotopy coproducts.

Next proposition follows from the fact that relative adjunctions are closed under composition and exponentiation.

1.31. Proposition. *If $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ is a relative left adjoint, then it commutes with homotopy colimits.*

2. Voevodsky homotopy colimits are realizable.

In this section we construct homotopy colimits and Kan extensions for relative categories $(\Delta^\circ\mathcal{C}, \mathcal{S})$ such that the class of weak equivalences $\mathcal{S}$ is a Δ-closed class in the sense of [V]. These are indeed examples of simplicial descent categories for which the simple functor is just the diagonal $\mathrm{D} : \Delta^\circ\Delta^\circ\mathcal{C} \to \Delta^\circ\mathcal{C}$. We will strongly use well-known results and techniques from simplicial homotopy theory, as well as others not so standard, such as Illusie's bisimplicial decalage ([I]).

2.1. Simplicial preliminaries. We assume the reader is familiar with the basics on simplicial objects, and is referred to the classical references [May] and [GZ] for further details.

Recall that the *simplicial category* Δ has as objects the ordered sets $[n] = \{0 < \cdots < n\}$, $n \geq 0$, and as morphisms the order preserving maps. The *face* and *degeneracy maps* are denoted, respectively, by $d^i : [n-1] \to [n]$ and $s^j : [n+1] \to [n]$.

If $\mathcal{D}$ is a category, $\Delta^\circ\mathcal{D}$ (resp. $\Delta^\circ\Delta^\circ\mathcal{D}$, $\Delta\mathcal{D}$) denotes the category of *simplicial* (resp. *bisimplicial*, *cosimplicial*) *objects* in $\mathcal{D}$. The *diagonal functor* $\mathrm{D} : \Delta^\circ\Delta^\circ\mathcal{D} \to \Delta^\circ\mathcal{D}$ is given by $\mathrm{D}(\{Z_{n,m}\}_{n,m\geq 0}) = \{Z_{n,n}\}_{n\geq 0}$. The *constant functor* $c_{\Delta^\circ} : \mathcal{D} \to \Delta^\circ\mathcal{D}$ is written simply as c. Given an object A of $\mathcal{D}$ and a simplicial object X, an *augmentation* $\epsilon : X \to A$ is just a simplicial morphism $\epsilon : X \to c(A)$. If X is a simplicial object, $c^{\mathrm{I}}(X)$ and $c^{\mathrm{II}}(X)$ denote the bisimplicial objects which are constant in the first and second index, respectively. That is, $c^{\mathrm{I}}(X)_{n,m} = X_m$ and $c^{\mathrm{II}}(X)_{n,m} = X_n$.

There is a combinatorial notion of homotopic morphisms $f \sim g$ in $\Delta^{\circ}\mathcal{D}$. If $\mathcal{D}$ has coproducts, this homotopy relation may be described as follows.

2.2. Definition. *Consider the natural action $\Delta^{\circ}\mathcal{D} \times \Delta^{\circ}Set \to \Delta^{\circ}\mathcal{D}$, sending $(X, K) \mapsto X \otimes K$, where*

$$(X \otimes K)_n = \coprod_{K_n} X_n \tag{1}$$

Recall that $\Delta[k]$ is the simplicial set with $\Delta[k]_n = Hom_{\Delta}([n],[k])$. Then, $X \otimes \Delta[1]$ is the simplicial cylinder of X, and the maps $d^0, d^1 : [0] \to [1]$ induce $d_0^X, d_1^X : X \to X \otimes \Delta[1]$. The simplicial morphisms $f, g : X \to Y$ are simplicially homotopic if there exists a homotopy $H : X \otimes \Delta[1] \to Y$ such that $Hd_0^X = f$ and $Hd_1^X = g$. Simplicial homotopy equivalences are defined as usual.

We will use the construction and properties of Illusie's *bisimplicial decalage dec* : $\Delta^{\circ}\mathcal{D} \to \Delta^{\circ}\Delta^{\circ}\mathcal{D}$ introduced in [I, p.7]. It is induced by the ordinal sum $\Delta \times \Delta \to \Delta$, $[n] + [m] = [n + m + 1]$.

2.3. Definition. *Given a simplicial object X, $dec(X)$ is the bisimplicial object given in bidegree (n, m) by $dec(X)_{n,m} = X_{n+m+1}$. The face and degeneracy maps of $dec(X)$ are defined as follows. On the one hand, $d_k^I : dec(X)_{n,m} \to dec(X)_{n-1,m}$ is $d_k : X_{n+m+1} \to X_{n+m}$, while s_k^I is $s_k : X_{n+m+1} \to X_{n+m+2}$. On the other hand, $d_k^{II} : dec(X)_{n,m} \to dec(X)_{n,m-1}$ is $d_{n+k+1} : X_{n+m+1} \to X_{n+m}$, and s^{II} is $s_{n+k+1} : X_{n+m+1} \to X_{n+m+2}$. There are two natural augmentations $\Lambda^I : dec(X) \to c^{\mathrm{I}}(X)$ and $\Lambda^{II} : dec(X) \to c^{\mathrm{II}}(X)$ given respectively by $\Lambda^I_{0,m} = d_0 : X_{m+1} \to X_m$ and $\Lambda^{II}_{n,0} = d_{n+1} : X_{n+1} \to X_n$.*

2.4. Proposition. *For each simplicial object X, the diagonals of the augmentations Λ^I and Λ^{II}, $\mathrm{D}(\Lambda^I), \mathrm{D}(\Lambda^{II}) : \mathrm{D}(dec(X)) \to X$ are simplicial homotopy equivalences which are simplicially homotopic, and the homotopies involved are natural in X.*

Proof. The first statement is [I, Proposition 1.6.2], while the last one follows from the proof given in loc. cit.. ∎

To close the preliminaries we recall from [BK, XI.5] the simplicial replacement of diagrams, which is the building block of Voevodsky homotopy colimits.

2.5. Definition. *The simplicial replacement functor $\amalg^I : \mathcal{D}^I \longrightarrow \Delta^{\circ}\mathcal{D}$ maps the diagram $X : I \to \mathcal{D}$ to the simplicial object $\amalg^I X$ given in degree n by*

$$\amalg^I_n X = \coprod_{\underline{i} = \{i_0 \to \cdots \to i_n\}} X(i_0)^{\underline{i}}$$

where $X(i_0)^{\underline{i}}$ is a copy of $X(i_0)$. The coproduct is indexed over the n-simplices of the simplicial nerve of I. The face map $d_k : \amalg^I_n X \to \amalg^I_{n-1} X$ is the coproduct of $1 : X(i_0)^{\underline{i}} \to$

$X(i_0)^{d_k(\underline{i})}$ *if* $k>0$, *and of* $X(i_0\to i_1): X(i_0)^{\underline{i}}\to X(i_1)^{d_0(\underline{i})}$ *if* $k=0$. *The degeneracy map* $s_k:\amalg_n^I X\to \amalg_{n+1}^I X$ *is the coproduct of the maps* $1: X(i_0)^{\underline{i}}\to X(i_0)^{s_k(\underline{i})}$. *The cosimplicial replacement functor* $\Pi^I:\mathcal{D}^I\longrightarrow \Delta\mathcal{D}$ *is the dual construction.*

Note that the colimit of X, if it exists, agrees with the colimit of $\amalg^I X$, giving rise to an augmentation $\amalg^I X\to \mathtt{colim}_I X$. To finish, let us note that $\amalg^I$ is natural in I. Indeed, given a morphism $(f,\tau): X\to Y$ between the diagrams $X: I\to\mathcal{D}$ and $Y: J\to \mathcal{D}$ - that is, a functor $f: I\to J$ and a natural transformation $\tau: X\to f^*Y$ - then the morphisms $\tau_{i_0}: X(i_0)^{\underline{i}}\to Y(f(i_0))^{f(\underline{i})}$ induce in a natural way

$$\amalg^\bullet(f,\tau):\amalg^I X\to \amalg^J Y \tag{2}$$

2.6. Voevodsky homotopy colimits. To begin with, let us recall the definition and first properties of Δ-closed classes.

2.7. Definition. *([V]) A class $\mathcal{S}$ of morphisms in $\Delta^\circ\mathcal{C}$ is Δ-closed if it satisfies the following axioms:*

0. The class $\mathcal{S}$ satisfies the 2-out-of-3 property.

1. The class $\mathcal{S}$ contains the simplicial homotopy equivalences.

2. If $F=F_{\cdot,\cdot}: Z_{\cdot,\cdot}\to T_{\cdot,\cdot}$ is a morphism of bisimplicial objects in $\mathcal{C}$ such that $F_{n,\cdot}\in\mathcal{S}$ (or $F_{\cdot,n}\in\mathcal{S}$) for all $n\geq 0$, the diagonal $\mathrm{D}(F)$ of F is in $\mathcal{S}$.

3. The class $\mathcal{S}$ is closed under finite coproducts. That is, $F\amalg G\in\mathcal{S}$ whenever F, $G\in\mathcal{S}$.

Note that hypothesis *0* always holds in our context, since we restrict ourselves to classes of weak equivalences which are saturated.

Also, we remark that assumption *3* is not included in Voevodsky's original definition. We opt to include it here because the resulting Δ-closed classes support a richer homotopical structure. For instance, they produce natural Brown structures of cofibrant objects ([Br]). This implies, in particular, the existence of a calculus of left fractions up to homotopy, and of cofiber sequences enjoying the usual properties.

2.8. Definition. *([V]) A morphism $F: X\to Y$ in $\Delta^\circ\mathcal{C}$ is a termwise coprojection if for each $n\geq 0$ there exists an object $A^{(n)}$ of $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} X_n \arrow[r, "F_n"] \arrow[dr] & Y_n \arrow[d, "\wr"] \\ & X_n\amalg A^{(n)} \end{tikzcd}$$

where $X_n\to X_n\amalg A^{(n)}$ is the canonical morphism.

2.9. Proposition. ([R, Proposition 4.9]) *Let $(\Delta^\circ\mathcal{C},\mathcal{S})$ be a relative category such that $\mathcal{S}$ is Δ-closed. Then $(\Delta^\circ\mathcal{C},\mathcal{S})$ is a Brown category of cofibrant objects, where the cofibrations are the termwise coprojections.*

2.10. Remark. By Lemma 2.25 and Proposition 2.9, under the previous assumptions $(\Delta^\circ\mathcal{C}, \mathcal{S})$ is moreover an ABC cofibration category in the sense of [RB]. As proved in loc. cit. this already guarantees the existence of homotopy colimits in $\Delta^\circ\mathcal{C}[\mathcal{S}^{-1}]$. We will see below that in addition they may be computed by Voevodsky's formula, and satisfy the cofinality property.

2.11. Definition. *[V, p. 11] The Voevodsky homotopy colimit*

$$\mathtt{hocolim}_I^{\mathrm{V}} : \Delta^\circ\mathcal{C}^I \longrightarrow \Delta^\circ\mathcal{C}$$

is the composition of the diagonal functor $\mathrm{D} : \Delta^\circ\Delta^\circ\mathcal{C} \to \Delta^\circ\mathcal{C}$ *with the simplicial replacement* $\amalg^I : \Delta^\circ\mathcal{C}^I \to \Delta^\circ\Delta^\circ\mathcal{C}$*. Given* $Z : J \to \Delta^\circ\mathcal{C}$*, then in degree* n

$$(\mathtt{hocolim}_I^{\mathrm{V}} Z)_n = \coprod_{i_0 \to \cdots \to i_n} Z_n(i_0)$$

If $f : I \to J$*, there is a natural morphism* $\mathtt{hocolim}_I^{\mathrm{V}} f^* Z \to \mathtt{hocolim}_J^{\mathrm{V}} Z$ *defined as the diagonal of* $\amalg(f, 1_{f^*Z})$ *(see (2)). Also, if* $\mathtt{colim}_I Z$ *exists, the augmentation* $\amalg^I Z \to \mathtt{colim}_I Z$ *induces a natural simplicial morphism* $\mathtt{hocolim}_I^{\mathrm{V}} Z \to \mathtt{colim}_I Z$*.*

Unless otherwise stated, we assume from now on that $\mathcal{S}$ is Δ-closed The rest of the section is devoted to proving the following two facts:

- (i) $\mathtt{hocolim}_I^{\mathrm{V}}$ is a realizable homotopy colimit, and it moreover produces pointwise homotopy left Kan extensions (Theorem 2.14).
- (ii) $\mathtt{hocolim}_I^{\mathrm{V}}$ satisfies the cofinality property (Corollary 2.23).

2.12. Homotopy Kan extensions. Recall that given a cocomplete category $\mathcal{D}$ and a functor $f : I \to J$ between small categories, the left adjoint $f_! : \mathcal{D}^I \to \mathcal{D}^J$ of f^* is

$$(f_! X)(j) = \mathtt{colim}_{(f/j)} u_j^* X$$

where $u_j : (f/j) \to I$ maps $\{f(i) \to j\}$ to i. Next we prove that replacing $\mathtt{colim}$ with $\mathtt{hocolim}^V$ in the above formula does produce a relative left adjoint of f^* in $\mathcal{R}el\mathcal{C}at$.

2.13. Definition. *Define* $f_!^{\mathrm{V}} : (\Delta^\circ\mathcal{C}^I, \mathcal{S}) \to (\Delta^\circ\mathcal{C}^J, \mathcal{S})$ *as*

$$(f_!^{\mathrm{V}} X)(j) = \mathtt{hocolim}_{(f/j)}^{\mathrm{V}} u_j^* X$$

2.14. Theorem. *Let* $f : I \to J$ *be a functor between small categories. Assume given a* Δ*-closed class* $\mathcal{S}$ *which is in addition closed under coproducts of cardinality* $\#I$*. Then,* $(f_!^{\mathrm{V}}, f^*)$ *is a relative adjoint pair between* $(\Delta^\circ\mathcal{C}^I, \mathcal{S})$ *and* $(\Delta^\circ\mathcal{C}^J, \mathcal{S})$*.*

Here $\#I$ means the cardinality of the set of morphisms of I. This ensures that all the $\mathtt{hocolim}_{(f/j)}^{\mathrm{V}}$ make sense and preserve weak equivalences. Note also that this property is implied by assumption *3* in case I is a finite category.

2.15. Remark. In particular, for $J = [0]$, the above theorem states that $\mathtt{hocolim}^{\mathrm{V}}_I$ is a realizable homotopy colimit.

The proof of previous theorem relies on the particular case of $f_!$ corresponding to $f = 1_I : I \to I$. The resulting $Q = 1^{\mathrm{V}}_! : \Delta^\circ\mathcal{C}^I \to \Delta^\circ\mathcal{C}^I$ is analogous to Dugger's 'replacement' functor [D, p. 30], that is,

$$(QX)(j) = \mathtt{hocolim}^{\mathrm{V}}_{(I/i)} u_i^* X$$

To begin with, let us construct a natural transformation $\rho : Q \to 1_{\Delta^\circ\mathcal{C}^I}$ which is a point-wise weak equivalence. Consider the triangle

$$\begin{tikzcd} (I/i) \arrow[r, "\pi"] \arrow[dr, "u_i^* X"'] \arrow[dr, phantom, shift left=3, "\overset{\tau}{\Rightarrow}"] & {[0]} \arrow[d, "X(i)"] \\ & \Delta^\circ\mathcal{C} \end{tikzcd}$$

where $\tau_{\alpha : i' \to i} = X(\alpha) : X(i') \to X(i)$. Then, $\rho_X : QX \to X$ is given by

$$\rho_X(i) = \mathtt{hocolim}^{\mathrm{V}}(\pi, \tau) : \mathtt{hocolim}^{\mathrm{V}}_{(I/i)} u_i^* X \to \mathtt{hocolim}^{\mathrm{V}}_{[0]} X(i) = X(i)$$

2.16. Lemma. *The natural transformation ρ is a point-wise weak equivalence.*

Proof. For each $i \in I$ the augmentation $\rho_X(i)$ has an extra degeneracy $s_{n+1} : \amalg_n^{(I/i)} X \to \amalg_{n+1}^{(I/i)} X$ which sends the component $X(i_0)$ with index $\underline{i} = \{i_0 \to \ldots \to i_n \to i\}$ to the component $X(i_0)$ with index $\{i_0 \to \ldots \to i_n \to i \to i\}$. Then, $\rho_X(i)$ is a simplicial homotopy equivalence and in particular it belongs to $\mathcal{S}$. ∎

Moreover, Q is a kind of 'resolution functor' for $f_!$:

2.17. Lemma. *Given $X \in \Delta^\circ\mathcal{C}^I$, the left Kan extension $f_!$ exists for any diagram of the form QX and*

$$f_! QX = f_!^{\mathrm{V}} X$$

Proof. The result is a direct consequence of the canonical isomorphism

$$\mathtt{colim}_{\alpha : f(i) \to j} \amalg^{(I/i)} v_i^* X \simeq \amalg^{(f/j)} u_j^* X$$

and the fact that colimits of simplicial objects are computed degreewise. ∎

Proof of Theorem 2.14. Note that $f_!^{\mathrm{V}} : (\Delta^\circ\mathcal{C}^I, \mathcal{S}) \to (\Delta^\circ\mathcal{C}^J, \mathcal{S})$ is a relative functor under the hypotheses made on f and $\mathcal{S}$. We will make use of Lemma 1.17 with $F = f_!$ and $G = f_*$. In view of Lemmas 2.16 and 2.17, it only remains to show that the relative natural transformations $f_!\rho Q$, $f_! Q\rho : f_! Q^2 \to f_! Q$ agree. But unravelling the definitions one finds that $(f_! Q^2 X)(j) = \mathtt{hocolim}_{\alpha : f(i) \to j} \mathtt{hocolim}_{(I/i)} u_i^* X$ is precisely the diagonal of $Dec(\amalg^{(f/j)} u_j^* X)$. Moreover, $(f_!\rho Q)(j)$ and $(f_! Q\rho)(j)$ correspond, respectively, to the diagonals of the augmentations Λ^{II} and Λ^I of Definition 2.3. Hence, by Proposition 2.4, we conclude that $f_!\rho Q$ and $f_! Q\rho$ are simplicially homotopic, and consequently they agree as relative natural transformations. ∎

2.18. Cofinality. To close the section we now show that $\mathtt{hocolim}^{\mathrm{V}}$ satisfies the cofinality property. This is a consequence of the following stronger property, that will be used later to show that a simplicial descent category (in particular our $(\Delta^{\circ}\mathcal{C},\mathcal{S})$) produces a right derivator.

2.19. Theorem. *Let $f : I \to J$ and $g : J \to K$ be functors between small categories and set $h = gf : I \to K$. Assume that for each $k \in K$ the natural transformation*

$$\mathtt{hocolim}^{\mathrm{V}}_{(k/h)} c_{(k/h)} \to \mathtt{hocolim}^{\mathrm{V}}_{(k/g)} c_{(k/g)}$$

induced by f belongs (pointwise) to $\mathcal{S}$. Then, the natural transformation

$$\mathtt{hocolim}^{\mathrm{V}}_{I} h^{*} \longrightarrow \mathtt{hocolim}^{\mathrm{V}}_{J} g^{*}$$

belongs (pointwise) to $\mathcal{S}$ as well.

For the proof we will need the following connection between the bar construction and Illusie's bisimplicial decalage.

2.20. Definition. *The simplicial two-sided bar construction associated with a bifunctor $F : I \times I^{\circ} \to \mathcal{D}$ is the simplicial object $\mathrm{W}(F)$ given by*

$$\mathrm{W}_n(F) = \coprod_{\underline{i}=\{i_0\to\cdots\to i_n\}} F(i_0,i_n)^{\underline{i}}$$

where $F(i_0,i_n)^{\underline{i}}$ is a copy of $F(i_0,i_n)$. The face map $d_k : \mathrm{W}_n(F) \to \mathrm{W}_{n-1}(F)$ is the coproduct of the morphisms $1 : F(i_0,i_n)^{\underline{i}} \to F(i_0,i_n)^{d_k(\underline{i})}$ if $0<k<n$, of $F(i_0 \to i_1, 1_{i_n}) : F(i_0,i_n)^{\underline{i}} \to F(i_1,i_n)^{d_0(\underline{i})}$ if $k=0$ and of $F(1_{i_0}, i_{n-1}\to i_n) : F(i_0,i_n)^{\underline{i}} \to F(i_0,i_{n-1})^{d_n(\underline{i})}$ if $k=n$. To finish, the degeneracy map $s_k : \mathrm{W}_n(F) \to \mathrm{W}_{n+1}(F)$ is the coproduct of $1 : F(i_0,i_n)^{\underline{i}} \to F(i_0,i_n)^{s_k(\underline{i})}$.

2.21. Example. Given a functor $f : I \to J$ and a diagram $Y : J \to \mathcal{D}$, we have an induced bifunctor $Y \otimes \mathrm{N}(\cdot/f) : J \times J^{\circ} \to \Delta^{\circ}\mathcal{D}$ defined as

$$\left(Y \otimes \mathrm{N}(\cdot/f)\right)(j,j') = Y(j) \otimes \mathrm{N}(j'/f)$$

Then $\mathrm{W}(X \otimes \mathrm{N}(\cdot/f))$ is the bisimplicial object given in bidegree (n,m) by

$$\mathrm{W}_{n,m}(X \otimes \mathrm{N}(\cdot/f)) = \coprod_{j_0\to\cdots\to j_n} \coprod_{j_n \to f(i_0)\to\cdots\to f(i_m)} Y(j_0)$$

It has two natural augmentations $\alpha : \mathrm{W}(Y \otimes \mathrm{N}(\cdot/f)) \to c^{\mathrm{I}}(\amalg^{I} f^{*}Y)$ and $\beta : \mathrm{W}(Y \otimes \mathrm{N}(\cdot/f)) \to c^{\mathrm{II}}(\amalg^{J} Y)$ given respectively by

$$\alpha_{n,m} : \coprod_{\substack{j_0 \to\cdots\to j_n \\ \downarrow \\ f(i_m)\leftarrow\cdots\leftarrow f(i_0)}} Y(j_0) \longrightarrow \coprod_{i_0\to\cdots\to i_m} Y(f(i_0)) \quad \beta_{n,m} : \coprod_{\substack{j_0 \to\cdots\to j_n \\ \downarrow \\ f(i_m)\leftarrow\cdots\leftarrow f(i_0)}} Y(j_0) \longrightarrow \coprod_{j_0\to\cdots\to j_n} Y(j_0)$$

When $f = 1_I : I \to I$, the resulting $\mathrm{W}(Y \otimes \mathrm{N}(\cdot/I))$, α and β admit the following equivalent description.

2.22. Lemma. *For any diagram $X : I \to \mathcal{D}$, $\mathrm{W}(X \otimes \mathrm{N}(\cdot/I))$ is canonically isomorphic to Illusie's decalage dec$(\amalg^I X)$ of $\amalg^I X$ in such a way that α corresponds to $\Lambda^I : \mathrm{W}(X \otimes \mathrm{N}(\cdot/I)) \to c^{\mathrm{I}}(\amalg^I X)$ and β corresponds to $\Lambda^{II} : \mathrm{W}(X \otimes \mathrm{N}(\cdot/I)) \to c^{\mathrm{II}}(\amalg^I X)$.*

Proof of Theorem 2.19. The assumption made on f and g means that for each $k \in K$ and $X \in \Delta^\circ\mathcal{C}$, the morphism $X \otimes \mathrm{N}(k/h) \to X \otimes \mathrm{N}(k/g)$ belongs to $\mathcal{S}$. Given $Y : K \to \Delta^\circ\mathcal{C}$, the natural morphism $\mathtt{hocolim}^{\mathrm{V}}_I h^*Y \longrightarrow \mathtt{hocolim}^{\mathrm{V}}_J g^*Y$ is by definition the diagonal of the bisimplicial morphism $\amalg^I_n h^*Y_m \to \amalg^J_n g^*Y_m$ induced by f. Since $\mathcal{S}$ is Δ-closed, we may assume that Y is constant in m, i.e. $Y : J \to \mathcal{C}$, and we must prove that $\amalg^I h^*Y \to \amalg^J g^*Y$ is in $\mathcal{S}$. Under the notations of Example 2.21, we have a commutative square

$$\begin{tikzcd}
\amalg^I h^*Y \arrow[rr] & & \amalg^J g^*Y \\
\mathrm{DW}(Y \otimes \mathrm{N}(\cdot/h)) \arrow[u, "\mathrm{D}(\alpha)"] \arrow[rr, "\Phi"] & & \mathrm{DW}(Y \otimes \mathrm{N}(\cdot/g)) \arrow[u, "\mathrm{D}(\alpha')"']
\end{tikzcd}$$

and it suffices to see that Φ, $\mathrm{D}(\alpha)$ and $\mathrm{D}(\alpha')$ belong to $\mathcal{S}$. By definition, Φ is the diagonal of the bisimplicial morphism

$$\phi_{n,m} : \coprod_{k_0 \to \cdots \to k_n} Y(k_0) \otimes \mathrm{N}_m(k_n/h) \to \coprod_{k_0 \to \cdots \to k_n} Y(k_0) \otimes \mathrm{N}_m(k_n/g)$$

But for fixed n, $\phi_{n,\cdot}$ is just the coproduct of the morphisms $Y(k_0) \otimes \mathrm{N}_m(k_n/h) \to Y(k_0) \otimes \mathrm{N}_m(k_n/g)$, that belong to $\mathcal{S}$ by hypothesis. Hence $\Phi = \mathrm{D}(\phi)$ is in $\mathcal{S}$. To finish, it remains to show that $\mathrm{D}(\alpha) \in \mathcal{S}$ (the same would hold for α'). Note that α is in bidegree (n, m) given by

$$\coprod_{i_0 \to \cdots \to i_m} \coprod_{k_0 \to \cdots \to k_n \to h(i_0)} Y(k_0) \longrightarrow \coprod_{i_0 \to \cdots \to i_m} Y(h(i_0))$$

so $\alpha_{\cdot,m}$ is the coproduct of the morphisms $\alpha^{\underline{i}}_{\cdot,m} : \amalg^{(K/h(i_0))} Y \to Y(h(i_0))$ induced by the final object of $(K/h(i_0))$. By Lemma 2.16, $\alpha^{\underline{i}}_{\cdot,m} = \rho_Y(h(i_0)) \in \mathcal{S}$. ∎

2.23. Corollary. *If $f : I \to J$ is a homotopy right cofinal functor, then for each diagram $Z : J \to \Delta^\circ\mathcal{C}$ the induced morphism $\mathtt{hocolim}^{\mathrm{V}}_I f^*Z \to \mathtt{hocolim}^{\mathrm{V}}_J Z$ is in $\mathcal{S}$. Consequently, $\mathtt{hocolim}^{\mathrm{V}}$ satisfies the cofinality property.*

Applying previous theorem to $Z = c_J(\Delta[0]) : J \to \Delta^\circ Set$, we obtain another proof for Quillen's Theorem A (cf. [Q]).

Proof. For each $j \in J$, $\mathrm{N}(j/f) \to \Delta[0]$ is by assumption a weak homotopy equivalence, and then so is the natural map $\mathrm{N}(j/f) \to \mathrm{N}(j/J)$. It follows from Lemma 2.24 below that for each $j \in J$, the induced morphism $X \otimes \mathrm{N}(j/f) \to X \otimes \mathrm{N}(j/J)$ is in $\mathcal{S}$. Hence, the hypothesis of previous theorem hold for f and $g = 1_J : J \to J$. We then conclude that the induced map $\mathtt{hocolim}^{\mathrm{V}}_I f^* \to \mathtt{hocolim}^{\mathrm{V}}_J$ is pointwise in $\mathcal{S}$ as required. ∎

2.24. Lemma. *If $f : L \to K$ is a map in $\Delta^\circ Set$ which is a weak homotopy equivalence, then for each simplicial object X it holds that $X \otimes f : X \otimes L \to X \otimes K$ is in $\mathcal{S}$.*

Proof. The proof uses an standard argument based on the Ex_∞ fibrant replacement of simplicial sets and anodyne extensions. The resolution functor Ex_∞ provides, for each simplicial set L, a natural weak homotopy equivalence $\epsilon_L : L \to Ex_\infty(L)$ such that $Ex_\infty(L)$ is fibrant. In addition, ϵ_L is an an anodyne extension (see [GZ, p. 68]). If $f : L \to K$ is a weak homotopy equivalence, we conclude that $Ex_\infty(f)$ is a weak homotopy equivalence between fibrant simplicial sets, and hence a simplicial homotopy equivalence. If X is a simplicial object of $\mathcal{C}$, it follows that $X \otimes Ex_\infty(f) \in \mathcal{S}$, and it suffices to prove that $X \otimes \epsilon_K$ and $X \otimes \epsilon_L$ are in $\mathcal{S}$. Therefore, we may assume that f is an anodyne extension. Recall that anodyne extensions form the smallest class $\mathcal{A}$ of inclusions of simplicial sets containing the horn-fillers $i_{k,n} : \Lambda^k[n] \to \Delta[n]$ for $0 \leq k \leq n$, and such that it is closed under retracts, cobase change, small coproducts and sequential colimits (see [GZ, IV.2]). Let W' be the class of morphisms consisting of the inclusions $f : K \to L$ of simplicial sets such that the resulting termwise coprojection $X \otimes f$ of $\Delta^\circ\mathcal{C}$ is in $\mathcal{S}$ for each simplicial object X. Since the horn-fillers $i_{k,n}$ are simplicial homotopy equivalences, they are in W'. Also, W' is closed under retracts and small coproducts because $\mathcal{S}$ is. Finally, W' is closed under cobase change and sequential colimits because trivial termwise coprojections in $(\Delta^\circ\mathcal{C}, \mathcal{S})$ are closed under them by Proposition 2.9 and Lemma 2.25. Consequently, $f \in \mathcal{A} \subset \mathrm{W}'$. ∎

2.25. Lemma. *Assume given a sequence $X^0 \overset{a^0}{\rightarrowtail} X^1 \overset{a^1}{\rightarrowtail} \cdots \overset{a^{m-1}}{\rightarrowtail} X^m \overset{a^m}{\rightarrowtail} \cdots$ of termwise coprojections in $\Delta^\circ\mathcal{C}$. Then $\mathtt{colim}_m X^m$ exists, and if in addition each $a^m \in \mathcal{S}$, the canonical morphism $X^0 \to \mathtt{colim}_m X^m$ is in $\mathcal{S}$ as well.*

Proof. Assume given termwise coprojections $a^m : X^m \rightarrowtail X^{m+1}$ for each $m \geq 0$. Being $\mathcal{C}$ closed under coproducts, $\mathtt{colim}_m X^m$ does exist in $\Delta^\circ\mathcal{C}$ because it exists degreewise. Denote by $\mathbb{N}$ the category given by the poset of natural numbers, and consider the commutative square

$$\begin{array}{ccc}
\mathtt{hocolim}^{\mathrm{V}}_{\mathbb{N}} X^0 = X^0 \otimes \mathrm{N}(\mathbb{N}) & \xrightarrow{\alpha'} & \mathtt{hocolim}^{\mathrm{V}}_{\mathbb{N}} X^\bullet \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X^0 & \xrightarrow{\quad\alpha\quad} & \mathtt{colim}_{\mathbb{N}} X^\bullet
\end{array}$$

where f and g are the natural morphisms from the Voevodsky homotopy colimit to the colimit of a diagram, while α and α' are induced by the morphism of diagrams $\Lambda : X^0 \to X^\bullet$ with $\Lambda^m = a^{m-1} \cdots a^0 : X^0 \to X^m$. If a^m is in $\mathcal{S}$ for all $m \geq 0$, so is Λ^m and then also α'. To finish, let us see that $g \in \mathcal{S}$. In this case, taking $X^k = X^0$ for all k we get that $f \in \mathcal{S}$ as well. Note that g is the diagonal of the bisimplicial morphism G given in bidegree (n, m) by $G_{n,m} : (\mathtt{hocolim}^{\mathrm{V}}_{\mathbb{N}} X^\bullet_n)_m \to colim_{\mathbb{N}} X^\bullet_n$. Then it suffices to prove that $G_{n,\cdot}$ is in $\mathcal{S}$ for each $n \geq 0$. Therefore, we may assume that our sequence $\{X^k, a^k\}_{k\geq 0}$

is $A^0 \to A^0 \amalg A^1 \to \cdots \to A^0 \coprod \cdots \coprod A^k \to \cdots$. Hence, $colim_k X^k = \coprod_{l\geq 0} A^l$, and $\{X^k, a^k\}_{k\geq 0}$ is the coproduct of the sequences

$$\tau_l A : \quad 0 \to \overset{l-1}{\cdots} \to 0 \to A^l \to A^l \to A^l \to \cdots$$

Since by definition $\mathtt{hocolim}_I^{\mathrm{V}}$ commutes with coproducts,

$$\mathtt{hocolim}_{\mathbb{N}}^{\mathrm{V}} X^\bullet = \coprod_{l\geq 0} \mathtt{hocolim}_{\mathbb{N}}^{\mathrm{V}} \tau_l A = \coprod_{l\geq 0} A^l \otimes \mathrm{N}(l/\mathbb{N})$$

Then $g = \coprod_{l\geq 0} g^{(l)} \in \mathcal{S}$, since $g^{(l)} : A^l \otimes \mathrm{N}(l/\mathbb{N}) \to A^l$ is a simplicial homotopy equivalence.

■

3. Characterization of realizable homotopy colimits.

We are now ready to prove the main result of the paper, characterizing homotopically cocomplete relative categories which are closed by coproducts.

3.1. Theorem. *For a relative category $(\mathcal{C}, \mathcal{W})$ closed under coproducts, the following are equivalent:*

i. *$(\mathcal{C}, \mathcal{W})$ has realizable homotopy colimits, which satisfy the cofinality property.*

ii. *$(\mathcal{C}, \mathcal{W})$ is a simplicial descent category.*

More concretely, if ii holds then $\mathtt{hocolim}_I = \mathbf{s}\amalg^I$ is a realizable homotopy colimit. Conversely, if i holds then $\mathbf{s} = \mathtt{hocolim}_{\Delta^\circ}$ is a simple functor for $(\mathcal{C}, \mathcal{W})$.

The dual result applies to relative categories $(\mathcal{C}, \mathcal{W})$ closed under products, and states that $(\mathcal{C}, \mathcal{W})$ has realizable homotopy limits satisfying the (dual) cofinality property if and only if $(\mathcal{C}, \mathcal{W})$ is a cosimplicial descent category.

The proof of previous theorem needs some preparatory material on simplicial descent categories and will be given later. Once this is done we show in Theorem 3.16 that, under the previous equivalent conditions, $(\mathcal{C}, \mathcal{W})$ produces a right derivator. Before going into further detail, let us first explore some consequences of Theorem 3.1. We begin with the following result about preservation of homotopy colimits by relative functors.

3.2. Corollary. *Let F be a relative functor between relative categories that satisfy the equivalent conditions of the previous theorem. Then F commutes with homotopy colimits if and only if it commutes with homotopy coproducts and Δ°-homotopy colimits.*

Proof. If $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ commutes with homotopy coproducts, then given $X : I \to \mathcal{C}$ and $n \geq 0$ the canonical morphism

$$\amalg_{i_0\to\cdots\to i_n} F(X(i_0)) \longrightarrow F(\amalg_{i_0\to\cdots\to i_n} X(i_0))$$

is a weak equivalence. Hence, the canonical morphism $\epsilon : \amalg^I F \to F\amalg^I$ is a pointwise weak equivalence. By assumption, $\mathtt{hocolim}^{\mathcal{C}}_{\Delta^\circ}\amalg^I$ and $\mathtt{hocolim}^{\mathcal{D}}_{\Delta^\circ}\amalg^I$ are I-homotopy colimits. Then ρ^I_F is a relative isomorphism since it is the composition of the weak equivalence $\mathtt{hocolim}^{\mathcal{C}}_{\Delta^\circ}\star\epsilon$ with the relative isomorphism $\rho^{\Delta^\circ}_F \star \amalg^I$. ■

3.3. Remark. Theorem 3.1 together with previous corollary can be stated as an equivalence of categories between the category formed by the relative categories satisfying hypothesis *i*, and the one formed by the relative categories satisfying hypothesis *ii*.

A second consequence of previous theorem concerns the relation between homotopy colimits and triangulated structures. Recall that if $(\mathcal{C},\mathcal{W})$ produces a stable Grothendieck derivator (in particular both homotopy colimits and homotopy limits are assumed to exist) then $\mathcal{C}^I[\mathcal{W}^{-1}]$ carries a natural triangulated structure. However, for a right derivator (that is, only homotopy colimits are required to exist) it is not clear how to construct cofiber sequences in $\mathcal{C}^I[\mathcal{W}^{-1}]$, and hence triangulated structures.

One of the advantages of working with the stronger notion of *realizable* homotopy colimit is that, by contrast, cofiber sequences can be defined in a natural way. Namely, if $(\mathcal{C},\mathcal{W})$ satisfies the hypothesis of Theorem 3.1 and $\mathcal{C}$ is a pointed category, the $\mathcal{C}one$ functor $Fl(\mathcal{C})\to\mathcal{C}$ is just $\mathcal{C}one(f)=\mathtt{hocolim}\{*\leftarrow X\stackrel{f}{\to}Y\}$ and the *suspension* is $\Sigma X=\mathcal{C}one(X\to *)$. Since $(\mathcal{C},\mathcal{W})$ is a simplicial descent category, [R, Corollary 5.6] implies that these cofiber sequences endow $\mathcal{C}[\mathcal{W}^{-1}]$ with a left triangulated structure, which is triangulated if $\Sigma:\mathcal{C}[\mathcal{W}^{-1}]\to\mathcal{C}[\mathcal{W}^{-1}]$ is an equivalence of categories.

3.4. Simplicial descent categories. Next we recall the definition of simplicial descent category, whose main feature is that it comes equipped with a *simple functor* $\mathbf{s}:\Delta^\circ\mathcal{C}\to\mathcal{C}$ that is a well behaved 'geometric realization'.

3.5. Definition. *Let $(\mathcal{C},\mathcal{W})$ be a relative category closed under finite coproducts. A simplicial descent structure on $(\mathcal{C},\mathcal{W})$ is a triple $(\mathbf{s}:\Delta^\circ\mathcal{C}\to\mathcal{C},\mu:\mathbf{s}\mathrm{D}\dashrightarrow\mathbf{ss},\lambda:\mathbf{s}c\dashrightarrow 1_{\mathcal{C}})$ satisfying the following axioms:*

(**S1**) *The canonical morphism $\mathbf{s}X\amalg\mathbf{s}(Y)\to\mathbf{s}(X\amalg Y)$ is in $\mathcal{W}$ for all X,Y in $\Delta^\circ\mathcal{C}$.*

(**S2**) *The simple functor $\mathbf{s}:(\Delta^\circ\mathcal{C},\mathcal{W})\to(\mathcal{C},\mathcal{W})$ is a relative functor.*

(**S3**) *$\mu:\mathbf{s}\mathrm{D}\dashrightarrow\mathbf{ss}$ is a relative isomorphism.*

(**S4**) *$\lambda:\mathbf{s}c\dashrightarrow 1_{\mathcal{C}}$ is a relative isomorphism, compatible with μ in the sense of (3.6).*

(**S5**) *For each object A of $\mathcal{C}$, $\mathbf{s}(d_0^A):\mathbf{s}(A)\to\mathbf{s}(A\otimes\Delta[1])$ is in $\mathcal{W}$.*

A simplicial descent category is a relative category closed under finite coproducts and endowed with a simplicial descent structure. It will be denoted by $(\mathcal{C},\mathcal{W},\mathbf{s},\mu,\lambda)$, $(\mathcal{C},\mathcal{W},\mathbf{s})$ or even by $(\mathcal{C},\mathcal{W})$ for shortness.

3.6. Compatibility between μ and λ. Given a simplicial object X, we have that $\mathbf{ss}c^{\mathrm{II}}(X)=\mathbf{s}(n\to\mathbf{s}c(X_n))$ and $\mathbf{ss}c^{\mathrm{I}}(X)=\mathbf{s}c\mathbf{s}(X)$. The compositions

$$\mathbf{s}(X)\xrightarrow{\mu_{c^{\mathrm{I}}(X)}}\mathbf{s}c\mathbf{s}(X)\xrightarrow{\lambda_{\mathbf{s}(X)}}\mathbf{s}(X)\qquad\qquad \mathbf{s}(X)\xrightarrow{\mu_{c^{\mathrm{II}}(X)}}\mathbf{ss}c(X)\xrightarrow{\mathbf{s}(\lambda_X)}\mathbf{s}(X)\tag{3}$$

give rise to relative isomorphisms of $\mathbf{s}$. Then, λ is said to be *compatible* with μ if these isomorphisms are the identity in $\mathcal{R}el\mathcal{C}at$.

3.7. Remark. We use here a slight variant of the notion given in [R], more suitable for the setting of relative categories. More concretely, the only difference is that the transformations μ and λ of (S3) and (S4) are now assumed to be relative isomorphisms instead of zigzags of natural weak equivalences as in loc. cit.

Previous axioms ensure that the simple functor has the correct homotopical meaning.

3.8. Theorem. *Let $(\mathcal{C}, \mathcal{W}, \mathbf{s})$ be a simplicial descent category. Then*
i. *The simple functor is a realizable homotopy colimit.*
ii. *$(\mathbf{s}, c)$ is a relative adjoint equivalence between $(\Delta^\circ\mathcal{C}, \mathcal{S} = \mathbf{s}^{-1}\mathcal{W})$ and $(\mathcal{C}, \mathcal{W})$.*

Proof. The proof is the same as the one of [R, Theorem 5.1]. ∎

By the next result, simplicial descent categories are closely related to Δ-closed classes.

3.9. Proposition.
- ***i.*** *Let $(\Delta^\circ\mathcal{C}, \mathcal{S})$ be a relative category closed under finite coproducts. Then $\mathcal{S}$ is Δ-closed if and only if $(\Delta^\circ\mathcal{C}, \mathcal{S}, \mathrm{D} : \Delta^\circ\Delta^\circ\mathcal{C} \to \Delta^\circ\mathcal{C})$ is a simplicial descent category.*
- ***ii.*** *If $(\mathcal{C}, \mathcal{W}, \mathbf{s})$ is a simplicial descent category, the class $\mathcal{S} = \mathbf{s}^{-1}\mathcal{W}$ of $\Delta^\circ\mathcal{C}$ is Δ-closed. If in addition $\mathcal{W}$ is closed under coproducts, then so is $\mathcal{S}$.*

Proof. The last statement of *ii* follows from Lemma 3.10. The remaining assertions of the proposition are proved in [R, Theorem 4.2]. ∎

3.10. Lemma. *Let $(\mathcal{C}, \mathcal{W}, \mathbf{s})$ be a simplicial descent category closed under coproducts. Then, the simple functor preserves all small coproducts up to weak equivalence. That is, given a family $\{X^\alpha\}_{\alpha\in\Lambda}$ of simplicial objects, the canonical morphism below is in $\mathcal{W}$*

$$\coprod_\alpha \mathbf{s}(X^\alpha) \longrightarrow \mathbf{s}(\coprod_\alpha X^\alpha)\ .$$

Proof. Being $\mathcal{W}$ closed under small coproducts, these are preserved by the localization functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$. The same applies to $\Delta^\circ\mathcal{C}[\mathcal{W}^{-1}]$. By Theorem 3.8 and Lemma 1.9, $\mathbf{s} : \Delta^\circ\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}]$ is a left adjoint, so it preserves coproducts. Then, $\coprod_\alpha \mathbf{s}(X^\alpha) \longrightarrow \mathbf{s}(\coprod_\alpha X^\alpha)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$, and hence a weak equivalence by saturation. ∎

Also, simplicial descent structures are inherited by diagram categories. The proof is a formal consequence of the axioms, and is left to the reader.

3.11. Proposition. *Let I be a small category and let $(\mathbf{s}, \mu, \lambda)$ be a simplicial descent structure on $(\mathcal{C}, \mathcal{W})$. Then, the triple $(\mathbf{s}^I, \mu^I, \lambda^I)$ defined pointwise is a simplicial descent structure on $(\mathcal{C}^I, \mathcal{W})$.*

3.12. Proof of Theorem 3.1. Let $(\mathcal{C}, \mathcal{W})$ be a relative category closed under coproducts. Assume that $(\mathcal{C}, \mathcal{W})$ has realizable homotopy colimits, which satisfy the cofinality property, and let us show that $(\mathcal{C}, \mathcal{W})$ has a simplicial descent structure with simple functor $\mathtt{hocolim}_{\Delta^\circ} : \Delta^\circ\mathcal{C} \to \mathcal{C}$.

(S1): This is proved in the same way as Lemma 3.10, using that $\mathtt{hocolim}_{\Delta^\circ} : \Delta^\circ\mathcal{C}[\mathcal{W}^{-1}] \to$

$\mathcal{C}[\mathcal{W}^{-1}]$ is a left adjoint by Lemma 1.15.

(S2): By hypothesis, $\mathtt{hocolim}_{\Delta^\circ} : (\Delta^\circ\mathcal{C}, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$ is a relative functor.

(S3): The diagonal $d : \Delta^\circ \to \Delta^\circ \times \Delta^\circ$ is a homotopy right cofinal functor (see [T, Lemma 5.33]). Then, $\mathtt{hocolim}(d) : \mathtt{hocolim}_{\Delta^\circ}\mathrm{D} \dashrightarrow \mathtt{hocolim}_{\Delta^\circ\times\Delta^\circ}$ is a relative isomorphism. It follows from Proposition 1.25 that there is a unique relative isomorphism $k : \mathtt{hocolim}_{\Delta^\circ\times\Delta^\circ} \dashrightarrow \mathtt{hocolim}_{\Delta^\circ}\mathtt{hocolim}_{\Delta^\circ}$ compatible with the adjunction morphisms. We define μ as the composition of k with $\mathtt{hocolim}(d)$.

(S4): Define $\lambda : \mathtt{hocolim}_{\Delta^\circ}\, c \dashrightarrow 1_{\mathcal{C}}$ as the adjunction morphism of $(\mathtt{hocolim}_{\Delta^\circ}, c)$. We claim that λ is a relative isomorphism. Note that the trivial functor $\pi : \Delta^\circ \to [0]$ is homotopy right cofinal because $(\pi/0) \equiv \Delta^\circ$ has an initial object. Then $\mathtt{hocolim}(\pi)$: $\mathtt{hocolim}_{\Delta^\circ}\pi^* = \mathtt{hocolim}_{\Delta^\circ} c \dashrightarrow \mathtt{hocolim}_{[0]}$ is a relative isomorphism. On the other hand, the adjunction morphism $u : \mathtt{hocolim}_{[0]} \dashrightarrow 1_{\mathcal{C}}$ of $(\mathtt{hocolim}_{[0]}, 1_{\mathcal{C}})$ is a relative isomorphism as well. Finally, the claim follows from the equality $\lambda = u\,\mathtt{hocolim}(\pi)$.

The compatibility between μ and λ follows from the Fubini property of $\mathtt{hocolim}$ (see Proposition 1.25) together with the fact that $d : \Delta^\circ \to \Delta^\circ \times \Delta^\circ$ composed with the projections $p_1, p_2 : \Delta^\circ \times \Delta^\circ \to \Delta^\circ$ is the identity.

(S5): Denote by $i_{[1]} : * \to \Delta^\circ$ the inclusion of the object $[1]$ in Δ°. Then $(i_{[1]})_!(A)$ exists and agrees with $A \otimes \Delta[1]$, for each $A \in \mathcal{C}$. Since it is a relative functor, we have that $(- \otimes \Delta[1], i^*_{[1]})$ is a relative adjunction, hence $\mathtt{hocolim}_{\Delta^\circ}(- \otimes \Delta[1])$ is isomorphic in $\mathcal{R}el\mathcal{C}at$ to the identity because it is relative left adjoint to $i^*_{[1]}c = 1_{\mathcal{C}}$.

Conversely, if *ii* holds then $\mathbf{s}\amalg^I$ is a realizable homotopy colimit by Proposition 3.14 below applied to the trivial functor $f : I \to [0]$. And $\mathbf{s}\amalg^I = \mathbf{s}\,\mathtt{hocolim}^{\mathrm{V}}_I\, c$ satisfies the cofinality property because, by Corollary 2.23, $\mathtt{hocolim}^{\mathrm{V}}_I$ does. $\square$

3.13. Definition. *Let I be a small category and $(\mathcal{C}, \mathcal{W}, \mathbf{s})$ be a simplicial descent category such that $(\mathcal{C}, \mathcal{W})$ is closed under coproducts (or drop this assumption if I is finite). Define $\mathtt{hocolim}_I : \mathcal{C}^I \to \mathcal{C}$ as the composition*

$$\mathcal{C}^I \xrightarrow{\ \amalg^I\ } \Delta^\circ\mathcal{C} \xrightarrow{\ \mathbf{s}\ } \mathcal{C}$$

of the simple functor with the simplicial replacement. Note that $\mathtt{hocolim}_I : (\mathcal{C}^I, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$ is a relative functor by (S2). More generally, given $f : I \to J$, the relative functor $f_! : (\mathcal{C}^I, \mathcal{W}) \to (\mathcal{C}^J, \mathcal{W})$ is defined as

$$(f_!X)(j) = \mathtt{hocolim}_{(f/j)} u_j^* X$$

where $u_j : (f/j) \to I$ maps $\{f(i) \to j\}$ to i.

3.14. Proposition. *Under the previous assumptions, $(f_!, f^*)$ is a relative adjunction.*

Proof. By Proposition 3.9, the class $\mathcal{S} = \mathbf{s}^{-1}\mathcal{W}$ is Δ-closed and it is closed under coproducts if $\mathcal{W}$ is. Given $f : I \to J$, consider the following diagram of relative functors

$$\begin{tikzcd}(\mathcal{C}^I, \mathcal{W}) \arrow[r, shift left, "c"] & (\Delta^\circ\mathcal{C}^I, \mathcal{S}) \arrow[l, shift left, "\mathbf{s}^I"] \arrow[r, shift left, "f_!^{\mathrm{V}}"] & (\Delta^\circ\mathcal{C}^J, \mathcal{S}) \arrow[l, shift left, "f^*"] \arrow[r, shift left, "\mathbf{s}^J"] & (\mathcal{C}^J, \mathcal{W}) \arrow[l, shift left, "c"]\end{tikzcd}$$

By Theorem 2.14, $(f_!^{\mathrm{V}}, f^*)$ is a relative adjunction. By Proposition 3.11, $(\mathcal{C}^I, \mathcal{W})$ and $(\mathcal{C}^J, \mathcal{W})$ are simplicial descent categories with simple functor defined pointwise. Note that $(\mathbf{s}^I)^{-1}\mathcal{W}$ agrees with the class of $\Delta^\circ\mathcal{C}^I$ defined pointwise by $\mathcal{S}$, and analogously for J. We conclude by Theorem 3.8 that the pairs $(c, \mathbf{s}^I)$ and $(\mathbf{s}^J, c)$ are relative adjoint equivalences of categories, and in particular relative adjunctions. It turns out that $\mathbf{s}^J\, f_!^{\mathrm{V}}\, c = f_!$ and $\mathbf{s}^I\, f^*\, c$ form a relative adjunction as well. But $\mathbf{s}^I f^* c = \mathbf{s}^I c f^*$ is isomorphic in $\mathcal{R}el\mathcal{C}at$ to f^* by (S4). Hence, $(f_!, f^*)$ is a relative adjunction. ∎

3.15. Grothendieck derivators. We study here the link between simplicial descent categories and Grothendieck derivators ([G]). Recall that a *prederivator* is a strict 2-functor $\mathbb{D} : cat^\circ \to \mathcal{C}at$, and that a *right derivator* (see [M, Definition 4.28]) is a prederivator $\mathbb{D}$ such that:

Der 1. Given A, B in *cat*, the functor $(i^*, j^*) : \mathbb{D}(A \amalg B) \to \mathbb{D}(A) \times \mathbb{D}(B)$ induced by the canonical inclusions $i : A \to A \amalg B$, $j : B \to A \amalg B$ is an equivalence of categories.

Der 2. Given A in *cat* and $a \in A$, denote also by $a : [0] \to A$ the functor $0 \mapsto a$. Then, a morphism F of $\mathbb{D}(A)$ such that a^*F is an isomorphism in $\mathbb{D}([0])$ for each $a \in A$ is an isomorphism in $\mathbb{D}(A)$.

Der 3d. Given $w : A \to B$ in *cat*, $w^* : \mathbb{D}(B) \to \mathbb{D}(A)$ has a left adjoint $w_! : \mathbb{D}(A) \to \mathbb{D}(B)$.

Der 4d. Given $w : A \to B$ in *cat* and $b \in B$, consider the diagram

$$\begin{tikzcd} (b/w) \arrow[r, "\pi"] \arrow[d, "u_b"'] \arrow[dr, phantom, "\swarrow\alpha"] & {[0]} \arrow[d, "b"] \\ A \arrow[r, "w"'] & B \end{tikzcd}$$

where $\alpha_{\tau : b \to w(a)} = \tau$. It gives rise by adjunction to the morphism $\pi_! u_b^* \to b^* w_!$ of $\mathbb{D}([0])$, which is assumed to be an isomorphism.

The following additional condition must be also satisfied. Given D in *cat*, denote by $p^D : D \to [0]$ the trivial functor. Consider a commutative triangle in *cat*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

such that for each $c \in C$, the induced morphism $p_!^{(v/c)} p^{(v/c)^*} \to p_!^{(w/c)} p^{(v/c)^*}$ is an isomorphism. Then, the induced morphism $p_{A!} v^* \to p_{B!} w^*$ is required to be an isomorphism.

In this notion *cat* may be more generally a suitable sub-2-category *dia* of *cat*. We will only use the cases $dia = cat$ and $dia = cat_f$, the 2-category of finite categories.

3.16. Theorem. *If $(\mathcal{C}, \mathcal{W})$ is a simplicial descent category, the 2-functor $\mathbb{D} : cat_f^\circ \to \mathcal{C}at$, $A \mapsto \mathcal{C}^{A^\circ}[\mathcal{W}^{-1}]$ is a right derivator. If moreover $(\mathcal{C}, \mathcal{W})$ is closed under coproducts, then $\mathbb{D} : cat^\circ \to \mathcal{C}at$, $I \mapsto \mathcal{C}^{A^\circ}[\mathcal{W}^{-1}]$ is a right derivator*

Proof. Der 1 is clear from the definition of $\mathbb{D}$. To see Der 2, consider a morphism F of $\mathbb{D}(I) = \mathcal{C}^I[\mathcal{W}^{-1}]$. By Lemma 3.17, $F = \tau F'\tau'$, where F' is a morphism in $\mathcal{C}^I$ and τ, τ' are isomorphisms in $\mathcal{C}^I[\mathcal{W}^{-1}]$. If $i^*F = F_i = \tau_i F'_i \tau'_i$ is an isomorphism for each $i \in I$, then F'_i is an isomorphism as well. Since $\mathcal{W}$ is saturated, $F'_i \in \mathcal{W}$ for each i. Hence, F' is an isomorphism in $\mathcal{C}^I[\mathcal{W}^{-1}]$ and consequently so is $F = \tau F'\tau'$. To see the remaining axioms, consider $f : I \to J$ in *cat*. By Proposition 3.14, $f_! : (\mathcal{C}^J, \mathcal{W}) \rightleftarrows (\mathcal{C}^I, \mathcal{W}) : f^*$ is a relative adjunction where $f_!$ is defined pointwise. By Lemma 1.15, $f_! : \mathcal{C}^I[\mathcal{W}^{-1}] \rightleftarrows \mathcal{C}^J[\mathcal{W}^{-1}] : f^*$ is an adjunction, so Der 3 and Der 4 hold. To finish, the last axiom follows from Theorem 2.19 and the fact that $p_!^A = \mathtt{hocolim}_{A^\circ}$ agrees with $\mathbf{s}\,\mathtt{hocolim}^{\mathrm{V}}_{A^\circ}\, c$. ■

3.17. Lemma. *Let $(\mathcal{C}, \mathcal{W})$ be a simplicial descent category. If I is a small category and F is a morphism in $\mathcal{C}^I[\mathcal{W}^{-1}]$, there exist isomorphisms τ and τ' of $\mathcal{C}^I[\mathcal{W}^{-1}]$ and a morphism F' of $\mathcal{C}^I$ such that $F = \tau F'\tau'$.*

Proof. Given a small category I, Proposition 3.11 implies that $(\mathcal{C}^I, \mathcal{W})$ is again a simplicial descent category. Therefore we may assume that $I = [0]$, and that F is a morphism of $\mathcal{C}[\mathcal{W}^{-1}]$. If $\mathbf{s} : \Delta^\circ\mathcal{C} \to \mathcal{C}$ is a simple functor on $(\mathcal{C}, \mathcal{W})$, then $(\Delta^\circ\mathcal{C}, \mathcal{S} = \mathbf{s}^{-1}\mathcal{W})$ is a Brown category of cofibrant objects by Propositions 3.9 and 2.9. In particular each morphism T of $\Delta^\circ\mathcal{C}[\mathcal{S}^{-1}]$ is represented by a length-two zigzag (see [Br, Theorem 1]). Consequently, $c(F) = w^{-1}T'$ in $\Delta^\circ\mathcal{C}[\mathcal{S}^{-1}]$ where T', w are morphisms of $\Delta^\circ\mathcal{C}$ and w is in $\mathcal{S}$. Hence, $\mathbf{s}c(F) = (\mathbf{s}(w))^{-1}\mathbf{s}(T')$. Using the relative isomorphism λ of (S4) we obtain the equality $\lambda_B\, \mathbf{s}c(F) = F\,\lambda_A$ in $\mathcal{C}[\mathcal{W}^{-1}]$. We conclude that $F = \lambda_B(\mathbf{s}(w))^{-1}\mathbf{s}(T')\lambda_A^{-1}$ and the statement holds for $\tau = \lambda_B(\mathbf{s}(w))^{-1}$, $F' = \mathbf{s}(T')$ and $\tau' = \lambda_A^{-1}$. ■

3.18. Remark. The above lemma says that the functor $\mathcal{C}^{I\times[1]}[\mathcal{W}^{-1}] \to (\mathcal{C}^I[\mathcal{W}^{-1}])^{[1]}$ is essentially surjective. Using a similar argument it may be proved that this functor is also full. The combination of these two properties is sometimes included as axiom Der 5 in the notion of Grothendieck derivator.

3.19. Remark. One could consider a notion of 'realizable' right derivator, i.e. a strict 2-functor $\mathbb{D} : cat^\circ \to \mathcal{R}el\mathcal{C}at$ (instead of $cat^\circ \to \mathcal{C}at$) satisfying the corresponding axioms. Then, in this context previous results come to say that for prederivators induced by relative categories closed under coproducts, to be a realizable right derivator is exactly the same thing than to be a simplicial descent category.

Recall that a morphism of right derivators is called *right exact* if it preserves homotopy left Kan extensions.

3.20. Corollary. *Let $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$ be a relative functor between relative categories satisfying the hypothesis of the previous theorem. If F commutes with homotopy coproducts and Δ°-homotopy colimits, the functors $F^{A^\circ} : \mathcal{C}^{A^\circ}[\mathcal{W}^{-1}] \to \mathcal{D}^{A^\circ}[\mathcal{W}^{-1}]$ produce a right exact morphism of right derivators.*

Proof. By [C, Proposition 2.6] it suffices to see that F preserves homotopy colimits, which holds by Corollary 3.2. ■

4. Examples.

In this last section we describe in some specific examples what the homotopy (co)limits obtained look like. We already deduced in Section 2 the formula $\mathtt{hocolim}_I = \mathrm{D}\amalg^I$ for relative categories $(\Delta^\circ\mathcal{C},\mathcal{S})$ where $\mathcal{S}$ is Δ-closed. For $(\Delta^\circ\mathcal{C},\mathcal{S}) = (\Delta^\circ Set,\mathcal{W})$, this recovers the original Bousfield-Kan formula for homotopy colimits given in [BK, XII.5.2]. Hirschhorn's generalization of Bousfield-Kan formula for model categories also fits in our setting.

4.1. Bousfield-Kan homotopy colimits for model categories. The fact that a model category $(\mathcal{M},\mathcal{W})$ has all *realizable* homotopy limits and colimits is not surprising and may be proved, for instance, by adapting the arguments in [C], [DHKS] or [CS] to the setting of relative categories.

It holds moreover that these realizable homotopy colimits are given by the Bousfield-Kan construction. Recall from [H, Chapter 19] that the (corrected) Bousfield-Kan homotopy colimit ${}_{\mathtt{c}}\mathtt{hocolim}_I^{BK} : \mathcal{M}^I \to \mathcal{M}$ is given by the formula

$$ {}_{\mathtt{c}}\mathtt{hocolim}_I^{BK} X = \int^i \widetilde{QX}(i) \otimes \mathrm{N}(i/I)^\circ \tag{4}$$

where $\widetilde{\ } : \mathcal{M} \to \Delta\mathcal{M}$ is a functorial cosimplicial frame, QX is a functorial cofibrant replacement of X and $\otimes : \Delta\mathcal{M} \times \Delta^\circ Set \to \mathcal{M}$ is given by $X \otimes K = \mathtt{colim}_{\Delta K}\, \pi_K^* X$.

4.2. Theorem. *Realizable homotopy colimits exist in any model category, and may be computed by the Bousfield-Kan formula* (4).

Proof. We sketch here a proof based on Theorem 3.1 and the results of [H]. If $\mathcal{M}_c$ denotes the full subcategory of $\mathcal{M}$ of cofibrant objects, then the restriction of ${}_{\mathtt{c}}\mathtt{hocolim}_I^{BK}$ to $\Delta^\circ\mathcal{M}_c$ gives a functor $\mathbf{s} = \mathtt{hocolim}_{\Delta^\circ}^{BK} : \Delta^\circ\mathcal{M}_c \to \mathcal{M}_c$. Using the results of [H], it may be proved that $\mathbf{s}$ is indeed a simple functor for $(\mathcal{M}_c,\mathcal{W})$. Since weak equivalences between cofibrant objects are closed under coproducts, Theorem 3.1 applies and tells us that realizable homotopy colimits exist in $(\mathcal{M}_c,\mathcal{W})$, and are given by the formula $\mathtt{hocolim}_{\Delta^\circ}^{BK}\amalg^I$. But it is easily seen that there is a relative isomorphism $\mathtt{hocolim}_I^{BK} \simeq \mathtt{hocolim}_{\Delta^\circ}^{BK}\amalg^I : (\mathcal{M}_c^I,\mathcal{W}) \to (\mathcal{M}_c,\mathcal{W})$. Consequently, the former is a realizable homotopy colimit for $(\mathcal{M}_c,\mathcal{W})$ as well. To finish, one uses that the inclusion $(\mathcal{M}_c,\mathcal{W}) \to (\mathcal{M},\mathcal{W})$ and a functorial cofibrant replacement $Q : (\mathcal{M},\mathcal{W}) \to (\mathcal{M}_c,\mathcal{W})$ are inverse equivalences of relative categories to conclude that ${}_{\mathtt{c}}\mathtt{hocolim}_I^{BK}$ is a realizable homotopy colimit. ∎

4.3. Remark. A direct consequence of previous theorem is the known fact that the Bousfield-Kan homotopy colimit (4) is a left derived functor of the colimit.

4.4. Remark. In analogy with the closed-under-coproducts case, in this case the following equation holds

$$\mathtt{hocolim}_I \simeq \mathtt{hocolim}_{\Delta^\circ} \amalg^I Q$$

4.5. Remark. Combining Theorems 4.2, 3.1 and 3.16 (and their duals) one gets another proof of the well-known fact that a model category produces a Grothendieck derivator (cf. [C]).

4.6. Further examples. Despite the abstract way in which simple functors are defined, in the examples they turn out to be familiar constructions which have already been used in a variety of applications, inside or outside homotopy theory.

Chain complexes. Positive (or uniformly bounded below) chain complexes on an abelian category $\mathcal{A}$ together with the quasi-isomorphisms as weak equivalences form a simplicial descent category where the simple functor $\mathbf{s} : \Delta^{\circ}C_{+}(\mathcal{A}) \to C_{+}(\mathcal{A})$ is just the total complex associated with a double complex. Hence, if $\mathcal{A}$ is an (AB4) abelian category then $I \mapsto \mathcal{D}_{+}(\mathcal{A}^{I^{\circ}})$ defines a right derivator with domain *cat*, for which $\mathtt{hocolim}_I = Tot\Pi^I$. Dually, if $\mathcal{A}$ is (AB4*) then $Tot\Pi^I$ computes $\mathtt{holim}_I$ in $C^{+}(\mathcal{A})$ for any small category I. A similar result is proved in [Ne, Lemma A.3.2] and states that $\mathtt{lim}^n : \mathcal{A}^I \to \mathcal{A}$ exists for any (AB4*) abelian category and may be computed as $\mathtt{lim}^n F = \mathrm{H}^n(\Pi^I F)$.

(Filtered) commutative differential graded algebras. If k is a field of characteristic 0, the relative category $(\mathbf{Cdga}_k, \mathcal{W})$ of (positively graded) commutative differential graded k-algebras and quasi-isomorphisms is a cosimplicial descent category (recall that it is indeed a model category). A formula for the simple functor is Navarro's Thom-Whitney simple ([N, (2.3)]), given by the end

$$\mathbf{s}_{TW}(A) = \int_n \mathrm{L}_n \otimes A^n$$

where L_n is the dg-algebra of polynomial differential forms on the n-simplex. This yields that $\mathbf{s}_{TW}\Pi^I$ is a formula for the homotopy limit, for any small category I.

An advantage of this approach is that adding filtrations to the picture does not take one outside the cosimplicial descent setting. If $\mathbf{FA}(k)$ denotes the category of filtered commutative differential graded k-algebras, recall from [N, (6.6)] that a decreasing multiplicative filtration λ of L induces $(\mathbf{s}_{TW}, \lambda) : \Delta\mathbf{FA}(k) \to \mathbf{FA}(k)$, $(A, F) \mapsto (\mathbf{s}_{TW}(A), \lambda * F)$ where $(\lambda * F)^k \mathbf{s}_{TW}(A)$ is defined over $\mathrm{L}_n \otimes A^m$ by $\sum_{i+j=k} \lambda^i L_n \otimes F^j A^m$.

For the trivial filtration ε and the bête filtration σ, one can prove using the results in loc. cit. that $(\mathbf{s}_{TW}, \varepsilon)$ and $(\mathbf{s}_{TW}, \sigma)$ are again simple functors for $(\mathbf{FA}(k), \mathrm{E}_1)$ and $(\mathbf{FA}(k), \mathrm{E}_2)$, respectively. Here E_i denotes the class of morphisms that induce an isomorphism between the ith-steps of the corresponding spectral sequences associated with the filtrations (in particular $\mathrm{E}_1 = \{$ graded quasi-isomorphisms$\}$). Hence these relative categories induce left derivators, where the homotopy limits over a small category I are given by $(\mathbf{s}_{TW}, \varepsilon)\Pi^I$ and $(\mathbf{s}_{TW}, \sigma)\Pi^I$, respectively.

Mixed Hodge complexes. This constitutes an interesting example in which finite homotopy limits exist, but usual finite limits do not exist. Recall that a *mixed Hodge complex* is a triple $K = ((K_{\mathbb{Q}}, \mathrm{W}), (K_{\mathbb{C}}, \mathrm{W}, \mathrm{F}), \alpha)$ where

1. $(K_{\mathbb{Q}},\mathrm{W})$ is a biregularly filtered positive cochain complex of $\mathbb{Q}$-vector spaces such that $\mathrm{H}^k(K_{\mathbb{Q}})$ is finite dimensional for all $k\geq 0$.

2. $(K_{\mathbb{C}},\mathrm{W},\mathrm{F})$ is a biregularly bifiltered positive cochain complex of $\mathbb{C}$-vector spaces, where W (resp. F) is an increasing (resp. decreasing) filtration, called the *weight* (resp. *Hodge*) filtration.

3. $\alpha=(\alpha_0,\alpha_1,(\widetilde{K},\widetilde{\mathrm{W}}))$ is a zigzag

$$(K_{\mathbb{C}},\mathrm{W}) \xleftarrow{\alpha_0} (\widetilde{K},\widetilde{\mathrm{W}}) \xrightarrow{\alpha_1} (K_{\mathbb{Q}},\mathrm{W})\otimes\mathbb{C}$$

   where α_i is a filtered quasi-isomorphism for $i=0,1$.

In addition, for each $n\geq 0$ the boundary map of the graded complex ${}_{\mathrm{W}}\mathbf{Gr}_n K_{\mathbb{C}}$ must be compatible with filtration F, and the induced filtration F on ${}_{\mathrm{W}}\mathbf{Gr}_n H^k K_{\mathbb{C}}$ is a Hodge structure of weight $n+k$. That is,

$${}_{\mathrm{F}}\mathbf{Gr}^p{}_{\overline{\mathrm{F}}}\mathbf{Gr}^q{}_{\mathrm{W}}\mathbf{Gr}_n H^k K_{\mathbb{C}} = 0 \text{ for } p+q\neq n+k\ .$$

A morphism $f=(f_{\mathbb{Q}},f_{\mathbb{C}},\widetilde{f}): K\to K'$ of mixed Hodge complexes consists of morphisms $f_{\mathbb{Q}}:(K_{\mathbb{Q}},\mathrm{W})\to(K'_{\mathbb{Q}},\mathrm{W}')$, $f_{\mathbb{C}}:(K_{\mathbb{C}},\mathrm{W},\mathrm{F})\to(K'_{\mathbb{C}},\mathrm{W}',\mathrm{F}')$ and $\widetilde{f}:(\widetilde{K},\widetilde{\mathrm{W}})\to(\widetilde{K}',\widetilde{\mathrm{W}}')$ of (bi)filtered complexes, such that the two squares built up using α, α', $f_{\mathbb{C}}$, $\widetilde{f}$ and $f_{\mathbb{Q}}\otimes\mathbb{C}$ commute. The resulting category $\mathcal{H}dg$ together with the class of weak equivalences

$$\mathrm{E}_{\mathcal{H}dg}=\{f=(f_{\mathbb{Q}},f_{\mathbb{C}},\widetilde{f})\mid f_{\mathbb{Q}} \text{ is a quasi-isomorphism of cochain complexes}\}$$

is by [R, Theorem 2.10] a cosimplicial descent category. Note that $\mathcal{H}dg$ is closed only under finite products because of the finite-dimensional assumption in (1).

The simple functor is Deligne's cosimplicial construction $\mathbf{s}_{\mathcal{H}dg}:\Delta\mathcal{H}dg\to\mathcal{H}dg$ ([De, (8.I.I5)]). It is defined on $K=((K_{\mathbb{Q}},\mathrm{W}),(K_{\mathbb{C}},\mathrm{W},\mathrm{F}),\alpha)$ by

$$\mathbf{s}_{\mathcal{H}dg}(K)=((\mathbf{s}(K_{\mathbb{Q}}),\delta\mathrm{W}),(\mathbf{s}(K_{\mathbb{C}}),\delta\mathrm{W},\mathbf{s}(\mathrm{F})))\ \text{, where}$$

$$\mathbf{s}(K_*)^n=\bigoplus_{p+q=n}K_*^{p,q}\ ;\quad (\delta\mathrm{W})_k(\mathbf{s}(K_*))^n=\bigoplus_{i+j=n}\mathrm{W}_{k+i}K_*^{i,j}\ ,\quad \text{if } * \text{ is } \mathbb{Q} \text{ or } \mathbb{C}$$
$$(\mathbf{s}(\mathrm{F}))^k(\mathbf{s}(K_{\mathbb{C}}))^n=\bigoplus_{p+q=n}\mathrm{F}^kK_{\mathbb{C}}^{p,q}\ .$$

Therefore, for a finite category I, we conclude by the previous results that $\mathbf{s}_{\mathcal{H}dg}\Pi^I$ computes $\mathtt{holim}_I$, and that $\mathbb{D}^{\mathcal{H}dg}: cat_f\to CAT\ ,\quad \mathbb{D}^{\mathcal{H}dg}(I)=\mathcal{H}dg^{I^\circ}[\mathrm{E}_{\mathcal{H}dg}^{-1}]$ is a left Grothendieck derivator.

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