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ON THE 3-REPRESENTATIONS OF GROUPS AND THE 2-CATEGORICAL TRACES

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ABSTRACT. To 2-categorify the theory of group representations, we introduce the notions of the 3-representation of a group in a strict 3-category and the strict 2-categorical action of a group on a strict 2-category. We also 2-categorify the concept of the trace by introducing the 2-categorical trace of a 1-endomorphism in a strict 3-category. For a 3-representation ρ of a group G and an element f of G, the 2-categorical trace $\mathbb{T}r_2\rho_f$ is a category. Moreover, the centralizer of f in G acts categorically on this 2-categorical trace. We construct the induced strict 2-categorical action of a finite group, and show that the 2-categorical trace $\mathbb{T}r_2$ takes an induced strict 2-categorical action into an induced categorical action of the initia groupoid. As a corollary, we get the 3-character formula of the induced strict 2-categorical action.

Contents

1. Introduction

The notion of a group acting on a category goes back to Grothendieck's Tohoku paper [15]. Recently Ganter, Kapranov [13] and Bartlett [5] categorified the concept of the trace of a linear transformation by introducing the notion of the category trace. This is a set associated to any endofunctor on a small category, and is a vector space in the linear case. Moreover, a functor commuting with the endofunctor defines a linear transformation on this vector space, whose ordinary trace defines a joint trace. This allowed these authors to define 2-characters. When a group acts on a k-linear category, the joint trace of a commuting pair of group elements is the 2-character of the categorical

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action. This is an analogue of the character of the representation of a group on a vector space and is a 2-class function. In general, an n-class function is a function defined on n-tuples of commuting elements of a group and invariant under simultaneous conjugation. Such functions already appear in equivariant Morava E-theory [16]. The theory of 2-representations was developed further in [6] [10] [11] [12] [14] [23] [26] etc..

During the past two decades an active direction of research has been the categorification of some algebraic, geometric or analytic concepts. For example, 2-vector spaces, 2-bundles (gerbes), 2-connections and 2-curvatures. All involve 2-categorical constructions and have various applications, such as a geometric definition of elliptic cohomology [1], 2-gauge theory [3] [4] and the 2-dimensional Langlands correspondence [17] [22]. It is believed that higher categorification is necessary for many geometric and physical applications. 3-categorical constructions already appear in the theory of 2-gerbes (3-bundles) [7] [8] and in 3-gauge theory [21] [24] [27], which involves more general **Gray**-categories. The purpose of this paper is to 2-categorify the theory of group representations and characters by introducing the notions of the 3-representation of a group in a 3-category, the strict 2-categorical action of a group on a 2-category and the 2-categorical trace. The problem of investigating representations of groups in higher categories has already been mentioned in [13].

A geometric motivation for considering higher representations of groups is as follows. Suppose that G is a Lie group and that H is a Lie subgroup. Let V be a finite dimensional representation of H. We can construct a homogeneous vector bundle $G \times_H V$ over the homogeneous space G/H as $G \times V$ modulo the equivalent relation

$$(g, v) \sim (gh, h^{-1}.v) \quad \text{for} \quad g \in G, h \in H, v \in V.$$

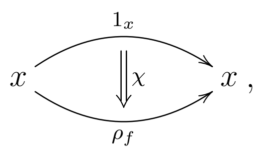
The space of sections of this bundle is exactly the space $\mathrm{Ind}_H^G V$ of the induced representation. When V is a 2- or 3-representation of H, a similar construction will give us a homogeneous 2- or 3-bundle over the homogeneous space G/H. This will provide us good examples of higher bundles in higher differential geometry and higher gauge theory. But for a higher representation π of the Lie group H, the functors $\pi(h)$ usually depend on $h \in H$ "discontinuously". Thus it is not easy to describe the space of "sections" of the resulting higher bundles. However, when G and H are finite, G/H is discrete, and so we have a clear picture. This is why we only consider 3-representations of a finite group in this paper.

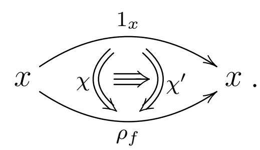
For simplicity, we only consider strict 2- and 3-categories. A 3-representation of a group G in a 3-category is given by a 1-isomorphism for each element of G, a 2-isomorphism for each pair of elements of G, and a 3-isomorphism for each triple of elements of G. These 3-isomorphisms must satisfy the 3-cocycle condition. This condition has a simple geometric interpretation: the composition of 3-isomorphisms corresponding to 5 tetrahedrons in the boundary of a 4-simplex is equal to the identity 3-arrow. Given a 2-category $\mathcal{V}$, a *strict 2-categorical action of G on $\mathcal{V}$* is given by an endofunctor of $\mathcal{V}$ for each element of G, a pseudonatural transformation between functors for each pair of elements of G, and a modification for each triple of elements of G. Details are given in Section 2.3-2.4.

Recall that given a 2-representation ϱ of a finite group G in a 2-category $\mathcal{V}$ and an element f of G, we have a 1-isomorphism

$$\varrho_f : x \to x,$$

where x is an object of $\mathcal{V}$ that G acts on. In [5] [13], the authors introduced the notion of the categorical trace $\mathbb{T}r\varrho_f$. This is the set of 2-arrows in $\mathcal{V}$, whose 1-source is the unit arrow 1_x and whose 1-target is ϱ_f. The centralizer of f in G acts on this set naturally. In our case, given a 3-representation ρ of G in a 3-category $\mathcal{C}$ and an element f of G, we have a 1-isomorphism $\rho_f : x \to x$ in $\mathcal{C}$. The 2-categorical trace $\mathbb{T}\mathrm{r}_2\rho_f$ is a category. Its objects are 2-arrows with 1-source the unit arrow 1_x and 1-target ρ_f, and its morphisms are 3-isomorphisms between such 2-isomorphisms:





Moreover, the centralizer of f in G, denoted by $C_G(f)$, acts categorically on the 2-categorical trace $\mathbb{T}\mathrm{r}_2\rho_f$ in the following sense. We can define an invertible functor ψ_g acting on $\mathbb{T}r_2\rho_f$ for each $g \in C_G(f)$, and for any $h, g \in C_G(f)$, define a natural isomorphism

$$\Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg}$$

between such functors on the category $\mathbb{T}r_2\rho_f$. This construction is given in Section 3. To prove the action to be categorical, we have to show the associativity in the definition of categorical action, i.e.,

$$\Gamma_{k,hg}\#(\psi_k \circ \Gamma_{h,g}) = \Gamma_{kh,g}\#(\Gamma_{k,h} \circ \psi_g) : \psi_k \circ \psi_h \circ \psi_g \longrightarrow \psi_{khg}, \tag{1}$$

for any $k, h, g \in C_G(f)$, where $\#$ is the composition of natural transformations between functors on the category $\mathbb{T}\mathrm{r}_2\rho_f$. This is the most difficult and technical part of this paper. By applying the 3-cocycle identity (15) repeatedly, we prove in Section 6 that

$$\{\psi_g, \Gamma_{h,g}\}_{g,h \in C_G(f)}$$

is a categorical action of the centralizer $C_G(f)$ on the category $\mathbb{T}\mathrm{r}_2\rho_f$.

An easy and interesting example of 3-representations is the 1-dimensional one, which is given by a 3-cocycle on a finite group G. A 3-*cocycle* is a function $c : G \times G \times G \longrightarrow k^*$ such that

$$c(g_3, g_2, g_1)c(g_4, g_3g_2, g_1)c(g_4, g_3, g_2) = c(g_4, g_3, g_2g_1)c(g_4g_3, g_2, g_1) \tag{2}$$

for any $g_4, \ldots, g_1 \in G$. Here k is a field of characteristic 0. Such a 3-cocycle gives us a strict action of G on a 2-category with only one object, one 1-arrow and the set of 2-arrows isomorphic to k^*. For an element f of G, its 2-categorical trace $\mathbb{T}\mathrm{r}_2\rho_f$ is a category with only one object and the set of 1-arrows isomorphic to k^*. For any h and g in the centralizer $C_G(f)$, we can construct an element $\Gamma_{h,g}$ (48) from the 3-cocycle c in (2) such that $\Gamma_{*,*}$ is a 2-cocycle on the centralizer. This can be proved quite easily and elementarily by using the condition (2) for 3-cocycles repeatedly in Section 6.1. This corresponds step by step to the proof of the general case carried out in Section 6.4. It can be viewed as a simple model of the proof of (1). The difficulty in the general case is that we have to handle diagrams, while in the 1-dimensional case we only need to handle element of the field k.

Suppose that $\mathcal{C}$ is a k-linear 3-category. Then $\mathbb{T}r_2\rho_f$ is also a k-linear category. If k, g and f are pairwise commutative, then ψ_k and ψ_g are k-linear endofunctors acting on $\mathbb{T}r_2\rho_f$. We define the 3-character of a 3-representation ρ to be

$$\chi_\rho(f,g,k) := \text{ the joint trace of functors } \psi_k \text{ and } \psi_g \text{ on } \mathbb{T}\mathrm{r}_2\rho_f.$$

It is the trace of the linear transformation induced by the functor ψ_k on the k-vector space $\mathbb{T}r\psi_g$.

Suppose that a subgroup H of a finite group G acts strictly 2-categorically on a 2-category $\mathcal{V}$. In Section 4, we define the induced 2-category $\mathrm{Ind}_H^G(\mathcal{V})$ and strict 2-categorical action of G on it. In Section 5, we calculate the 2-categorical trace of the induced strict 2-categorical action as

$$\mathbb{T}r_2(\mathrm{Ind}_H^G\rho) = \mathrm{Ind}_{\Lambda(H)}^{\Lambda(G)}\mathbb{T}r_2(\rho), \tag{3}$$

where $\Lambda(H)$ and $\Lambda(G)$ are initia groupoids associated to groups H and G, respectively. As a corollary, we derive the 3-character of the induced strict 2-categorical action, which coincides with the formula in [16] for n-characters when $n = 3$. These results are the generalization of induced categorical action and the 2-character formula in [13].

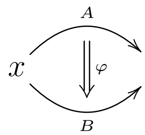
It would be interesting to investigate the m-representation of a group in an m-category, the m-cocycle condition and $(m-1)$-categorical trace for a positive integer $m > 3$.

I would like to thank the anonymous referee for his/her many inspiring and valuable suggestions.

2. The 3-representations of groups

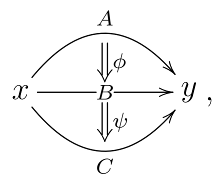
2.1. Strict 2-categories. A 2-*category* is a category enriched over the category of all small categories. In particular, a strict 2-category $\mathcal{C}$ consists of collections $\mathcal{C}_0$ of objects, $\mathcal{C}_1$ of arrows and $\mathcal{C}_2$ of 2-arrows, together with

- functions $s_n, t_n : \mathcal{C}_i \to \mathcal{C}_n$ for all $0 \le n < i \le 2$, called *n-source* and *n-target*,
- functions $\#_n : \mathcal{C}_{n+1} \times \mathcal{C}_{n+1} \to \mathcal{C}_{n+1}$ for all $n = 0, 1$, called *vertical composition*,
- a function $\#_0 : \mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2$, called the *horizontal composition*,
- a function $1_* : \mathcal{C}_i \to \mathcal{C}_{i+1}$ for $i = 0, 1$, called the *identity*.

For a 1-arrow $x \xrightarrow{A} y$, its 0-source and 0-target are x and y, respectively. For a 2-arrow 

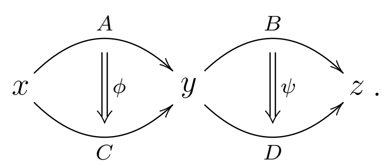
in $\mathcal{C}_2$, its 1-source and 1-target are $x \xrightarrow{A} y$ and $x \xrightarrow{B} y$, respectively, while its 0-source and 0-target are x and y, respectively.

Two 1-arrows A and A' are called 0-*composable* if the 0-target of A coincides with the 0-source of A'. In this case, their vertical composition is $A\#_0A' : x \xrightarrow{A} y \xrightarrow{A'} z$. Two 2-arrows ϕ and ψ are called 1-*composable* if the 1-target of ϕ coincide with the 1-source of ψ. In this case, their vertical composition $\phi\#_1\psi$ is

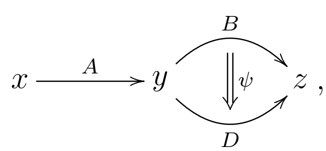


where $A = s_1(\phi)$, $B = t_1(\phi) = s_1(\psi)$, $C = t_1(\psi)$, $x = s_0(\phi) = s_0(\psi)$, $y = t_0(\phi) = t_0(\psi)$. In general, two arrows are composable if the target matching condition is satisfied.

Two 2-arrows ϕ and ψ are called *horizontally composable* (0-*composable*) if the 0-target of ϕ coincides with the 0-source of ψ. In this case, their horizontal composition $\phi\#_0\psi$ is



In particular, when $\phi = 1_A$ we call $1_A\#_0\psi$ *whiskering from left by* 1-*arrow* A, and denote it by

$A\#_0\psi :$ 

Similarly, we define *whiskering from right by a* 1-*arrow.*

The identities satisfy

$$\begin{array}{llll} 1_x\#_0A = A = A\#_01_y, & \text{for any} & 1-\text{arrow} & A : x \longrightarrow y; \\ 1_A\#_1\phi = \phi = \phi\#_11_B, & \text{for any} & 2-\text{arrow} & \phi : A \Longrightarrow B. \end{array} \tag{4}$$

The composition $\#_p$ satisfies the *associativity*

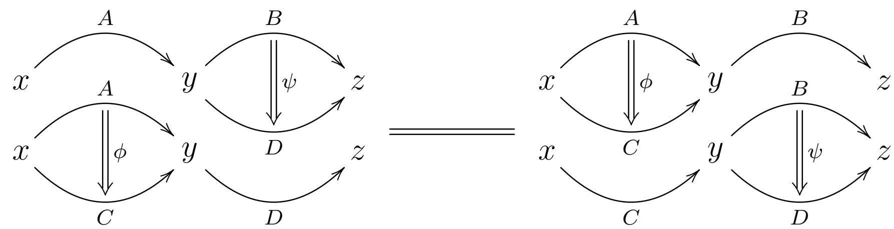
$$(\phi\#_p\psi)\#_p\omega = \phi\#_p(\psi\#_p\omega), \tag{5}$$

if the corresponding arrows are p-composable, for $p = 0$ or 1.

The horizontal composition satisfies the *interchange law*:

$$(A\#_0\psi)\#_1(\phi\#_0D) = \phi\#_0\psi = (\phi\#_0B)\#_1(C\#_0\psi). \tag{6}$$

Namely,



the vertical composition of left two 2-arrows coincides with the vertical composition of right two 2-arrows. They are both equal to the horizontal composition $\phi\#_0\psi$. The interchange law allows us to change the order of compositions of 2-arrows, up to whiskerings. This is essentially the paste theorem for 2-categories (cf. §2.13 in [18]).

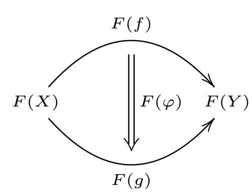
The interchange law (6) is a special case of the following more general *compatibility condition* for different compositions. If $(\beta, \beta'), (\gamma, \gamma') \in \mathcal{C}_k \times \mathcal{C}_k$ are p-composable and $(\beta, \gamma), (\beta', \gamma') \in \mathcal{C}_k \times \mathcal{C}_k$ are q-composable, $p, q = 0, 1$, then we have

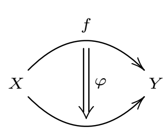
$$(\beta\#_p\beta')\#_q(\gamma\#_p\gamma') = (\beta\#_q\gamma)\#_p(\beta'\#_q\gamma'). \tag{7}$$

The the left-hand side of the interchange law (6) is exactly the compatibility condition (7) with $p = 0, q = 1, \beta = 1_A, \beta' = \psi, \gamma = \phi, \gamma' = 1_D$, by using the property (4) of identities. (4) (5) and (7) are the main axioms that a strict 2-category satisfies.

A 1-arrow $A : x \to y$ is called *invertible* or a 1-*isomorphism*, if there exists another 1-arrow $B : y \to x$ such that $1_x = A\#_0B$ and $B\#_0A = 1_y$. A strict 2-category in which every 1-arrow is invertible is called a *strict* 2-*groupoid*. A 2-arrow $\varphi : A \Rightarrow B$ is called *invertible* or a 2-*isomorphism* if there exists another 2-arrow $\psi : B \Rightarrow A$ such that $\psi\#_1\varphi = 1_B$ and $\varphi\#_1\psi = 1_A$. ψ is uniquely determined and called the *inverse of* φ.

Let $\mathcal{S}$ and $\mathcal{T}$ be two strict 2-categories. A *(strict)* 2-*functor* $F : \mathcal{S} \to \mathcal{T}$ is an assignment of a 2-arrow

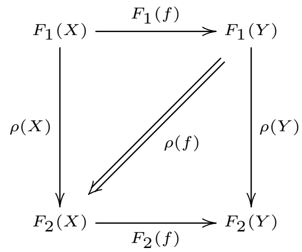


to each 2-arrow 

such that F preserves compositions $\#_p$ and identities. More explicitly, we have

- $F(\varphi\#_1\psi) = F(\varphi)\#_1F(\psi)$ and $F(1_f) = 1_{F(f)}$ for all composable 2-arrows φ and ψ and any 0- or 1-arrow f;
- $F(g)\#_0F(f) = F(g\#_0f)$ for all composable 1-arrows g and f, and $F(\varphi)\#_0F(\psi) = F(\varphi\#_0\psi)$ for all horizontally composable 2-arrows φ and ψ.

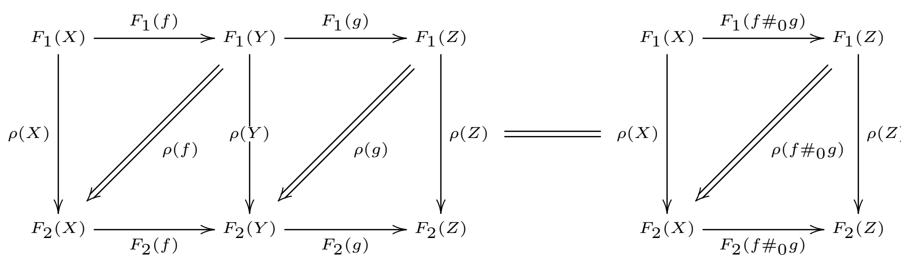
Let F_1 and F_2 be two 2-functors from $\mathcal{S}$ to $\mathcal{T}$. A *pseudonatural transformation* $\rho : F_1 \to F_2$ is an assignment of a 1-arrow $\rho(X)$ in $\mathcal{T}$ to each object X in $\mathcal{S}$ and a 2-isomorphism $\rho(f)$



(8)

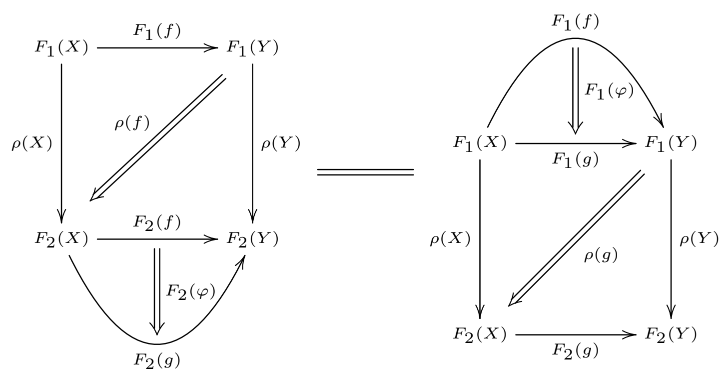
in $\mathcal{T}$ to each 1-arrow $f : X \to Y$ in $\mathcal{S}$ such that they satisfy two axioms

- The composition of 1-arrows in $\mathcal{S}$:



; (9)

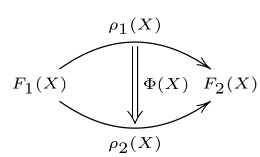
- The compatibility with 2-arrows:



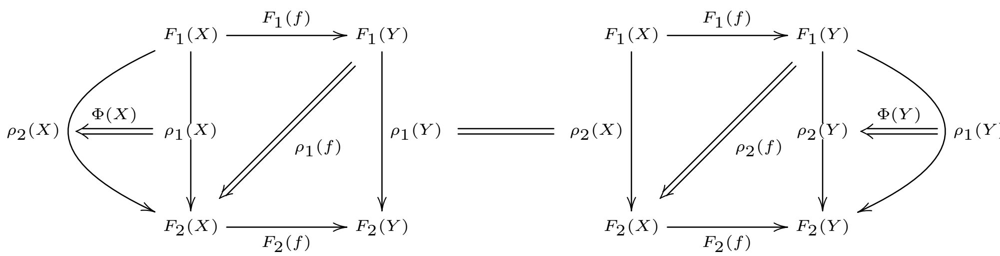
(10)

for any 2-arrow $\varphi : f \Rightarrow g$.

Let $F_1, F_2 : \mathcal{S} \to \mathcal{T}$ be two strict 2-functors and let $\rho_1, \rho_2 : F_1 \to F_2$ be pseudonatural transformations. A *modification* $\Phi : \rho_1 \Longrightarrow \rho_2$ is an assignment of a 2-arrow



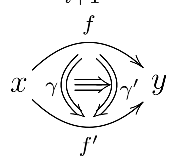
in $\mathcal{T}$ to any object X in $\mathcal{S}$, which satisfies



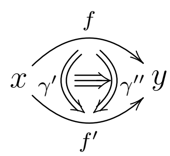
. (11)

2.2. Strict 3-categories. A 3-*category* is a category enriched over the category of all small strict 2-categories. In particular, a *strict* 3-*category* $\mathcal{C}$ consists of collections $\mathcal{C}_0$ of objects, $\mathcal{C}_1$ of 1-arrows, $\mathcal{C}_2$ of 2-arrows, and $\mathcal{C}_3$ of 3-arrows, together with

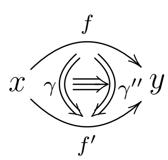
- functions $s_n, t_n : \mathcal{C}_i \to \mathcal{C}_n$ for all $0 \leq n < i \leq 3$, called *n-source* and *n-target*,
- functions $\#_n : \mathcal{C}_{n+1} \times \mathcal{C}_{n+1} \to \mathcal{C}_{n+1}$ for all $n = 0, 1, 2$, called *vertical composition*,
- a function $\#_p : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$, $p + 2 \leq i$, called the *horizontal composition*,
- a function $1_* : \mathcal{C}_i \to \mathcal{C}_{i+1}$ for $i = 0, 1$, called *identity*.

For a 3-arrow φ : 

, its 2-source and 2-target are γ and γ' respectively.

The 3-arrows φ and φ' : 

are 2-composable, and their composition $\varphi \#_2 \varphi'$ is



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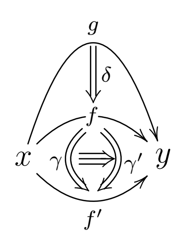
In a strict 3-category, 0-, 1- and 2-arrows behave as in a 2-category. We call two 3-arrows φ and ψ *horizontally p-composable* if the p-target of φ coincides with the p-source of ψ, $p = 0, 1$, and denote their horizontal composition as $\varphi \#_p \psi$.

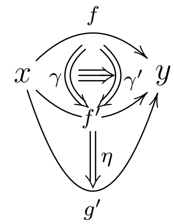
For a 2-arrow δ, 3-arrows 1_δ and φ are *horizontally* 1-*composable* if the 1-target of δ coincides with the 1-source of φ. In this case,

$$\delta \#_1 \varphi := 1_\delta \#_1 \varphi$$

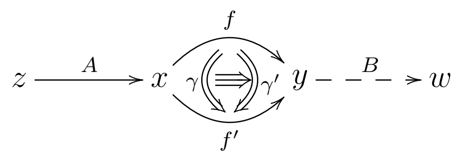


,

is called *whiskering from above by a 2-arrow* δ. It is similar to define whiskering from below:



There is also *whiskering from left (or right) by a* 1*-arrow* $A\#_0\varphi := 1_{1_A}\#_0\varphi$ (or $\varphi\#_0 B$):



The properties of identities, the associativity and the compatibility condition for different compositions, similar to (4) (5) and (7) for a strict 2-category, also hold in a strict 3-category. See page 8 of [19] for an explicit definition of a strict m-category.

A *strict* 3*-functor* (or a *functor*) is a map preserving compositions and identities.

2.3. Remark. *In a strict* 3*-category, the interchange law (6) for the horizontal composition of* 2*-arrows is also satisfied. But in general, a* 3*-category does not satisfy the interchange law.* **Gray***-categories are the greatest possible semi-strictification of* 3*-categories, and appear naturally in* 3*-gauge theory [27]. The* 3*-representation in a* **Gray***-category is more natural, but is much more complicated. So we restrict to the* 3*-representation in strict* 3*-categories in this paper.*

In a strict 3-category $\mathcal{C}$, a 1-arrow $B : x \to y$ is called *a* 1*-isomorphism* if there exists 1-arrow $C : y \to x$ such that there exist 2-isomorphisms $u : 1_y \Longrightarrow C\#_k B$ and $v : 1_x \Longrightarrow B\#_k C$. We call C a *quasi-inverse to* B, and vise versa. However, when $k = 2$ or 3, we call a k-arrow *a* k*-isomorphism* if it is strictly invertible.

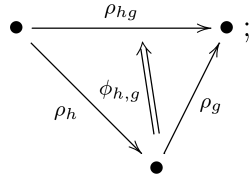
2.4. The 3-representations of a group in a strict 3-category . Let $\mathcal{C}$ be a strict 3-category and let G be a group. G can be viewed as a strict 3-category with only one object $\bullet$, G as the set of 1-arrows $g : \bullet \to \bullet$, the set of 2-arrows consisting of the identities of 1-arrows, and the set of 3-arrows consisting of the identities of 2-arrows. A 3*-representation of a group* G *in* $\mathcal{C}$ is a weak functor ρ from G to $\mathcal{C}$ in the following sense. We have

(1) an object x of $\mathcal{C}$;

(2) for each $g \in G$, a 1-isomorphism $\rho_g : x \to x$;

(3) for each $h, g \in G$, a 2-isomorphism $\phi_{h,g} : \rho_h\rho_g \Longrightarrow \rho_{hg}$ (here and in the following

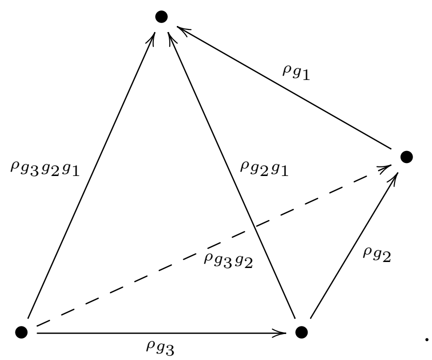
we write $\rho_h \#_0 \rho_g$ as $\rho_h\rho_g$ for simplicity), corresponding to the 2-cell



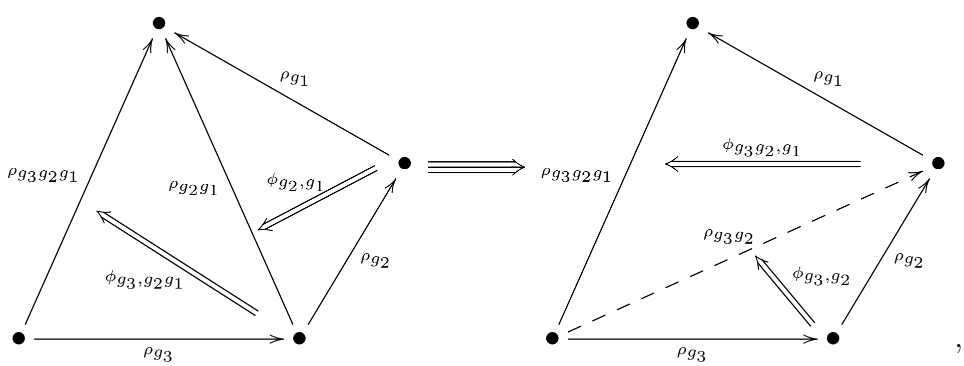
(4) for each $g_3, g_2, g_1 \in G$, a 3-isomorphism, called the *associator*,

$$\Phi_{g_3,g_2,g_1} : (\rho_{g_3}\#_0\phi_{g_2,g_1})\#_1\phi_{g_3,g_2g_1} \Longrrightarrow (\phi_{g_3,g_2}\#_0\rho_{g_1})\#_1\phi_{g_3g_2,g_1}, \tag{12}$$

corresponding to the 3-cell

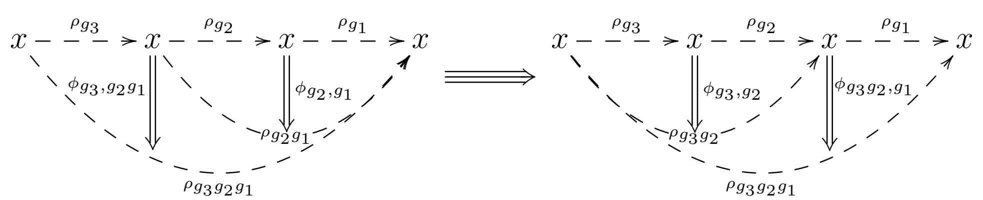


It can be viewed as exchanging the diagonals of the quadrilateral:



(13)

which can also be drawn in the following form:



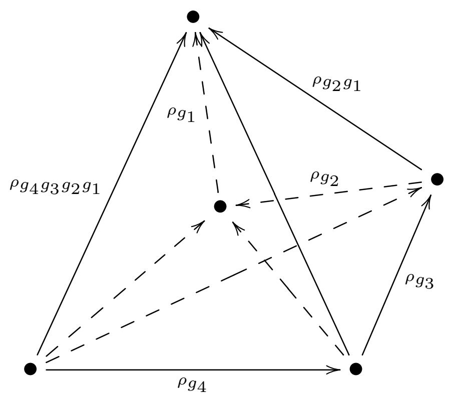
(14)

(5) a 2-isomorphism $\phi_1 : \rho_1 \Longrightarrow 1_x$;

such that the following conditions are satisfied:

- $\phi_{1,g} = \phi_1 \#_0 \rho_g$, $\phi_{g,1} = \rho_g \#_0 \phi_1$.
- the 3-*cocycle condition* that for any $g_4, \ldots, g_1 \in G$, we have

$$\begin{aligned} &\{[\rho_{g_4} \#_0 \Phi_{g_3,g_2,g_1}] \#_1 \phi_{g_4,g_3g_2g_1}\} \#_2 \{[\rho_{g_4} \#_0 \phi_{g_3,g_2} \#_0 \rho_{g_1}] \#_1 \Phi_{g_4,g_3g_2,g_1}\} \\ &\qquad \#_2 \{[\Phi_{g_4,g_3,g_2} \#_0 \rho_{g_1}] \#_1 \phi_{g_4g_3g_2,g_1}\} \\ &= \{[(\rho_{g_4}\rho_{g_3}) \#_0 \phi_{g_2,g_1}] \#_1 \Phi_{g_4,g_3,g_2g_1}\} \#_2 \{[\phi_{g_4,g_3} \#_0 (\rho_{g_2}\rho_{g_1})] \#_1 \Phi_{g_4g_3,g_2,g_1}\}. \end{aligned} \tag{15}$$



Equivalently, the composition of the 3-isomorphisms represented by 5 tetrahedrons above in the boundary of a 4-simplex is the identity. This comes from the fact that the boundary of the corresponding 4-simplex in the 3-category G is the identity 3-arrow.

2.5. Remark. *(1) For simplicity, we assume in this paper that* $\rho_1 = 1_x$ *and that* ϕ_1 *is the identity.*

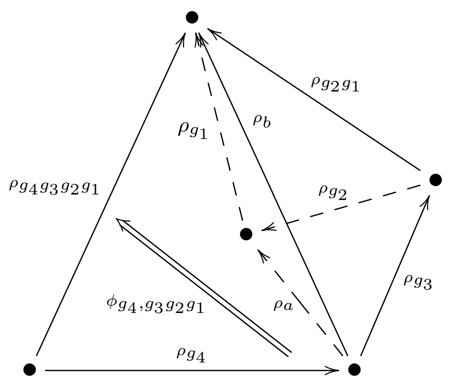
(2) The 3*-cocycle* $\{\Phi_{g_3,g_2,g_1}\}$ *defines an element of the* 3*-dimensional non-abelian cohomology. A first attempt at an explicit description of the* 3*-dimensional non-abelian cohomology of a group goes back to Dedecker [9]. See section 4 of [7] for* 3*-dimensional non-abelian Čech cocycles, which can be used to construct a* 2*-gerbe.*

2.6. The 3-cocycle condition . We will give a clear geometric description of the 3-cocycle condition (15) in terms of 5 tetrahedrons in the boundary of a 4-simplex above. This is equivalent to triviality of the 3-holonomy. See section 5 C of [27] for the 3-holonomy in the lattice 3-gauge theory (the cubical case), where 3-gauge theory from the point of view of **Gray**-categories is investigated.

In the left-hand side of the 3-cocycle condition (15), the first 3-isomorphism is

$$A_1 = [\rho_{g_4} \#_0 \Phi_{g_3,g_2,g_1}] \#_1 \phi_{g_4,g_3g_2g_1}. \tag{16}$$

Here Φ_{g_3,g_2,g_1} is a 3-isomorphism whiskered from left by the 1-isomorphism ρ_{g_4}, and $\rho_{g_4} \#_0 \Phi_{g_3,g_2,g_1}$ is whiskered from below by the 2-isomorphism $\phi_{g_4,g_3g_2g_1}$. A_1 corresponds to the 3-cell

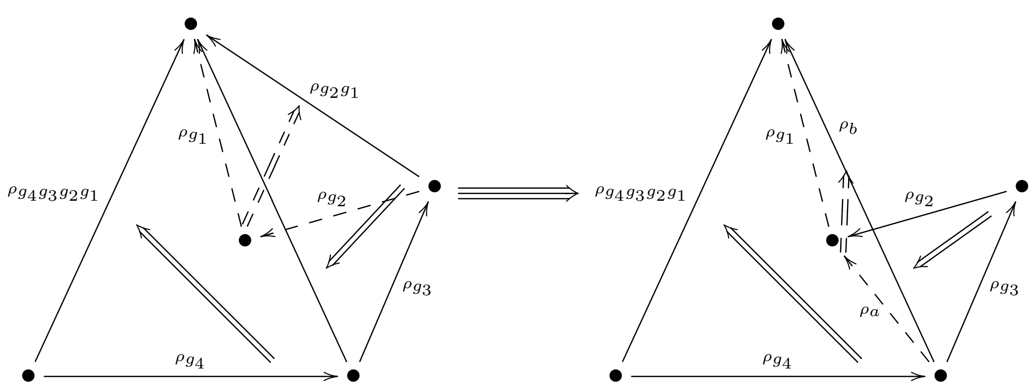


The 3−arrow A_1

whose 2-source and 2-target are the 2-isomorphisms

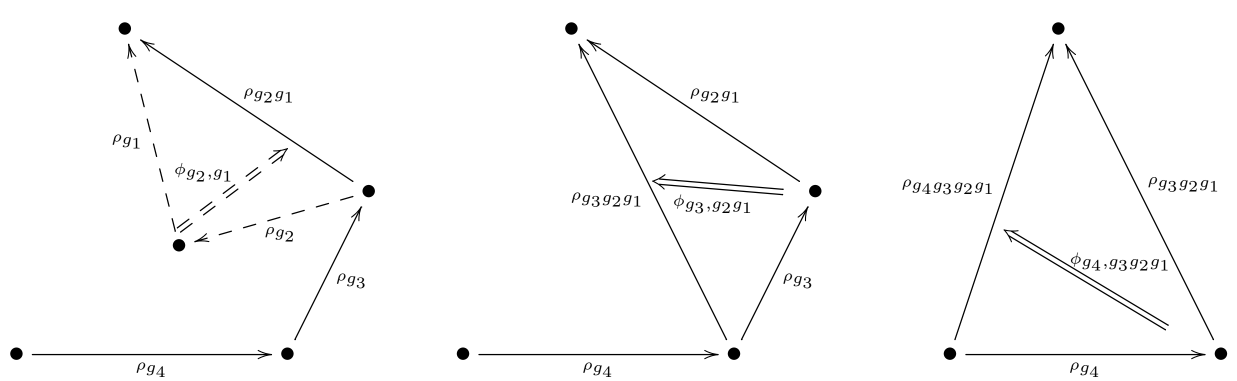
$$\begin{aligned} s_2(A_1) &= [(\rho_{g_4}\rho_{g_3})\#_0\phi_{g_2,g_1}]\#_1[\rho_{g_4}\#_0\phi_{g_3,g_2g_1}]\#_1\phi_{g_4,g_3g_2g_1} : \rho_{g_4}\rho_{g_3}\rho_{g_2}\rho_{g_1} \longrightarrow \rho_{g_4g_3g_2g_1}, \\ t_2(A_1) &= [\rho_{g_4}\#_0\phi_{g_3,g_2}\#_0\rho_{g_1}]\#_1[\rho_{g_4}\#_0\phi_{g_3g_2,g_1}]\#_1\phi_{g_4,g_3g_2g_1} : \rho_{g_4}\rho_{g_3}\rho_{g_2}\rho_{g_1} \longrightarrow \rho_{g_4g_3g_2g_1}, \end{aligned} \tag{17}$$

corresponding to 2-cells

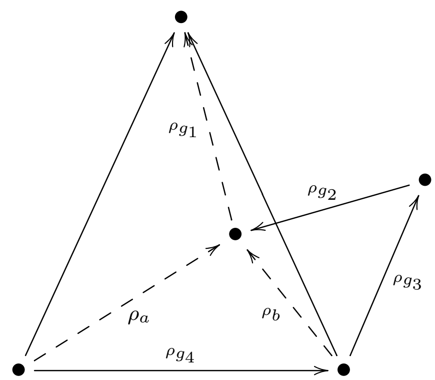


The 2−arrow $s_2(A_1)$ The 2−arrow $t_2(A_1)$

respectively, where $\rho_a := \rho_{g_3g_2}, \rho_b := \rho_{g_3g_2g_1}$. It is fundamental in this paper to write down the p-arrow corresponding to p-cells as whiskered vertical compositions. For example, $s_2(A_1)$ in (17) is the composition of the following three whiskered 2-isomorphisms.



The second 3-isomorphism in the left-hand side of the 3-cocycle condition (15) is $A_2 = [\rho_{g_4} \#_0 \phi_{g_3,g_2} \#_0 \rho_{g_1}] \#_1 \Phi_{g_4,g_3g_2,g_1}$, corresponding to the 3-cell

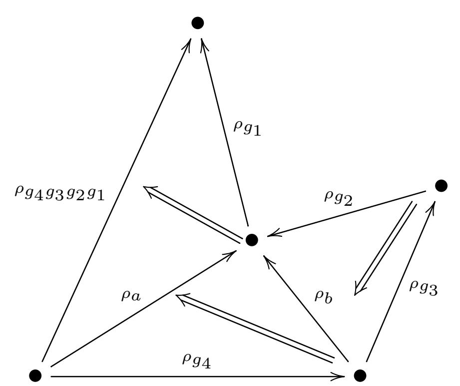


The 3−arrow A_2

(here $\rho_a := \rho_{g_4g_3g_2}, \rho_b := \rho_{g_3g_2}$) with 2-source $s_2(A_2) = t_2(A_1)$ in (17) and 2-target

$$t_2(A_2) = [\rho_{g_4} \#_0 \phi_{g_3,g_2} \#_0 \rho_{g_1}] \#_1 [\phi_{g_4,g_3g_2} \#_0 \rho_{g_1}] \#_1 \phi_{g_4g_3g_2,g_1} \tag{18}$$

corresponding to 2-cells

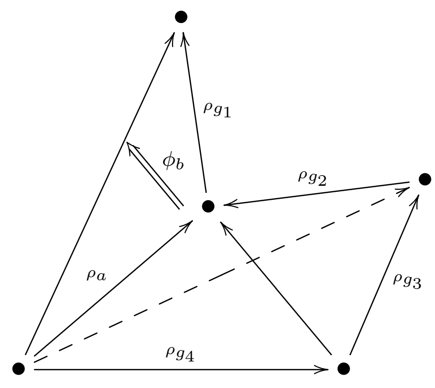


The 2−arrow $t_2(A_2)$

And the third 3-isomorphism in the left-hand side of the 3-cocycle condition (15) is

$$A_3 = [\Phi_{g_4,g_3,g_2} \#_0 \rho_{g_1}] \#_1 \phi_{g_4g_3g_2,g_1},$$

corresponding to the 3-cell

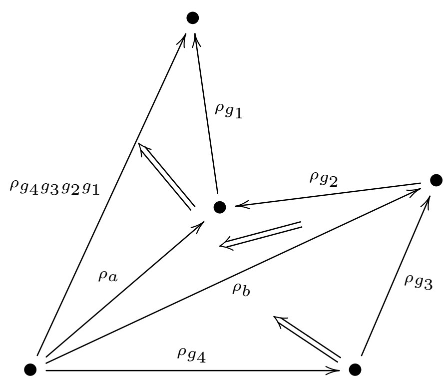


The 3−arrow A_3

(here $\rho_a := \rho_{g_4g_3g_2}$, $\phi_b := \phi_{g_4g_3g_2,g_1}$) with 2-source $s_2(A_3) = t_2(A_2)$ in (18) and 2-target

$$t_2(A_3) = [\phi_{g_4,g_3}\#_0(\rho_{g_2}\rho_{g_1})]\#_1[\phi_{g_4g_3,g_2}\#_0\rho_{g_1}]\#_1\phi_{g_4g_3g_2,g_1}, \tag{19}$$

corresponding to the 2-cells



, (20)

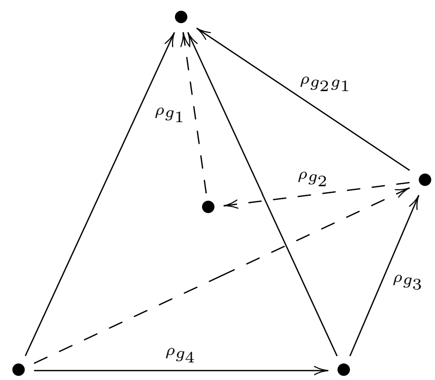
The 2−arrow $t_2(A_3)$

where $\rho_a := \rho_{g_4g_3g_2}$, $\rho_b := \rho_{g_4g_3}$. Then the composition $A_1\#_2A_2\#_2A_3$ of 3-isomorphisms is the left-hand side of the 3-cocycle condition (15), whose 2-source is $s_2(A_1)$ in (17) and 2-target is $t_2(A_3)$ in (19).

On the right-hand side of the 3-cocycle condition (15), the first 3-isomorphism is

$$A_1' = [(\rho_{g_4}\rho_{g_3})\#_0\phi_{g_2,g_1}]\#_1\Phi_{g_4,g_3,g_2g_1},$$

corresponding to the 3-cell

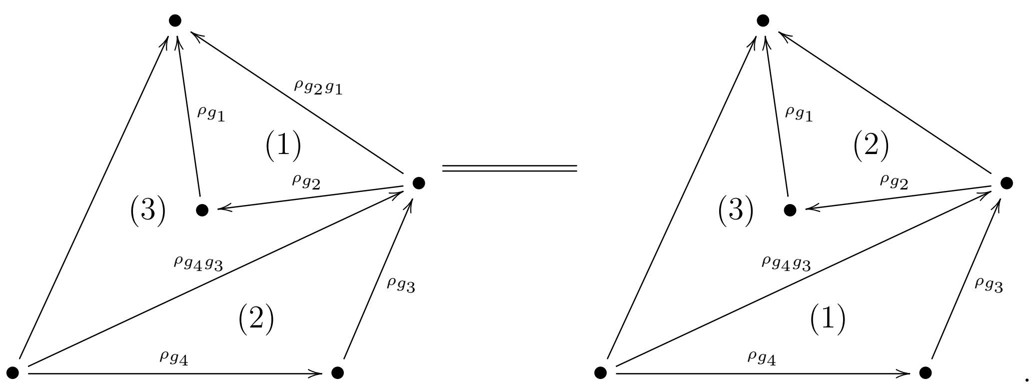


The 3−arrow A'_1

with 2-source $s_2(A_1)$ in (17) and 2-target

$$t_2(A'_1) = [(\rho_{g_4}\rho_{g_3})\#_0\phi_{g_2,g_1}]\#_1[\phi_{g_4,g_3}\#_0\rho_{g_2g_1}]\#_1\phi_{g_4g_3,g_2g_1}, \tag{21}$$

corresponding to the left 2-cells in the following diagram:

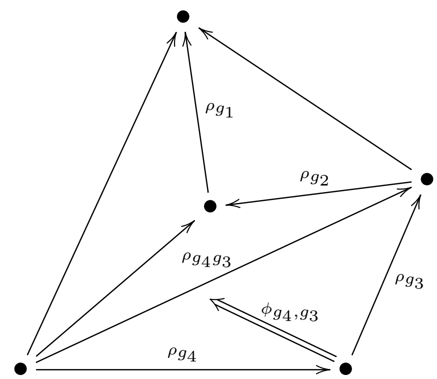


By the interchange law (6) for horizontal compositions, we can interchange 2-isomorphism (1) and (2) identically in the left 2-cells above to get the 2-isomorphism

$$s_2(A'_2) = [\phi_{g_4,g_3}\#_0(\rho_{g_2}\rho_{g_1})]\#_1[\rho_{g_4g_3}\#_0\phi_{g_2,g_1}]\#_1\phi_{g_4g_3,g_2g_1}, \tag{22}$$

corresponding to the right 2-cells above. The last 3-isomorphism is

$$A'_2 = [\phi_{g_4,g_3}\#_0(\rho_{g_2}\rho_{g_1})]\#_1\Phi_{g_4g_3,g_2,g_1}$$



The 3−arrow A_2'

whose 2-target is exactly the 2-isomorphism $t_2(A_3)$ in (19)-(20).

It is not easy to draw several 3-cells corresponding to the composition of 3-arrows in a 3-category $\mathcal{C}$. For this reason, let us consider the associated 2-category $\mathcal{C}^+$ such that

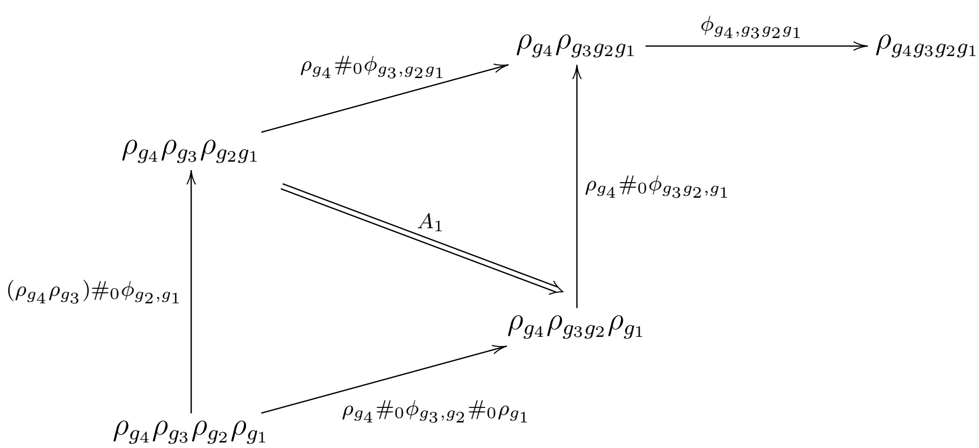
$$(\mathcal{C}^+)_i := \mathcal{C}_{i+1},$$

and i-source and i-target are s_{i+1} and t_{i+1}, $i = 0, 1, 2$, respectively. Functions $\widetilde{\#}_p : \mathcal{C}_k^+ \times \mathcal{C}_k^+ \longrightarrow \mathcal{C}_k^+$ are described by arrows $\#_{p+1} : \mathcal{C}_{k+1} \times \mathcal{C}_{k+1} \longrightarrow \mathcal{C}_{k+1}$, and identities $\widetilde{1} : \mathcal{C}_{k-1}^+ \to \mathcal{C}_k^+$ are defined in a similar manner. $\mathcal{C}^+$ is a strict 2-category since $Hom_{\mathcal{C}}(x, y)$ is a strict 2-category for any objects x, y of $\mathcal{C}$, by the fact that a strict 3-category is a category enriched over the category of all small strict 2-categories. We also define $\mathcal{C}^{++}$ to be the category with

$$(\mathcal{C}^{++})_i := \mathcal{C}_{i+2}$$

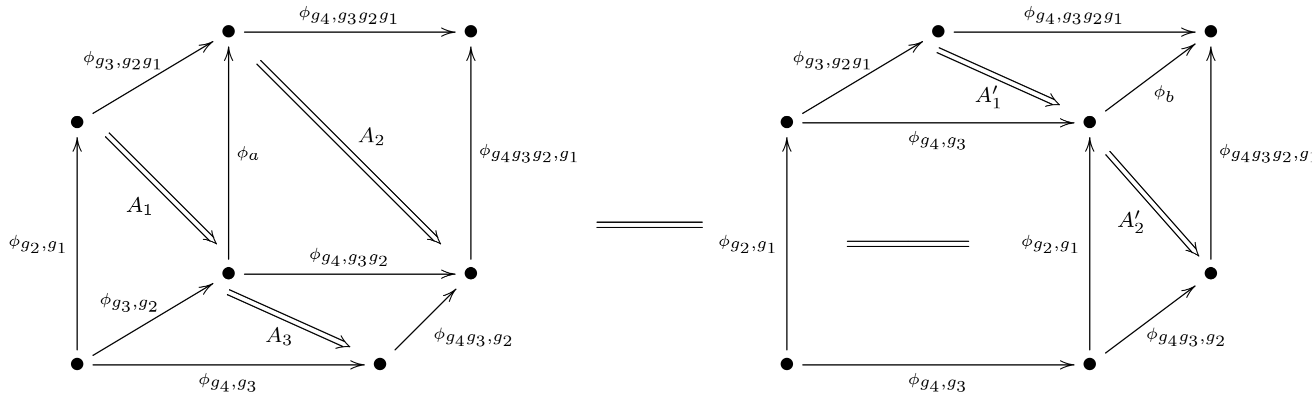
and the i-source and i-target are now s_{i+2} and t_{i+2}, $i = 0, 1$, respectively. The function $\widetilde{\#}_0 : \mathcal{C}_1^{++} \times \mathcal{C}_1^{++} \longrightarrow \mathcal{C}_1^{++}$ becomes $\#_2 : \mathcal{C}_3 \times \mathcal{C}_3 \longrightarrow \mathcal{C}_3$. $\mathcal{C}^{++}$ is a category by the same reason.

In the corresponding strict 2-category $\mathcal{C}^+$, 3-isomorphism A_1 in (16) is represented by the following 2-isomorphism:



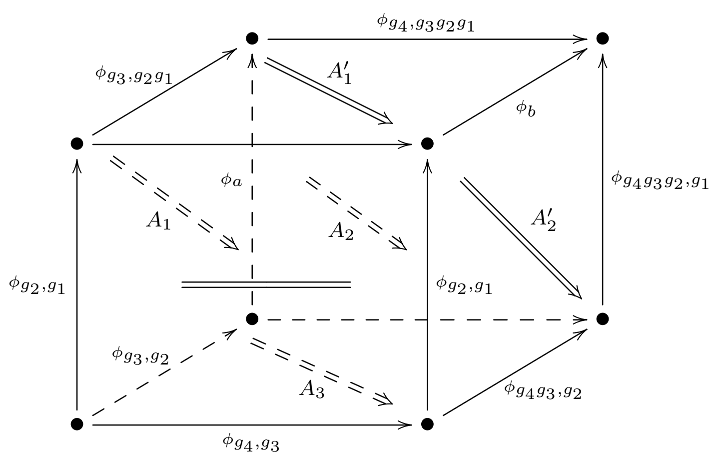
(23)

Here the upper and lower boundaries in (23) (as 1-arrows in $\mathcal{C}^+$) represent the source $s_2(A_1)$ and target $t_2(A_1)$ in (17) (as 2-isomorphisms in $\mathcal{C}$) respectively. To draw the picture neatly, we omit the whiskering parts. Then the 3-cocycle condition (15) can be expressed simply as an identity of 2-isomorphisms in $\mathcal{C}^+$ as follows:



(24)

where $\phi_a := \phi_{g_3g_2,g_1}$, $\phi_b := \phi_{g_4g_3,g_2g_1}$. Here $\bullet$'s above represent 1-isomorphisms in $\mathcal{C}$. The 2-isomorphisms in (17), (18), (19), (21) and (22) are represented by 1-isomorphisms in (24). Now the 3-cocycle condition (24) can be viewed as the commutativity of the 2-isomorphisms in the boundary of the following cube in $\mathcal{C}^+$:



(25)

2.7. Remark. *(1) In the upper boundaries of diagrams in (24), the number of group elements in the second subscripts of $\phi_{*,*}$'s is increasing: g_1, g_2g_1 $g_3g_2g_1$, while in the lower boundaries it is the number of group elements in the first subscripts of $\phi_{*,*}$'s which are increasing: g_4, g_4g_3 $g_4g_3g_2$.*

(2) (24) or (25) is similar to the pentagon condition of bicategories, but here we actually have more complicated whiskering (cf. (23)).

Given a strict 2-category $\mathcal{V}$, there exists an *associated* 3-*category* $\mathcal{V}^*$ for which $\mathcal{V}^*_0$ consists of one object $\mathcal{V}$, $\mathcal{V}^*_1$ consists of all functors from $\mathcal{V}$ to $\mathcal{V}$, $\mathcal{V}^*_2$ consists of all pseudo-natural transformations and $\mathcal{V}^*_3$ consists of all modifications. This is a 3-category. Because only 3-representations of a group in a strict 3-category are developed, we have to consider a strict 3-subcategory $\mathcal{W}$ of $\mathcal{V}^*$ for a strict 2-categories $\mathcal{V}$. We call a 3-representation of G in such a strict 3-subcategory $\mathcal{W}$ a *strict 2-categorical action of G on $\mathcal{V}$*. In particular, we have an endofunctor $\rho_g : \mathcal{V} \to \mathcal{V}$ for each $g \in G$, a pseudonatural transformation $\phi_{h,g} : \rho_h \#_0 \rho_g \Longrightarrow \rho_{hg}$ for each $h, g \in G$, and a modification Φ_{g_3,g_2,g_1} (the associator in (12)) for each $g_3, g_2, g_1 \in G$. Here $\rho_h \#_0 \rho_g$ is the composition of functors:

$$\rho_h \#_0 \rho_g(w) := \rho_h(\rho_g(w))$$

for $w \in \mathcal{V}$. By the definition of 3-representations, the endofunctor ρ_g, the pseudonatural transformation $\phi_{h,g}$ and the modification Φ_{g_3,g_2,g_1} must all be invertible in $\mathcal{W} \subset \mathcal{V}^*$.

For example, for the 2-category $\mathcal{V}$ used in the 1-dimensional 3-representation in Subsection 3.8, its $\mathcal{V}^*$ is a strict 3-category. For the general action of G on a 2-category $\mathcal{V}$, we need to develop 3-representation of a group in a **Gray**-category, since the semi-strictification of a 3-category is a **Gray**-category.

When a 2-category $\mathcal{V}$ is viewed as a 3-category with only identity 3-arrow, a 3-representation of G in $\mathcal{V}$ is a *2-representation* if the the associator 3-isomorphism in (12) is the identity, so that the 3-cocycle condition (15) holds trivially. This coincides with the definition of the 2-representation in the strict sense in section 2.2 of [13]. And for a category $\mathcal{V}$, a 2-representation of G in the 2-category $\mathcal{V}^*$ is a *categorical action of G on $\mathcal{V}$*.

3. The 2-categorical traces of 3-representations

3.1. The 2-categorical trace of a 1-endomorphism. Let $\mathcal{C}$ be a 3-category, $x \in \mathcal{C}$ and $A : x \to x$ be a 1-endomorphism. Then A is an object of the 2-category $\mathrm{Hom}_{\mathcal{C}}(x, x)$. The 2-*categorical trace* of A is defined as

$$\mathbb{T}r_2(A) = \mathrm{Hom}_{\mathcal{C}}(1_x, A),$$

which is a category. This is a subcategory of $\mathcal{C}^{++}$.

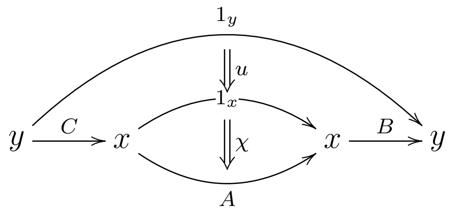
Let $A : x \to x$ be a 1-endomorphism for $x \in \mathcal{C}_0$, and let the 1-arrow $C : y \to x$ be a quasi-inverse to a 1-arrow $B : x \to y$. Then for any 2-arrow $\chi : 1_x \Longrightarrow A$ in $\mathbb{T}r_2(A)_0$, the composition

$$1_y \xrightarrow{u} C\#_0 B = C\#_0 1_x \#_0 B \xrightarrow{C\#_0\chi\#_0 B} C\#_0 A\#_0 B$$

defines a functor

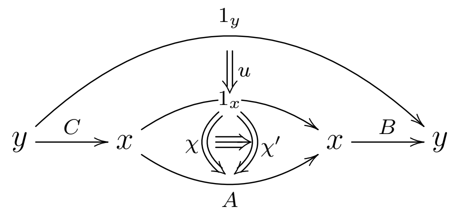
$$\begin{aligned} \Psi(C, B, u) : \mathbb{T}r_2(A)_0 &\longrightarrow \mathbb{T}r_2(C\#_0 A\#_0 B)_0, \\ (\chi : 1_x \Longrightarrow A) &\longmapsto u\#_1[C\#_0\chi\#_0 B], \end{aligned}$$

corresponding to the diagram



and for any 3-arrow $\gamma : \chi \Longrightarrow \chi'$ in $\mathbb{T}r_2(A)_1$, we have

$$\begin{aligned} \mathbb{T}r_2(A)_1 &\longrightarrow \mathbb{T}r_2(C\#_0A\#_0B)_1, \\ \gamma &\longmapsto u\#_1[C\#_0\gamma\#_0B], \end{aligned}$$



3.2. Proposition. *$\Psi(C, B, u) : \mathbb{T}r_2(A) \longrightarrow \mathbb{T}r_2(C\#_0A\#_0B)$ is a functor.*

Proof. For 2-arrows $\chi, \chi', \widetilde{\chi} : 1_x \Longrightarrow A$ and 3-arrows $\gamma : \chi \Longrightarrow \chi'$, $\widetilde{\gamma} : \chi' \Longrightarrow \widetilde{\chi}$, we have the composition $\gamma\#_2\widetilde{\gamma} : \chi \Longrightarrow \widetilde{\chi}$. Then by using repeatedly the compatibility condition (7) for compositions, we find

$$\begin{aligned} \Psi(C, B, u)(\gamma)\#_2\Psi(C, B, u)(\widetilde{\gamma}) &= \{u\#_1[C\#_0\gamma\#_0B]\}\#_2\{u\#_1[C\#_0\widetilde{\gamma}\#_0B]\} \\ &= u\#_1[C\#_0(\gamma\#_2\widetilde{\gamma})\#_0B] \\ &= \Psi(C, B, u)(\gamma\#_2\widetilde{\gamma}). \end{aligned}$$

Thus $\Psi(C, B, u)$ is a functor. ∎

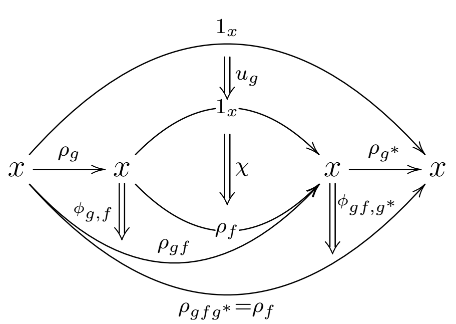
3.3. The 2-categorical trace $\mathbb{T}\mathrm{r}_2\rho_f$. Let ρ be a 3-representation of G in a 3-category $\mathcal{C}$. Fix an object x in $\mathcal{C}$ that G acts on. For $f \in G$, let $\rho_f : x \to x$ be a 1-isomorphism in $\mathcal{C}$. Recall that $\mathbb{T}\mathrm{r}_2\rho_f$ is a category whose objects are 2-arrows with source 1_x and target ρ_f and the morphisms are 3-arrows between them. In the sequel, we will use the notation

$$g^* := g^{-1}$$

for simplicity. For any g commuting with f and a 2-arrow $\chi : 1_x \Longrightarrow \rho_f$ in $(\mathbb{T}\mathrm{r}_2\rho_f)_0$, we define a 2-arrow $\psi_g(\chi) : 1_x \Longrightarrow \rho_f$ by

$$\psi_g(\chi) := u_g\#_1\left[\rho_g\#_0\chi\#_0\rho_{g^*}\right]\#_1\left[\phi_{g,f}\#_0\rho_{g^*}\right]\#_1\phi_{gf,g^*}. \tag{26}$$

This is given by the composition of 2-arrows in the following diagram

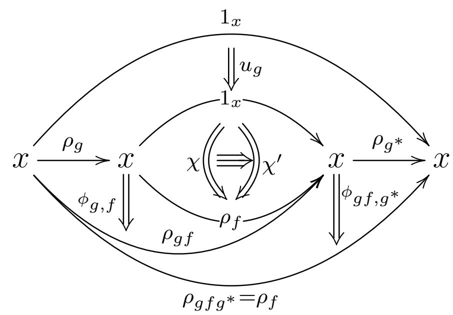


(27)

where $u_g = \phi^{-1}_{g,g^*} : 1_x \Longrightarrow \rho_g \rho_{g^*}$. For a 3-arrow $\Theta : \chi \Rrightarrow \chi'$, we define $\psi_g(\Theta)$ as a 3-arrow whiskered by corresponding 2-isomorphisms in (27). In other words,

$$\psi_g(\Theta) = u_g \#_1 [\rho_g \#_0 \Theta \#_0 \rho_{g^*}] \#_1 [(\phi_{g,f} \#_0 \rho_{g^*}) \#_1 \phi_{gf,g^*}] : \psi_g(\chi) \Rrightarrow \psi_g(\chi') \tag{28}$$

is a 3-arrow corresponding to the diagram



in the 3-category $\mathcal{C}$. Then ψ_g defines an endofunctor ψ_g on $\mathbb{T}\mathrm{r}_2\rho_f$ by the proof of Proposition 3.2. Namely, we have

$$\psi_g(\Theta \widetilde{\#}_0 \Theta') = \psi_g(\Theta) \widetilde{\#}_0 \psi_g(\Theta')$$

for any 3-arrow $\Theta' : \chi' \Rrightarrow \chi''$, where $\widetilde{\#}_0$ is the composition in the category $\mathcal{C}^{++}$ ($\widetilde{\#}_0 = \#_2$).

In Section 3.4, we will construction a natural isomorphism $\Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg}$ for given $g, h \in C_G(f)$. It gives us natural isomorphisms $\Gamma_{g^*,g} : \psi_{g^*} \circ \psi_g \longrightarrow \psi_1$ and $\Gamma_{g,g^*} : \psi_g \circ \psi_{g^*} \longrightarrow \psi_1$. Thus ψ_g for each $g \in C_G(f)$ is an equivalence of the category $\mathbb{T}\mathrm{r}_2\rho_f$.

3.4. The adjoint 2-isomorphisms. For a 2-isomorphism $x \underset{\chi_2}{\overset{\chi_1}{\Downarrow \phi}} y$ in a 2-category $\mathcal{V}$, we define the *adjoint 2-isomorphism* $\phi^\dagger$ to be $y \underset{\chi_2^{-1}}{\overset{\chi_1^{-1}}{\Downarrow \phi^\dagger}} x$ by the composition of arrows

$$y \xrightarrow{\chi_1^{-1}} x \underset{\chi_1}{\overset{\chi_2}{\Downarrow \phi^{-1}}} y \xrightarrow{\chi_2^{-1}} x \,. \tag{29}$$

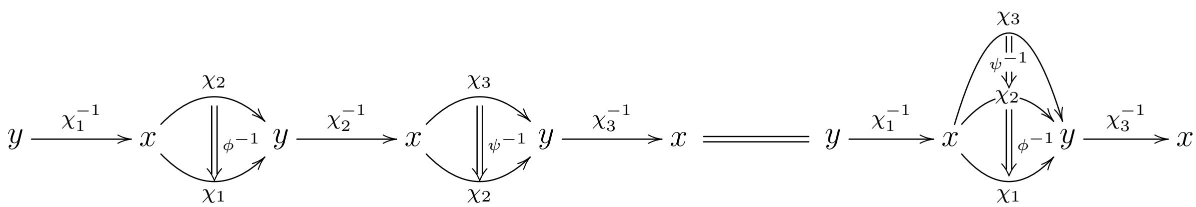
This is a 2-isomorphism with inverted 1-source and 1-target. This operation will be used later. See also section 2 of [20] for the definition of similar adjoint 2-arrows, but ϕ^{-1} in (29) is replaced there by ϕ.

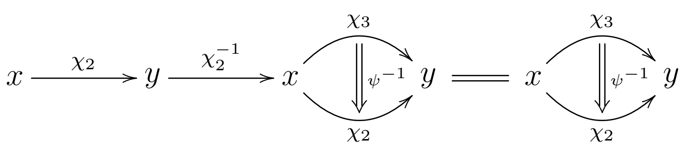
3.5. Proposition. *(1) For any pair of 2-isomorphisms* $x \overset{\chi_1}{\underset{\chi_2}{\Downarrow \phi}} y$ *and* $x \overset{\chi_2}{\underset{\chi_3}{\Downarrow \psi}} y$ *, we have* $(\phi \#_1 \psi)^\dagger = \phi^\dagger \#_1 \psi^\dagger$.

(2) For any 1-isomorphism $\chi_0 : z \longrightarrow x$*, we have* $(\chi_0 \#_0 \phi)^\dagger = \phi^\dagger \#_0 \chi_0^{-1}$*; and for 1-isomorphism* $\widetilde{\chi}_0 : y \longrightarrow z$*, we have* $(\phi \#_0 \widetilde{\chi}_0)^\dagger = \widetilde{\chi}_0^{-1} \#_0 \phi^\dagger$.

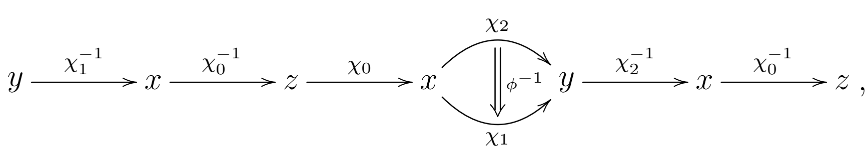
(3) For a 2-isomorphism $y \overset{\widetilde{\chi}_1}{\underset{\widetilde{\chi}_2}{\Downarrow \widetilde{\phi}}} z$*, we have* $(\phi \#_0 \widetilde{\phi})^\dagger = \widetilde{\phi}^\dagger \#_0 \phi^\dagger$*, i.e.,* $z \overset{\widetilde{\chi}_1^{-1}}{\underset{\widetilde{\chi}_2^{-1}}{\Downarrow \widetilde{\phi}^\dagger}} y \overset{\chi_1^{-1}}{\underset{\chi_2^{-1}}{\Downarrow \phi^\dagger}} x$ *.*

Proof. (1) $\phi^\dagger \#_1 \psi^\dagger = (\phi \#_1 \psi)^\dagger$ follows from



by  and the interchange law (6) for horizontal compositions.

(2) follows from the fact that $(\chi_0 \#_0 \phi)^\dagger$ is



since $\chi_0^{-1} \#_0 \chi_0$ is equal to the identity 1_x.

(3) Note that $\phi \#_0 \widetilde{\phi} = (\chi_1 \#_0 \widetilde{\phi}) \#_1 (\phi \#_0 \widetilde{\chi}_2)$ by using the interchange law (6) . We see that

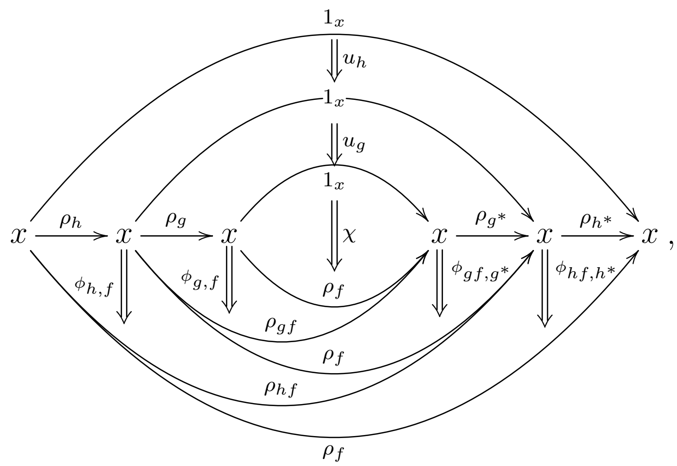
$$\left(\phi \#_0 \widetilde{\phi}\right)^\dagger = \left(\chi_1 \#_0 \widetilde{\phi}\right)^\dagger \#_1 (\phi \#_0 \widetilde{\chi}_2)^\dagger = \left(\widetilde{\phi}^\dagger \#_0 \chi_1^{-1}\right) \#_1 \left(\widetilde{\chi}_2^{-1} \#_0 \phi^\dagger\right) = \widetilde{\phi}^\dagger \#_0 \phi^\dagger$$

by using (1), (2) and the interchange law (6) again. ∎

3.6. The categorical action of the centralizer of f on $\mathbb{T}\mathrm{r}_2\rho_f$. To construct a categorical action of the centralizer $C_G(f)$ of f on the category $\mathbb{T}\mathrm{r}_2\rho_f$, let us write down the composition law for the functors ψ_h and ψ_g,

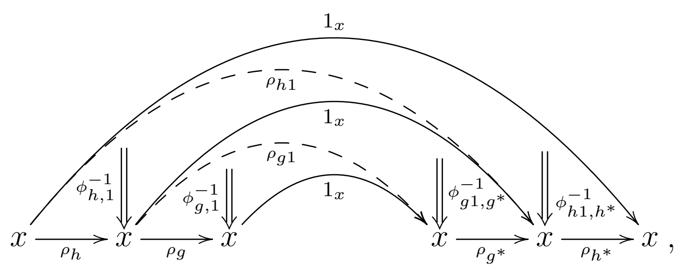
$$\psi_h \circ \psi_g : \mathbb{T}\mathrm{r}_2\rho_f \longrightarrow \mathbb{T}\mathrm{r}_2\rho_f,$$

where $h, g \in C_G(f)$. For a fixed $\chi \in (\mathbb{T}\mathrm{r}_2\rho_f)_0$ and $\Theta \in (\mathbb{T}\mathrm{r}_2\rho_f)_1$, by using the definition (26)-(28) of ψ_* twice, we see that $\psi_h \circ \psi_g(\chi) = \psi_h(\psi_g(\chi))$ is the composition of 2-arrows in $\mathcal{C}$ in the following diagram:



(30)

and $\psi_h \circ \psi_g(\Theta) = \psi_h(\psi_g(\Theta))$ is a 3-arrow in $\mathcal{C}$ defined similarly. Recall that we assume $\rho_{g1} = \rho_g 1_x$ and $\rho_{h1} = \rho_h 1_x$. The upper half part of (30) is the same as the lower half with f replaced by 1_x and 2-isomorphisms inverted:



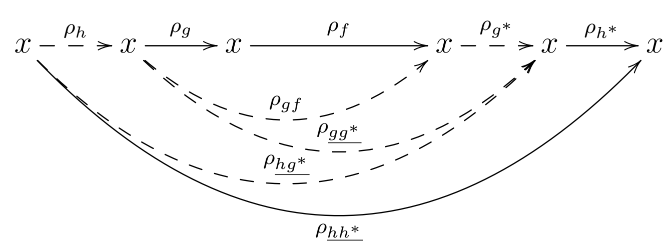
(31)

namely, we have $u_h = \phi^{-1}_{h1,h^*}\#_1[\phi^{-1}_{h,1}\#_0\rho_{h^*}]$ and similar identity for u_g. Note that $\phi_{h,1}$ and $\phi_{g,1}$ are identities by our assumptions in Remark 2.2 (1).

Now let us write down the natural isomorphism

$$\Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg}$$

between functors on the category $\mathbb{T}\mathrm{r}_2\rho_f$. The lower half of diagram (30) is



(32)

Here and in the following, for simplicity, we will use the notation

$$\rho_{\underline{g_1 g_2}} := \rho_{g_1 \dots g_2},$$

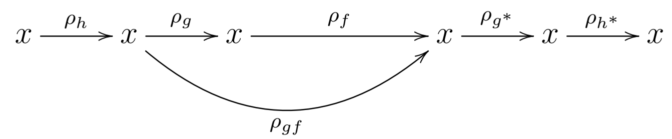
i.e., we omit the group elements between g_1 and g_2 in the sequence h, g, f, g^*, h^* in diagram (32).

Recall that the associator 3-isomorphism Φ_{g_3,g_2,g_1} in (12)-(13) can be drawn in the form (14). By definition, the 3-isomorphism

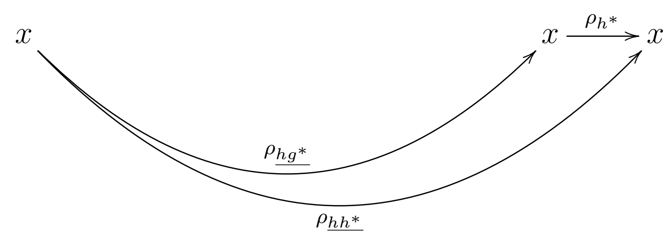
$$\hat{\Lambda}_1 = \gamma_1 \#_1 [\Phi_{h,gf,g^*} \#_0 \rho_{h^*}] \#_1 \gamma_2, \tag{33}$$

is the associator $\Phi_{h,gf,g^*} \#_0 \rho_{h^*}$ whiskered by two 2-isomorphisms

$\gamma_1 = [\rho_h \#_0 \phi_{g,f} \#_0 (\rho_{g^*} \rho_{h^*})]$:

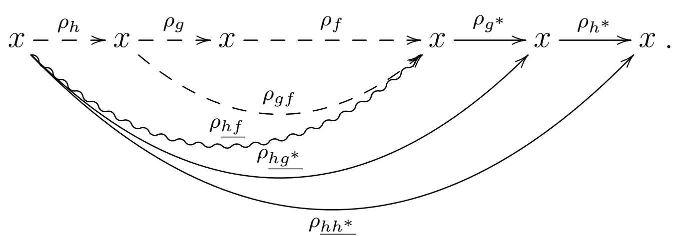


$\gamma_2 = \phi_{\underline{hg^*},h^*}$: (34)



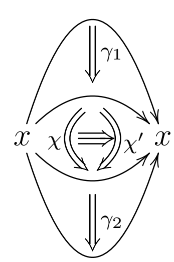
from above and below, respectively. This replaces the diagonal $\rho_{\underline{gg^*}}$ of the dotted quadrilateral in diagram (32) by the wavy diagonal $\rho_{\underline{hf}}$ of the same quadrilateral in the following diagram:

(35)



$\hat{\Lambda}_1$ in (33) is the following 3-isomorphism

(36)

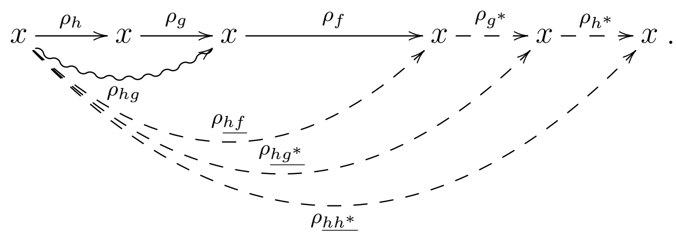


where χ is the 2-arrow corresponding to the dotted quadrilateral in diagram (32), χ' is the 2-arrow corresponding to the same quadrilateral in diagram (35) with the diagonal changed, and 2-arrows γ_1 and γ_2 are given by (34).

The 3-isomorphism

$$\hat{\Lambda}_2 = [\Phi_{h,g,f}\#_0(\rho_{g^*}\rho_{h^*})]\#_1\{[\phi_{\underline{hf},g^*}\#_0\rho_{h^*}]\#_1\phi_{\underline{hg^*},h^*}\}, \tag{37}$$

as a whiskered associator (14), then changes the diagonal ρ_{gf} of the dotted-wavy quadrilateral in diagram (35) to the wavy diagonal ρ_{hg} of the same quadrilateral in the following diagram:

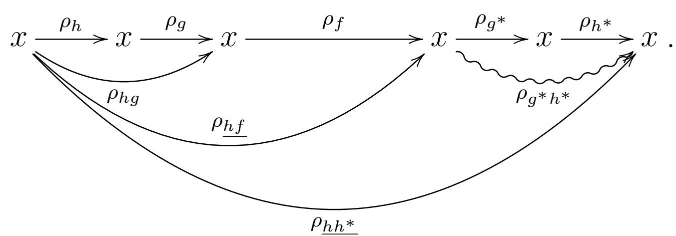


(38)

Similarly, the 3-isomorphism

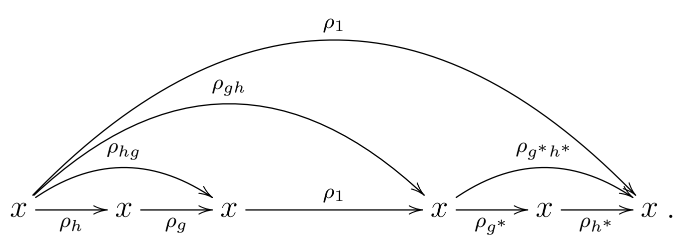
$$\hat{\Lambda}_3 = \{[\phi_{h,g}\#_0(\rho_f\rho_{g^*}\rho_{h^*})]\#_1[\phi_{hg,f}\#_0(\rho_{g^*}\rho_{h^*})]\}\#_1\Phi^{-1}_{\underline{hf},g^*,h^*}, \tag{39}$$

which is the whiskered associator $\Phi^{-1}_{\underline{hf},g^*,h^*}$, changes the diagonal ρ_{hg^*} of the dotted quadrilateral in diagram (38) to the wavy diagonal $\rho_{g^*h^*}$ of the same quadrilateral in the following diagram:



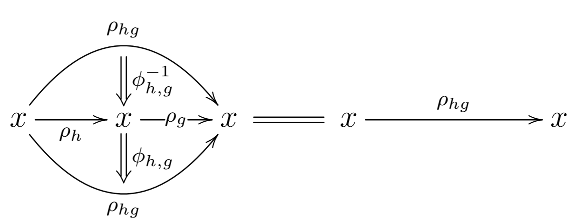
(40)

Recall that the upper half of diagram (30) is the same as the lower half with f replaced by 1 and 2-isomorphisms inverted. So by the corresponding 3-isomorphisms, denoted by $\hat{\Lambda}'_1, \hat{\Lambda}'_2, \hat{\Lambda}'_3$, the upper half of (31) is changed to



(41)

Note that



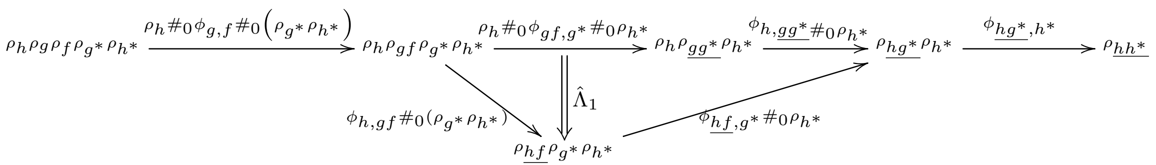
(42)

and the part involving $\rho_{g^*}\rho_{h^*}$ is also cancelled. As a result, the composition of (41) and (40), together with 2-arrow $\chi : 1_x \Longrightarrow \rho_f$, gives us the diagram (27) with g replaced by gh. This is exactly $\psi_{gh}(\chi)$. Therefore, the composition of suitable whiskered 3-isomorphisms $\hat{\Lambda}_1', \hat{\Lambda}_2', \hat{\Lambda}_3'$, $\hat{\Lambda}_1, \hat{\Lambda}_2$ and $\hat{\Lambda}_3$ gives a natural isomorphism $\Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg}$ such that for $\chi \in (\mathbb{T}r_2\rho_f)_0$

$$\Gamma_{h,g}(\chi) : \psi_h(\psi_g(\chi)) \Longrightarrow \psi_{hg}(\chi)$$

is a 3-isomorphism in $\mathcal{C}$.

It is not easy to draw 3-arrows $\hat{\Lambda}_j$'s in the 3-category $\mathcal{C}$. But in the 2-category $\mathcal{C}^+$, the first 3-arrow $\hat{\Lambda}_1$ in (33) can be drawn as the 2-isomorphism corresponding to the following diagram:



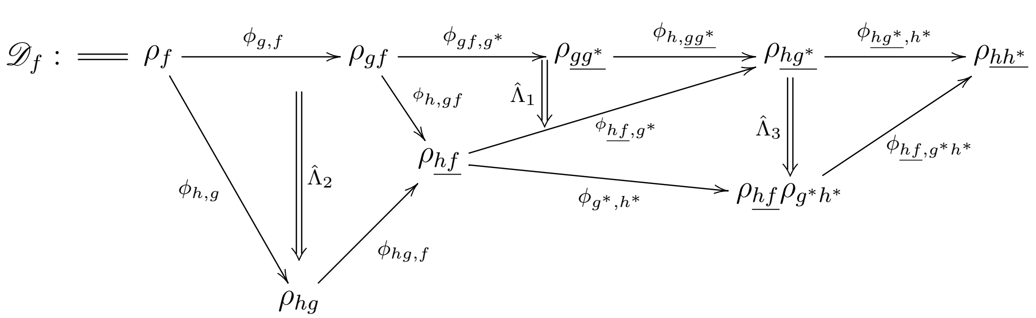
Here the upper path

$$\rho_h\rho_g\rho_f\rho_{g^*}\rho_{h^*} \xrightarrow{\rho_h\#_0\phi_{g,f}\#_0(\rho_{g^*}\rho_{h^*})} \rho_h\rho_{gf}\rho_{g^*}\rho_{h^*} \xrightarrow{\rho_h\#_0\phi_{gf,g^*}\#_0\rho_{h^*}} \cdots\cdots$$

corresponds to the 2-isomorphisms in $\mathcal{C}$ in (32) (the lower half of $\psi_h(\psi_g(\chi))$), while the lower paths corresponds to the 2-isomorphisms in $\mathcal{C}$ in (35) (the lower half of $\psi_{hg}(\chi)$)). And the 2-isomorphism $\hat{\Lambda}_1$ corresponds to the 3-isomorphism in $\mathcal{C}$ in (33). Since $\mathbb{T}\mathrm{r}_2\rho_f$ is a subcategory of $\mathcal{C}^{++}$, diagrams in the 2-category $\mathcal{C}^+$ are sufficient for our purpose. In the sequel, to simplify diagrams,

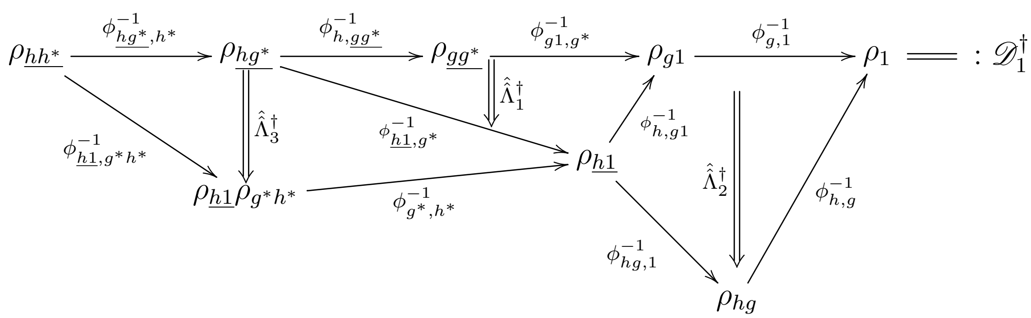
$$\rho_h \cdots \rho_{\underline{g_1g_2}} \cdots \rho_{h^*} \quad \text{is simply written as} \quad \rho_{\underline{g_1g_2}},$$

as an object in the 2-category $\mathcal{C}^+$. For simplicity, we also omit the whiskering part of 1-isomorphisms $\phi_{*,*}$'s in diagrams. The 3-isomorphisms $\hat{\Lambda}_1 : (32) \Longrightarrow (35)$, $\hat{\Lambda}_2 : (35) \Longrightarrow (38)$ and $\hat{\Lambda}_3 : (38) \Longrightarrow (40)$ in the 3-category $\mathcal{C}$ correspond to 2-isomorphisms in the 2-category $\mathcal{C}^+$ in the following diagram:



(43)

respectively. Just as for the upper half of diagram (30), diagram (31) is changed to diagram (41). In $\mathcal{C}^+$, this is the composition of 2-isomorphisms given by the following diagram

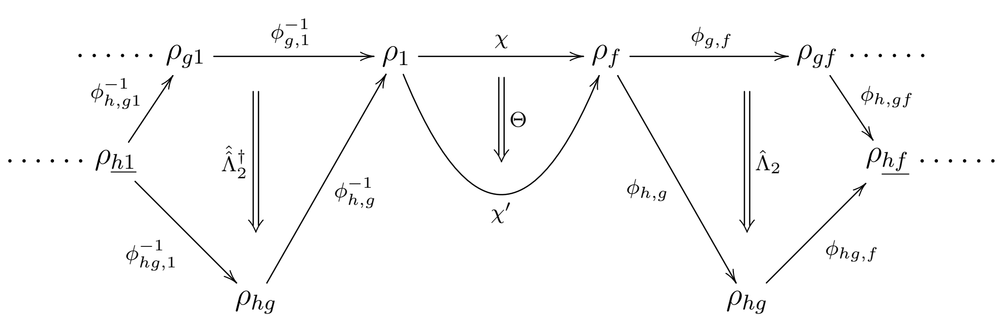


(44)

where $\hat{\hat{\Lambda}}_j$ is the 2-isomorphism previously denoted by $\hat{\Lambda}_j$ (with f replaced by 1), and $\hat{\hat{\Lambda}}_j^\dagger$ (previously denoted by $\hat{\Lambda}'_j$) is the 2-isomorphism adjoint to $\hat{\hat{\Lambda}}_j$, defined in §3.4 . Recall that the adjoint 2-isomorphism is the inverse one with 1-source and 1-target inverted. We apply the adjoint operation to diagram (43) to get diagram (44), the mirror-symmetric diagram of (43), by using Proposition 3.5. Given $\chi : 1_x \Longrightarrow \rho_f$, we connect the diagrams (44) and (43) to get $\Gamma_{h,g}(\chi)$ as a 2-isomorphism in $\mathcal{C}^+$:

$$\mathscr{D}_1^\dagger \xrightarrow{\chi} \mathscr{D}_f. \tag{45}$$

For objects $\chi, \chi' \in (\mathbb{T}r_2\rho_f)_0$ and a morphism $\Theta : \chi \to \chi'$ in $(\mathbb{T}r_2\rho_f)_1$ (i.e., a 3-arrow in $\mathcal{C}$), $\Gamma_{h,g}(\Theta)$ is also a 3-arrow. We connect diagrams (43) and (44) to get $\Gamma_{h,g}(\Theta)$ as the following diagram in the 2-category $\mathcal{C}^+$:



. (46)

Note that $\psi_h \circ \psi_g(\chi)$ in (30) is the upper boundary of diagram (46) and $\psi_{hg}(\chi')$ is the lower boundary of diagram (46). $\Gamma_{h,g}(\chi)$ is the diagram (46) with the 2-arrow $\Theta : \chi \Longrightarrow \chi'$ deleted, but 1-arrow $\chi : \rho_1 \longrightarrow \rho_f$ remains, whereas $\Gamma_{h,g}(\chi')$ is the diagram (46) with the 2-arrow $\Theta : \chi \Longrightarrow \chi'$ deleted, but 1-arrow $\chi' : \rho_1 \longrightarrow \rho_f$ remains. Applying the interchange law (6) to the diagram (46), we see that $\Gamma_{h,g}$ is a natural isomorphism in the category $\mathbb{T}\mathrm{r}_2\rho_f \subset \mathcal{C}^{++}$, i.e. the diagram

$$\begin{array}{ccc}
\psi_h \circ \psi_g(\chi) & \xrightarrow{\psi_h\circ\psi_g(\Theta)} & \psi_h \circ \psi_g(\chi') \\
\Big\downarrow{\scriptstyle \Gamma_{h,g}(\chi)} & & \Big\downarrow{\scriptstyle \Gamma_{h,g}(\chi')} \\
\psi_{hg}(\chi) & \xrightarrow[\psi_{hg}(\Theta)]{} & \psi_{hg}(\chi')
\end{array}$$

is commutative, where the 2-arrows $\psi_h \circ \psi_g(\Theta)$ and $\psi_{hg}(\Theta)$ in $\mathcal{C}^+$ are Θ whiskered by 1-isomorphisms corresponding to the upper and lower boundaries of diagram (46), respectively.

3.7. Theorem. *$\{\psi_g, \Gamma_{h,g}\}_{g,h\in C_G(f)}$ is a categorical action of the centralizer $C_G(f)$ on the category $\mathbb{T}\mathrm{r}_2\rho_f$.*

This theorem will be proved in Section 6 by checking the associative law (1) for $\Gamma_{*,*}$, which is an identity of natural transformations between functors on $\mathbb{T}\mathrm{r}_2\rho_f$. Note that in (1), $s_0(\Gamma_{k,hg}) = \psi_k \circ \psi_{hg} = t_0(\psi_k \circ \Gamma_{h,g})$. So the composition of natural transformations used in (1) is in the usual order, not in the natural order which we assumed in Remark 2.1 (1).

3.8. 1-dimensional 3-representations. We fix a field k of characteristic 0 containing all roots of unity. Let $\mathcal{A}$ be a 2-category with only one object, one 1-arrow and 2-arrows $\mathcal{A}_2 \cong k^*$. Fix a 3-cocycle c satisfying the condition (2). Let ϱ^c be the strict 2-categorical action of G on $\mathcal{A}$ as follows: ϱ^c_g is the identity functor for each $g \in G$;

$$\phi_{h,g} : 1_{\mathcal{A}} = \varrho^c_h \varrho^c_g \Longrightarrow \varrho^c_{hg} = 1_{\mathcal{A}}$$

is also the identity pseudonatural isomorphism for any $h, g \in G$; and

$$\Phi_{g_3,g_2,g_1} : id = (\varrho^c_{g_3} \#_0 \phi_{g_2,g_1}) \#_1 \phi_{g_3,g_2g_1} \Longrightarrow (\phi_{g_3,g_2} \#_0 \varrho^c_{g_1}) \#_1 \phi_{g_3g_2,g_1} = id, \tag{47}$$

is a modification determined by the element $c(g_3, g_2, g_1) \in k^*$ for any $g_3, g_2, g_1 \in G$. Then the 3-cocycle condition (24) for Φ is reduced to the equation (2). The cohomology classes of 3-cocycles are classified by $H^3(G, k^*)$.

For $f \in G$, it is easy to see that $\mathbb{T}\mathrm{r}_2\varrho^c_f$ is a category with a single object given by the identity pseudonatural isomorphism $\chi_0 : 1_{\mathcal{A}} \to \varrho^c_f = 1_{\mathcal{A}}$, and morphisms $(\mathbb{T}\mathrm{r}_2\rho_f)_1 \cong k^*$ (an element of k^* provides a modification). For $g \in C_G(f)$, $\psi_g : \mathbb{T}\mathrm{r}_2\varrho^c_f \to \mathbb{T}\mathrm{r}_2\varrho^c_f$ is the identity functor by the definitions (26)-(28). And

$$\Gamma_{h,g} : \chi_0 = \psi_h \circ \psi_g(\chi_0) \longrightarrow \psi_{hg}(\chi_0) = \chi_0$$

is a natural isomorphism given by the element (also denoted by $\Gamma_{h,g}$ by abuse of notations)

$$\Gamma_{h,g} = \frac{c(h, gf, g^*)c(h, g, f)c(\underline{hf}, g^*, h^*)^{-1}}{c(h, g, g^*)c(h, g, 1)c(\underline{hg}, g^*, h^*)^{-1}}. \tag{48}$$

This element is obtained by replacing Φ_{g_3,g_2,g_1} by the element $c(g_3,g_2,g_1)$ and all other isomorphisms by 1 in $\hat{\Lambda}_j$'s in (33) (37) (39), and using the adjoint operation (29), .

3.9. PROPOSITION. *Γ given by (48) is a 2-cocycle on the centralizer $C_G(f)$.*

This proposition will be proved in Section 6.1.

3.10. REMARK. *There exists a transgression map that maps a 3-cocycle c on a finite group G to a 2-cocycle on the inertia groupoid of G [26]. It is given by*

$$C_{h,g} := \frac{c(h,g,f)c(hgfg^{-1}h^{-1},h,g)}{c(h,gfg^{-1},g)}$$

for given $f \in G$ (cf. Remark 3.17 in [14]). Note that for $h,g \in C_G(f)$ we have $C_{h,g} := c(h,g,f)c(f,h,g)/c(h,f,g)$. So our 2-cocycle $\Gamma_{h,g}$ in (48) is different from the transgressed one. On the other hand, our 2-cocycle is only defined for elements which commute with a given element f, not on the entire inertia groupoid of G.

Let ϱ be a categorical action of a finite group G on a k-linear category $\mathcal{W}$. For a commuting pair of elements g and f in G, the 2-*character* $\chi_\varrho(f,g)$ of a categorical action ϱ is the joint trace of functors ϱ_f and ϱ_g, i.e., the trace of the linear transformation induced by the functor ϱ_g on the categorical trace $\mathbb{T}r\varrho_f$ (a k-vector space, which we assume to be finite dimensional).

Now let ρ be a strict 2-categorical action of a finite group G on a k-linear 2-category $\mathcal{V}$. Then $\mathbb{T}r_2\rho_f$ is a k-linear category and ψ defines a categorical action of the centralizer of f in G on it by Theorem 3.7. If $k,g,f \in G$ are pairwise commutative, we define *the* 3-*character* of the 2-categorical action ρ to be

$$\chi_\rho(f,g,k) := \chi_\psi(g,k), \tag{49}$$

the joint trace of functors ψ_g and ψ_k acting on the k-linear category $\mathbb{T}r_2\rho_f$, i.e., the trace of the linear transformation induced by the functor ψ_k on the k-vector space $\mathbb{T}r\psi_g$, which we assume to be finite dimensional.

By using the 2-character formula for 1-dimensional 2-representation in proposition 5.1 of [13], the 3-character of the 3-representation ϱ^c for pairwise commutative $k,g,f \in G$ is given by

$$\chi_{\varrho^c}(f,g,k) = \frac{\Gamma_{k,g}\Gamma_{kg,k^{-1}}}{\Gamma_{k,1}\Gamma_{k,k^{-1}}},$$

where the expressions $\Gamma_{*,*}$'s are defined by (48). It can also be derived from diagram (27).

4. The induced strict 2-categorical action on the induced 2-category

4.1. THE INDUCED 2-CATEGORY. Let $H \subset G$ be a subgroup of a finite group G and let $\rho : H \to \mathcal{V}^*$ be a strict 2-categorical action of H on a strict 2-category $\mathcal{V}$ (cf. definitions at the end of Section 2.4). $\mathrm{Ind}_H^G(\mathcal{V})$ is a strict 2-category where

• objects are maps $\vartheta : G \longrightarrow \mathcal{V}_0$ together with a 1-isomorphism

$$u_{g,h} : \vartheta(gh) \longrightarrow \rho_{h^*}\vartheta(g)$$

for each $g \in G, h \in H$, satisfying the condition:

(1) $u_{g,1} : \vartheta(g) \longrightarrow \rho_1\vartheta(g)$ coincides with $\phi_1^{-1}[\vartheta(g)]$;

(2) for each $g \in G, h_1, h_2 \in H$, we have a 2-isomorphism:

$$\begin{array}{ccc} \vartheta(gh_1h_2) & \xrightarrow{\;u_{gh_1,h_2}\;} & \rho_{h_2^*}\vartheta(gh_1) \\ {\scriptstyle u_{g,h_1h_2}}\Big\downarrow & \Swarrow & \Big\downarrow{\scriptstyle \rho_{h_2^*}u_{g,h_1}} \\ \rho_{(h_1h_2)^*}\vartheta(g) & \xrightarrow[\;\phi^{-1}_{h_2^*,h_1^*}\vartheta(g)\;]{} & \rho_{h_2^*}\rho_{h_1^*}\vartheta(g) \end{array}$$

• 1-arrows $F : (\vartheta, u) \to (\vartheta', u')$ between objects;

• 2-arrows $\gamma : F \to \widetilde{F}$.

For $k \in G$, the action $(\mathrm{ind}_H^G\rho)_k$ on the 2-category $\mathrm{Ind}_H^G(\mathcal{V})$ is given by

$$\left[(\mathrm{ind}_H^G\rho)_k\vartheta\right](g) = \vartheta(k^{-1}g), \qquad \left[(\mathrm{ind}_H^G\rho)_k u\right]_{g,h} = u_{k^{-1}g,h},$$

for an object (ϑ, u) in $\mathrm{Ind}_H^G(\mathcal{V})$. And $(\mathrm{ind}_H^G\rho)_k(F)$ for a 1-arrows $F : (\vartheta, u) \to (\vartheta', u')$ and $(\mathrm{ind}_H^G\rho)_k(\gamma)$ for a 2-arrow $\gamma : F \to \widetilde{F}$ can be defined similarly. In general, each commutative diagram in the definition of the induced category in section 7.1 of [13] is replaced by a 2-isomorphism.

We will not write down the definition of the induced 2-category $\mathrm{Ind}_H^G(\mathcal{V})$ explicitly. It is a little bit complicated. Since we only work on finite groups, we can simply identify $\mathrm{Ind}_H^G(\mathcal{V})$ with $\mathcal{V}^m$ as a 2-category, where m is the index of H in G. For a strict 2-category $\mathcal{V}$, $\mathcal{V}^m$ is also a strict 2-category with

$$\begin{aligned} \text{objects} \quad \mathcal{V}_0^m &:= \{(x_1, \ldots, x_m); x_j \in \mathcal{V}_0\}, \\ p-\text{arrows} \quad \mathcal{V}_p^m &:= \{(\gamma_1, \ldots, \gamma_m) : (x_1, \ldots, x_m) \to (y_1, \ldots, y_m); \mathcal{V}_p \ni \gamma_j : x_j \to y_j\}, \end{aligned}$$

$p = 1, 2$. The compositions are defined as

$$(\ldots, \gamma_j, \ldots)\#_p(\ldots, \gamma_j', \ldots) := (\ldots, \gamma_j\#_p\gamma_j', \ldots), \tag{50}$$

if γ_j and γ_j' are p-composable. The axioms for functions $\#_p$ and identities of $\mathcal{V}^m$ are obviously satisfied. The identification $\mathrm{Ind}_H^G(\mathcal{V}) \cong \mathcal{V}^m$ can be obtained by choosing a system of representatives

$$\mathscr{R} = \{r_1, \ldots, r_m\}$$

of left cosets of H in G, and associating to each map $\vartheta : G \to \mathcal{V}_0$ an object $(\vartheta(r_1), \dots, \vartheta(r_m))$ in $\mathcal{V}_0^m$.

Let $a_{jk} : \mathcal{V} \to \mathcal{V}$ be functors such that the $m \times m$ matrix $F = (a_{jk})$ has only one nonvanishing entry in each row or column. Then F defines a strict functor from $\mathcal{V}^m$ to $\mathcal{V}^m$ by

$$F(\dots, \delta_j, \dots) = \left(\dots, \sum_k a_{jk}(\delta_k), \dots \right),$$

where we write $\sum_k a_{jk}(\delta_k)$ formally for $\delta_k \in \mathcal{V}_p$, $p = 0, 1, 2$, since there exists only one term in this sum. But when the 2-category is k-linear, such sums exist. If $\widetilde{F} = (\widetilde{a}_{jk}) : \mathcal{V}^m \to \mathcal{V}^m$ is another such functor, then we have

$$(F \#_0 \widetilde{F})_{jk} := \sum_l a_{jl} \widetilde{a}_{lk}.$$

Moreover, a pseudonatural transformation $\phi : F \to \widetilde{F}$ is given by an $m \times m$ matrix $\phi = (\phi_{jk})$ with $\phi_{jk} : a_{jk} \to \widetilde{a}_{jk}$ a pseudonatural transformation between functors on $\mathcal{V}$. Let $\widetilde{\phi} = (\widetilde{\phi}_{jk}) : \widetilde{F} \to \widetilde{\widetilde{F}}$ be another pseudonatural transformation. Then their composition is $\phi \#_1 \widetilde{\phi} := (\phi_{jk} \#_1 \widetilde{\phi}_{jk})$.

4.2. The induced strict 2-categorical action . Suppose that ρ is a strict 2-categorical action of H on the 2-category $\mathcal{V}$. For $f \in G$, we define $(\mathrm{ind}_H^G \rho)_f$ to be a functor from $\mathcal{V}^m$ to $\mathcal{V}^m$ as follows. It is an $m \times m$ matrix whose entries are functors from $\mathcal{V}$ to $\mathcal{V}$, i.e., the (j, i)-entry is

$$\left[(\mathrm{ind}_H^G \rho)_f \right]_{ji} = \begin{cases} \rho_h, & \text{if } \quad f r_i = r_j h, \quad \text{for} \quad h \in H, \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

This corresponds to the fact that for a map $\vartheta : G \to \mathcal{V}_0$, we have $[(\mathrm{ind}_H^G \rho)_f(\vartheta)](r_j) = \vartheta(f^{-1} r_j)$ and $\vartheta(f^{-1} r_j) = \vartheta(r_i h^{-1}) \to \rho_h \vartheta(r_i)$. It is clear that only one entry in each row or column of the $m \times m$ matrix $(\mathrm{ind}_H^G \rho)_f$ is nonvanishing. Then,

$$(\mathrm{ind}_H^G \rho)_f(\dots, \delta_j, \dots) = \left(\dots, \sum \left((\mathrm{ind}_H^G \rho)_f \right)_{ji} (\delta_i), \dots \right), \tag{52}$$

where $\delta_j \in \mathcal{V}_p$, for $p = 0, 1, 2$.

For simplicity, from now on the induced object will be denoted by the hatted one, e.g. $\mathrm{ind}_H^G \rho$ is denoted by $\widehat{\rho}$. The composition functor $\widehat{\rho}_{g_2}$ and $\widehat{\rho}_{g_1}$ is defined as

$$(\widehat{\rho}_{g_2} \widehat{\rho}_{g_1})_{ki} = \begin{cases} \rho_{h_2} \rho_{h_1}, & \text{if } \quad g_1 r_i = r_j h_1, g_2 r_j = r_k h_2, \quad \text{for} \quad \text{some} \quad h_1, h_2 \in H, \\ 0, & \text{otherwise.} \end{cases} \tag{53}$$

Thus $\widehat{\rho}_{g_2} \widehat{\rho}_{g_1}$ can be viewed as the product of two $m \times m$ matrices of functors. On the other hand,

$$(\widehat{\rho}_{g_2 g_1})_{ki} = \rho_{h_2 h_1} \tag{54}$$

since $(g_2g_1)r_i = r_k(h_2h_1)$ by (53). We define the pseudonatural transformation (as a 2-isomorphism in $(\mathcal{V}^m)^*$)

$$\widehat{\phi}_{g_2,g_1} : \widehat{\rho}_{g_2}\widehat{\rho}_{g_1} \Longrightarrow \widehat{\rho}_{g_2g_1},$$

as the $m \times m$ matrix whose (k,i)-entry is the 2-isomorphism

$$(\widehat{\phi}_{g_2,g_1})_{ki} = \phi_{h_2,h_1} : \rho_{h_2}\rho_{h_1} \longrightarrow \rho_{h_2h_1}, \tag{55}$$

and all other entries vanish. For $g_1, g_2, g_3 \in G$, the 3-isomorphism in $(\mathcal{V}^m)^*$

$$\widehat{\Phi}_{g_3,g_2,g_1} : [\widehat{\rho}_{g_3}\#_0\widehat{\phi}_{g_2,g_1}]\#_1\widehat{\phi}_{g_3,g_2g_1} \Longrightarrow [\widehat{\phi}_{g_3,g_2}\#_0\widehat{\rho}_{g_1}]\#_1\widehat{\phi}_{g_3g_2,g_1}$$

is a modification. Write

$$g_3r_k = r_lh_3$$

for some $h_3 \in H$. Then we have

$$[\widehat{\rho}_{g_3}\#_0\widehat{\phi}_{g_2,g_1}]_{li} = \rho_{h_3}\#_0\phi_{h_2,h_1} \quad \text{and} \quad [\widehat{\phi}_{g_3,g_2}\#_0\widehat{\rho}_{g_1}]_{li} = \phi_{h_3,h_2}\#_0\rho_{h_1}, \tag{56}$$

etc.. We define $\widehat{\Phi}_{g_3,g_2,g_1}$ as an $m \times m$ matrix whose (l,i)-entry is the modification (as a 3-isomorphism in $\mathcal{V}^*$)

$$(\widehat{\Phi}_{g_3,g_2,g_1})_{li} = \Phi_{h_3,h_2,h_1} : [\rho_{h_3}\#_0\phi_{h_2,h_1}]\#_1\phi_{h_3,h_2h_1} \Longrightarrow [\phi_{h_3,h_2}\#_0\rho_{h_1}]\#_1\phi_{h_3h_2,h_1},$$

and all other entries vanish.

For $g_4 \in G$, write

$$g_4r_l = r_th_4$$

for some $h_4 \in H$. The (t,i)-entry of the $m \times m$ matrix $[\widehat{\rho}_{g_4}\#_0\widehat{\Phi}_{g_3,g_2,g_1}]\#_1\widehat{\phi}_{g_4,g_3g_2g_1}$ is the modification

$$[\rho_{h_4}\#_0\Phi_{h_3,h_2,h_1}]\#_1\phi_{h_4,h_3h_2h_1}$$

of $\mathcal{V}$, and similarly we obtain other terms in the 3-cocycle condition (15) for $\widehat{\Phi}$. So the 3-cocycle condition (15) for $\widehat{\Phi}$ is reduced to the 3-cocycle condition for Φ. Note that functors or pseudonatural transformation or modification we consider are matrices, of which entries are in a strict 3-subcategory $\mathcal{W}$ of $\mathcal{V}^*$. It follows from the strictness of $\mathcal{W}$ that $\widehat{\rho}$ is a strict 2-categorical action of G on $\mathcal{V}^m \approx \mathrm{Ind}_H^G(\mathcal{V})$.

5. The 3-character of the induced strict 2-categorical action

5.1. The 2-categorical trace of the induced strict 2-categorical action. As above ρ is a strict 2-categorical action of H on the 2-category $\mathcal{V}$. Let $\mathcal{R}$ be a system of representatives of G/H. We have the decomposition

$$\mathcal{R} = \mathcal{R}' \cup \mathcal{R}'',$$

where $\mathcal{R}' := \{r \in \mathcal{R}; r^{-1}fr \in H\}$, $\mathcal{R}'' := \{r \in \mathcal{R}; r^{-1}fr \notin H\}$. For a fixed element f of G, the decomposition

$$[f]_G \cap H = [h_1]_H \cup \cdots [h_n]_H$$

induces a decomposition

$$\mathcal{R}' = \bigcup_{i=1}^{n} \mathcal{R}_i \qquad \text{with} \qquad \mathcal{R}_i = \left\{ r \in \mathcal{R}; r^{-1}fr \in [h_i]_H \right\}.$$

For fixed i, we pick $r_i \in \mathcal{R}_i$ and write $h_i = r_i^{-1}fr_i$. For $r \in \mathcal{R}_i$, we have $r^{-1}fr = h^{-1}h_ih$ for some $h \in H$. From now on, by replacing r by rh^{-1} in the representatives of $\mathcal{R}_i \subset G/H$, we can assume

$$r^{-1}fr = h_i \qquad \text{for} \quad \text{all} \qquad r \in \mathcal{R}_i. \tag{57}$$

Denote

$$m_i := |\mathcal{R}_i|, \qquad m' := |\mathcal{R}'| = \sum_{i=1}^{n} m_i, \qquad m'' := |\mathcal{R}''|, \qquad m := m' + m''.$$

It follows from the definition (51)-(52) of $\widehat{\rho}_f$ that

$$\widehat{\rho}_f = \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0n} \\ A_{10} & A_{11} & 0 & \cdots & 0 \\ A_{20} & 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n0} & 0 & 0 & \cdots & A_{nn} \end{pmatrix}, \qquad A_{ii} = \begin{pmatrix} \rho_{h_i} & & \\ & \ddots & \\ & & \rho_{h_i} \end{pmatrix}_{m_i \times m_i}, \tag{58}$$

where $i = 1, \ldots, n$, and A_{00} is a off-diagonal $m'' \times m''$ matrix. So an object of $\mathbb{T}r_2\widehat{\rho}_f$ is a pseudonatural transformation $\chi : 1_{\mathcal{V}^m} \to \widehat{\rho}_f$ of the form

$$\chi = \begin{pmatrix} 0_{m'' \times m''} & & & \\ & \ddots & & \\ & & D_i & \\ & & & \ddots \end{pmatrix}, \qquad D_i = \begin{pmatrix} \chi_{m_1+\cdots+m_{i-1}+1} & & \\ & \ddots & \\ & & \chi_{m_1+\cdots+m_i} \end{pmatrix}, \tag{59}$$

where $\chi_{m_1+\cdots+m_{i-1}+\alpha} : 1_{\mathcal{V}} \to \rho_{h_i}$ is an object of $\mathbb{T}r_2\rho_{h_i}$, $\alpha = 1, \ldots, m_i$. Also morphisms in $\mathbb{T}r_2\widehat{\rho}_f$ are diagonal. So we have

$$\mathbb{T}r_2\widehat{\rho}_f = \bigoplus_{i=1}^{n} (\mathbb{T}r_2\rho_{h_i})^{m_i}.$$

5.2. LEMMA. ([13], Lemma 7.7) *Left multiplication with r_i^{-1} maps $\mathcal{R}_i$ into a system of representatives of $C_G(h_i)/C_H(h_i)$.*

For $g \in C_G(f)$ and $r \in \mathcal{R}_i$, we write

$$gr = \widetilde{r}h, \tag{60}$$

for some $\widetilde{r} \in \mathcal{R}$ and $h \in H$. Also, r is uniquely determined by $\widetilde{r}$ for fixed g. Then

$$\widetilde{r}^{-1} f \widetilde{r} = hr^{-1}g^{-1}fgrh^{-1} = hh_ih^{-1}$$

by (57). Hence $\widetilde{r} \in \mathcal{R}_i$ and so $\widetilde{r}^{-1} f \widetilde{r} = h_i$ by assumption (57). It follows that $h \in C_H(h_i)$. Then

$$\begin{array}{rlcl} gr & = \widetilde{r}h & \text{gives} & (\widehat{\rho}_g)_{\widetilde{r}r} = \rho_h, \\ fr & = rh_i & \text{gives} & (\widehat{\rho}_f)_{rr} = \rho_{h_i}, \\ g^{-1}\widetilde{r} & = rh^{-1} & \text{gives} & (\widehat{\rho}_{g^{-1}})_{r\widetilde{r}} = \rho_{h^*}, \end{array}$$

and all other entries vanish. Thus

$$(\widehat{\rho}_g \widehat{\rho}_f \widehat{\rho}_{g^*})_{\widetilde{r}\widetilde{r}} = \rho_h \rho_{h_i} \rho_{h^*}$$

and all other entries in the last $(m' \times m')$-block vanish (see (58)).

We denote by $\widetilde{\psi}$ the categorical action of the centralizer $C_G(f)$ of f on the category $\mathbb{T}r_2\widehat{\rho}_f$. By definition (26), $\widetilde{\psi}_g$ for $g \in C_G(f)$ is an invertible functor as follows. For a pseudonatural transformation $\text{diag}(\ldots, \chi_r, \ldots) = \chi : 1_{\mathcal{V}^m} \to \widehat{\rho}_f$ in (59), $\widetilde{\psi}_g(\chi)$ is a pseudonatural transformation given by

$$\text{diag}(1_{\mathcal{V}}, \ldots, 1_{\mathcal{V}}) \xrightarrow{\widehat{\phi}^{-1}_{g,g^*}} \widehat{\rho}_g \widehat{\rho}_{g^*} \xrightarrow{\widehat{\rho}_g \#_0 \chi \#_0 \widehat{\rho}_{g^*}} \widehat{\rho}_g \widehat{\rho}_f \widehat{\rho}_{g^*} \xrightarrow{\widehat{\phi}_{g,f} \#_0 \widehat{\rho}_{g^*}} \widehat{\rho}_{gf} \widehat{\rho}_{g^*} \xrightarrow{\widehat{\phi}_{gf,g^*}} \widehat{\rho}_{gfg^*} = \widehat{\rho}_f,$$

where the first m'' diagonal terms of $\widetilde{\psi}_g(\chi)$ must vanish, and other diagonal terms are

$$\begin{array}{rl} \left(\widehat{\phi}^{-1}_{g,g^*}\right)_{\widetilde{r}\widetilde{r}} & = \phi^{-1}_{h,h^*} : 1_{\mathcal{V}} \to \rho_h \rho_{h^*}, \\ (\widehat{\rho}_g \#_0 \chi \#_0 \widehat{\rho}_{g^*})_{\widetilde{r}\widetilde{r}} & = (\widehat{\rho}_g)_{\widetilde{r}r} \#_0 \chi_{rr} \#_0 (\widehat{\rho}_{g^*})_{r\widetilde{r}} = \rho_h \#_0 \chi_r \#_0 \rho_{h^*} : \rho_h \rho_{h^*} \to \rho_h \rho_{h_i} \rho_{h^*}, \\ \left(\widehat{\phi}_{g,f} \#_0 \widehat{\rho}_{g^*}\right)_{\widetilde{r}\widetilde{r}} & = \phi_{h,h_i} \#_0 \rho_{h^*} : \rho_h \rho_{h_i} \rho_{h^*} \to \rho_{hh_i} \rho_{h^*} \\ \left(\widehat{\phi}_{gf,g^*}\right)_{\widetilde{r}\widetilde{r}} & = \phi_{hh_i,h^*} : \rho_{hh_i} \rho_{h^*} \to \rho_{hh_ih^*} = \rho_{h_i}. \end{array}$$

All other entries vanish by definitions (54)-(55). Therefore, $\widetilde{\psi}_g(\chi)$ is a diagonal $m \times m$ matrix of pseudonatural transformations, whose $(\widetilde{r}, \widetilde{r})$-entry for $\widetilde{r} \in \mathcal{R}'$ is

$$\left(\widetilde{\psi}_g(\chi)\right)_{\widetilde{r}\widetilde{r}} = \phi^{-1}_{h,h^*} \#_1 [\rho_h \#_0 \chi_r \#_0 \rho_{h^*}] \#_1 [\phi_{h,h_i} \#_0 \rho_{h^*}] \#_1 \phi_{hh_i,h^*} : 1_{\mathcal{V}} \to \rho_{h_i}, \tag{61}$$

and vanishes for all $\widetilde{r} \in \mathcal{R}''$.

Now denote by $\psi^{(i)}$ the categorical action of the centralizer $C_H(h_i)$ on the category $\mathbb{T}r_2\rho_{h_i}$, which is constructed from the strict 2-categorical action ρ of H on $\mathcal{V}$. Recall that by definition (26), we have a functor $\psi_h^{(i)}$ for each $h \in C_H(h_i)$. For $h \in C_H(h_i)$ and a pseudonatural transformation $\omega : 1_{\mathcal{V}} \to \rho_{h_i}$, the pseudonatural transformation $\psi_h^{(i)}(\omega)$ is again by definition (26) the composition of the following pseudonatural transformations between functors:

$$1_{\mathcal{V}} \xrightarrow{\phi_{h,h^*}^{-1}} \rho_h\rho_{h^*} \xrightarrow{\rho_h\#_0\omega\#_0\rho_{h^*}} \rho_h\rho_{h_i}\rho_{h^*} \xrightarrow{\phi_{h,h_i}\#_0\rho_{h^*}} \rho_{hh_i}\rho_{h^*} \xrightarrow{\phi_{hh_i,h^*}} \rho_{hh_ih^*} = \rho_{h_i}.$$

Then we see that (61) can be written as

$$\left(\widetilde{\psi}_g\left(\chi\right)\right)_{\widetilde{r}\widetilde{r}} = \psi_h^{(i)}(\chi_r) : 1_{\mathcal{V}} \to \rho_{h_i}, \tag{62}$$

with $r, \widetilde{r} \in R_i$ and h determined by (60). Namely, the resulting $\widetilde{r}$-th diagonal term is the image of the r-th diagonal term under the action of the functor $\psi_h^{(i)}$.

Note that we have the identification

$$\mathrm{Ind}_{C_H(h_i)}^{C_G(h_i)}\mathbb{T}r_2\rho_{h_i} \cong (\mathbb{T}r_2\rho_{h_i})^{m_i}, \tag{63}$$

since $|C_G(h_i)/C_H(h_i)| = m_i$ by Lemma 5.2, and that (60) is equivalent to

$$(r_i^{-1}gr_i)(r_i^{-1}r) = (r_i^{-1}\widetilde{r})h, \qquad h \in C_H(h_i). \tag{64}$$

The coset $C_G(h_i)/C_H(h_i)$ are represented by $r_i^{-1}r$ for $r \in \mathcal{R}_i$ by Lemma 5.2 again, and an element of $C_G(h_i)$ can always be written as $r_i^{-1}gr_i$ for some $g \in C_G(f)$. As above we denote by $\widehat{\psi^{(i)}}$ the induced action of the centralizer $C_G(h_i)$ of h_i on the category $\mathrm{Ind}_{C_H(h_i)}^{C_G(h_i)}\mathbb{T}r_2\rho_{h_i}$. Recall the definition (51)-(52) of the induced action. So the action of $r_i^{-1}gr_i \in C_G(h_i)$ on the induced category (63) is given by the functor $\widehat{\psi^{(i)}}_{r_i^{-1}gr_i}$ on $(\mathbb{T}r_2\rho_{h_i})^{m_i}$ with

$$\left(\widehat{\psi^{(i)}}_{r_i^{-1}gr_i}(\chi)\right)_{r_i^{-1}\widetilde{r},r_i^{-1}r} = \psi_h^{(i)}\left(\chi_{r_i^{-1}r}\right) : 1_{\mathcal{V}} \to \rho_{h_i}. \tag{65}$$

for $\chi \in (\mathbb{T}r_2\rho_{h_i})^{m_i}$, where h is given by (64), and all other entries vanish. Here we use the expressions $r_i^{-1}r$ as indices of the components of $(\mathbb{T}r_2\rho_{h_i})^{m_i}$. Comparing (62) with (65), we find that the action of $g \in C_G(f)$ on $(\mathbb{T}r_2\rho_{h_i})^{m_i}$ coincides with the induced action of $r_i^{-1}gr_i \in C_G(h_i)$ on it, and so the action of the centralizer $C_G(f)$ on $\mathbb{T}r_2\widehat{\rho}_f$ decomposes into actions on

$$\bigoplus_i(\mathbb{T}r_2\rho_{h_i})^{m_i} = \bigoplus_i\mathrm{Ind}_{C_H(h_i)}^{C_G(h_i)}\mathbb{T}r_2\rho_{h_i}. \tag{66}$$

Recall that the *initia groupoid* $\Lambda(G)$ of a group G has as objects, the elements of G, and for two such elements u and v, there is one morphism in $\Lambda(G)$ from u to v for every $g \in G$ such that $v = gug^{-1}$. Note that the initia groupoid $\Lambda(G)$ is equivalent to the groupoid with the set of objects consisting of the conjugacy classes $[g_i]$ and the set of morphisms consisting of $g : [g_i] \to [g_i]$ for $g \in C_G(g_i)$. Therefore the above result can be summarized as follows.

5.3. THEOREM. *Let $\mathcal{V}$ be a k-linear 2-category. The 2-categorical trace $\mathbb{T}r_2$ takes induced strict 2-categorical action into the induced categorical action of the associated initia groupoids, i.e. (3) holds.*

5.4. REMARK. *Even for the categorical action, Section 4 above and the present subsection provide some details not written down explicitly in section 7.2 of [13].*

5.5. THE 3-CHARACTER FORMULA. Recall the 2-character formula for an induced categorical action.

5.6. THEOREM. ([18], Corollary 7.6) *Let ϱ be a categorical action of a subgroup H of a finite group G on a k-linear category $\mathcal{W}$. Suppose that $\mathbb{T}r\,\varrho_h$ is finite dimensional for each $h \in H$. Then the 2-character of the induced categorical action of G is given by*

$$\chi_{\mathrm{ind}}(f,g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}(f,g)s \in H \times H}} \chi_{\varrho}(s^{-1}fs, s^{-1}gs) \tag{67}$$

for $g \in C_G(f)$.

We now state:

5.7. THEOREM. *Let H be a subgroup of a finite group G and let ρ be a strict 2-categorical action of H on the 2-category $\mathcal{V}$. Let ψ be the categorical actions of the centralizers on the 2-categorical trace. Suppose that $\mathbb{T}r\psi_h$ is finite dimensional for each $h \in H$. Then the 3-character of the induced strict 2-categorical action of G is given by*

$$\chi_{\mathrm{ind}}(f,g,k) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}(f,g,k)s \in H \times H \times H}} \chi_{\rho}(s^{-1}fs, s^{-1}gs, s^{-1}ks) \tag{68}$$

for f, g and k pairwise commutative.

PROOF. By the decomposition (66) of the action of $C_G(f)$ on $\mathbb{T}r_2\widehat{\rho}_f$ and (62)-(65), we have

$$\chi_{\mathrm{ind}}(f,g,k) = \sum_{i=1}^{n} \chi_{\widehat{\psi^{(i)}}}(r_i^{-1}gr_i, r_i^{-1}kr_i).$$

Now apply Theorem 5.6 to the categorical action $\widehat{\psi^{(i)}}$ (65) of $C_G(h_i)$, which is induced from the categorical action $\psi^{(i)}$ of $C_H(h_i)$ on $\mathbb{T}r_2\rho_{h_i}$, to get

$$\chi_{\mathrm{ind}}(f,g,k) = \sum_{i=1}^{n} \frac{1}{|C_H(h_i)|} \sum_{\substack{t \in C_G(h_i) \\ t^{-1}r_i^{-1}(g,k)r_it \in C_H(h_i) \times C_H(h_i)}} \chi_{\psi^{(i)}}(t^{-1}r_i^{-1}gr_it, t^{-1}r_i^{-1}kr_it).$$

Recall that $\psi^{(i)}$ is the categorical action of $C_H(h_i)$ on $\mathbb{T}r_2\rho_{h_i}$ constructed from the strict 2-categorical action ρ of H. So we have

$$\chi_{\psi^{(i)}}(t^{-1}r_i^{-1}gr_it, t^{-1}r_i^{-1}kr_it) = \chi_{\rho}(h_i, t^{-1}r_i^{-1}gr_it, t^{-1}r_i^{-1}kr_it)$$

by the definition of the 3-character (49) for the strict 2-categorical action ρ of group H. Moreover, the decomposition of the action of $C_G(f)$ on $\mathbb{T}r_2\widehat{\rho}_f$ in Section 5.1 is independent of the choice of $h_i \in [h_i]_H$, conjugacy class of h_i in H. Therefore,

$$\chi_{\text{ind}}(f,g,k) = \sum_{h\in H} \frac{1}{|[h]_H|}\frac{1}{|C_H(h)|} \sum_{\substack{s^{-1}fs=h, s\in G,\\ s^{-1}(g,k)s\in C_H(h)\times C_H(h)}} \chi_\rho(h, s^{-1}gs, s^{-1}ks).$$

Here we have used the fact that $h_i = s^{-1}fs = s^{-1}r_ih_ir_i^{-1}s$ if and only if $r_i^{-1}s \in C_G(h_i)$. Note that for $s \in G$, we have $s^{-1}gs$ (resp. $s^{-1}ks$) $\in H$ if and only if $s^{-1}gs$ (resp. $s^{-1}ks$) $\in C_H(h)$ since g and k commute with $f = shs^{-1}$. The 3-character formula (68) follows. ∎

6. The categorical action of the centralizer of f on $\mathbb{T}\text{r}_2\rho_f$

6.1. A MODEL: THE 1-DIMENSIONAL CASE. Let us prove by using the condition (2) for 3-cocycles repeatedly that the expression Γ given in (48) is a 2-cocycle on the centralizer $C_G(f)$. This proof corresponds step by step to that of the general case carried out in Section 6.4.

Proof of Proposition 3.9. By the definition of $\Gamma_{*,*}$ in (48), we see that

$$\Gamma_{h,g}\Gamma_{k,hg} = \frac{\Pi_f}{\Pi_1},$$

where

$$\Pi_f := c(h,gf,g^*)c(h,g,f)c(hgf,g^*,h^*)^{-1} \cdot c(k,\underline{hf},g^*h^*)c(k,hg,f)c(\underline{kf},g^*h^*,k^*)^{-1}, \tag{69}$$

and Π_1 is just Π_f with f replaced by 1. Similarly, we have

$$\Gamma_{k,h}\Gamma_{kh,g} = \frac{\Pi'_f}{\Pi'_1},$$

where

$$\Pi'_f = c(k,hf,h^*)\mathbf{c}(\mathbf{k},\mathbf{h},\mathbf{f})c(khf,h^*,k^*)^{-1} \cdot \mathbf{c}(\mathbf{kh},\mathbf{gf},\mathbf{g}^*)c(kh,g,f)c(\underline{kf},g^*,(kh)^*)^{-1}, \tag{70}$$

and Π'_1 is just Π'_f with f replaced by 1.

Apply the 3-cocycle condition (2) to the product of the two boldface terms in (70) with $g_4 = k, g_3 = h, g_2 = gf, g_1 = g^*$ to get

$$\begin{aligned}\Pi'_f = \;& c(h,gf,g^*)c(k,hgf,g^*)c(k,h,gf)\\ & \cdot c(k,hf,h^*)\mathbf{c}(\mathbf{khf},\mathbf{h}^*,\mathbf{k}^*)^{-1}c(kh,g,f)\mathbf{c}(\underline{\mathbf{kf}},\mathbf{g}^*,(\mathbf{kh})^*)^{-1}.\end{aligned} \tag{71}$$

Here the second line above is the right-hand side of (70) with the two boldface terms deleted. Note that $\underline{kf}g^* = khf$. Apply the 3-cocycle condition (2) to the product of the two boldface terms in (71) with $g_4 = \underline{kf}, g_3 = g^*, g_2 = h^*, g_1 = k^*$ to get

$$\begin{aligned} \Pi'_f = \; & c(g^*, h^*, k^*)^{-1} c(\underline{kf}, g^*h^*, k^*)^{-1} \mathbf{c}(\underline{\mathbf{kf}}, \mathbf{g}^*, \mathbf{h}^*)^{-\mathbf{1}} \\ & \cdot c(h, gf, g^*) \mathbf{c}(\mathbf{k}, \underline{\mathbf{hf}}, \mathbf{g}^*) c(k, h, gf) \cdot \mathbf{c}(\mathbf{k}, \mathbf{hf}, \mathbf{h}^*) \cdot c(kh, g, f), \end{aligned} \tag{72}$$

Here the second line above is the right-hand side of (71) with the two boldface terms deleted. Apply the 3-cocycle condition (2) to the product of the three boldface terms in (72) with $g_4 = k$, $g_3 = \underline{hf}$, $g_2 = g^*$,$g_1 = h^*$ to get

$$\begin{aligned} \Pi'_f = \; & c(k, \underline{hf}, g^*h^*) c(\underline{hf}, g^*, h^*)^{-1} \\ & \cdot c(g^*, h^*, k^*)^{-1} c(\underline{kf}, g^*h^*, k^*)^{-1} c(h, gf, g^*) \mathbf{c}(\mathbf{k}, \mathbf{h}, \mathbf{gf}) \mathbf{c}(\mathbf{kh}, \mathbf{g}, \mathbf{f}), \end{aligned} \tag{73}$$

by $k\underline{hf} = \underline{kf}$ and $\underline{hf}g^* = hf$. Here the second line above is the right-hand side of (72) with the three boldface terms deleted. Apply the 3-cocycle condition (2) to the product of the two boldface terms in (73) with $g_4 = k$, $g_3 = h$, $g_2 = g$, $g_1 = f$ to get

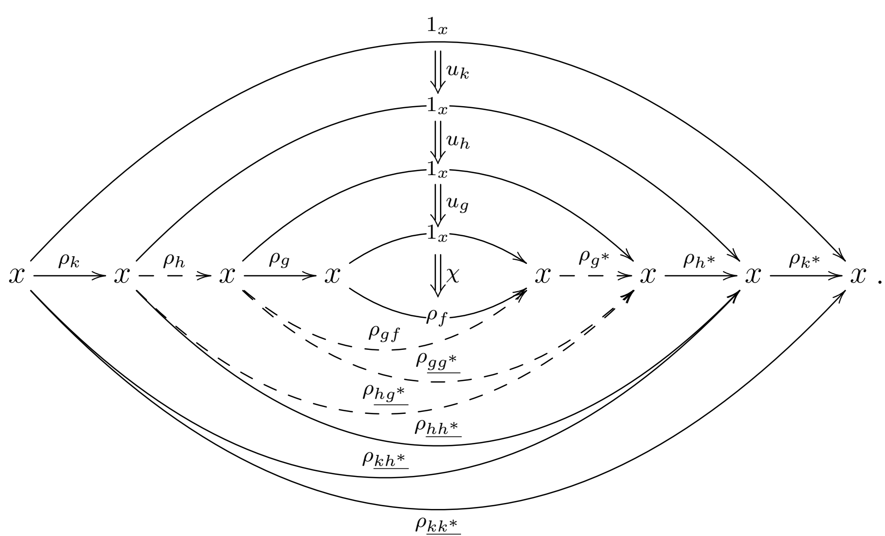
$$\begin{aligned} \Pi'_f = \; & c(h, g, f) \cdot c(k, hg, f) \mathbf{c}(\mathbf{k}, \mathbf{h}, \mathbf{g}) \\ & \cdot c(k, \underline{hf}, g^*h^*) c(\underline{hf}, g^*, h^*)^{-1} \mathbf{c}(\mathbf{g}^*, \mathbf{h}^*, \mathbf{k}^*)^{-\mathbf{1}} c(\underline{kf}, g^*h^*, k^*)^{-1} c(h, gf, g^*). \end{aligned} \tag{74}$$

For $f = 1$ in (74), we see that Π'_1 also has the product $c(k, h, g) c(g^*, h^*, k^*)^{-1}$ of the two boldface terms, which is independent of f. They are cancelled in Π'_f / Π'_1. So we get

$$\frac{\Pi'_f}{\Pi'_1} = \frac{\Pi_f}{\Pi_1}$$

by comparing (74) and (69). Proposition 3.9 is proved.

6.2. THE NATURAL ISOMORPHISM $\Gamma_{k,hg} \# (\psi_k \circ \Gamma_{h,g})$. Let us write down the natural isomorphism $\Gamma_{k,hg} \# (\psi_k \circ \Gamma_{h,g})$. For a fixed $\chi \in (\mathbb{T}\mathrm{r}_2 \rho_f)_0$, by using the definition of compositions in (30) twice, we see that $\psi_k \circ \psi_h \circ \psi_g(\chi)$ is the composition of the following 2-arrows:



(75)

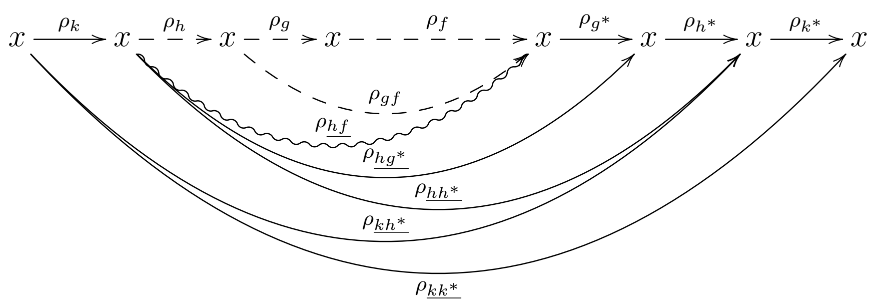
Let us calculate the 3-isomorphism

$$[\Gamma_{k,hg}\#(\psi_k \circ \Gamma_{h,g})](\chi) : \psi_k \circ \psi_h \circ \psi_g(\chi) \Longrightarrow \psi_{khg}(\chi) \tag{76}$$

for a fixed 2-arrow $\chi \in \mathbb{T}r_2\rho_f \subset \mathcal{C}^{++}$. We consider the lower half part of (75) first. The 3-isomorphism

$$\Lambda_1 = \diamondsuit\#_1[\rho_k\#_0\Phi_{h,gf,g^*}\#_0(\rho_{h^*}\rho_{k^*})]\#_1\diamondsuit, \tag{77}$$

the associator Φ_{h,gf,g^*} (14) whiskered by 2-isomorphisms $\diamondsuit$ which we do not write down explicitly, changes the diagonal $\rho_{\underline{gg^*}}$ of the dotted quadrilateral in (75) to the wavy diagonal $\rho_{\underline{hf}}$ of the same quadrilateral in the following diagram:

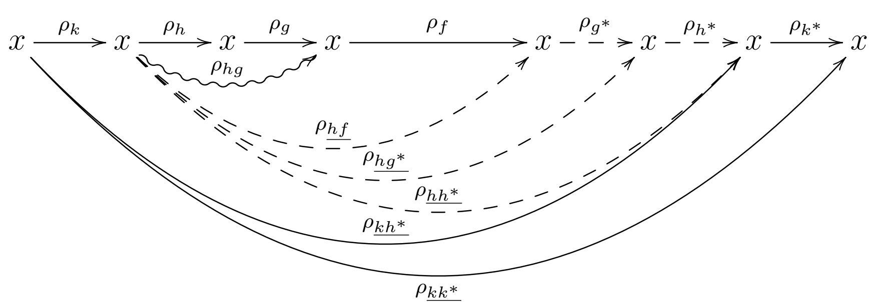


(78)

This is a 3-arrow as (36). The 2-arrows outside the quadrilateral are fixed as the whiskering parts. The 3-isomorphism

$$\Lambda_2 = \diamondsuit\#_1[\rho_k\#_0\Phi_{h,g,f}\#_0(\rho_{g^*}\rho_{h^*}\rho_{k^*})]\#_1\diamondsuit, \tag{79}$$

as a whiskered associator (14), changes the diagonal ρ_{gf} of the above dotted-wavy quadrilateral to the wavy diagonal ρ_{hg} of the same quadrilateral in the following diagram:

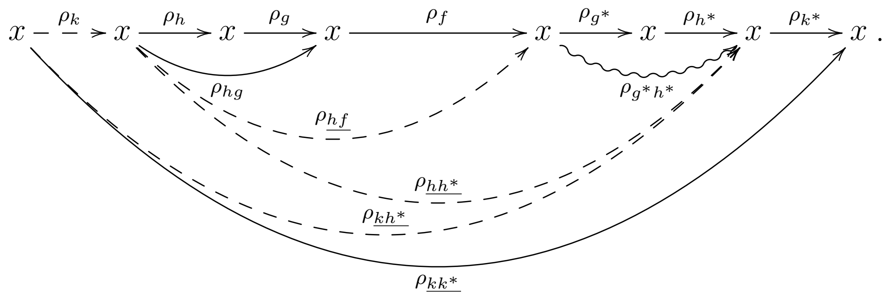


(80)

The 3-isomorphism

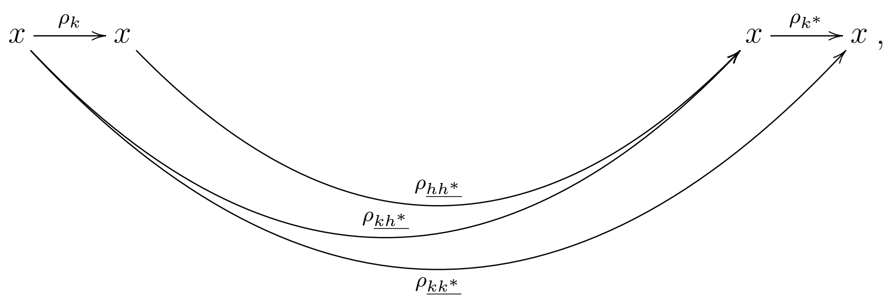
$$\Lambda_3 = \diamondsuit\#_1[\rho_k\#_0\Phi^{-1}_{\underline{hf},g^*,h^*}\#_0\rho_{k^*}]\#_1\diamondsuit, \tag{81}$$

as a whiskered associator (14), changes the diagonal $\rho_{\underline{hg^*}}$ of the above dotted quadrilateral to the wavy diagonal $\rho_{g^*h^*}$ of the same quadrilateral in the following diagram:



(82)

Note that the diagrams (78), (80) and (82) are exactly the diagrams (35), (38) and (40) by adding from below to each of these the arrows:



By definition, the composition $\Lambda_1 \#_2 \Lambda_2 \#_2 \Lambda_3$ is the 3-isomorphism

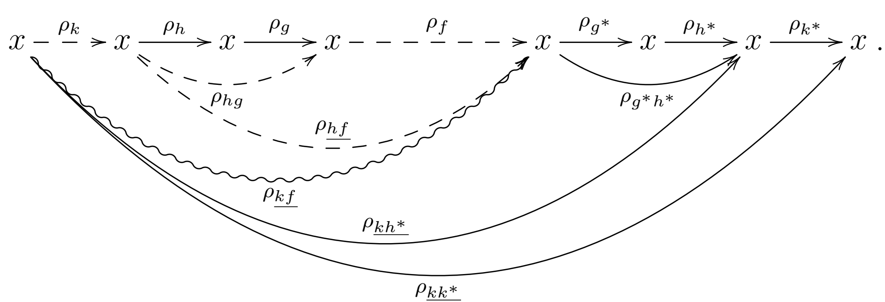
$$[\psi_k \circ \Gamma_{h,g}](\chi) : \psi_k \circ \psi_h \circ \psi_g(\chi) \Longrightarrow \psi_k \circ \psi_{hg}(\chi)$$

corresponding to the lower half of (75).

The 3-isomorphism

$$\Lambda_4 = \diamondsuit \#_1 [\Phi_{k,\underline{hf},g^*h^*} \#_0 \rho_{k^*}] \#_1 \diamondsuit,$$

as a whiskered associator (14), changes the diagonal $\rho_{\underline{hh^*}}$ of the dotted-wavy quadrilateral in (82) to the wavy diagonal $\rho_{\underline{kf}}$ of the same quadrilateral in the following diagram:

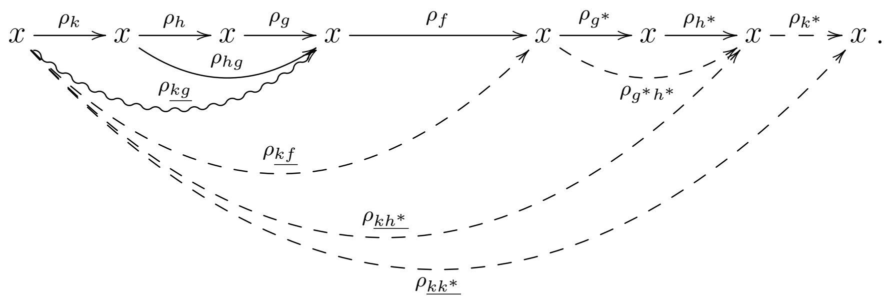


(83)

The 3-isomorphism

$$\Lambda_5 = \diamondsuit \#_1 \left[\Phi_{k,hg,f} \#_0 \left(\rho_{g^*} \rho_{h^*} \rho_{k^*}\right)\right] \#_1 \diamondsuit,$$

as a whiskered associator (14), changes the diagonal $\rho_{\underline{hf}}$ of the above dotted-wavy quadrilateral to the wavy diagonal $\rho_{\underline{kg}}$ of the same quadrilateral in the following diagram:

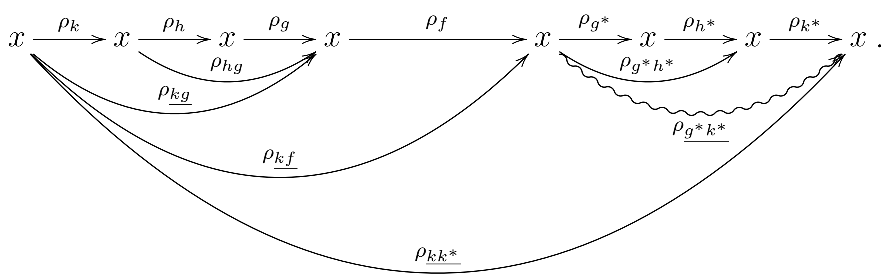


(84)

The 3-isomorphism

$$\Lambda_6 = \Diamond \#_1 \Phi^{-1}_{\underline{kf},g^*h^*,k^*},$$

as a whiskered associator (14), changes the diagonal $\rho_{\underline{kh^*}}$ of the above dotted quadrilateral to the wavy diagonal $\rho_{\underline{g^*k^*}}$ of the same quadrilateral in the following diagram:



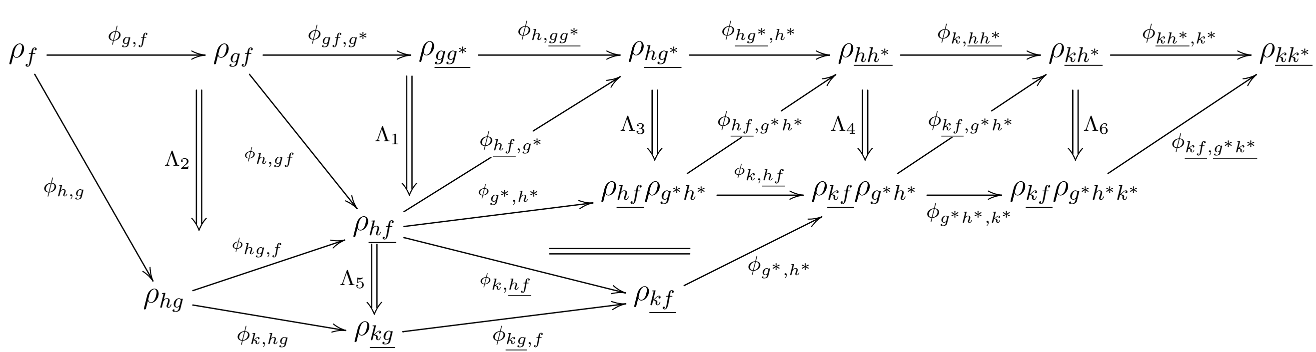
(85)

The composition $\Lambda_4 \#_2 \Lambda_5 \#_2 \Lambda_6$ is the 3-isomorphism

$$\Gamma_{k,hg}(\chi) : \psi_k \circ \psi_{hg}(\chi) \Longrightarrow \psi_{khg}(\chi)$$

corresponding to the lower half of (75).

In the 2-category $\mathcal{C}^+$, the composition $\Lambda_1 \#_2 \cdots \#_2 \Lambda_6$ of 3-isomorphisms corresponds to the following diagram $\mathscr{D}^l_f :=$

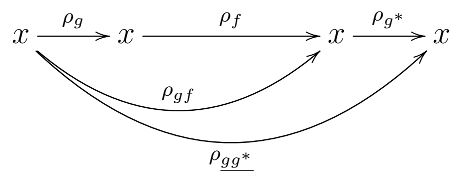


(86)

where the symbol = in this diagram follows from the interchange law (6) for a horizontal composition: the commutativity of $\phi_{k,\underline{hf}}$ and ϕ_{g^*,h^*}. Note that the part involving $\Lambda_1, \Lambda_2, \Lambda_3$ is just the diagram (43). Let $\mathscr{D}_1^l$ be the corresponding diagram in $\mathcal{C}^+$ with f replaced by 1, by using adjoint operations as in (44). Then as in (45), the 2-isomorphism in $\mathcal{C}^+$ corresponding to the morphism $[\Gamma_{k,hg}\#(\psi_k \circ \Gamma_{h,g})](\chi)$ in $\mathbb{T}r_2\rho_f$ is

$$\mathscr{D}_1^l \xrightarrow{\chi} \mathscr{D}_f^l. \tag{87}$$

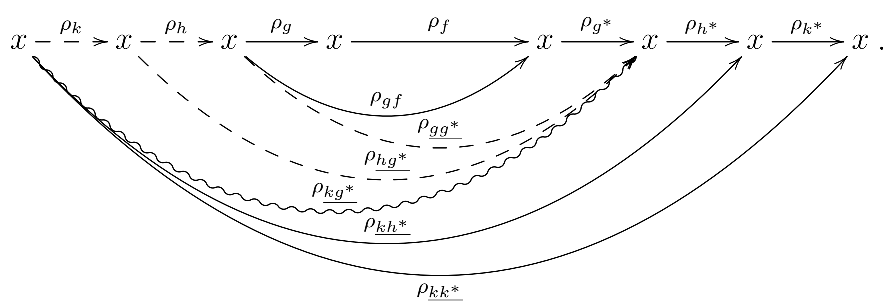
6.3. The natural isomorphism $\Gamma_{kh,g}\#(\Gamma_{k,h} \circ \psi_g)$. To calculate $\Gamma_{k,h} \circ \psi_g$, we fix the part



in the lower half of (75), which corresponds to ψ_g. The 3-isomorphism

$$\widetilde{\Lambda}_1 = \Diamond\#_1\Phi_{k,\underline{hg^*},h^*}\#_1\Diamond, \tag{88}$$

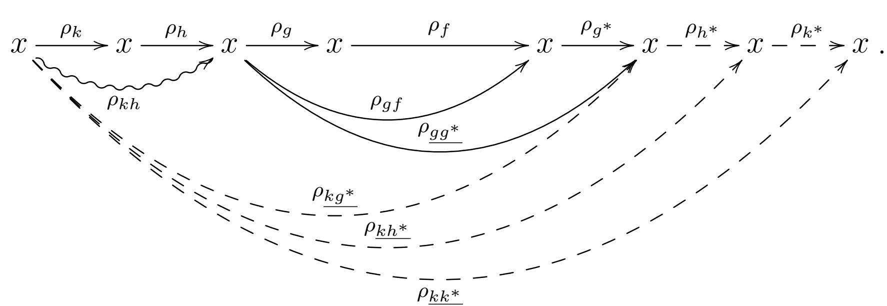
as a whiskered associator (14), changes the 1-isomorphism $\rho_{\underline{hh^*}}$ in the lower part of (75) to the wavy diagonal $\rho_{\underline{kg^*}}$ of the same quadrilateral in the following diagram:



The 3-isomorphism

$$\widetilde{\Lambda}_2 = \Diamond\#_1[\Phi_{k,h,\underline{gg^*}}\#_0(\rho_{h^*}\rho_{k^*})]\#_1\Diamond, \tag{89}$$

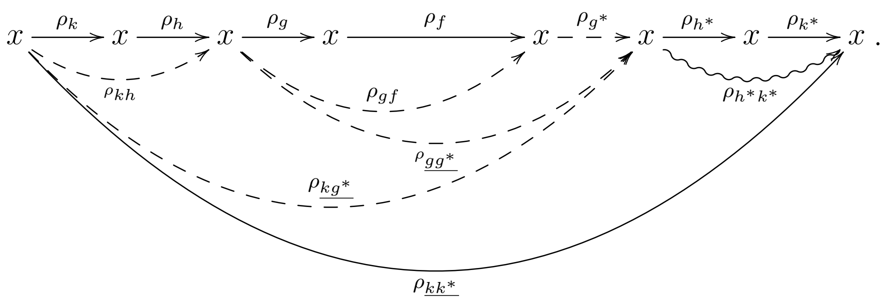
as a whiskered associator (14), changes the diagonal $\rho_{\underline{hg^*}}$ of the above dotted-wavy quadrilateral to the wavy diagonal ρ_{kh} of the same quadrilateral in the following diagram:



The 3-isomorphism

$$\widetilde{\Lambda}_3 = \Diamond \#_1 \Phi^{-1}_{\underline{kg^*}, h^*, k^*}, \tag{90}$$

as a whiskered associator (14), changes the diagonal $\rho_{\underline{kh^*}}$ of the above dotted quadrilateral to the wavy diagonal $\rho_{h^*k^*}$ of the same quadrilateral in the following diagram:



(91)

The composition $\widetilde{\Lambda}_1 \#_2 \widetilde{\Lambda}_2 \#_2 \widetilde{\Lambda}_3$ is the 3-isomorphism

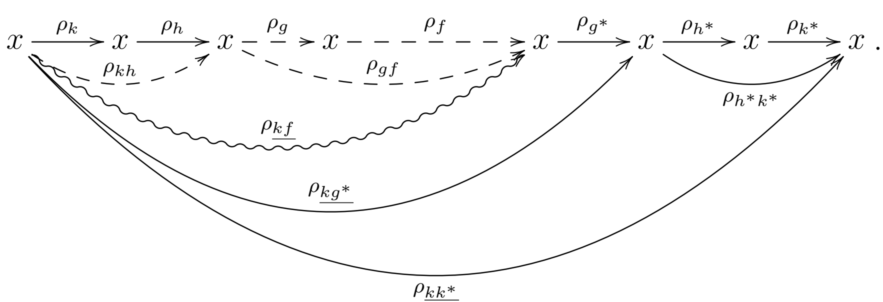
$$\Gamma_{k,h} \circ \psi_g : \psi_k \circ \psi_h \circ \psi_g(\chi) \Longrightarrow \psi_{kh} \circ \psi_g(\chi),$$

corresponding to the lower half of (75).

The 3-isomorphism

$$\widetilde{\Lambda}_4 = \Diamond \#_1 [\Phi_{kh, gf, g^*} \#_0 (\rho_{h^*} \rho_{k^*})] \#_1 \Diamond, \tag{92}$$

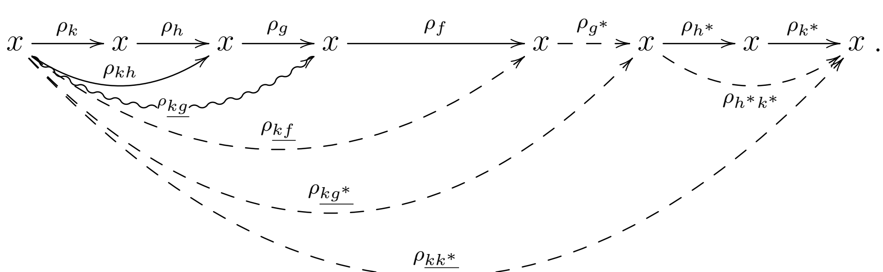
as a whiskered associator (14), changes the diagonal $\rho_{\underline{gg^*}}$ of the dotted quadrilateral in (91) to the wavy diagonal $\rho_{\underline{kf}}$ of the same quadrilateral in the following diagram:



The 3-isomorphism

$$\widetilde{\Lambda}_5 = \Diamond \#_1 [\Phi_{kh, g, f} \#_0 (\rho_{g^*} \rho_{h^*} \rho_{k^*})] \#_1 \Diamond, \tag{93}$$

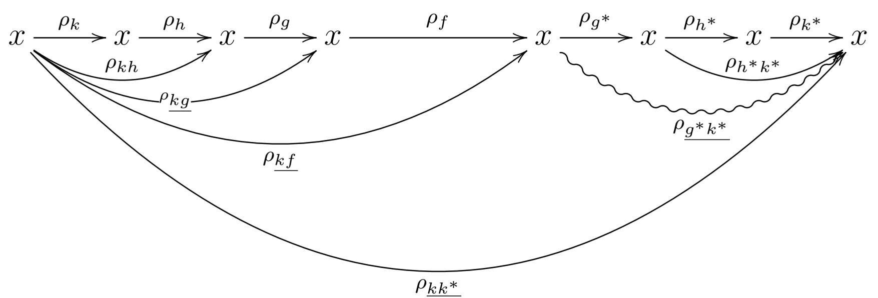
as a whiskered associator (14), changes the diagonal ρ_{gf} of the above dotted quadrilateral to the wavy diagonal $\rho_{\underline{kg}}$ of the same quadrilateral in the following diagram:



At last, the 3-isomorphism

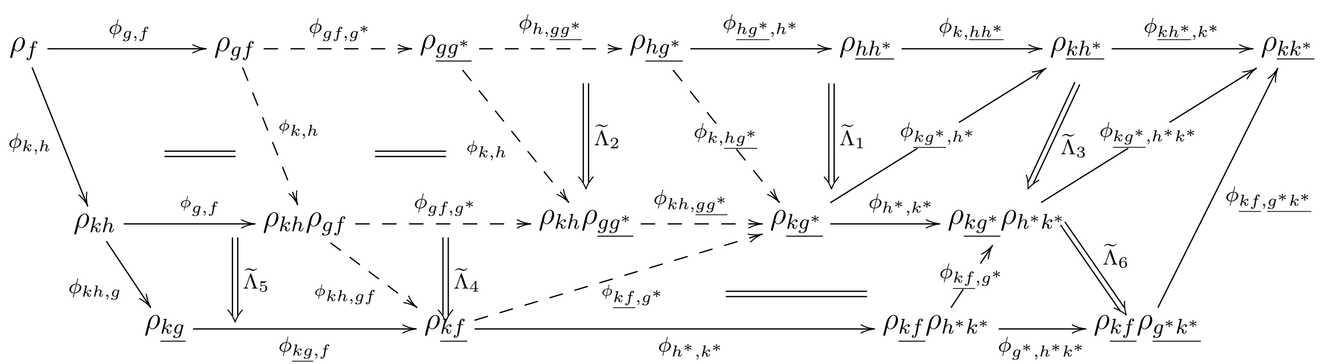
$$\widetilde{\Lambda}_6 = \Diamond \#_1 \Phi^{-1}_{\underline{kf}, g^*, h^*k^*}, \tag{94}$$

as a whiskered associator (14), changes the diagonal $\rho_{\underline{kg^*}}$ of the above dotted quadrilateral to the wavy diagonal $\rho_{\underline{g^*k^*}}$ of the same quadrilateral in the following diagram:



The composition $\widetilde{\Lambda}_4 \#_2 \widetilde{\Lambda}_5 \#_2 \widetilde{\Lambda}_6$ is the 3-isomorphism $\Gamma_{kh,g}(\chi) : \psi_{kh} \circ \psi_g(\chi) \Longrightarrow \psi_{khg}(\chi)$ corresponding to the lower half of (75).

The composition of $\widetilde{\Lambda}_1 \#_2 \cdots \#_2 \widetilde{\Lambda}_6$ in the 2-category $\mathcal{C}^+$ is the following diagram $\mathscr{D}^r_f :=$



(95)

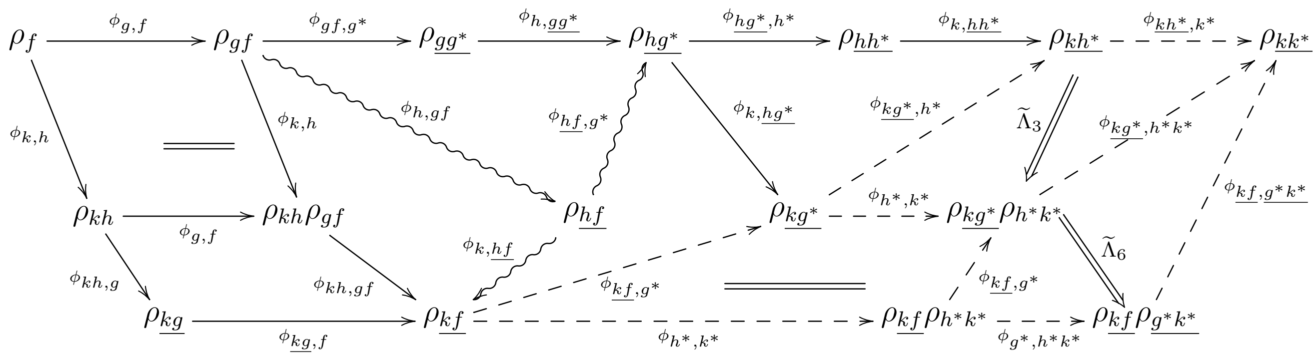
Let $\mathscr{D}^r_1$ be the corresponding diagram in $\mathcal{C}^+$ with f replaced by 1, by using adjoint operations as in (44). Then the 2-isomorphism in $\mathcal{C}^+$ corresponding to the morphism $[\Gamma_{kh,g} \# (\Gamma_{k,h} \circ \psi_g)](\chi)$ in $\mathbb{T}r_2\rho_f$ is

$$\mathscr{D}^r_1 \xrightarrow{\chi} \mathscr{D}^r_f. \tag{96}$$

6.4. The proof of the associativity. Let us show the identity (1), i.e., that diagrams $\mathscr{D}^l_1 \xrightarrow{\chi} \mathscr{D}^l_f$ in (87) and $\mathscr{D}^r_1 \xrightarrow{\chi} \mathscr{D}^r_f$ in (96) are identical in the 2-category $\mathcal{C}^+$, by using the 3-cocycle identity (24) repeatedly. This proof corresponds to that of the 1-dimensional case in Section 6.1 step by step.

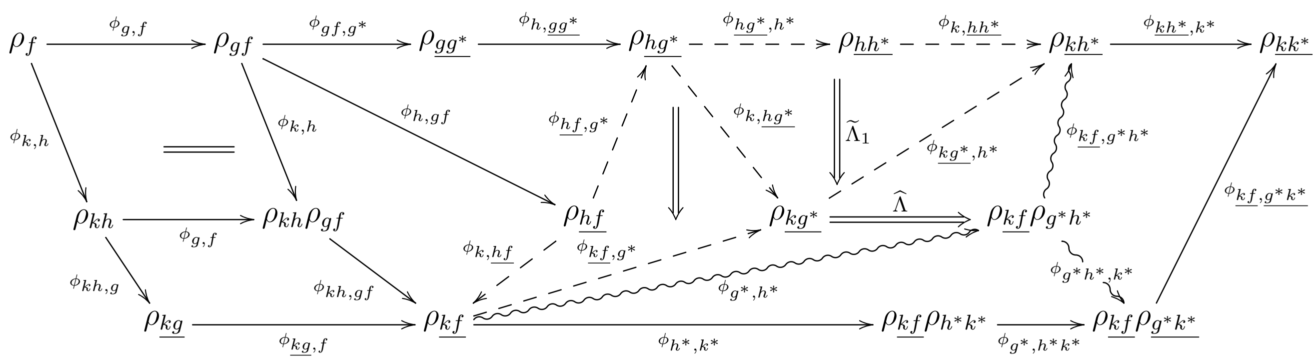
Apply the 3-cocycle identity (24) to the dotted diagram in (95) with $g_4 = k, g_3 =$

$h, g_2 = gf, g_1 = g^*$ to get wavy isomorphisms in the following diagram



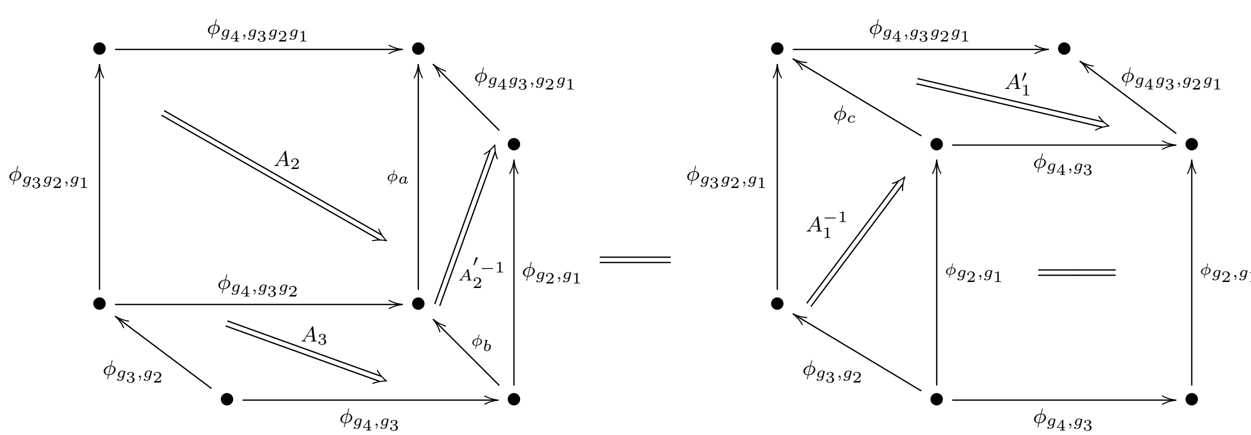
(97)

Note that $\widetilde{\Lambda}_3$ in (90) and $\widetilde{\Lambda}_6$ in (94) are the inverse of associators. Apply the 3-cocycle identity, the inverse version of (24) (the lower and upper boundaries are exchanged), to the above dotted diagram with $g_4 = \underline{kf}, g_3 = g^*, g_2 = h^*, g_1 = k^*$ to get wavy isomorphisms in the following:

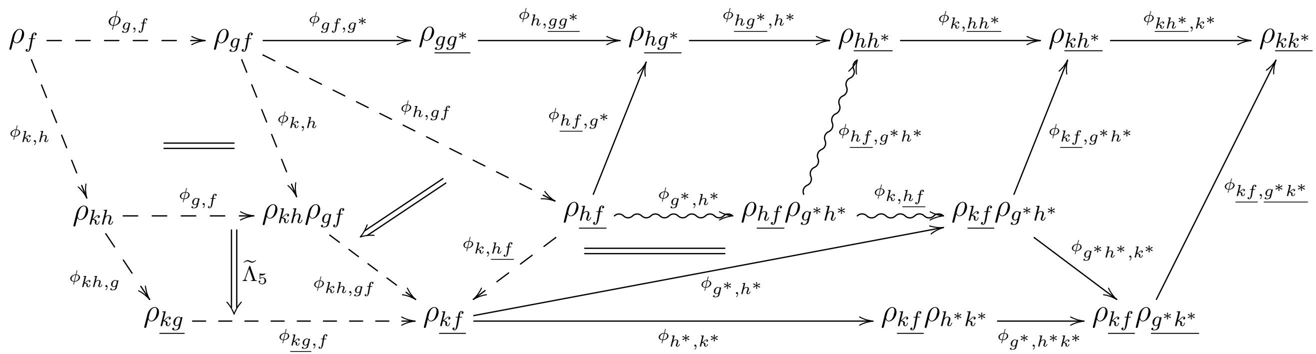


(98)

where $\widehat{\Lambda}$ is the inverse of a whiskered associator. Note that the commutative cube in (25) implies the following identity.

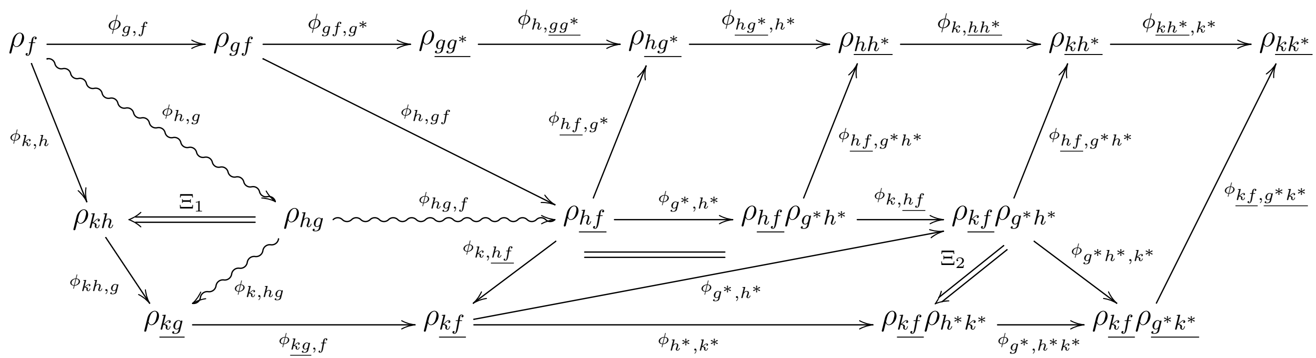


where $\phi_a := \phi_{g_4g_3g_2,g_1}, \phi_b := \phi_{g_4g_3,g_2}, \phi_c := \phi_{g_3,g_2g_1}$. The left-hand side is the back, bottom and right (this 2-isomorphism is inverted) faces of the cube in (25), while the right-hand side is the left (this 2-isomorphism is inverted), top and front faces of the cube. Apply this identity to the dotted-wavy diagram in (98) with $g_4 = k$, $g_3 = \underline{hf}$, $g_2 = g^*$,$g_1 = h^*$ to get wavy isomorphisms in the following:



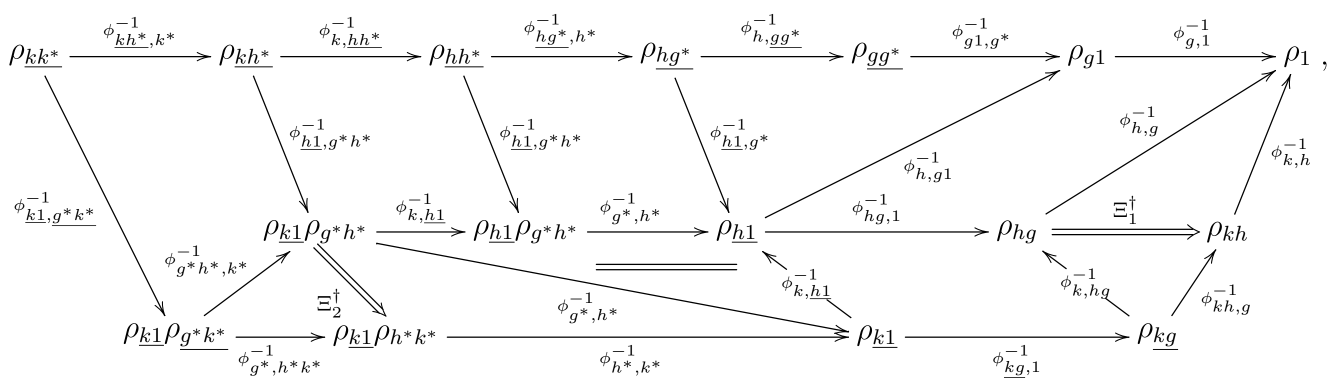
(99)

Apply the 3-cocycle identity (24) to the above dotted diagram with $g_4 = k$, $g_3 = h$, $g_2 = g$, $g_1 = f$ to get wavy isomorphisms in the following diagram $\widetilde{\mathscr{D}}^r_f :=$



(100)

With f replaced by 1, by using adjoint operations as in (44), the diagram $\mathscr{D}^r_1$ corresponding to the upper half is identically changed to the following diagram $\widetilde{\mathscr{D}}^r_1 :=$

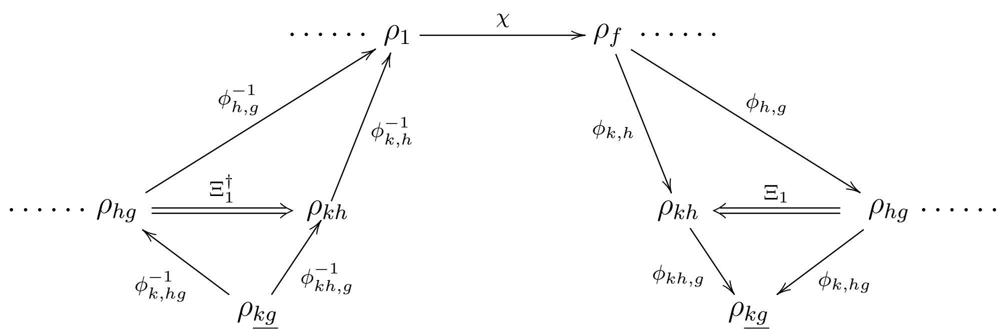


(101)

where $\Xi^\dagger_j$ is the adjoint of Ξ_j, $j = 1, 2$. Then the whole diagram $\mathscr{D}^r_1 \xrightarrow{\chi} \mathscr{D}^r_f$ in (96) is identically changed to

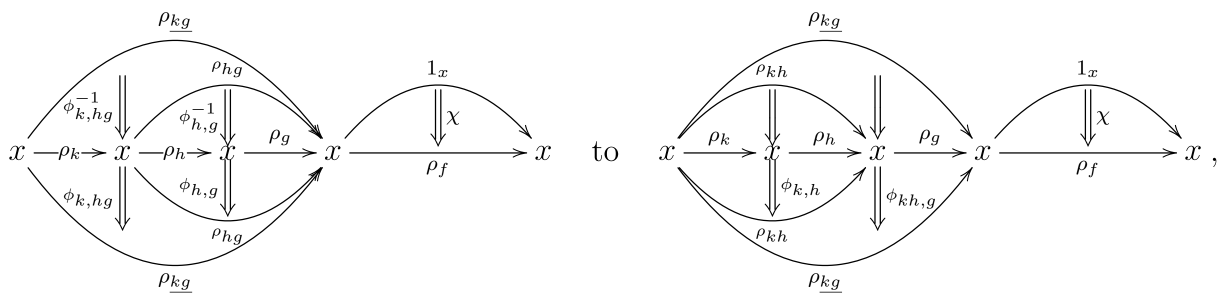
$$\widetilde{\mathscr{D}}^r_1 \xrightarrow{\chi} \widetilde{\mathscr{D}}^r_f,$$

namely,

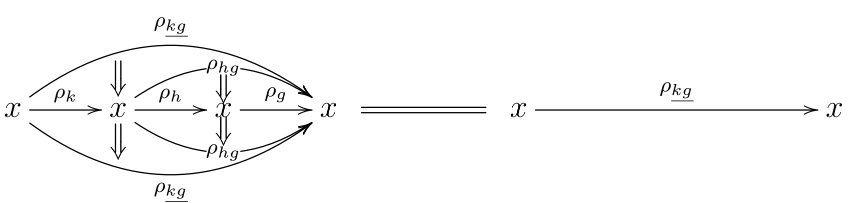


(102)

Note that (100) is exactly $\mathscr{D}^l_f$ in (86) with two extra 2-isomorphisms Ξ_1 and Ξ_2. But by definition, the 2-isomorphisms Ξ_1 and $\Xi_1^\dagger$ are the associators (13) corresponding to the 3-isomorphisms in $\mathcal{C}$, which change



and we have



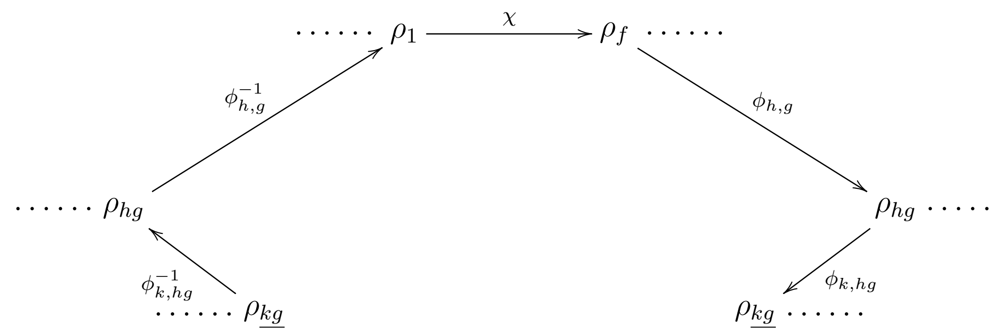
by cancellation (42). So Ξ_1 and $\Xi_1^\dagger$ are cancelled. More precisely, as a 3-isomorphism, $\Xi_1^\dagger \#_0 \chi \#_0 \Xi_1$ is

$$(\Xi_1^\dagger \#_0 \chi)\#_1(\Xi_1 \#_0 \chi) = (\Xi_1^\dagger \#_1 \Xi_1)\#_0 \chi = 1_{\rho_{\underline{kg}}} \#_0 \chi,$$

and we have

$$(\phi^{-1}_{k,hg} \#_0 \phi^{-1}_{h,g}) \#_0 (\phi_{h,g} \#_0 \phi_{k,hg}) \#_0 \chi = (\phi^{-1}_{k,hg} \#_0 \phi^{-1}_{h,g}) \#_0 \chi \#_0 (\phi_{h,g} \#_0 \phi_{k,hg})$$

in the 2-category $\mathcal{C}^+$, up to whiskering, by the interchange law. Namely, $\widetilde{\mathscr{D}}_1^r \xrightarrow{\chi} \widetilde{\mathscr{D}}_f^r$ in (102) is identical to



Similarly, the 2-isomorphisms Ξ_2 (100) and $\Xi_2^\dagger$ in (101) are also cancelled. The resulting diagram is exactly the diagram $\mathscr{D}_1^l \xrightarrow{\chi} \mathscr{D}_f^l$ in (87). This completes the proof of Theorem 3.7.

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