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NORMALIZERS, CENTRALIZERS AND ACTION ACCESSIBILITY

J. R. A. GRAY

Abstract. We give several reformulations of action accessibility in the sense of D. Bourn and G. Janelidze. In particular we prove that a pointed exact protomodular category is action accessible if and only if for each normal monomorphism $\kappa : X \to A$ the normalizer of $\langle \kappa, \kappa \rangle : X \to A \times A$ exists. This clarifies the connection between normalizers and action accessible categories established in a joint paper of D. Bourn and the author, in which it is proved that for pointed exact protomodular categories the existence of normalizers implies action accessibility. In addition we prove a pointed exact protomodular category with coequalizers is action accessible if centralizers of normal monomorphisms exist, and the normality of unions holds.

1. Introduction

Recall for a pointed category $\mathbb{C}$ a split extension is a diagram

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

where κ is the kernel of α and $\alpha\beta = 1_B$. A morphism of split extensions is a diagram

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \end{array}$$

where the top and bottom are split extensions (the domain and codomain respectively), $\kappa' f = g\kappa$, $\alpha' g = h\alpha$, and $\beta' h = g\beta$. We will denote by $\mathbf{SPLEXT}(\mathbb{C})$ the category of split extensions, and by $\mathbf{SPLEXT}_X(\mathbb{C})$ the category with objects those split extensions with kernel X, and with morphisms those morphisms of split extensions where the morphism between their kernels is 1_X. An extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

is said to be faithful if it is a sub-terminal object in $\mathbf{SPLEXT}_X(\mathbb{C})$, that is, there is at most one morphism from any object to it in this category. A pointed protomodular

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category $\mathbb{C}$ is a called action accessible [7] if for each X the category $\mathbf{SPLEXT}_X(\mathbb{C})$ has enough sub-terminal objects, that is, each object admits a morphism into a sub-terminal object.

It is well known that each split extension of groups

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

is determined by a morphism

$$B \to \mathrm{Aut}(X).$$

This can be understood categorically as the fact that when $\mathbb{C}$ is the category of groups, the category $\mathbf{SPLEXT}_X(\mathbb{C})$ has a terminal object. Semi-abelian categories which satisfy this property are called action representative and were introduced and studied in [2] (see also [3]) by F. Borceux, G. Janelidze and G. M. Kelly. Action accessible categories were introduced by D. Bourn and G. Janelidze in [7] as a weakening of action representable categories so as to include the category of rings as an example but still to allow, amongst other things, centralizers of equivalence relations to be constructed in a similar way as in the category of groups.

In [4] D. Bourn studied further how the existence of centralizers of equivalence relations is related to the concept of action accessibility (as well as *groupoid accessibility*). In particular it was shown that a split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

in an action accessible category is faithful if and only if the inverse image of the centralizer of the kernel pair of α along β is indiscrete.

A. S. Cigoli and S. Mantovani showed in [8] that a pointed exact protomodular category $\mathbb{C}$ is action accessible if and only if what they called non-symmetric centralizers exist in $\mathbb{C}$. In addition in a talk by A. S. Cigoli on their joint work, it was explained that under certain conditions a category $\mathbb{C}$ is action accessible provided that certain centralizers of subobjects exist and satisfy certain properties (see Theorem 2.15 and the paragraph before it).

In Section 2 we recall many of these results relating to the existence of centralizers of subobjects and add other equivalent formulations (see Theorem 2.15). In addition we show that a pointed exact protomodular category with coequalizers is action accessible if it has centralizers of normal monomorphisms which are normal and the normality of unions holds in $\mathbb{C}$. Recall from [3] that the normality of unions holds in $\mathbb{C}$ if for any subobjects A, B and C of an object D, if A is normal in both B and C, then it is normal in the join of B and C (in D).

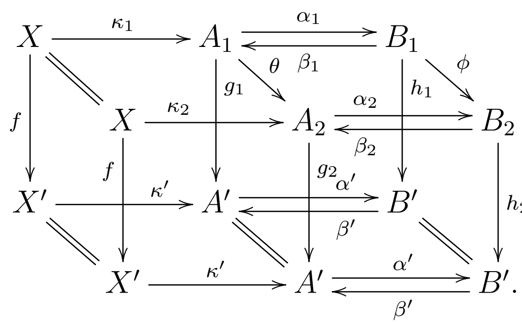
Let us denote by $\mathbf{K}$ the functor sending a split extension in $\mathbf{SPLEXT}(\mathbb{C})$ to its kernel. Recall that the normalizer [9] of a monomorphism is defined as the terminal object in the category of factorizations as a normal monomorphism (i.e. a kernel) followed by a monomorphism i.e. the normalizer of $S \leq X$ is the largest subobject of X in which S is normal. D. Bourn together with the author introduced in [6] another notion of normalizer (using a different notion of normal) which can be defined for a finitely complete category and which coincides with the definition given above in a pointed exact protomodular category. In the same paper it was shown that $\mathbb{C}$ has normalizers in the sense of [6], and hence in the sense defined above (when $\mathbb{C}$ is pointed exact protomodular), if and only if for each split extension

$$X' \xrightarrow{\kappa'} A' \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} B' \tag{1}$$

and for each monomorphism $f : X \to X'$ there exists a **K**-precartesian lifting of f to (1) . Such a **K**-precartesian lifting can be seen to be the terminal object in the following subcategory of the category of morphisms of **SPLEXT**($\mathbb{C}$). The objects are morphisms of split extensions of the form

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B'; \end{array}$$

and morphisms are morphisms of the form



As mentioned above, in the paper [8], A. S. Cigoli and S. Mantovani showed that a pointed exact protomodular category is action accessible if and only if the category has what they called non-symmetric centralizers. In the context of a pointed exact protomodular action accessible category $\mathbb{C}$ it can be seen that an equivalence relation $r_1, r_2 : R \to A$ with common section $s : A \to R$ has a non-symmetric centralizer if and only if the category of relations on

$$X \xrightarrow{k} R \underset{s}{\overset{r_1}{\rightleftarrows}} A$$

in $\mathbf{SPLEXT}_X(\mathbb{C})$ has a terminal object (this is essentially Proposition 4.1 of [8]). Using the isomorphism of categories between subobjects of products and relations (considered as parallel pairs) it can be seen that the existence of such a terminal object is equivalent to the existence of a **K**-precartesian lifting of $\langle 1, 1\rangle : X \to X \times X$ to

$$X \times X \xrightarrow{k \times k} R \times R \underset{s \times s}{\overset{r_1 \times r_1}{\rightleftarrows}} A \times A.$$

From this observation and what was recalled previously it follows that a pointed exact protomodular category with normalizers is action accessible [6]. In Section 3 of this paper we show that a pointed exact protomodular category is action accessible if and only if each normal monomorphism composed with the diagonal morphism has a normalizer (see Theorem 3.1). The proof of the fact that the existence of normalizers implies action accessibility given in [6] was different from what was described above and made use of a simple property studied by D. Bourn in [5] which is equivalent to the existence of centralizers for Mal'tsev categories.

2. Action accessibility, centralizers and the normality of unions

In this section we recall various results from the papers [8] and [4] in order to give reformulations of action accessibility. Note that a diagram

$$\begin{array}{ccc} C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

where $\gamma\delta = 1_D$, $\alpha\beta = 1_B$, $q\gamma = \alpha p$ and $\beta q = p\delta$ will be called a split pullback if the diagram consisting of rightward and downward directed arrows is a pullback. Recall that a pair of morphisms $f : A \to C$ and $g : B \to C$ commute in a (weakly) unital category $\mathbb{C}$ [12, 1] if there exists a (necessarily unique) morphism $\varphi : A\times B \to C$, where $(A\times B, \pi_1, \pi_2)$ is the product of A and B, making the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\langle 1,0\rangle} & A\times B & \xleftarrow{\langle 0,1\rangle} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle g} & \\ & & C, & & \end{array}$$

in which $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are the unique morphisms such that $\pi_1\langle 1, 0\rangle = 1_A$, $\pi_2\langle 1, 0\rangle = 0$, $\pi_1\langle 0, 1\rangle = 0$ and $\pi_2\langle 0, 1\rangle = 1$, commute.

2.1. Lemma. [8], Lemma 2.3. *Let $\mathbb{C}$ be a pointed protomodular category and let*

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B', \end{array}$$

be a morphism of split extensions. For each morphism $s' : S' \to B'$ such that $\beta' s'$ commutes with κ' the morphism $s : S \to B$ obtained from the pullback

$$\begin{array}{ccc} S & \xrightarrow{s} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ S' & \xrightarrow[s']{} & B' \end{array}$$

is such that βs commutes with κ.

2.2. Lemma. [8], Lemma 2.4, Proposition 2.5. *Let $\mathbb{C}$ be a pointed protomodular category. For each morphism of split extensions*

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \end{array}$$

the kernel of h, $\mathbf{ker}(h) : \mathbf{Ker}(h) \to B$ has the properties:

(a) $\beta\mathbf{ker}(h)$ commutes with κ;

(b) $\beta\mathbf{ker}(h)$ is a normal monomorphism.

When the codomain of the morphism is faithful then $\mathbf{ker}(h)$ has the additional property:

(c) if $s : S \to B$ is a morphism such that βs commutes with κ, then there exists a unique morphism $\overline{s}$ such that $s = \mathbf{ker}(h)\overline{s}$.

2.3. Remark. The above proofs hold for morphisms of split extensions in an arbitrary pointed category with finite limits provided the square on the right is a split pullback.

2.4. Definition. *For any morphisms $f : A \to C$ and $g : B \to C$ the* centralizer of f relative to g *is the morphism $z_{f,g} : Z_{f,g} \to B$ with the properties:*

(a) $gz_{f,g}$ commutes with f;

(b) for each morphism $s : S \to B$ such that gs commutes with f there exists a unique morphism $\overline{s}$ such that $z_{f,g}\overline{s} = s$.

The centralizer of f relative to 1_C will be denoted by z_f (rather than $z_{f,1_C}$) and called the centralizer of f.

2.5. Corollary. [8], Proposition 2.5. *Let $\mathbb{C}$ be a pointed protomodular action accessible category. For each split extension*

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the centralizer $z_{\kappa,\beta}$ of κ relative to β exists, and $\beta z_{\kappa,\beta}$ is normal.

Recall that a monomorphism is called protosplit if it is normal and its cokernel is a split epimorphism [3]. Using the same construction as in [8] we obtain:

2.6. Proposition. *Let $\mathbb{C}$ be a unital category with cokernels. The following are equivalent:*

(a) for each split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the centralizer of κ relative to β exists;

(b) $\mathbb{C}$ has centralizers of normal monomorphisms;

(c) $\mathbb{C}$ has centralizers of protosplit monomorphisms.

Proof. The implication $(b) \Rightarrow (c)$ follows from the fact that each protosplit monomorphism is a normal monomorphism.
$(a) \Rightarrow (b)$: Let $n : N \to X$ be a normal monomorphism and consider the morphism of split extensions

$$\begin{array}{ccccc} N & \xrightarrow{k} & R & \underset{s}{\overset{r_1}{\rightleftarrows}} & X \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \langle r_1, r_2\rangle} & & \downarrow{\scriptstyle 1_X} \\ X & \xrightarrow{\langle 0,1\rangle} & X \times X & \underset{\langle 1,1\rangle}{\overset{\pi_1}{\rightleftarrows}} & X \end{array}$$

determined by its denormalization (the kernel pair of the cokernel of n). Since 0 commutes with any morphism and $\langle r_1, r_2\rangle$ is a monomorphism it follows (see e.g. [1]) that for each morphism $u : U \to X$, su commutes with k if and only if n commutes with u. It follows that the morphism $z_{k,s} : Z_{k,s} \to X$ is the centralizer of n.
$(c) \Rightarrow (a)$: For each split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

it is easy to check that t obtained by the pullback

$$\begin{array}{ccc} T & \xrightarrow{\overline{\beta}} & Z_\kappa \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle z_\kappa} \\ B & \xrightarrow[\beta]{} & A \end{array}$$

is the centralizer of κ relative to β. ■

2.7. Corollary. *Let $\mathbb{C}$ be a unital category with cokernels. The following are equivalent:*

(a) for each split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the centralizer of κ relative to β exists and the composite $\beta z_{\kappa,\beta}$ is normal;

(b) $\mathbb{C}$ *has centralizers of normal monomorphisms which are normal and have the property that if their cokernel splits, then the intersection with each splitting is normal.*

(c) $\mathbb{C}$ *has centralizers of protosplit monomorphisms which are normal and have the property that their intersection with each splitting of their cokernel is normal.*

Proof. The proof of Proposition 2.6 can easily be extended to prove that the above statements are equivalent. ∎

The existence of centralizers of normal monomorphisms which are normal was proved for a homological action accessible categories in [7].

2.8. Corollary. *Let* $\mathbb{C}$ *be a homological action accessible category with cokernels. Centralizers of normal monomorphisms exist, are normal, and have the property that if their cokernel splits, then the intersection with each splitting is normal.*

Proof. The proof follows from Corollary 2.5 and Corollary 2.7 ∎

Recall that in a category with finite limits, for each diagram

$$\begin{array}{ccc} D & \underset{\epsilon}{\overset{\delta}{\rightleftarrows}} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

in which p and q are regular epimorphisms, $\alpha\beta = 1_B$, $\delta\epsilon = 1_E$, $p\epsilon = \beta q$ and $q\delta = \alpha p$, the diagram

$$\begin{array}{ccc} D & \xrightarrow{\delta} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\alpha} & B \end{array}$$

is a pushout (since the induced morphism between the kernel pair of p and the kernel pair of q is a (split) epimorphism.)

2.9. Lemma. *Let* $\mathbb{C}$ *be a regular subtractive category [10]. Each morphism*

$$\begin{array}{ccccc} X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

can be decomposed as a composite

$$\begin{array}{ccccc} X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{\sigma} & C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

where p, q and r are regular epimorphisms, and l, m and n are monomorphisms.

Proof. Let

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

be a morphism of split extensions. Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{\sigma} & C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array} \qquad (*)$$

where each of the vertical morphisms from left to right is the factorization as a regular epimorphism followed by a monomorphism of f, g and h respectively, and σ ,γ and δ are the induced morphisms between them. The top part of the diagram can be extended to a 3x3 lemma diagram (with zeros omitted)

$$\begin{array}{ccccc} \mathbf{Ker}(p) & \xrightarrow{\overline{\kappa}} & \mathbf{Ker}(q) & \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} & \mathbf{Ker}(r) \\ {\scriptstyle \mathbf{ker}(p)}\downarrow & & \downarrow{\scriptstyle \mathbf{ker}(q)} & & \downarrow{\scriptstyle \mathbf{ker}(r)} \\ X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{\sigma} & C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D \end{array}$$

where the top two lines and all columns are exact (in the sense of Z. Janelidze in [11]). It follows [11] that the bottom row is exact and since σ is a monomorphism that the bottom row is a split extension. Therefore, the diagram (*) gives the desired decomposition. ∎

Following Z. Janelidze in [11] we will call a category normal if it is pointed, regular, and each regular epimorphism is normal. The following result is a generalization for normal subtractive categories of Corollary 2.8 of [8], we omit the proof as it essentially the same.

2.10. Corollary. *Let $\mathbb{C}$ be a normal subtractive action accessible category (where we have dropped the pointed protomodularity requirement in the definition of action accessible category). For each split extension*

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

There exists a morphism of split extensions

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{\sigma} & C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D \end{array}$$

with codomain faithful, and with q and r normal epimorphisms.

It easily follows from the definition of an eccentric extension in [5] that a split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

in a pointed protomodular category is eccentric if and only if $z_{\kappa,\beta} = 0$. In this paper we will use this as the definition of an eccentric extension. The following proposition is essentially the same as [4] Corollary 4.1 we omit the proof.

2.11. Proposition. *Let $\mathbb{C}$ be a homological action accessible category. Eccentric extensions are faithful.*

Recall from [1]

2.12. Proposition. *Let $\mathbb{C}$ be a pointed protomodular category and let $f : A \to C$ and $g : B \to C$ be normal monomorphisms in $\mathbb{C}$. If the intersection $A \cap B$ is trivial then f and g commute.*

2.13. Proposition. *Let $\mathbb{C}$ be a pointed protomodular category. For each split pullback*

$$\begin{array}{ccc} A \times_B B' & \underset{\langle \beta h, 1\rangle}{\overset{\pi_2}{\rightleftarrows}} & B' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle h} \\ A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

where h is a normal monomorphism, $\langle \beta h, 1\rangle$ is normal if and only if βh commutes with the kernel of α

Proof. Let $\kappa : X \to A$ be the kernel of α. It follows that $\langle \kappa, 0\rangle : X \to A \times_B B'$ is the kernel of $\pi_2 : A \times_B B' \to B'$. Suppose that $\langle \beta h, 1\rangle$ is a normal monomorphism. Since by Proposition 2.12 the morphisms $\langle \kappa, 0\rangle$ and $\langle \beta h, 1\rangle$ commute, it follows that $\kappa = \pi_1\langle \kappa, 0\rangle$ and $\beta h = \pi_1\langle \beta h, 1\rangle$ commute. Conversely suppose that βh commutes with κ and that φ is the morphism showing they commute. Since the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1,0\rangle} & X \times B' & \underset{\langle 0,1\rangle}{\overset{\pi_2}{\rightleftarrows}} & B' \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

is a morphism of split extensions and $\mathbb{C}$ is protomodular it follows that the square on the right is a split pullback. It easily follows that $\langle \beta h, 1\rangle$ is a normal monomorphism. ∎

Next we show that a weak form of normality of unions holds in any homological action accessible category.

2.14. Proposition. *Let $\mathbb{C}$ be a homological action accessible category. For each split pullback*

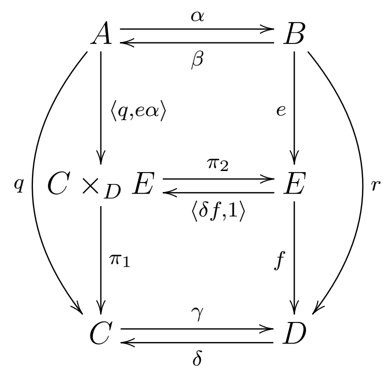
$$\begin{array}{ccc} A \times_B B' & \underset{\langle \beta h,1\rangle}{\overset{\pi_2}{\rightleftarrows}} & B' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle h} \\ A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

if h and $\langle \beta h, 1\rangle$ are normal, then βh is normal.

Proof. Let $\kappa : X \to A$ be the kernel of α. Since by Proposition 2.13 it follows that κ and βh commute and since by Corollary 2.5 $z_{\kappa,\beta}$ exists, it follows that there exists a unique morphism $h' : B' \to Z_{\kappa,\beta}$ such that $h = z_{\kappa,\beta} h'$. By Corollary 2.10 and Corollary 2.5 it follows that there exists a morphism

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{\sigma} & C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D \end{array}$$

with codomain faithful, and with q and r normal epimorphisms, such that the kernel of r is $z_{\kappa,\beta}$. Consider the diagram



in which e is the cokernel of h, and f is the unique morphism such that $fe = r$ (which exists since $rh = rz_{\kappa,\beta}h' = 0h' = 0$) . Since the outer arrows of the diagram above form a split pullback and the bottom square is a split pullback it follows that the top square is also a split pullback, this means that βh is the kernel of $\langle q, e\alpha\rangle$ and is therefore normal as required. ∎

As pointed out to me by D. Bourn, the fact that the conditions (a) and (b) of the following theorem are equivalent follows from the results in [4], the same fact appeared in a talk by A. S. Cigoli on joint work with S. Mantovani entitled *Action accessibility and centralizers* at the CT conference in 2010.

2.15. THEOREM. *Let $\mathbb{C}$ be a homological category. The following are equivalent:*

(a) $\mathbb{C}$ is action accessible;

(b) (i) for each split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the centralizer of κ relative to β exists and $\beta z_{\kappa,\beta}$ is a normal monomorphism;

(ii) Eccentric extensions are faithful;

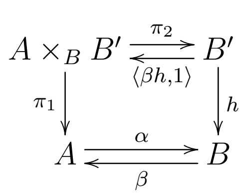
(c) (i) $\mathbb{C}$ has centralizers of normal monomorphisms;

(ii) the intersection of the centralizer of a protosplit monomorphism and any splitting of its cokernel is normal;

(iii) Eccentric extensions are faithful;

(d) (i) $\mathbb{C}$ has centralizers of normal monomorphisms which are normal;

(ii) for each split pullback



if h and $\langle \beta h, 1\rangle$ are normal, then βh is normal;

(iii) Eccentric extensions are faithful.

Proof. The equivalence of (b) and (c) follows from Corollary 2.7. The implication $(a) \Rightarrow (d)$ follows from Corollary 2.8 and Propositions 2.11 and 2.14. The implication $(d) \Rightarrow (b)$ follows from Corollary 2.7 and Proposition 2.13. To complete the proof we will show that $(b) \Rightarrow (a)$. Let

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

be a split extension. Consider the diagram

$$\begin{array}{ccc}
Z_{\kappa,\beta} & \underset{1_{Z_{\kappa,\beta}}}{\overset{1_{Z_{\kappa,\beta}}}{\rightleftarrows}} & Z_{\kappa,\beta} \\
{\scriptstyle \beta z_{\kappa,\beta}}\downarrow & & \downarrow{\scriptstyle z_{\kappa,\beta}} \\
A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\
C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D
\end{array}$$

in which r and q are the cokernels of $z_{\kappa,\beta}$ and $\beta z_{\kappa,\beta}$ respectively and γ and δ are the induced morphisms between them. Since $\beta z_{\kappa,\beta}$ is a normal monomorphism it follows that the square at the bottom is a split pullback and so can be completed as a morphism of split extensions

$$\begin{array}{ccccc}
X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\
{\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\
X & \xrightarrow{\sigma} & C & \underset{\delta}{\overset{\gamma}{\rightleftarrows}} & D.
\end{array}$$

Since in the diagram

$$\begin{array}{ccc}
Z_{\sigma,\delta} \times_D B & \xrightarrow{\pi_2} & B \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle r} \\
Z_{\sigma,\delta} & \xrightarrow[z_{\sigma,\delta}]{} & D
\end{array}$$

by Lemma 2.1, $\beta\pi_2$ commutes with κ, it follows by the universal property of $z_{\kappa,\beta}$ that there exists a unique morphism $t : Z_{\sigma,\delta} \times_D B \to Z_{\kappa,\beta}$ such that $\pi_2 = z_{\kappa,\beta}t$. Since $z_{\sigma,\delta}\pi_1 = r\pi_2 = rz_{\kappa,\beta}t = 0t = 0 = 0\pi_1$ and since π_1 is a (regular) epimorphism (being the pullback of a regular epimorphism), it follows that $z_{\sigma,\delta} = 0$ and so by assumption the split extension

$$X \xrightarrow{\sigma} C \underset{\delta}{\overset{\gamma}{\rightleftarrows}} D$$

is faithful, and every split extension admits a morphism into a faithful split extension as required. ∎

The following result is a generalization for subtractive categories of Lemma 3.1 of [8] we omit the proof:

2.16. Lemma. *Let $\mathbb{C}$ be a regular subtractive category. Each parallel pair of morphisms*

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle g}\downarrow\downarrow{\scriptstyle g'} & & {\scriptstyle h}\downarrow\downarrow{\scriptstyle h'} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

can be decomposed as a composite

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\overline{\kappa}} & \overline{A} & \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} & \overline{B} \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle m}\downarrow\downarrow{\scriptstyle m'} & & {\scriptstyle n}\downarrow\downarrow{\scriptstyle n'} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

where e and f are regular epimorphisms, and the pairs m and m', and n and n' are jointly monomorphic.

2.17. Corollary. *Let $\mathbb{C}$ be a regular subtractive category. A split extension*

$$K \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

is faithful if and only if for each parallel pair of morphisms

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle g}\downarrow\downarrow{\scriptstyle g'} & & {\scriptstyle h}\downarrow\downarrow{\scriptstyle h'} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

where the pairs g and g', and h and h' are jointly monomorphic, $g = g'$ and $h = h'$.

2.18. Proposition. *Let $\mathbb{C}$ be a pointed exact protomodular category with coequalizers in which the normality of unions holds. Eccentric extensions are faithful.*

Proof. Let

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

be an eccentric extension. It follows from Corollary 2.17 and protomodularity that it is sufficient to show for each pair of jointly monomorphic morphisms of split extensions

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle g}\downarrow\downarrow{\scriptstyle g'} & & {\scriptstyle h}\downarrow\downarrow{\scriptstyle h'} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B. \end{array}$$

that $h = h'$. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & & \\ {\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle u} & \searrow^{\langle g,g'\rangle} & \\ A & \xrightarrow{e} & R & & \\ & \searrow_{\langle 1,1\rangle} & & \searrow^{\langle r_1,r_2\rangle} & \\ & & & & A \times A \end{array}$$

in which $r_1, r_2 : R \to A$ is the kernel pair of the coequalizer of $g, g' : A' \to A$, and e and u are the unique morphisms such that $r_1 e = 1_A$ and $r_2 e = 1_A$, and $r_1 u = g$ and $r_2 u = g'$. Since e and u are jointly extremal-epimorphic (indeed if they factor through some monomorphism $m : S \to R$, then it will make S a reflexive relation and hence an effective equivalence relation contained in R and containing A'), it follows by the normality of unions $k = e\kappa = u\kappa'$ is a normal monomorphism. Since in the pullback

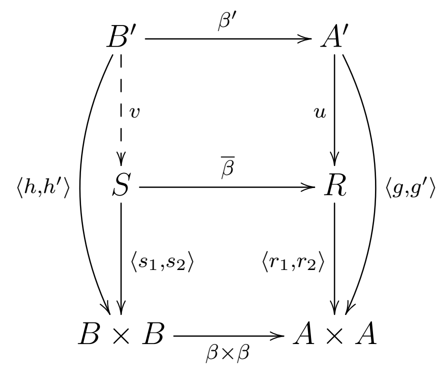
$$\begin{array}{ccc} Y & \xrightarrow{\lambda} & R \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \langle r_1,r_2\rangle} \\ A & \xrightarrow[\langle 0,1\rangle]{} & A \times A \end{array}$$

λ is a normal monomorphism with intersection with k equal to 0, it follows by Proposition 2.12 that k and λ commute. Therefore since $(\beta \times \beta)\langle 0, 1\rangle = \langle 0, 1\rangle\beta$ it follows that in the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\mu} & S & \xrightarrow{\overline{\beta}} & R \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \langle s_1,s_2\rangle} & & \downarrow{\scriptstyle \langle r_1,r_2\rangle} \\ B & \xrightarrow[\langle 0,1\rangle]{} & B \times B & \xrightarrow[\beta\times\beta]{} & A \times A \end{array}$$

where both the left and right hand squares are pullbacks, $\overline{\beta}\mu$ commutes with k. It follows that $\beta\eta = r_2\overline{\beta}\mu$ commutes with $\kappa = r_2 k$ and therefore by assumption $\eta = 0$. Since $\langle s_1, s_2\rangle : S \to B \times B$ is the preimage of an equivalence relation it is also an equivalence relation, and since $\eta = 0$ is the normalization of $\langle s_1, s_2\rangle : S \to B \times B$ it follows that

$s_1 = s_2$. Therefore the unique morphism v making the diagram



commute, forces $h = s_1 v = s_2 v = h'$ as required. ∎

2.19. Theorem. *Let $\mathbb{C}$ be a pointed exact protomodular category with coequalizers. If $\mathbb{C}$ satisfies the normality of unions and has centralizers of normal monomorphisms which are normal, then $\mathbb{C}$ is action accessible.*

Proof. The proof follows from Theorem 2.15 and Proposition 2.18 since the diagram consisting of leftward and downward directed arrows in Condition (d)(ii) of Theorem 2.15 is a union by protomodularity. ∎

3. Action accessibility and normalizers

It was shown in [6] that for a pointed exact protomodular category action accessibility follows from the existence of normalizers. In this section we show that for a pointed exact protomodular category action accessibility is equivalent to the existence of certain normalizers.

3.1. Theorem. *Let $\mathbb{C}$ be a pointed exact protomodular category. The following are equivalent:*

(a) $\mathbb{C}$ is action accessible;

(b) for each normal monomorphism $\kappa : X \to A$ the normalizer of $\langle \kappa, \kappa \rangle$ exists;

(c) for each protosplit monomorphism $\kappa : X \to A$ the normalizer of $\langle \kappa, \kappa \rangle$ exists;

(d) for each split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the category of relations in $\mathbf{SPLEXT}_X(\mathbb{C})$ on

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

has a terminal object;

(e) *for each equivalence relation*

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \quad (\text{with } s: A \to R)$$

the category of relations in $\mathbf{SPLEXT}_X(\mathbb{C})$ *on*

$$X \xrightarrow{k} R \underset{s}{\overset{r_1}{\rightleftarrows}} A,$$

where k *is the kernel of* r_1, *has a terminal object;*

(f) *for each split extension*

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the category of parallel morphisms in $\mathbf{SPLEXT}_X(\mathbb{C})$ *with codomain*

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

has a terminal object;

(g) *for each equivalence relation*

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \quad (\text{with } s: A \to R)$$

the category of parallel morphisms in $\mathbf{SPLEXT}_X(\mathbb{C})$ *with codomain*

$$X \xrightarrow{k} R \underset{s}{\overset{r_1}{\rightleftarrows}} A,$$

where k *is the kernel of* r_1, *has a terminal object;*

(h) *for each split extension*

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

there is a $\mathbf{K}$*-precartesian lifting of* $\langle 1, 1\rangle : X \to X \times X$ *to*

$$X \times X \xrightarrow{\kappa\times\kappa} A \times A \underset{\beta\times\beta}{\overset{\alpha\times\alpha}{\rightleftarrows}} B \times B.$$

(i) *for each equivalence relation*

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \quad (\text{with } s: A \to R)$$

there is a $\mathbf{K}$*-precartesian lifting of* $\langle 1, 1\rangle : X \to X \times X$ *to*

$$X \times X \xrightarrow{k\times k} R \times R \underset{s\times s}{\overset{r_1\times r_1}{\rightleftarrows}} A \times A,$$

where k *is the kernel of* r_1.

PROOF. The implications $(b) \Rightarrow (c)$, $(d) \Rightarrow (e)$, $(f) \Rightarrow (g)$, and $(h) \Rightarrow (i)$ follow trivially, and the implication $(c) \Rightarrow (h)$ follows from Proposition 2.4 and Lemma 2.7 in [6]. The implications $(d) \Leftrightarrow (f)$ and $(e) \Leftrightarrow (g)$ follow from Lemma 2.16, and the implications $(f) \Leftrightarrow (h)$ and $(g) \Leftrightarrow (i)$ follow from the fact that for any category with binary products there is an isomorphism of categories between relations on an object and monomorphisms into the product of that object with itself. Therefore the proof will be completed if we show that $(a) \Rightarrow (f)$, $(d) \Rightarrow (a)$, and $(i) \Rightarrow (b)$.

$(a) \Rightarrow (f)$: Let

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

be a split extension. Since $\mathbb{C}$ is action accessible there exists a morphism

$$\begin{array}{ccccc} X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\overline{\kappa}} & \overline{A} & \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} & \overline{B} \end{array}$$

with codomain a faithful extension. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{\kappa}} & R & \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightleftarrows}} & S \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\overline{\kappa}} & \overline{A} & \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} & \overline{B} \end{array}$$

in which (S, s_1, s_2) and (R, r_1, r_2) are the kernel pair of h and g respectively, $\tilde{\alpha}$ and $\tilde{\beta}$ are the induced morphisms between the kernel pairs, and $\tilde{\kappa}$ is the kernel of $\tilde{\alpha}$. Let

$$\begin{array}{ccccc} X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle g_1}\downarrow\downarrow{\scriptstyle g_2} & & {\scriptstyle h_1}\downarrow\downarrow{\scriptstyle h_2} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

be a parallel pair of morphism. Since the extension

$$X \xrightarrow{\overline{\kappa}} \overline{A} \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} \overline{B}$$

is faithful it follows that in the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\
{\scriptstyle 1_X}\downdownarrows{\scriptstyle 1_X} & & {\scriptstyle g_1}\downdownarrows{\scriptstyle g_2} & & {\scriptstyle h_1}\downdownarrows{\scriptstyle h_2} \\
X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\
{\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
X & \xrightarrow{\overline{\kappa}} & \overline{A} & \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightleftarrows}} & \overline{B}
\end{array}$$

$gg_1 = gg_2$ and $hh_1 = hh_2$ so by the universal properties of the kernel pairs (R, r_1, r_2) and (S, s_1, s_2), there exists a unique morphism

$$\begin{array}{ccccc}
X & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \\
{\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
X & \xrightarrow{\tilde{\kappa}} & R & \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightleftarrows}} & S
\end{array}$$

such that $g_1 = r_1 p$, $g_2 = r_2 p$, $h_1 = s_1 q$ and $h_2 = s_2 q$. This proves that

$$\begin{array}{ccccc}
X & \xrightarrow{\tilde{\kappa}} & R & \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightleftarrows}} & S \\
{\scriptstyle 1_X}\downdownarrows{\scriptstyle 1_X} & & {\scriptstyle r_1}\downdownarrows{\scriptstyle r_2} & & {\scriptstyle s_1}\downdownarrows{\scriptstyle s_2} \\
X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B
\end{array}$$

is the terminal object in the category of parallel morphisms in $\mathbf{SPLEXT}_X(\mathbb{C})$ with codomain

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B.$$

$(d) \Rightarrow (a)$: Let

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

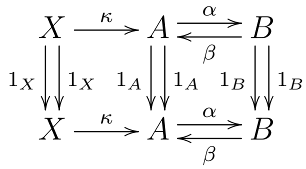
be a split extension and let

$$\begin{array}{ccccc}
X & \xrightarrow{\tilde{\kappa}} & R & \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightleftarrows}} & S \\
{\scriptstyle 1_X}\downdownarrows{\scriptstyle 1_X} & & {\scriptstyle r_1}\downdownarrows{\scriptstyle r_2} & & {\scriptstyle s_1}\downdownarrows{\scriptstyle s_2} \\
X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B
\end{array}$$

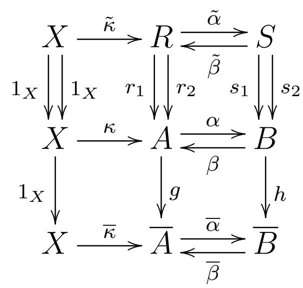
be the terminal object in the category of relations in $\mathbf{SPLEXT}_X(\mathbb{C})$ with codomain

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B.$$

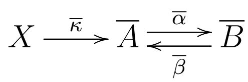
Note that since



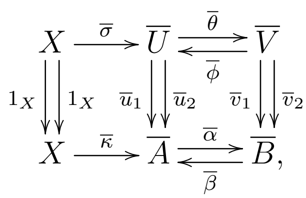
is an object in the same category it follows that (R, r_1, r_2) and (S, s_1, s_2) are effective equivalence relations and hence the pairs r_1, r_2 and s_1, s_2 admit coequalizers. Consider the diagram



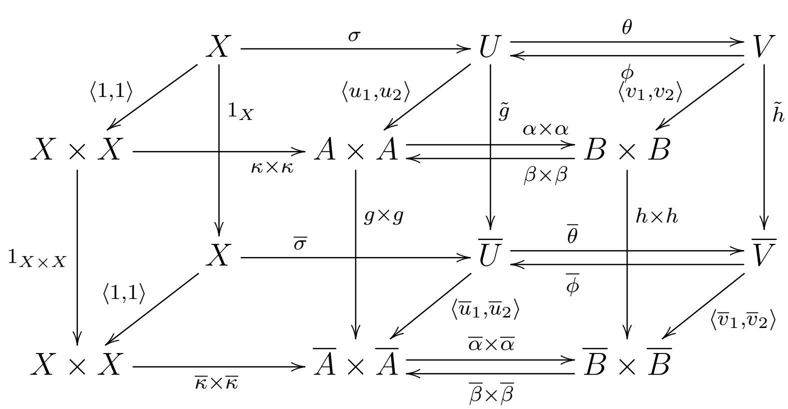
in which g and h are the coequalizers of r_1 and r_2 and s_1 and s_2 respectively, and $\overline{\kappa}$ is the kernel of $\overline{\alpha}$ (which has domain X since in an exact category, every equivalence relation is effective and every regular epimorphism is an effective descent morphism, the lower right hand side square is a split pullback since so are upper ones). We will show that the split extension



is faithful. It follows from Corollary 2.17 that it is sufficient to show that, for each relation



in $\mathbf{SPLEXT}_X(\mathbb{C})$, $\overline{v}_1 = \overline{v}_2$. By forming the pullback



in **SPLEXT**($\mathbb{C}$) we obtain a new relation

$$\begin{array}{ccccc} X & \xrightarrow{\sigma} & U & \underset{\phi}{\overset{\theta}{\rightleftarrows}} & V \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle u_1}\downarrow\downarrow{\scriptstyle u_2} & & {\scriptstyle v_1}\downarrow\downarrow{\scriptstyle v_2} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

which therefore factors through the relation

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{\kappa}} & R & \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightleftarrows}} & S \\ {\scriptstyle 1_X}\downarrow\downarrow{\scriptstyle 1_X} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

via a unique morphism

$$\begin{array}{ccccc} X & \xrightarrow{\sigma} & U & \underset{\phi}{\overset{\theta}{\rightleftarrows}} & V \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{\tilde{\kappa}} & R & \underset{\tilde{\beta}}{\overset{\tilde{\alpha}}{\rightleftarrows}} & S. \end{array}$$

It follows that $\overline{v}_1\tilde{h} = hv_1 = hs_1v = hs_2v = hv_2 = \overline{v}_2\tilde{h}$ and similarly $\overline{u}_1\tilde{g} = \overline{u}_2\tilde{g}$ and therefore, since $\tilde{g}$ and $\tilde{h}$ being pullbacks of regular epimorphism are (regular) epimorphisms, that $\overline{u}_1 = \overline{u}_2$ and $\overline{v}_1 = \overline{v}_2$.
$(i) \Rightarrow (b)$: Let $\kappa : X \to A$ be a normal monomorphism and let $\gamma : A \to C$ be a morphism which it is the kernel of. By forming the pullback

$$\begin{array}{ccc} A \times_C A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \gamma} \\ A & \xrightarrow[\gamma]{} & C \end{array}$$

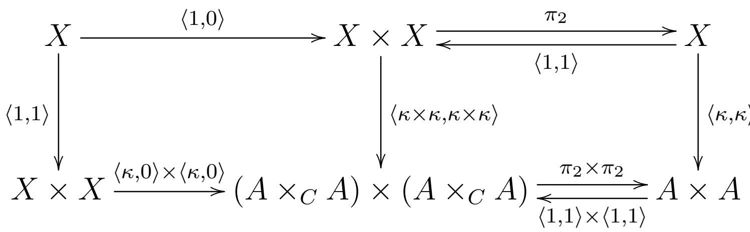
we obtain the split extension

$$X \xrightarrow{\langle \kappa,0\rangle} A \times_C A \underset{\langle 1,1\rangle}{\overset{\pi_2}{\rightleftarrows}} A.$$

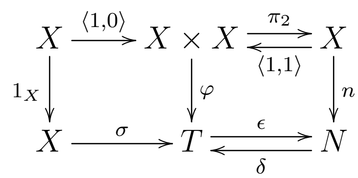
Let

$$\begin{array}{ccccc} X & \xrightarrow{\sigma} & T & \underset{\delta}{\overset{\epsilon}{\rightleftarrows}} & N \\ {\scriptstyle \langle 1,1\rangle}\downarrow & & \downarrow{\scriptstyle \langle\langle t_{1,1},t_{1,2}\rangle,\langle t_{2,1},t_{2,2}\rangle\rangle} & & \downarrow{\scriptstyle \langle m_1,m_2\rangle} \\ X \times X & \xrightarrow{\langle \kappa,0\rangle\times\langle \kappa,0\rangle} & (A \times_C A) \times (A \times_C A) & \underset{\langle 1,1\rangle\times\langle 1,1\rangle}{\overset{\pi_2\times\pi_2}{\rightleftarrows}} & A \times A \end{array}$$

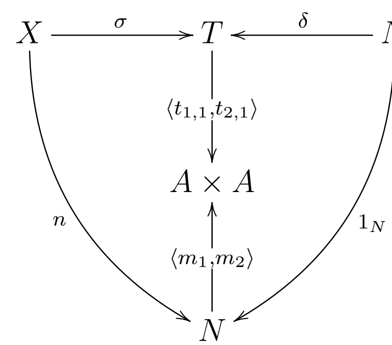
be a **K**-precartesian lifting. Since the diagram



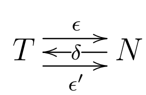
is a lifting of the same morphism to the same extension there exists a unique morphism



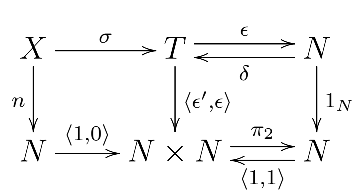
such that $\langle \kappa \times \kappa, \kappa \times \kappa\rangle = \langle\langle t_{1,1}, t_{1,2}\rangle, \langle t_{2,1}, t_{2,2}\rangle\rangle\varphi$ and $\langle \kappa, \kappa\rangle = \langle m_1, m_2\rangle n$. We will show that $(N, n, \langle m_1, m_2\rangle)$ is the normalizer of $\langle \kappa, \kappa\rangle$. It is easy to check that the diagram



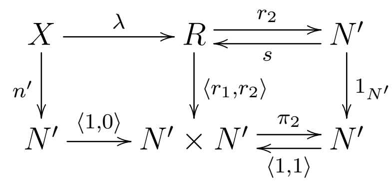
commutes, and therefore since $\langle m_1, m_2\rangle$ is a monomorphism and σ and δ are jointly strongly epimorphic, there exist a unique morphism $\epsilon' : T \to N$ such that $\langle m_1, m_2\rangle\epsilon' = \langle t_{1,1}, t_{2,1}\rangle$, and hence $\epsilon'\delta = 1_N$ and $\epsilon'\sigma = n$. Since $\langle\langle t_{1,1}, t_{1,2}\rangle, \langle t_{2,1}, t_{2,2}\rangle\rangle$ is a monomorphism and $\pi_1 \times \pi_1, \pi_2 \times \pi_2 : (A \times_C A) \times (A \times_C A) \to A \times A$ are jointly monomorphic it follows that $\epsilon, \epsilon' : T \to N$ are jointly monomorphic. Therefore since $\epsilon\delta = 1_N = \epsilon'\delta$ and since $\mathbb{C}$ is a Mal'tsev category, it follows that



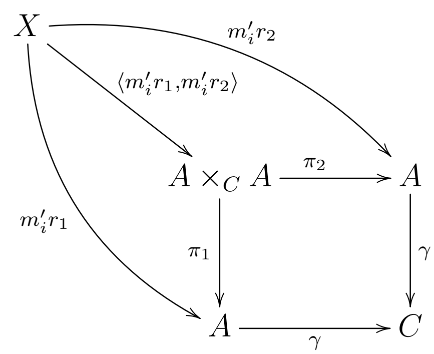
is an equivalence relation. It follows that since the diagram



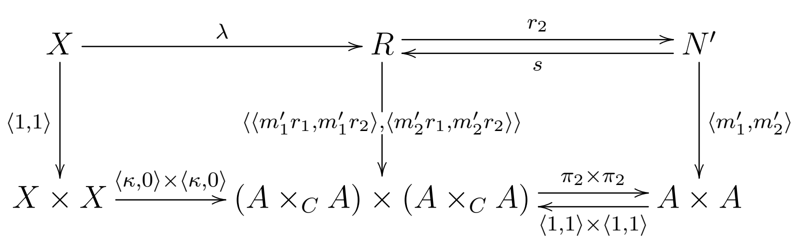
is a morphism of split extensions, n is normal. Let $(N', n', \langle m'_1, m'_2\rangle)$ be a factorization of $\langle \kappa, \kappa\rangle$ as a normal monomorphism followed by a monomorphism, and let



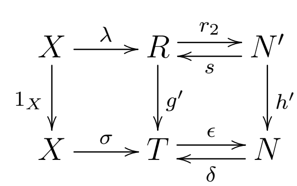
be the morphism of split extensions determined by the denormalization of n'. Since λ and s are jointly epimorphic, $\gamma m'_i r_1 \lambda = \gamma m'_i n' = \gamma\kappa = 0 = \gamma 0 = \gamma m'_i 0 = \gamma m'_i r_2 \lambda$, and $\gamma m'_i r_1 s = \gamma m'_i = \gamma m'_i r_2 s$ it follows that $\gamma m'_i r_1 = \gamma m'_i r_2$. Therefore for each i in $\{1, 2\}$ there exists a unique morphism $\langle m'_i r_1, m'_i r_2\rangle$ making the diagram



commute. Since the diagram



is a morphism of split extensions there exists a unique morphism



such that $\langle\langle m'_1 r_1, m'_1 r_2\rangle, \langle m'_2 r_1, m'_2 r_2\rangle\rangle = \langle\langle t_{1,1}, t_{1,2}\rangle, \langle t_{2,1}, t_{2,2}\rangle\rangle g'$ and $\langle m'_1, m'_2\rangle = \langle m_1, m_2\rangle h'$. This proves that $(N, n, \langle m_1, m_2\rangle)$ is the normalizer of $\langle \kappa, \kappa\rangle$. ∎

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Mathematics Division
Department of Mathematical Sciences
Stellenbosch University
Private Bag X1
7602 Matieland
South Africa
Email: jamesgray@sun.ac.za

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