THE LINEARITY OF TRACES IN MONOIDAL CATEGORIES AND BICATEGORIES

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ABSTRACT. We show that in any symmetric monoidal category, if a weight for colimits is absolute, then the resulting colimit of any diagram of dualizable objects is again dualizable. Moreover, in this case, if an endomorphism of the colimit is induced by an endomorphism of the diagram, then its trace can be calculated as a linear combination of traces on the objects in the diagram. The formal nature of this result makes it easy to generalize to traces in homotopical contexts (using derivators) and traces in bicategories. These generalizations include the familiar additivity of the Euler characteristic and Lefschetz number along cofiber sequences, as well as an analogous result for the Reidemeister trace, but also the orbit-counting theorem for sets with a group action, and a general formula for homotopy colimits over EI-categories.

Contents

1. Introduction

In this paper, we study the following question: given a diagram in a category, when can the "size" of its colimit be calculated in terms of the "size" of the objects occurring in the diagram? Such a question might pertain to various notions of "size", such as cardinality, dimension, or Euler characteristic. Here are a few well-known facts that can be interpreted as answers to instances of this question.

(i) If X and Y are finite sets, then we have an obvious formula for the cardinality of their disjoint union:
$$\#(X \sqcup Y) = \#X + \#Y.$$

(ii) More generally, for finite CW complexes X and Y, the Euler characteristic of their disjoint union is the sum of their Euler characteristics:
$$\chi(X \sqcup Y) = \chi(X) + \chi(Y).$$

(iii) Similarly, if X and Y are finite-dimensional vector spaces, we have an analogous formula for the dimension of their sum:
$$\dim(X \oplus Y) = \dim(X) + \dim(Y).$$

(iv) If $X \hookrightarrow Y$ and $X \hookrightarrow Z$ are injections of finite sets, then we have the "inclusion-exclusion" formula for the cardinality of their pushout:
$$\#(Y +_X Z) = \#Y + \#Z - \#X.$$

(v) More generally, if $Y \leftarrow X \rightarrow Z$ is an arbitrary span of finite CW complexes, then there is a similar formula for the Euler characteristic of their *homotopy* pushout:
$$\chi(Y +_X^h Z) = \chi(Y) + \chi(Z) - \chi(X).$$

(vi) As a particular case of (v), if $Z = \star$ is the one-point space and $X \rightarrow Y$ is the inclusion of a subcomplex, then the homotopy pushout is homotopy equivalent to the quotient Y/X, and we have
$$\chi(Y/X) = \chi(Y) - \chi(X).$$

(vii) If X is a finite-dimensional chain complex and $\dim(X) = \sum_n (-1)^n \dim(X_n)$ is its *graded dimension*, then there is an obvious formula for the graded dimension of its suspension:
$$\dim(\Sigma X) = -\dim(X).$$

(viii) Similarly, if X is a finite CW complex, then we have an analogous formula for the Euler characteristic of its suspension:

$$\chi(\Sigma X) = -\chi(X).$$

(ix) If G is a finite group and X a finite G-set, then we have the orbit-counting theorem (a.k.a. Burnside's lemma or the Cauchy-Frobenius lemma) for the cardinality of its quotient:

$$\#(X/G) = \frac{1}{\#G} \sum_{g \in G} \#(X^g).$$

Here X^g is the set of fixed points of $g \in G$ acting on X.

(x) If $e \colon X \to X$ is an idempotent linear operator on a finite-dimensional vector space (i.e. a projection), then the dimension of its cokernel is equal to its trace:

$$\dim(\mathrm{cok}(e)) = \mathrm{tr}(e).$$

(xi) The cardinality (or Euler characteristic) of the empty set is zero:

$$\#\emptyset = \chi(\emptyset) = 0$$

as is the dimension of the zero vector space:

$$\dim(0) = 0.$$

In all cases, the formulas have a common shape: the size of a colimit is expressed as a *linear combination* of the sizes of its inputs (or other related trace-like invariants). The first general theory of such formulas was described by Leinster [Lei08]: he showed that if A is a finite category that admits a *weighting*, which is a function $k \colon \mathrm{ob}(A) \to \mathbb{Q}$ satisfying certain properties, then the formula

$$\# \mathrm{colim}(X) = \sum_a k_a \cdot \# X_a$$

holds whenever $X \colon A \to \mathrm{Set}$ is a finite coproduct of representables. This includes examples (i), (iv), the special case of (ix) when the action is free, and a similar special case of (x). However, it applies only to finite sets, thus excluding the algebraic or homotopical examples; nor does it deal with the case of non-free actions.

Our original motivation to study this question came from a generalization of (vi) to a statement about *Lefschetz numbers*. In fact, all of the above formulas can be similarly generalized to become statements about a trace-like invariant of an endomorphism, which reduce to the previous statements in the case of identity maps. Specifically:

- For an endomorphism $f \colon X \to X$ of a finite set, we can consider the cardinality $\#\mathrm{Fix}(f)$ of the set of fixed points of f. When $f = \mathrm{id}_X$ this reduces to $\#X$.

- For an endomorphism $f\colon X \to X$ of a finite-dimensional vector space, we can consider its *trace* in the usual sense. When $f = \mathrm{id}_X$ this reduces to $\dim(X)$.

- For an endomorphism $f\colon X \to X$ of a finite-dimensional manifold, we can consider its *Lefschetz number*. When $f = \mathrm{id}_X$ this reduces to $\chi(X)$.

All of the above formulas remain true if cardinalities, dimensions, and Euler characteristics are replaced by fixed-point counts, traces, and Lefschetz numbers. More specifically, given a *natural* endomorphism of a diagram, there is an induced endomorphism of its colimit, and we have formulas calculating trace-like invariants of the latter in terms of the corresponding trace-like invariants of the objects in the diagram. For example:

- If $X \hookrightarrow Y$ and $X \hookrightarrow Z$ are injections of finite sets and we have endofunctions $f\colon Y \to Y$ and $g\colon Z \to Z$ which agree and induce an endomorphism when restricted to X, then there is an induced endofunction $h\colon Y +_X Z \to Y +_X Z$, and we have

$$\#\mathrm{Fix}(h) = \#\mathrm{Fix}(f) + \#\mathrm{Fix}(g) - \#\mathrm{Fix}(f|_X).$$

- If e is an idempotent linear operator on a finite-dimensional vector space X, and $f\colon X \to X$ is any linear operator that commutes with e, then there is an induced operator $g\colon \mathrm{cok}(e) \to \mathrm{cok}(e)$ and we have

$$\mathrm{tr}(g) = \mathrm{tr}(e \circ f).$$

- If $X \hookrightarrow Y$ is an inclusion of finite CW complexes, and $f\colon Y \to Y$ is an endomorphism such that $f(X) \subseteq X$, then there is an induced endomorphism f/X of the quotient Y/X, and we have

$$L(f/X) = L(f) - L(f|_X) \tag{1.1}$$

  where L denotes the Lefschetz number.

Eq. (1.1) is better known when written in the following way:

$$L(f) = L(f|_X) + L(f/X) \tag{1.2}$$

In this form it is known as the *additivity* of the Lefschetz number.

In [May01], May gave a very general proof of (1.2), using the fact that the Lefschetz number is an instance of an abstract notion of *trace* that can be defined for an endomorphism of a *dualizable object* in any symmetric monoidal category. All of the above "size-like" and "trace-like" invariants can be put into this framework, sometimes by first mapping them into another category. Namely:

- A vector space is dualizable just when it is finite-dimensional, and in that case the categorical trace of an endomorphism is precisely the classical trace.

- If a space X is a finite CW complex, then its *suspension spectrum* is dualizable in the stable homotopy category, and in that case the categorical trace of an endomorphism is precisely the Lefschetz number.

- A finite set can either be regarded as a finite CW complex and mapped into the stable homotopy category, or else regarded as the basis of a vector space. In either case, the resulting categorical trace gives precisely the number of fixed points of an endofunction.

Certain properties of this abstract categorical trace are well-known and easy to prove. For instance, if the monoidal category is semi-additive (i.e. finite products and coproducts coincide naturally), then the trace is additive on direct sums; this implies (i), (ii), and (iii), and a nullary version of it implies (xi). The trace is also *cyclic*; this fairly easily implies (x).

May showed an analogous, but more complicated, general result: if the symmetric monoidal category is *triangulated* in a way compatible with its monoidal structure, then the categorical trace is additive along distinguished triangles, in the sense of (1.2). This implies (vi) and (v), and thereby (iv). (It is also fairly easy to see that May's axioms for compatibility between a triangulation and a monoidal structure imply (vii) and (viii).)

Our original motivation was a desire to extend May's result to an additivity theorem for the *Reidemeister trace*, a fixed-point invariant that refines the Lefschetz number. Unlike the Lefschetz number, the Reidemeister trace is not a categorical trace in a symmetric monoidal category, but it is an instance of a more general kind of abstract trace that takes place in a *bicategory* [Pon10, PS13]. We found that the most natural way to do this was to set up a general theory that applies to colimits of potentially arbitrary shapes, and indeed potentially arbitrary *weights*, which turns out to include *all* the above examples.

Recall that in enriched category theory, we consider not just ordinary colimits but *weighted* colimits: if A is a small category describing the shape of our diagram and $\mathbf{V}$ is a symmetric monoidal category, then a **weight** is a functor $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$. (Ordinary "unweighted" colimits are the special case when Φ is constant at the unit object.) Such a weight Φ is said to be **absolute** if Φ-weighted colimits are preserved by every $\mathbf{V}$-functor; for instance, finite coproducts are absolute in vector spaces. The simplest case of our general theorem is then:

1.3. Theorem. *Let $\mathbf{V}$ be a closed, cocomplete, semi-additive, symmetric monoidal category. If A is a finite category, $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ is absolute, and $X\colon A \to \mathbf{V}$ is a diagram such that each X_a is dualizable, then the weighted colimit $\mathrm{colim}^{\Phi}(X)$ is also dualizable, and we have a formula for its formal Euler characteristic (the trace of its identity map):*

$$\chi(\mathrm{colim}^{\Phi}(X)) = \sum_{[\alpha]} \phi_{[\alpha]} \operatorname{tr}(X_\alpha)$$

More generally, for any endo-natural-transformation $f\colon X \to X$ of such an X, we have a similar formula for its trace:

$$\operatorname{tr}(\mathrm{colim}^{\Phi}(f)) = \sum_{[\alpha]} \phi_{[\alpha]} \operatorname{tr}(X_\alpha \circ f_a).$$

We call this theorem a *linearity formula*, because it expresses the trace associated to the colimit as a *linear combination* of traces associated to the input diagram. The sum is indexed by "conjugacy classes" of endomorphisms $\alpha\colon a \to a$ in the category A (we will define these later). In most of the above examples, the only endomorphisms are identities, so it reduces to a sum over objects of A. In particular, Theorem 1.3 has the following specializations.

- If $\mathbf{V}$ is pointed with zero object 0, then 0 is dualizable and $\chi(0) = 0$, giving example (xi).

- If $\mathbf{V}$ is semi-additive and X and Y are dualizable, then $\chi(X \oplus Y) = \chi(X) + \chi(Y)$. This implies examples (i), (ii), and (iii).

- In any $\mathbf{V}$, if X is dualizable and $e : X \to X$ is idempotent, then $\chi(\mathrm{cok}(e)) = \mathrm{tr}(e)$, giving example (x). Here e itself serves as the only relevant "conjugacy class".

- If $\mathbf{V}$ is semi-additive, and X is dualizable with an action of a finite group G whose cardinality is invertible in $\mathbf{V}$ (e.g. if $\mathbf{V}$ is rational vector spaces), then

$$\chi(X/G) = \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}(X(g)).$$

  This implies example (ix). Here the "conjugacy classes" are ordinary conjugacy classes in G.

All of these apply also to traces of nonidentity morphisms.

However, Theorem 1.3 does not apply as stated to the homotopical examples, including (vi) and the motivating case (1.1), since homotopy colimits are not particular weighted colimits.[1] We need a version of it that applies to a "natively homotopical" context, and for this we find it most convenient to use *derivators*. A derivator is an enhancement of a homotopy category with just enough information to determine homotopy limits and colimits by universal properties, which is exactly what we need for this theorem. Derivators are often also easier to work with for formal results of this sort than other models of homotopy theory, such as model categories or $(\infty, 1)$-categories.

Thus, after proving Theorem 1.3 as stated, we prove an analogous theorem for closed symmetric monoidal derivators.

1.4. THEOREM. *Let $\mathscr{V}$ be a closed, semi-additive, symmetric monoidal derivator. If A is a finite category, $\Phi\colon A^{\mathrm{op}} \to \mathscr{V}$ is absolute and has a coefficient decomposition, and $X\colon A \to \mathscr{V}$ is a diagram such that each X_a is dualizable, then the weighted colimit $\mathrm{colim}^{\Phi}(X)$ is also dualizable, and for any endomorphism $f\colon X \to X$ we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \sum_{[\alpha]} \phi_{[\alpha]} \, \mathrm{tr}(X_\alpha \circ f_a).$$

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[1]They can be calculated in examples *using* certain weighted colimits, but the relevant weights are not absolute, and the "dualizability" is also only up to homotopy.

As before, the sum is again over conjugacy classes in A; the condition that Φ "has a coefficient decomposition" is technical and practically always satisfied. Theorem 1.4 has the following specializations:

- All the examples of Theorem 1.3 mentioned above also apply to derivators.

- If $\mathscr{V}$ is stable and $i : X \to Y$ is a map between dualizable objects, then its cofiber object $\mathsf{C}(i)$ is also dualizable, and $\chi(\mathsf{C}(i)) = \chi(Y) - \chi(X)$. This gives example (vi), while the generalization to traces gives (1.1).

- If $\mathscr{V}$ is stable and $Y \leftarrow X \to Z$ is a span of dualizable objects, then its homotopy pushout $Y +_X^h Z$ is dualizable, and $\chi(Y +_X^h Z) = \chi(Y) + \chi(Z) - \chi(X)$. This implies examples (v) and (iv).

- More generally, if $\mathscr{V}$ is stable and $X : A \to \mathscr{V}$ is *any* diagram with A "homotopy finite" (see §7) and each X_a dualizable, then $\mathrm{colim}(X)$ is dualizable, and we have

$$\chi(\mathrm{colim}(X)) = \sum_a \chi(X_a) \cdot \sum_{k \geq 0} (-1)^k \cdot \# \left\{ \begin{array}{c} \text{composable strings of nonidentity} \\ \text{arrows of length } k \text{ starting at } a \end{array} \right\} \quad (1.5)$$

- If $\mathscr{V}$ is stable and rational, and $X : A \to \mathscr{V}$ is a diagram with each X_a dualizable and A a finite EI-category (i.e. every endomorphism is an isomorphism), then $\mathrm{colim}(X)$ is dualizable, and we have

$$\chi(\mathrm{colim}(X)) = \sum_{[a]} \sum_C \chi(X_C) \cdot \sum_k (-1)^k \sum_{[\vec{\alpha}]} \sum_{\vec{C}} \frac{\# \vec{C}}{\# \mathrm{Aut}(\vec{\alpha})} \quad (1.6)$$

where $\vec{\alpha}$ ranges over composable strings of noninvertible arrows of length k starting at a, C ranges over conjugacy classes in $\mathrm{Aut}_A(a)$, and $\vec{C}$ ranges over conjugacy classes of "automorphisms of $\vec{\alpha}$" (see §9) restricting at a to C.

As before, all of these also apply to traces of nonidentity morphisms.

These formulas also appear in the literature in various forms. As mentioned before, (1.1) was proven abstractly by [May01] for monoidal homotopy categories arising from a model structure, and then again in [GPS14a] for stable monoidal derivators, using essentially the same method as May. Our proof uses the basic definitions relating to monoidal derivators from [GPS14a], but the underlying idea of the proof is quite different from that of [May01] — and much more general, since it applies to colimits other than just cofibers.

On the other hand, when applied to diagrams of sets (via their suspension spectra), the formula (1.5) reproduces a large subclass of the formulas for cardinalities of colimits from [Lei08]. (Curiously, however, there are some examples to which both our theory and Leinster's apply, but yield different formulas.)

Finally, while this paper was in preparation, Gallauer [GAdS14] independently obtained a formula equivalent to (1.6) by other methods. His approach relies on many

explicit computations in derivators, while we use categorical abstraction to package such computations into conceptual facts. (Much of this packaging was already done in [Gro13, GPS14b, GPS14a]; what remains is mostly isolated in §15 of this paper.) We expect that it would be possible to reduce both approaches to similar ideas, but in practice our paper proposes a very different perspective.

Even with Theorem 1.4 under our belts, however, we still have not captured all of the examples of interest. For example, we cannot yet describe the additivity of the Reidemeister trace, since that is a trace in a bicategory (in the sense of [Pon10, PS13]) rather than in a symmetric monoidal category. However, it is completely straightforward to generalize Theorems 1.3 and 1.4 to bicategories and even to *derivator bicategories* (bicategories whose hom-categories are derivators). This is a significant advantage of our approach to additivity over others such as [May01] and [GAdS14]. In the end, our most general linearity formula is the following.

1.7. THEOREM. *Let $\mathscr{W}$ be a closed, locally semi-additive, derivator bicategory. Let R and S be objects of $\mathscr{W}$, let A be a finite category, let $\Phi\colon A^{\mathrm{op}} \to \mathscr{W}(R,R)$ be absolute and have a coefficient decomposition, and let $X\colon A \to \mathscr{W}(R,S)$ be a diagram such that each X_a is a dualizable 1-cell. Then the weighted colimit $\mathrm{colim}^{\Phi}(X)$ is also a dualizable 1-cell, and for any endomorphism $f\colon X \to X$ we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \sum_{[\alpha]} \phi_{[\alpha]} \, \mathrm{tr}(X_\alpha \circ f_a).$$

Note that in the bicategorical case, our colimits are "local colimits" in a hom-category (or hom-derivator) $\mathscr{W}(R,S)$, while the "weight" Φ is a diagram of 1-cells. We recover "unweighted" colimits by taking Φ to be constant at the unit 1-cell $\mathbb{I}_R$. All the examples mentioned above generalize directly to the bicategorical context; here are a couple examples to give the idea.

- If $\mathscr{W}$ is locally semi-additive and $X, Y \in \mathscr{W}(R,S)$ are dualizable 1-cells, then $X \oplus Y$ is dualizable, and $\chi(X \oplus Y) = \chi(X) + \chi(Y)$.

- If $\mathscr{W}$ is locally stable and $i\colon X \to Y$ is a morphism of dualizable 1-cells in $\mathscr{W}(R,S)$, then its cofiber is dualizable, and $\chi(\mathsf{C}(i)) = \chi(Y) - \chi(X)$.

In particular, from the second example we can obtain a formula for the Reidemeister trace analogous to (1.1): for $i\colon X \hookrightarrow Y$ an inclusion of dualizable spaces and $f\colon Y \to Y$ such that $f(X) \subseteq X$, we have
$$R(f) - i(R(f|_X)) = R_{Y|X}(f),$$
where $R(f)$ is the Reidemeister trace of f and $R_{Y|X}(f)$ is the "relative Reidemeister trace" [Pon11]. However, this application requires a bit of work to construct the relevant derivator bicategory (which is a generalization of the ordinary bicategory of parametrized spectra from [MS06]). Since the focus of this paper is categorical rather than topological, we postpone this work to the companion paper [PS14a], which is logically dependent on this one; and give only a brief sketch of the proof in §11.

The generalization to bicategorical traces has one further advantage: it yields a uniqueness statement for linearity formulas. Namely, if a linearity formula for some type of colimit can be shown to exist (by any method) and is sufficiently general (in particular, it must apply to bicategories as well as monoidal categories), then it *must* arise from Theorem 1.7. This is a satisfying general statement that our approach does not "miss" any linearity formulas.

We now summarize the organization of the paper. Since it is fairly long, we have divided it into four parts, with the intent that the earlier sections should stand on their own and convey the important ideas, while advanced or technical aspects are postponed to later sections.

Part 1 is about linearity formulas in symmetric monoidal categories, and contains all the essential ideas regarding our approach to linearity. We begin in §2 with a review of traces in symmetric monoidal categories and bicategories, including the notion of *shadow* from [Pon10, PS13] that enables the definition of bicategorical trace. Of particular note is the *composition theorem* for bicategorical trace, Theorem 2.6, which is easy to prove formally but directly gives rise to our linearity formulas.

The next two sections §§3–4 treat Theorem 1.3 in the symmetric monoidal case. In §3 we describe the general theorem (with the proof of one technical lemma postponed), then in §4 we show how it applies to a number of examples. Also in §4 we recall the technical tool of *base change objects* (representable profunctors), and use it to prove the missing lemma and construct several more examples, including the orbit-counting theorem.
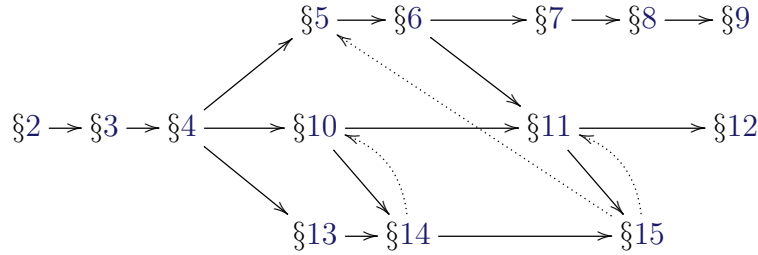
In Part 2, we generalize to monoidal derivators. We begin in §5 with the general theory, with one technical lemma postponed to Part 4. Then in §6 we apply the theory to the main new class of examples: *stable* monoidal derivators (such as classical stable homotopy theory), and the linearity formula (1.1). These two sections are the essence of our theory of linearity for derivators; the rest of Part 2 consists of expanding their application by reducing more complicated colimits in derivators to simpler ones, and could be omitted without loss of continuity. In §7, we obtain a general formula for all *homotopy finite* colimits, which often agrees with Leinster's formula. In §8 we generalize the orbit-counting theorem to derivators, and in §9 we combine these results to obtain formulas for colimits over *EI-categories* in rational stable derivators.

Part 3 is about traces in bicategories. In §10 we describe the theory for ordinary (i.e. non-derivator) bicategorical traces. This section could be read immediately after Part 1, and contains no especially new examples of traces. Then in §11 we introduce derivator bicategories and prove the corresponding linearity theorem. Since this version of the theorem includes all the previous versions as special cases, it is not technically necessary to build up to it in stages. However, it is easier to understand the ideas in simple cases first and then to introduce generalizations one by one. In §12 we prove the uniqueness statement for linearity formulas, establishing that the approach in this paper captures all similar linearity expressions. As remarked previously, the generalization to derivator bicategories is an essential part of this result.

Finally, in Part 4, we discuss base change objects for monoidal derivators and derivator

bicategories. This allows us to complete the identifications of the traces described in Parts 2 and 3. In §13 we describe a general structure for base change objects based on [Shu08]; then we apply it to bicategories in §14 and derivator bicategories in §15.

Shown below is a dependency graph of the paper. Solid arrows denote the order in which sections may be read; dotted arrows denote partial logical dependencies.

$$\S5 \longrightarrow \S6 \longrightarrow \S7 \longrightarrow \S8 \longrightarrow \S9$$

$$\S2 \longrightarrow \S3 \longrightarrow \S4 \longrightarrow \S10 \longleftrightarrow \S11 \longleftrightarrow \S12$$

$$\S13 \longrightarrow \S14 \longrightarrow \S15$$

Part 1: Linearity in monoidal categories

In this first part of the paper, we describe linearity explicitly and concretely in the simplest case: symmetric monoidal categories.

2. Traces in monoidal categories and bicategories

Let $\mathbf{V}$ be a closed symmetric monoidal category with unit object $\mathbb{S}$, monoidal product $\otimes$, and internal hom $\triangleright$. The latter means we have natural isomorphisms

$$\mathbf{V}(X \otimes Y, Z) \cong \mathbf{V}(X, Y \triangleright Z).$$

We refer to the internal-hom $X \triangleright \mathbb{S}$ as the **canonical dual** of X and write it as DX. There is a canonical **evaluation** map $\epsilon \colon DX \otimes X \to \mathbb{S}$, defined by adjunction from the identity of DX. See [DP80, LMSM86].

We say that an object X is **dualizable** if the canonical map

$$\mu_{X,U} \colon U \otimes DX \longrightarrow X \triangleright U \tag{2.1}$$

(whose adjunct is $U \otimes DX \otimes X \xrightarrow{\mathrm{id}_U \otimes \epsilon} U \otimes \mathbb{S} \cong U$) is an isomorphism for all objects U. It is sufficient to require this for $U = X$.

This definition of duality is convenient in concrete examples, but there is an equivalent characterization that tends to be more convenient for studying traces.

2.2. THEOREM. *An object X is dualizable if and only if there is an object Y of $\mathbf{V}$ and morphisms*

$$\mathbb{S} \xrightarrow{\eta} X \otimes Y \ \text{ and } \ Y \otimes X \xrightarrow{\epsilon} \mathbb{S}$$

so that the composites

$$X \cong \mathbb{S} \otimes X \xrightarrow{\eta \otimes \mathrm{id}} X \otimes Y \otimes X \xrightarrow{\mathrm{id} \otimes \epsilon} X \otimes \mathbb{S} \cong X$$

$$Y \cong Y \otimes \mathbb{S} \xrightarrow{\text{id} \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\epsilon \otimes \text{id}} \mathbb{S} \otimes Y \cong Y$$

are identity maps.

SKETCH OF PROOF. If X is dualizable, let $Y = DX$, ϵ the evaluation as above, and η the composite $\mathbb{S} \to X \triangleright X \xleftarrow{\cong} X \otimes DX$. Conversely, by composing with η and ϵ we have natural isomorphisms $\mathbf{V}(Z \otimes X, U) \cong \mathbf{V}(Z, U \otimes Y)$, whence the Yoneda lemma gives $Y \cong DX$ by an isomorphism inducing (2.1). ∎

By analogy with the evaluation ϵ, we call η the **coevaluation**.

Using this characterization, we define the **trace** of an endomorphism $f \colon X \to X$ of a dualizable object to be the composite

$$\mathbb{S} \xrightarrow{\eta} X \otimes DX \xrightarrow{f \otimes \text{id}} X \otimes DX \xrightarrow{\cong} DX \otimes X \xrightarrow{\epsilon} \mathbb{S}.$$

We denote the trace of the identity morphism of X by $\chi(X) = \mathrm{tr}(\mathrm{id}_X)$ and call it the **Euler characteristic** of X.

More generally, we may consider a closed *bicategory* $\mathbf{W}$, with unit objects $\mathbb{I}_B \in \mathbf{W}(B, B)$, bicategorical composition product $\odot$ and internal-homs $\triangleright$ and $\triangleleft$. We write $\odot$ in diagrammatic order, so that if $X \in \mathbf{W}(A, B)$ and $Y \in \mathbf{W}(B, C)$ we have $X \odot Y \in \mathbf{W}(A, C)$, and we orient $\triangleright$ and $\triangleleft$ so that the adjunction isomorphisms preserve cyclic order:

$$\mathbf{W}(A, C)(X \odot Y, Z) \cong \mathbf{W}(A, B)(X, Y \triangleright Z) \cong \mathbf{W}(B, C)(Y, Z \triangleleft X).$$

Any monoidal category can be regarded as a bicategory with only one object. Another important example to keep in mind is the bicategory whose objects are (noncommutative) rings, whose morphisms are bimodules, with $\mathbb{I}_B = {}_B B_B$ and $\odot$ the usual tensor product of bimodules, and $\triangleright$ and $\triangleleft$ the usual hom-bimodules.

In a closed bicategory $\mathbf{W}$, if $X \in \mathbf{W}(A, B)$ is a 1-cell, we refer to the internal-hom $X \triangleright \mathbb{I}_B \in \mathbf{W}(B, A)$ as the **canonical right dual**, written $D_r X$. There is again an evaluation map $\epsilon \colon D_r X \odot X \to \mathbb{I}_B$. We say that X is **right dualizable** if the analogous map

$$\mu_{X,U} \colon U \odot D_r X \longrightarrow X \triangleright U \tag{2.3}$$

is an isomorphism for all 1-cells U. Again, it suffices to require this for $U = X$. As in the symmetric monoidal case, there are numerous other characterizations of dualizability; one will be relevant here.

2.4. THEOREM. *An object $X \in \mathbf{W}(A, B)$ is right dualizable if and only if there is an object $Y \in \mathbf{W}(B, A)$ and morphisms*

$$\mathbb{I}_A \xrightarrow{\eta} X \odot Y \ \text{ and } \ Y \odot X \xrightarrow{\epsilon} \mathbb{I}_B$$

so that the composites

$$X \cong \mathbb{I}_A \odot X \xrightarrow{\eta \odot \text{id}} X \odot Y \odot X \xrightarrow{\text{id} \odot \epsilon} X \odot \mathbb{I}_B \cong X$$

$$Y \cong Y \odot \mathbb{I}_A \xrightarrow{\text{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\epsilon \odot \text{id}} \mathbb{I}_B \odot Y \cong Y$$

are identity maps.

The proof is analogous to the monoidal case and it shows that $Y \cong D_r X$. See for instance [MS06, §16.4].

To define the trace in a symmetric monoidal category we used the symmetry isomorphism. The bicategories we are interested in do not have the same kind of symmetry, but we can introduce similar structure that will allow us to define a trace. A **shadow** for $\mathbf{W}$ is a collection of functors

$$\langle\!\langle - \rangle\!\rangle \colon \mathbf{W}(A, A) \to \mathbf{T}$$

for all objects A of $\mathbf{W}$, where $\mathbf{T}$ is some fixed category, together with natural isomorphisms $\langle\!\langle X \odot Y \rangle\!\rangle \cong \langle\!\langle Y \odot X \rangle\!\rangle$ that are compatible with the unit and associativity isomorphisms of $\mathbf{W}$; see [PS13, Defn. 4.1] for details. For an object A, we write $\langle\!\langle A \rangle\!\rangle = \langle\!\langle \mathbb{I}_A \rangle\!\rangle$. In the example of rings and bimodules, the most natural choice for the shadow of an A-A-bimodule is the quotient abelian group that coequalizes the right and left actions of A. If a monoidal category is *symmetric*, then its identity functor is a shadow for the corresponding one-object bicategory.

If $X \in \mathbf{W}(A, B)$ is right dualizable and $\mathbf{W}$ is equipped with a shadow, then the **trace** of a 2-cell $f \colon X \to X$ is the composite

$$\langle\!\langle A \rangle\!\rangle \xrightarrow{\eta} \langle\!\langle X \odot D_r X \rangle\!\rangle \xrightarrow{f \odot \text{id}} \langle\!\langle X \odot D_r X \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle D_r X \odot X \rangle\!\rangle \xrightarrow{\epsilon} \langle\!\langle B \rangle\!\rangle.$$

This general definition is due to [Pon10] and was studied abstractly in [PS13]. In particular, if A and B are noncommutative rings and X is an A-B-bimodule, then this yields the *Hattori-Stallings trace* of f.

More generally, a *twisted* endomorphism $f \colon Q \odot X \to X \odot P$ also has a trace, defined to be the composite

$$\langle\!\langle Q \rangle\!\rangle \xrightarrow{\eta} \langle\!\langle Q \odot X \odot D_r X \rangle\!\rangle \xrightarrow{f \odot \text{id}} \langle\!\langle X \odot P \odot D_r X \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle D_r X \odot X \odot P \rangle\!\rangle \xrightarrow{\epsilon} \langle\!\langle P \rangle\!\rangle.$$

Originally, traces were only defined for untwisted endomorphisms, but there are many examples where the source and target twisting is essential, such as the Reidemeister trace to be discussed in §11 and [PS14a].

The advantage of formulating traces abstractly in this way is that general theorems become easy to prove in the abstract context, but can reduce to quite nontrivial results in examples. This is the case for our linearity formulas, which follow more or less directly (once the framework is set up correctly) from abstract theorems about compositions of dualizable objects.

For instance, the following theorem is easy to prove, but it can be a source of many dual pairs that would otherwise be nontrivial to construct, as observed in [MS06]. We will also use it in this way, to conclude that colimits of certain shapes are dualizable.

2.5. THEOREM. *If $Y \in \mathbf{W}(B, C)$ and $X \in \mathbf{W}(A, B)$ are right dualizable, then $Y \odot X$ is right dualizable, and we have $D_r(Y \odot X) \cong D_r X \odot D_r Y$.*

In this case, if $g \colon Q \odot Y \to Y \odot P$ and $f \colon P \odot X \to X \odot L$ are two 2-cells, we have the composite
$$(\mathrm{id}_Y \odot f)(g \odot \mathrm{id}_X) \colon Q \odot Y \odot X \longrightarrow Y \odot X \odot L$$
and we can ask about its trace. This can be identified by a straightforward diagram chase.

2.6. THEOREM. [PS13, Prop. 7.5] *In the above situation, we have*

$$\mathrm{tr}\left((\mathrm{id}_Y \odot f)(g \odot \mathrm{id}_X)\right) = \mathrm{tr}(f) \circ \mathrm{tr}(g).$$

Theorem 2.6 is the origin of all our linearity formulas. The basic idea is that given X and f, we choose Y so that $Y \odot X$ is the colimit of X. With $g = \mathrm{id}_Y$, the left-hand side of Theorem 2.6 is then the trace of $\mathrm{colim}(f)$, while the right-hand side expresses it as a composite of a "row vector" with a "column vector", hence a linear combination of the components of the trace of f.

2.7. REMARK. Reflecting our interests here, we will make limited explicit use of twisted traces. There will be none in the first two parts and only target twisting in the later parts. Despite this, many of the results in the paper stated for untwisted or partially twisted traces extend to the case of more general twisting.

3. Linearity in monoidal categories

For this section, let $\mathbf{V}$ be a complete and cocomplete closed symmetric monoidal category, with tensor product $\otimes$, unit object $\mathbb{S}$, and internal-hom $\triangleright$. Then we can construct the following closed bicategory $\mathbf{Prof}(\mathbf{V})$:

- Its objects are small categories A, B, C, ...

- Its 1-cells are $\mathbf{V}$-profunctors (a.k.a. distributors, bimodules, or just "modules"). A $\mathbf{V}$-profunctor $H \colon A \nrightarrow B$ is defined to be a functor $B^{\mathrm{op}} \times A \to \mathbf{V}$.

- Its 2-cells are morphisms of profunctors, i.e. natural transformations.

- The composite of profunctors $H \colon A \nrightarrow B$ and $K \colon B \nrightarrow C$ is the coend

$$(H \odot K)(c, a) = \int^{b \in B} H(b, a) \otimes K(c, b).$$

- The unit 1-cell $\mathbb{I}_A \colon A \nrightarrow A$ consists of copowers of the unit object $\mathbb{S}$ by the homsets of A:

$$\mathbb{I}_A(a, a') = A(a, a') \cdot \mathbb{S}.$$

- The right hom of profunctors $H\colon B \nrightarrow C$ and $K\colon A \nrightarrow C$ is the end

$$(H \rhd K)(b, a) = \int_{c \in C} H(c, b) \rhd K(c, a)$$

  and similarly for the left hom $\lhd$.

- It has a shadow valued in $\mathbf{V}$, defined by

$$\langle\!\langle H \rangle\!\rangle = \int^{a \in A} H(a, a).$$

Let $\mathbb{1}$ denote the terminal category, with one object and one (identity) morphism. Then $\mathbf{V}$-profunctors $A \nrightarrow \mathbb{1}$ are equivalent to $\mathbf{V}$-functors $A \to \mathbf{V}$, while profunctors $\mathbb{1} \nrightarrow A$ are equivalent to $\mathbf{V}$-functors $A^{\mathrm{op}} \to \mathbf{V}$. The latter sort of functor is the one traditionally used as a *weight* for colimits in enriched category theory. In the special case of $\mathbf{V}$ itself, the traditional definition of weighted colimits is equivalent to the following.

3.1. DEFINITION. For functors $X\colon A \to \mathbf{V}$ and $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$, the Φ-**weighted colimit of** X is the composite of profunctors

$$\mathrm{colim}^{\Phi}(X) = \Phi \odot X = \int^{a \in A} \big(\Phi(a) \otimes X(a) \big)$$

regarded as an object of $\mathbf{V}$.

If Φ is constant at the unit object $\mathbb{S}$, then it is easy to identify the Φ-weighted colimit of X with its ordinary colimit. This is the case we generally care most about, but it is conceptually helpful to consider the general case. In particular, as we will see in Example 4.4, including weighted colimits is what unifies "additivity formulas" with "multiplicativity formulas".

3.2. REMARK. In fact, in enriched category theory one additionally considers colimits where the diagram shape A is a $\mathbf{V}$-enriched category; see for instance [Kel82]. The definition of $\mathbf{Prof}(\mathbf{V})$ and everything we do with it can also be generalized to this situation; this follows from the general theorems in Part 4. However, the case of unenriched A suffices for the examples here.

Now by Theorem 2.6, if X and Φ are right dualizable when regarded as profunctors, then $\mathrm{colim}^{\Phi}(X)$ is dualizable in $\mathbf{V}$. To avoid confusion, we introduce new names for profunctor right dualizability of X and Φ, which are of very different sorts.

3.3. DEFINITION.

- A functor $X\colon A \to \mathbf{V}$ is **pointwise dualizable** if it is right dualizable in $\mathbf{Prof}(\mathbf{V})$ when regarded as a profunctor $A \nrightarrow \mathbb{1}$.

- A functor $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ is **absolute** if it is right dualizable in $\mathbf{Prof}(\mathbf{V})$ when regarded as a profunctor $\mathbb{1} \nrightarrow A$.

Then Theorem 2.5 immediately implies:

3.4. THEOREM. *If $X\colon A \to \mathbf{V}$ is pointwise dualizable and $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ is absolute, then* $\mathrm{colim}^{\Phi}(X)$ *is dualizable.*

By analogy with [PS12, PS14b], these notions might also be called *fiberwise dualizable* and *totally dualizable*. However, when thinking of Φ as a weight for colimits, the term "absolute" is common; it refers to the fact that in this case Φ-weighted colimits in any $\mathbf{V}$-category are preserved by any $\mathbf{V}$-functor (see [Str83]). (The word *Cauchy* is also in use, since when metric spaces are regarded as enriched categories as in [Law74], convergence of Cauchy sequences becomes an example.) Similarly, when talking about diagrams (rather than fibrations, as in [PS14b]), the adjective "pointwise" seems more intuitive than "fiberwise". It is further justified by the following result that is closely related to [PS14b, Cor. 4.4] and [GPS14a, Lem. 11.5].

3.5. LEMMA. *A functor $X\colon A \to \mathbf{V}$ is pointwise dualizable if and only if each object $X(a)$ is dualizable in $\mathbf{V}$.*

PROOF. For any $U \in \mathbf{Prof}(\mathbf{V})(B, \mathbb{1})$, $a \in A$, and $b \in B$, the (b, a) component of the morphism $\mu_{X,U}$ from (2.3) is $\mu_{X(a),U(b)}$. But $\mu_{X,U}$ is an isomorphism if and only if all its components are. ∎

Now Theorem 2.6 gives us a formula for traces.

3.6. THEOREM. *If $X\colon A \to \mathbf{V}$ is pointwise dualizable and $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ is absolute, then for any $f\colon X \to X$, we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \mathrm{tr}(f) \circ \mathrm{tr}(\mathrm{id}_{\Phi}). \tag{3.7}$$

PROOF. Identify $\mathrm{colim}^{\Phi}(f)$ with $(\mathrm{id}_{\Phi} \odot f)\colon \Phi \odot X \to \Phi \odot X$ and then apply Theorem 2.6. ∎

Note that the shadow of $\mathbb{1}$ is $\mathbb{S}$; thus this theorem asserts that the trace of $\mathrm{colim}^{\Phi}(f)$ is a composite $\mathbb{S} \xrightarrow{\mathrm{tr}(\mathrm{id}_{\Phi})} \langle\!\langle A \rangle\!\rangle \xrightarrow{\mathrm{tr}(f)} \mathbb{S}$. In order for this to be useful, we need to be able to calculate $\mathrm{tr}(f)$ and $\mathrm{tr}(\mathrm{id}_{\Phi})$ more concretely. We start with a description of $\langle\!\langle A \rangle\!\rangle$. Since colimits commute with colimits (including copowers), we have

$$\langle\!\langle A \rangle\!\rangle = \int^{a \in A} \left(A(a, a) \cdot \mathbb{S} \right) \cong \left(\int^{a \in A} A(a, a) \right) \cdot \mathbb{S}.$$

The set $\int^{a \in A} A(a, a)$ is the disjoint union of all the endomorphism sets $A(a, a)$, quotiented by the relation $\alpha \circ \beta \sim \beta \circ \alpha$ for any α, β (which need not be endomorphisms themselves). We call this relation *conjugacy*, since when A is a group regarded as a one-object category, it becomes precisely the relation of conjugacy, and $\int^{a \in A} A(a, a)$ is the set of conjugacy classes.

Thus, $\langle\!\langle A \rangle\!\rangle$ is the copower of $\mathbb{S}$ by the set of conjugacy classes of A, and we have coprojections

$$A(a, a) \cdot \mathbb{S} \longrightarrow \langle\!\langle A \rangle\!\rangle$$

that send the copy of $\mathbb{S}$ in the domain corresponding to each $\alpha \in A(a, a)$ to the copy in the codomain corresponding to its conjugacy class. Since these coprojections are jointly epimorphic, $\mathrm{tr}(f)$ is determined by one morphism $\mathrm{tr}(f)_{[\alpha]} \colon \mathbb{S} \to \mathbb{S}$ for each conjugacy class $[\alpha]$ of A. The following lemma identifies these morphisms.

3.8. LEMMA. [The component lemma for symmetric monoidal categories] *For any morphism* $\alpha \in A(a, a)$, $\mathrm{tr}(f)_{[\alpha]}$ *is the trace of the composite*

$$X(a) \xrightarrow{X_\alpha} X(a) \xrightarrow{f_a} X(a) \ .$$

Since the trace is invariant under cyclic permutation, any representative for the conjugacy class can be chosen to compute this trace.

PROOF. We will prove this in §4 on page 617.   ∎

Thus, we have a complete computation of $\mathrm{tr}(f)$ for any endomorphism f of a pointwise dualizable $X \colon A \to \mathbf{V}$.

The other ingredient in (3.7) is $\mathrm{tr}(\mathrm{id}_\Phi)$. This is a morphism $\mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ which depends on Φ; we call it the **coefficient vector** of Φ. This name is inspired by the case of most interest: when A is finite (or, slightly more generally, has only finitely many conjugacy classes) and $\mathbf{V}$ is **semi-additive** (i.e. finite products and coproducts coincide naturally, and are called *biproducts* or *direct sums* and written as $X \oplus Y$). In this case, $\langle\!\langle A \rangle\!\rangle$ is a direct sum of copies of $\mathbb{S}$ indexed by the conjugacy classes of A, and so $\mathrm{tr}(\mathrm{id}_\Phi)$ really is a "column vector" whose entries are morphisms $\mathbb{S} \to \mathbb{S}$. (Note that when $\mathbf{V}$ is semi-additive, $\mathbf{V}(\mathbb{S}, \mathbb{S})$ is a commutative semiring which acts on every homset of $\mathbf{V}$.) We denote the entries of this column vector by $\phi_{[\alpha]}$, and call them the **coefficients** of Φ.

Similarly, in this case $\mathrm{tr}(f)$ is a "row vector" whose entries are the traces $\mathrm{tr}(f_a \circ X_\alpha)$, giving Theorem 3.6 a more familiar form.

3.9. COROLLARY. *If* $\mathbf{V}$ *is semi-additive,* A *has finitely many conjugacy classes, and* $\Phi \colon A^{\mathrm{op}} \to \mathbf{V}$ *is absolute, then we have*

$$\mathrm{tr}(\mathrm{colim}^\Phi(f)) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot \mathrm{tr}\left(f_a \circ X_\alpha\right). \tag{3.10}$$

for any pointwise dualizable $X \colon A \to \mathbf{V}$ *and* $f \colon X \to X$.

This is the origin of our term **linearity formula**. In particular, when f is the identity morphism, we have

$$\chi(\mathrm{colim}^\Phi(X)) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot \mathrm{tr}\left(X_\alpha\right).$$

4. Examples

In order to obtain concrete examples, we need to identify some absolute weights and calculate their coefficient vectors. There is no general way to do this: the question of which weights are absolute depends heavily on what $\mathbf{V}$ we choose, even for simple categories A.

Most of the examples we can describe at this point are fairly trivial, in the sense that their linearity formula can be proven easily in more direct ways. Thus, while it is satisfying to have a general theory, it may seem at this point that it doesn't buy us very much. This is largely true in the non-homotopical case, although in Examples 4.2 and 4.12 we get a little simplification, amounting to the fact that it suffices to prove the linearity formula in a few particularly simple cases. This will also be true in the homotopical examples to be considered in §5 and beyond, but in that case it is a much bigger win.

4.1. EXAMPLE. Let A be the empty category, and $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ the unique functor; then Φ-weighted colimits are initial objects. For any category B, there is a unique profunctor $U\colon B \nrightarrow A$, and the map $\mu_{\Phi,U}\colon U \odot D_r\Phi \to \Phi \triangleright U$ is the unique map from the initial to the terminal object of $\mathbf{Prof}(\mathbf{V})(B, \mathbb{1})$. To say that this is an isomorphism when $B = \mathbb{1}$ is by definition to say that $\mathbf{V}$ is *pointed*, and this in turn implies the corresponding statement for general B. When $\mathbf{V}$ is pointed, its joint initial and terminal object is called the *zero object* and denoted 0.

Thus, this Φ is absolute just when $\mathbf{V}$ is pointed. There is a unique functor $X\colon A \to \mathbf{V}$, which is trivially pointwise dualizable; hence its colimit, which is the zero object of $\mathbf{V}$, is dualizable. Finally, the shadow of A is the zero object 0, so the trace of the unique endomorphism of 0 is the composite $\mathbb{S} \to 0 \to \mathbb{S}$, i.e. the zero endomorphism of $\mathbb{S}$. It is quite trivial to prove all this directly, but it serves as a good beginning example to see the general theory working.

4.2. EXAMPLE. Let A be the discrete category with two objects a and b. Then a diagram $X\colon A \to \mathbf{V}$ consists of a pair of objects X_a and X_b, and is pointwise dualizable just when X_a and X_b are dualizable.

Let $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ be constant at $\mathbb{S}$. Then $\mathrm{colim}^{\Phi}(X) = X_a + X_b$, i.e. Φ-weighted colimits are binary coproducts. Now the right dual $D_r\Phi$ is given by

$$
\begin{aligned}
(D_r\Phi)_a &= \int_x \Phi(x) \triangleright \mathbb{I}_A(x, a) \\
&\cong \Big(\Phi(a) \triangleright \mathbb{I}_A(a, a)\Big) \times \Big(\Phi(b) \triangleright \mathbb{I}_A(b, a)\Big) \\
&\cong \Big(\mathbb{S} \triangleright \mathbb{S}\Big) \times \Big(\mathbb{S} \triangleright \emptyset\Big) \\
&\cong \mathbb{S} \times \emptyset
\end{aligned}
$$

and similarly $(D_r\Phi)_b \cong \emptyset \times \mathbb{S}$, where $\emptyset$ is the initial object of $\mathbf{V}$.

If $\mathbf{V}$ is pointed, then $\emptyset = 0$ and $\mathbb{S} \times 0 \cong \mathbb{S}$, so that $D_r\Phi$ is also constant at $\mathbb{S}$. Thus, for $U \in \mathbf{Prof}(\mathbf{V})(C, A)$ we have

$$(U \odot D_r\Phi)_c = U_{c,a} + U_{c,b} \qquad \text{and}$$
$$(\Phi \triangleright U)_c = U_{c,a} \times U_{c,b}$$

while $\mu_{\Phi,U}$ is the canonical morphism

$$U_{c,a} + U_{c,b} \longrightarrow U_{c,a} \times U_{c,b}$$

whose components $U_{c,a} \to U_{c,a}$ and $U_{c,b} \to U_{c,b}$ are the identity and whose components $U_{c,a} \to U_{c,b}$ and $U_{c,b} \to U_{c,a}$ are zero morphisms. To say that this is an isomorphism when $C = \mathbb{1}$ is by definition to say that $\mathbf{V}$ is semi-additive, and this in turn implies the corresponding statement for general C.

Thus, when $\mathbf{V}$ is semi-additive, this Φ is absolute, and so binary coproducts of dualizable objects are dualizable. We have $\langle\!\langle A \rangle\!\rangle \cong \mathbb{S} \oplus \mathbb{S}$, and for a pointwise dualizable X and $f \colon X \to X$, Lemma 3.8 implies that $\mathrm{tr}(f) \colon \mathbb{S} \oplus \mathbb{S} \longrightarrow \mathbb{S}$ is the row vector composed of $\mathrm{tr}(f_a)$ and $\mathrm{tr}(f_b)$. Thus, we have

$$\mathrm{tr}(f_a \oplus f_b) = \phi_a \cdot \mathrm{tr}(f_a) + \phi_b \cdot \mathrm{tr}(f_b)$$

for some $\phi_a, \phi_b \in \mathbf{V}(\mathbb{S}, \mathbb{S})$. Knowing that such ϕ_a and ϕ_b exist, and are the same for all X and f, enables us to calculate them easily. Namely, let $X_a = 0$ and $X_b = \mathbb{S}$ and let f be the identity. Then $\mathrm{tr}(f_a) = 0$ by the previous example, and $\mathrm{tr}(f_b) = 1$ since it is the identity; while $X_a \oplus X_b \cong \mathbb{S}$ and $f_a \oplus f_b = 1$, so that $\mathrm{tr}(\mathrm{colim}^\Phi f) = 1$ as well. Thus, $1 = \phi_a \cdot 0 + \phi_b \cdot 1$, so $\phi_b = 1$. Similarly, $\phi_a = 1$, so our linearity formula is

$$\mathrm{tr}(f_a \oplus f_b) = \mathrm{tr}(f_a) + \mathrm{tr}(f_b).$$

As before, of course, it is fairly easy to prove this directly.

4.3. EXAMPLE. The formal analysis of Example 4.2 applies equally well when A is *any* discrete category. Semi-additivity of $\mathbf{V}$ again implies that all *finite* coproducts are absolute, with an analogous linearity formula. Examples of $\mathbf{V}$ for which *infinite* coproducts are absolute arise somewhat more rarely, but they do exist. For instance, if $\mathbf{V}$ is the category of suplattices (i.e. posets with all suprema, and supremum-preserving functions), then coproducts of arbitrary cardinality are absolute, and we have an analogous linearity formula:

$$\mathrm{tr}\left(\bigoplus_a f_a\right) = \sum_a \mathrm{tr}(f_a).$$

In this case, $\mathbb{S}$ is the two-element lattice, and $\mathbf{V}(\mathbb{S}, \mathbb{S})$ is a two-element set, while sums of morphisms are pointwise suprema. Thus, traces carry very little information. Informally, while traces in the additive case "count" fixed points, traces in the suplattice case merely record whether any fixed point exists.

4.4. EXAMPLE. Let $A = \mathbb{1}$ be the terminal category. Then $X\colon A \to \mathbf{V}$ is just an object of $\mathbf{V}$, and is pointwise dualizable just when that object is dualizable. Similarly, $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ is also just an object of $\mathbf{V}$, and is absolute just when that object is dualizable, while $\mathrm{colim}^{\Phi}(X)$ is just the tensor product $\Phi \otimes X$.

The shadow of $\mathbb{1}$ is just the unit object $\mathbb{S}$, and the trace of $f\colon X \to X$ is just its ordinary trace in $\mathbf{V}$. The trace of id_{Φ} is also obviously just its trace in $\mathbf{V}$, giving the unique coefficient ϕ. Thus the linearity formula reduces to

$$\mathrm{tr}(\Phi \otimes f) = \mathrm{tr}(f) \circ \mathrm{tr}(\mathrm{id}_{\Phi}),$$

which is (a special case of) the usual *multiplicativity* formula for traces, [PS14b]. Thus we see that *linearity* includes both *additivity* (as the special case when all coefficients are 1) and *multiplicativity* (as the special case when there is only one term). The "twisted multiplicativity" of [PS14b] is also a sort of linearity, but using a more complicated bicategory.

4.5. EXAMPLE. Let $\mathbf{V}$ be the category of $\mathbb{Z}$-graded objects in an additive symmetric monoidal category $\mathbf{U}$, with the usual tensor product:

$$(X \otimes Y)_n = \bigoplus_{k+m=n} X_k \otimes Y_m$$

and the symmetry isomorphism that maps $X_k \otimes Y_m$ to $Y_m \otimes X_k$ by $(-1)^{km}$. Let S^n denote the graded object that is the unit object $\mathbb{S}$ of $\mathbf{U}$ in degree n and 0 in all other degrees. Then S^n is dualizable (indeed, invertible) with dual S^{-n}, and inspecting the definition of trace yields $\mathrm{tr}(\mathrm{id}_{S^n}) = (-1)^n$.

Hence, Example 4.4 implies that if X is dualizable, so is $S^1 \otimes X$, and the trace of $S^1 \otimes f$ is the negative of the trace of f. Of course, $S^1 \otimes X$ is just the "suspension" of X, with $(S^1 \otimes X)_n = X_{n-1}$. A similar argument applies to chain complexes in an additive symmetric monoidal category.

Before we give more examples of Theorem 3.6 we introduce some important general examples of dualizable profunctors. For any profunctor $H\colon B \nrightarrow D$ and any functors $f\colon A \to B$ and $g\colon C \to D$, we have an induced profunctor $H(g,f)\colon A \nrightarrow C$ defined by

$$(H(g,f))(c,a) = H(gc, fa).$$

In particular, taking H to be the identity profunctor $\mathbb{I}_B$ and g to be the identity functor, we have a profunctor $\mathbb{I}_B(\mathrm{id}_B, f)\colon A \nrightarrow B$, which we generally denote by $B(\mathrm{id}, f)$. Similarly, we have $B(f, \mathrm{id})\colon B \nrightarrow A$; these two are defined by

$$(B(\mathrm{id}, f))(b,a) = B(b, fa) \cdot \mathbb{S} \qquad \text{and} \qquad (B(f, \mathrm{id}))(a,b) = B(fa, b) \cdot \mathbb{S}.$$

These are called **representable profunctors** or **base change objects**. The following facts about them are well-known and easy to prove. Abstractly, they say that $\mathbf{Prof}(\mathbf{V})$ is a *proarrow equipment* [Woo82] or a *framed bicategory* [Shu08]; we will return to this point of view in §13.

4.6. PROPOSITION. $B(\mathrm{id}, f)$ *is right dualizable, with right dual* $B(f, \mathrm{id})$. *The evaluation has components*

$$\int^{a\in A} (B(fa, b) \cdot \mathbb{S}) \otimes (B(b', fa) \cdot \mathbb{S}) \longrightarrow B(b', b) \cdot \mathbb{S}$$

given by composition in B, *while the coevaluation has components*

$$A(a, a') \cdot \mathbb{S} \longrightarrow B(fa, fa') \cdot \mathbb{S} \cong \int^{b\in B} (B(b, fa') \cdot \mathbb{S}) \otimes (B(fa, b) \cdot \mathbb{S})$$

given by the action of f *on arrows.*

4.7. PROPOSITION. *For any* $\mathbf{V}$-*profunctor* $H \colon B \nrightarrow D$ *and functors* $f \colon A \to B$ *and* $g \colon C \to D$, *we have*

$$H(\mathrm{id}, f) \cong B(\mathrm{id}, f) \odot H$$
$$H(g, \mathrm{id}) \cong H \odot D(g, \mathrm{id}).$$

We can also explicitly calculate traces with respect to this dual pair.

4.8. PROPOSITION. *If* $\nu \colon f \to f$ *is a natural transformation, then the trace of the induced endomorphism* $\nu \colon B(\mathrm{id}, f) \to B(\mathrm{id}, f)$ *is the map*

$$\langle\!\langle A \rangle\!\rangle = \int^{a\in A} A(a, a) \cdot \mathbb{S} \longrightarrow \int^{b\in B} B(b, b) \cdot \mathbb{S} = \langle\!\langle B \rangle\!\rangle$$

induced by the diagonals of the following commutative squares:

$$
\begin{array}{ccc}
A(a, a) & \xrightarrow{\ \nu_a \times f\ } & B(fa, fa) \times B(fa, fa) \\
{\scriptstyle f \times \nu_a}\downarrow & & \downarrow{\scriptstyle \mathrm{comp}} \\
B(fa, fa) \times B(fa, fa) & \xrightarrow[\mathrm{comp}]{} & B(fa, fa)
\end{array}
$$

In particular, the trace of the identity map $\mathrm{id}_{B(\mathrm{id}, f)}$ *is induced by the maps* $f \colon A(a, a) \to B(fa, fa)$.

PROOF. By inspection of the definition of traces and the description of the evaluation and coevaluation in Proposition 4.6. ∎

4.9. EXAMPLE. For any $a \in A$ there is a functor $a \colon \mathbb{1} \to A$ and the profunctor $A(\mathrm{id}, a)$ is an absolute weight. By Proposition 4.7, the colimit of $X \colon A \to \mathbf{V}$ weighted by $A(\mathrm{id}, a)$ is just $X(a)$, which is dualizable whenever X is pointwise dualizable. By Proposition 4.8, the coefficient vector of $A(\mathrm{id}, a)$ is the map $\mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ induced by the identity morphism of a, so that the linearity formula becomes the obvious fact that $\mathrm{tr}(f_a) = \mathrm{tr}(f_a)$.

We can obtain less trivial examples by invoking the following easy fact.

4.10. PROPOSITION. *Any retract of a right dualizable 1-cell in a closed bicategory is again right dualizable. That is, if $X, Y \in \mathbf{W}(A, B)$ and we have $r\colon X \to Y$ and $s\colon Y \to X$ with $rs = \mathrm{id}_Y$, and X is right dualizable, then so is Y. Moreover, if $\mathbf{W}$ has a shadow, then the trace of any $f\colon Y \to Y$ is equal to the trace of $sfr\colon X \to X$.*

PROOF. If Y is a retract of X, then $\mu_{Y,U}$ from (2.3) is a retract of $\mu_{X,U}$ for any U, and a retract of an isomorphism is an isomorphism. The statement about traces follows from cyclicity ([PS13, Corollary 7.3]), since $\mathrm{tr}(f) = \mathrm{tr}(frs) = \mathrm{tr}(sfr)$. ∎

This immediately gives rise to the following somewhat tautological example.

4.11. EXAMPLE. Let A be the category generated by a single object a and a single idempotent $\alpha\colon a \to a$. Then a functor $A \to \mathbf{V}$ consists of an object X together with an idempotent $e\colon X \to X$, and is pointwise dualizable just when X is dualizable. Similarly, a functor $\Phi\colon A^{\mathrm{op}} \to \mathbf{V}$ is also just an object with an idempotent. If we take Φ to be the unit object $\mathbb{S}$ with the identity idempotent, then a Φ-weighted colimit of (X, e) is a splitting of the idempotent e.

To see Φ is absolute, first observe that the representable profunctor $A(\mathrm{id}, a)\colon \mathbb{1} \nrightarrow A$ is absolute by Proposition 4.6. Concretely, the value of $A(\mathrm{id}, a)$ on the single object is the coproduct $\mathbb{S} + \mathbb{S}$, equipped with the idempotent induced by α, which is the fold map followed by the first coprojection. The splitting of this idempotent is precisely our weight Φ, so by Proposition 4.10, Φ is also absolute. Moreover, its coefficient vector $\mathrm{tr}(\mathrm{id}_\Phi)$ is the trace of the idempotent $\alpha\colon A(\mathrm{id}, a) \to A(\mathrm{id}, a)$. Since A has two conjugacy classes, the identity and the idempotent, this coefficient vector is a morphism

$$\phi\colon \mathbb{S} \to \{\mathrm{id}_a, \alpha\} \cdot \mathbb{S} \cong \mathbb{S} + \mathbb{S}.$$

By Proposition 4.8, this map is induced by the action of the functor $a\colon \mathbb{1} \to A$ (which yields the first coprojection) followed by composing with α. Thus, it is just the second coprojection.

Now suppose $e\colon X \to X$ is an idempotent and we have $f\colon X \to X$ such that $fe = ef$, so that f is an endomorphism of $(X, e)\colon A \to \mathbf{V}$. Then if Y is a splitting of (X, e), with section $s\colon Y \to X$ and retraction $r\colon X \to Y$, the induced endomorphism of Y is the composite rfs, and the general linearity formula says that $\mathrm{tr}(rfs)$ is the composite

$$\mathbb{S} \xrightarrow{\ \phi\ } \mathbb{S} + \mathbb{S} \xrightarrow{\ [\mathrm{tr}(f), \mathrm{tr}(fe)]\ } \mathbb{S}.$$

Since ϕ is the second coprojection, this yields $\mathrm{tr}(fe)$. This also follows directly from the cyclicity of ordinary traces. In an additive context, we may say that the coefficients of Φ are $\phi_{[\mathrm{id}_a]} = 0$ and $\phi_{[\alpha]} = 1$, but in this case this formula holds whether or not $\mathbf{V}$ is additive.

For a less trivial example, let G be a finite group and $A = \mathbf{B}G$ the corresponding one-object groupoid. Then a functor $\mathbf{B}G \to \mathbf{V}$ consists of an object X with a left action by G, and is pointwise dualizable just when X is dualizable. If we take $\Phi\colon \mathbf{B}G^{\mathrm{op}} \to \mathbf{V}$ to be $\mathbb{S}$ with the trivial right G-action, then $\mathrm{colim}^\Phi(X)$ is the quotient X/G.

For absoluteness of this weight, we need an additional condition on $\mathbf{V}$. If $\mathbf{V}$ is semi-additive, then the monoid $\mathbf{V}(\mathbb{S}, \mathbb{S})$ of endomorphisms of the unit object is a semiring. Additionally, there is a homomorphism $\mathbb{N} \to \mathbf{V}(\mathbb{S}, \mathbb{S})$ sending n to $\overbrace{\mathrm{id}_{\mathbb{S}} + \mathrm{id}_{\mathbb{S}} + \cdots + \mathrm{id}_{\mathbb{S}}}^{n}$. We say that $\mathbf{V}$ is n-**divisible** if the image of n in $\mathbf{V}(\mathbb{S}, \mathbb{S})$ is (multiplicatively) invertible, and write $\frac{1}{n}$ for its inverse. This implies that for any morphism $h \colon X \to Y$ in $\mathbf{V}$, there is a morphism $k \colon X \to Y$ such that $h = \overbrace{k + k + \cdots + k}^{n}$. Specifically, k is the composite

$$X \cong \mathbb{S} \otimes X \xrightarrow{\frac{1}{n} \otimes h} \mathbb{S} \otimes Y \cong Y.$$

We denote this morphism k by $\frac{1}{n} \cdot h$.

4.12. THEOREM. *Suppose G is a finite group, $X \colon \mathbf{B}G \to \mathbf{V}$ is pointwise dualizable, and $f \colon X \to X$ is a natural transformation. If $\mathbf{V}$ is semi-additive and $\#G$-divisible, then*

$$\mathrm{tr}(f/G) = \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}(f \circ X(g))$$

where $X(g)$ is the action of $g \in G$ on X.

This a fundamental example of our approach. We will extend this result to derivators in §8.

PROOF. The unique representable $\mathbf{B}G(\mathrm{id}, a)$ is the copower $G \cdot \mathbb{S} \cong \bigoplus_{g \in G} \mathbb{S}$, with the right G-action that permutes the summands, i.e. $g \in G$ sends the hth summand to the $(hg)^{\mathrm{th}}$. If $\mathbb{S}$ has the trivial G-action, the fold map

$$r = [\mathrm{id}]_{g \in G} \colon \bigoplus_{g \in G} \mathbb{S} \longrightarrow \mathbb{S}$$

is G-equivariant, i.e. is a morphism in $\mathbf{V}^{\mathbf{B}G^{\mathrm{op}}}$. The diagonal $(\mathrm{id})_{g \in G} \colon \mathbb{S} \to \bigoplus_{g \in G} \mathbb{S}$ is also G-equivariant for these actions; let s be the morphism

$$s = \tfrac{1}{\#G} \cdot (\mathrm{id})_{g \in G} \colon \mathbb{S} \longrightarrow \bigoplus_{g \in G} \mathbb{S}.$$

Then the composite $rs \colon \mathbb{S} \to \mathbb{S}$ is

$$\sum_{g \in G} \frac{1}{\#G} = \#G \cdot \frac{1}{\#G} = \mathrm{id}_{\mathbb{S}}.$$

Hence, our weight Φ is a retract of $\mathbf{B}G(\mathrm{id}, a)$ in $\mathbf{V}^{\mathbf{B}G^{\mathrm{op}}} = \mathbf{Prof}(\mathbf{V})(\mathbb{1}, \mathbf{B}G)$, and thus is absolute. Therefore, if X is a dualizable object with any left G-action, its quotient X/G is also dualizable, and we have a linearity formula for traces.

Since $\mathrm{id}_\Phi = rs$ as above, by the cyclicity of traces, the coefficient vector $\mathrm{tr}(\mathrm{id}_\Phi)$ is equal to $\mathrm{tr}(sr)$. Inspecting the definitions of s and r, we see that sr is the G-equivariant endomorphism of $\bigoplus_{g\in G} \mathbb{S}$ determined by the $(\#G \times \#G)$-matrix where all entries are $\frac{1}{\#G}$. To calculate the trace of this endomorphism, note that the dual representable $\mathbf{B}G(a, \mathrm{id})$ is $\bigoplus_{g\in G} \mathbb{S}$ with the *left* G-action where $g \in G$ sends the hth summand to the $(gh)^{\text{th}}$, while the unit profunctor $\mathbb{I}_{\mathbf{B}G}$ is $\bigoplus_{g\in G} \mathbb{S}$ with both right and left G-actions. By Proposition 4.6, the coevaluation

$$\mathbb{S} \longrightarrow \int^{\mathbf{B}G} \left(\bigoplus_{g\in G}\mathbb{S} \otimes \bigoplus_{h\in G}\mathbb{S} \right) \xrightarrow{\sim} \int^{\mathbf{B}G} \bigoplus_{g,h\in G}\mathbb{S}$$

picks out the image of the $(e, e)^{\text{th}}$ summand, whereas the evaluation

$$\bigoplus_{g\in G}\mathbb{S} \otimes \bigoplus_{h\in G}\mathbb{S} \xrightarrow{\sim} \bigoplus_{g,h\in G}\mathbb{S} \longrightarrow \bigoplus_{g\in G}\mathbb{S}$$

maps the $(g, h)^{\text{th}}$ summand to the $(gh)^{\text{th}}$ summand. Thus, the trace of sr:

$$\mathbb{S} \longrightarrow \int^{\mathbf{B}G} \bigoplus_{g,h\in G}\mathbb{S} \xrightarrow{sr\otimes\mathrm{id}} \int^{\mathbf{B}G} \bigoplus_{k,\ell\in G}\mathbb{S} \longrightarrow \int^{\mathbf{B}G} \bigoplus_{m\in G}\mathbb{S}$$

is induced (after passage to $\int^{\mathbf{B}G}$) by the composite

$$\mathbb{S} \xrightarrow{(\delta_{g,e}\cdot\delta_{h,e})_{g,h}} \bigoplus_{g,h\in G}\mathbb{S} \xrightarrow{\left(\frac{1}{\#G}\cdot\delta_{h,\ell}\right)_{g,h,k,\ell}} \bigoplus_{k,\ell\in G}\mathbb{S} \xrightarrow{(\delta_{k\ell,m})_{k,\ell,m}} \bigoplus_{m\in G}\mathbb{S} \tag{4.13}$$

in which the δ's are Kronecker's. Multiplying these matrices, we see that the mth component of (4.13) is

$$\sum_{g,h,k,\ell} \left(\delta_{g,e} \cdot \delta_{h,e} \cdot \frac{1}{\#G} \cdot \delta_{h,\ell} \cdot \delta_{k\ell,m} \right) = \frac{1}{\#G}.$$

Since passage to $\int^{\mathbf{B}G}$ simply identifies the kth and $(k')^{\text{th}}$ summands when k and k' are in the same conjugacy class of G, the trace of sr is the vector

$$\mathbb{S} \xrightarrow{\left(\frac{\#C}{\#G}\right)_C} \bigoplus_C \mathbb{S}$$

where C ranges over conjugacy classes in G.

In other words, the coefficient vector of Φ assigns to a conjugacy class C the number $\frac{\#C}{\#G}$. Thus, our linearity formula is

$$\mathrm{tr}(f/G) = \sum_C \frac{\#C}{\#G} \cdot \mathrm{tr}(f \circ X(C))$$

where $X(C)$ denotes the action on X of some element of C — by cyclicity of traces, it doesn't matter which. We can split this up as a sum over elements of G rather than conjugacy classes to recover the description in the statement of the theorem. ∎

In particular, if f is the identity morphism of X, then we have

$$\chi(X/G) = \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}(X(g)). \tag{4.14}$$

This is a generalization of the orbit-counting theorem (a.k.a. Burnside's lemma or the Cauchy-Frobenius lemma). Namely, suppose $\mathbf{V} = R\text{-}\mathbf{Mod}$ for a commutative ring R in which $\#G$ is invertible, Z is a finite G-set, and $X = R[Z]$ with the induced G-action. Then $X/G \cong R[Z/G]$, so that $\chi(X/G) = \#(Z/G)$; while $\mathrm{tr}(X(g))$ is the number of fixed points of g acting on Z. Thus (4.14) reduces exactly to the orbit-counting theorem:

$$\#(Z/G) = \frac{1}{\#G} \sum_{g \in G} \#(Z^g).$$

4.15. REMARK. Note that in the previous example, the pointwise trace of id_X is the "row vector" with entries indexed by conjugacy classes of G, which assigns to each conjugacy class the trace of its action on X. In other words, it is the *character* of the group representation X.

4.16. REMARK. In addition to colimits weighted by profunctors $\Phi \colon \mathbb{1} \nrightarrow A$, we may consider those weighted by arbitrary profunctors $\Phi \colon B \nrightarrow A$. In this case the "colimit" of $X \colon A \to \mathbf{V}$, defined as before using the tensor product of functors, is a diagram $B \to \mathbf{V}$ rather than a single object. For instance, if $\theta \colon G \to H$ is a group homomorphism, with corresponding functor $\mathbf{B}\theta \colon \mathbf{B}G \to \mathbf{B}H$, then the colimit of $X \colon \mathbf{B}G \to \mathbf{V}$ weighted by the representable $\mathbf{B}H(\mathbf{B}\theta, \mathrm{id})$ is the *induced representation* of X along θ. If moreover G and H are finite and $\mathbf{V}$ is $\#G$-divisible, then a computation similar to Theorem 4.12 produces the formula for the character of an induced representation:

$$\mathrm{tr}(\mathrm{colim}^{\mathbf{B}H(\theta,\mathrm{id})}(X)(h)) = \frac{1}{\#G} \sum_{\substack{k \in H \\ k^{-1}hk = \theta(g)}} \mathrm{tr}(X(g))$$

Finally, we can use the base change profunctors to prove Lemma 3.8. Recall the statement, which applies in the situation of a pointwise dualizable $X \colon A \to \mathbf{V}$ and an endomorphism $f \colon X \to X$.

RESTATEMENT OF LEMMA 3.8. [The component lemma for symmetric monoidal categories]. *For any morphism $\alpha \in A(a,a)$, $\mathrm{tr}(f)_{[\alpha]}$ is the trace of the composite*

$$X(a) \xrightarrow{X_\alpha} X(a) \xrightarrow{f_a} X(a) \,. \tag{4.17}$$

Here $\mathrm{tr}(f)_{[\alpha]}$ denotes the composite of $\mathrm{tr}(f) \colon \langle\!\langle A \rangle\!\rangle \to \mathbb{S}$ with the coprojection $\mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ induced by the conjugacy class $[\alpha]$.

PROOF. Let a also denote the functor $\mathbb{1} \to A$ that picks out the object $a \in A$. Then by Proposition 4.6, the profunctor $A(\mathrm{id}, a)\colon \mathbb{1} \nrightarrow A$ is right dualizable, and we have an endomorphism $A(\mathrm{id}, \alpha)\colon A(\mathrm{id}, a) \to A(\mathrm{id}, a)$ induced by composition with α. Therefore, by Theorem 2.6, the trace of the composite

$$A(\mathrm{id}, a) \odot X \xrightarrow{A(\mathrm{id},\alpha)\odot\mathrm{id}} A(\mathrm{id}, a) \odot X \xrightarrow{\mathrm{id}\odot f} A(\mathrm{id}, a) \odot X \qquad (4.18)$$

is equal to the composite

$$\mathbb{S} \xrightarrow{\mathrm{tr}(A(\mathrm{id},\alpha))} \langle\!\langle A \rangle\!\rangle \xrightarrow{\mathrm{tr}(f)} \mathbb{S}.$$

However, under the isomorphism $A(\mathrm{id}, a) \odot X \cong X(a)$ of Proposition 4.7, (4.18) is identified with (4.17). Finally, Proposition 4.8 tells us that $\mathrm{tr}(A(\mathrm{id}, \alpha))$ is the coprojection $\mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ induced by $[\alpha]$.   ∎

4.19. REMARK. For simplicity, we have assumed that our monoidal category $\mathbf{V}$ is complete, cocomplete, and closed. However, it follows for formal reasons that the same linearity formulas hold even if $\mathbf{V}$ admits only the particular colimits in question. For instance, using the methods of [Kel82, §3.11], we can embed any $\mathbf{V}$ in a complete and cocomplete closed monoidal category $\mathbf{V}'$, by a functor that preserves limits, tensor products, and any relevant colimits, hence also dualizability and traces.

Part 2: Linearity in derivators

In this second part of the paper, we extend the approach to linearity from Part 1 to homotopical situations, in which we must replace colimits by *homotopy* colimits. (Recall that our motivation for this generalization is a desire to capture the familiar additivity of the Lefschetz number, (1.1).) There are many axiomatic frameworks for homotopy theory, such as model categories and $(\infty, 1)$-categories, but the one which we find most convenient is *derivators*, which were invented by Grothendieck [Gro90], Franke [Fra], and Heller [Hel88] and studied further by [Mal, Cis03, Gro13].

§5 and §6 contain the general results and basic examples. The remaining sections in this part, §§7–9, contain more examples, including the linearity formulas for homotopy finite colimits and EI-categories. (A reader who is mainly interested in additivity on cofiber sequences, such as for the applications to the Lefschetz number and Reidemeister trace, should feel free to skip these sections.) To minimize the background required for this part of the paper, we postpone some details and proofs until Part 4.

5. Linearity in monoidal derivators

We begin by recalling some of the basic notions of derivator theory; see [Gro13, GPS14b, GPS14a] for details. A **derivator** is a 2-functor $\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$, where $\mathcal{C}at$ and $\mathcal{C}AT$

denote the 2-categories of small and not necessarily small categories, respectively, that satisfies four axioms. One of these is that for any functor $u\colon A \to B$, the functor

$$u^* \coloneqq \mathscr{D}(u)\colon \mathscr{D}(B) \to \mathscr{D}(A)$$

has a left adjoint $u_!$ and a right adjoint u_*.

We think of the category $\mathscr{D}(A)$ as the homotopy category of A-shaped homotopy coherent diagrams in some homotopy theory that $\mathscr{D}$ represents, and refer to its objects as **coherent diagrams**. With this in mind, we refer to u^* as *restriction along* u and think of $u_!$ and u_* as corresponding *(homotopy) Kan extensions*. Each object a of a category A defines a functor $a\colon \mathbb{1} \to A$, where $\mathbb{1}$ denotes the terminal category. Thus, we have restriction functors $a^*\colon \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$ which take $X \in \mathscr{D}(A)$ to $X_a \coloneqq a^*X$; together these define a functor $\mathscr{D}(A) \to \mathscr{D}(\mathbb{1})^A$. We call $\mathscr{D}(\mathbb{1})$ the **underlying category** of $\mathscr{D}$, and the image of a coherent diagram $X \in \mathscr{D}(A)$ under this functor its **underlying diagram**. (This motivated our abuse of notation in Theorem 1.4 where we wrote $X\colon A \to \mathscr{V}$ rather than the more correct $X \in \mathscr{V}(A)$.) Another axiom of a derivator is that the underlying diagram functor is **conservative**, i.e. a morphism $X \to Y$ in $\mathscr{D}(A)$ is an isomorphism just when its image in $\mathscr{D}(\mathbb{1})^A$ is an isomorphism.

The third axiom of a derivator is that it takes coproducts in $\mathcal{C}at$ to products in $\mathcal{C}AT$, i.e. $\mathscr{D}(\coprod_i A_i) \simeq \prod_i \mathscr{D}(A_i)$. The final axiom is more technical and allows us to compute with the functors $u_!$ and u_*. Essentially it says that they are "pointwise" Kan extensions: each object $(u_!X)_a$ is a colimit (i.e. a left Kan extension to $\mathbb{1}$) over the comma category (u/a). We will not make much explicit use of this axiom; instead we will rely on results from [GPS14b, GPS14a] that build on it.

Now if $\mathcal{M}ONCAT$ denotes the 2-category of monoidal categories and strong monoidal functors, a **monoidal derivator** is a 2-functor $\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{M}ONCAT$ satisfying two further conditions. The first is that the composite of $\mathscr{D}$ with the forgetful 2-functor $\mathcal{M}ONCAT \to \mathcal{C}AT$ is a derivator. The second is that the monoidal product is *cocontinuous* in each variable, a compatibility condition between the monoidal product and left adjoints $u_!$ [GPS14a, Definition 3.19]. A monoidal derivator is **symmetric** if it lifts to the 2-category of symmetric monoidal categories and symmetric monoidal functors. Finally, there is also a notion of *closed* monoidal derivator [GPS14a, Definition 8.5], but we will not need to use the details.

If $\mathscr{D}$ is a monoidal derivator, then we write the tensor product of $\mathscr{D}(A)$ as $\otimes_A$ and its unit object as $\mathbb{S}_A$. When $A = \mathbb{1}$ we sometimes omit the subscripts. In particular, $\mathbb{S} \coloneqq \mathbb{S}_{\mathbb{1}}$ is "the unit object of $\mathscr{D}$". Note that since $\pi_A^*\colon \mathscr{D}(\mathbb{1}) \to \mathscr{D}(A)$ is strong monoidal, where $\pi_A\colon A \to \mathbb{1}$ is the unique functor, we have in particular $\mathbb{S}_A \cong \pi_A^*(\mathbb{S})$ for any A; a coherent diagram in the image of π_A^* is said to be **constant**.

The simplest examples of derivators are **represented**, where $y(\mathscr{C})(A) \coloneqq \mathscr{C}^A$ for some complete and cocomplete ordinary category $\mathscr{C}$. If $\mathscr{C}$ is also a closed monoidal category, then $y(\mathscr{C})$ is a closed monoidal derivator. Most other interesting examples arise from homotopy categories of model categories, where we have $\mathscr{H}o(\mathscr{C})(A) \coloneqq \mathscr{C}^A[(\mathscr{W}^A)^{-1}]$ for $\mathscr{W}$ the weak equivalences in $\mathscr{C}$. If $\mathscr{C}$ is a (closed) monoidal model category, then $\mathscr{H}o(\mathscr{C})$

is a closed monoidal derivator. (One can also pass directly to a derivator from an $(\infty, 1)$-category; see [GPS14b, Example 2.5] or [RV15, §5.3].)

The *classical homotopy derivator* $\mathscr{H}o(\mathbf{Top})$ of spaces under weak homotopy equivalence is *universal*, in the sense that it acts on every other derivator. We denote this action by $\circledast\colon \mathscr{H}o(\mathbf{Top}) \times \mathscr{D} \to \mathscr{D}$. It is easiest to define by

$$|NA| \circledast X := (\pi_A)_!(\pi_A)^* X$$

for $X \in \mathscr{D}(\mathbb{1})$ and $A \in \mathcal{C}at$, where $|NA|$ denotes the geometric realization of the nerve of A. Since every space is weakly homotopy equivalent to $|NA|$ for some A, this suffices to define the action $\circledast$. (It is not obvious that $(\pi_A)_!(\pi_A)^* X$ depends only on the weak homotopy type of $|NA|$; this is a theorem of Heller [Hel88] and Cisinski [Cis06].)

For the rest of this section (and until the end of §6), let $\mathscr{V}$ be a closed symmetric monoidal derivator. We first intend to mimic the construction of the bicategory $\mathbf{Prof}(\mathbf{V})$ from §3. This requires the definition of *coend* in a derivator, which was introduced in [GPS14a]. Let $\mathrm{tw}(A)$ be the **twisted arrow category** of A. The objects of $\mathrm{tw}(A)$ are the morphisms of A and the morphisms from $a_1 \xrightarrow{f_1} b_1$ to $a_2 \xrightarrow{f_2} b_2$ in $\mathrm{tw}(A)$ are pairs of morphisms $b_1 \xrightarrow{h} b_2$ and $a_2 \xrightarrow{g} a_1$ such that $f_2 = hf_1g$. This category has source and target projections $(s, t)\colon \mathrm{tw}(A) \to A^{\mathrm{op}} \times A$. Therefore, its opposite has projections $(t^{\mathrm{op}}, s^{\mathrm{op}})\colon \mathrm{tw}(A)^{\mathrm{op}} \to A^{\mathrm{op}} \times A$.

Now if $\mathscr{V}$ is a derivator, the **coend** of $X \in \mathscr{V}(A^{\mathrm{op}} \times A)$ is defined by

$$\int^A X := (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^* X,$$

where $\pi_{\mathrm{tw}(A)^{\mathrm{op}}}\colon \mathrm{tw}(A)^{\mathrm{op}} \to \mathbb{1}$ is the unique functor. (We will use π throughout this paper to denote projection maps.) It was shown in [GPS14a, Theorem 5.9] that we can construct a closed bicategory $\mathbf{Prof}(\mathscr{V})$ from a closed symmetric monoidal derivator $\mathscr{V}$ as follows:

- Its objects are small categories.

- Its hom-category from A to B is $\mathscr{V}(A \times B^{\mathrm{op}})$.

- Its composition functors $\odot$ are the composites

$$\mathscr{V}(A \times B^{\mathrm{op}}) \times \mathscr{V}(B \times C^{\mathrm{op}})$$

$$\xrightarrow{\pi^*_{B \times C^{\mathrm{op}}} \times \pi^*_{A \times B^{\mathrm{op}}}} \mathscr{V}(A \times B^{\mathrm{op}} \times B \times C^{\mathrm{op}}) \times \mathscr{V}(A \times B^{\mathrm{op}} \times B \times C^{\mathrm{op}})$$

$$\xrightarrow{\otimes_{A \times B^{\mathrm{op}} \times B \times C^{\mathrm{op}}}} \mathscr{V}(A \times B^{\mathrm{op}} \times B \times C^{\mathrm{op}})$$

$$\xrightarrow{\int^B} \mathscr{V}(A \times C^{\mathrm{op}}).$$

- The identity 1-cell of a small category B is

$$\mathbb{I}_B = (t, s)_! \mathbb{S}_{\mathrm{tw}(B)} \cong (t, s)_! (\pi_{\mathrm{tw}(B)})^* \mathbb{S}_{\mathbb{1}} \in \mathscr{V}(B \times B^{\mathrm{op}})$$

where $\mathbb{S}_{\mathbb{1}}$ is the monoidal unit of $\mathscr{V}(\mathbb{1})$.

(In the terminology of [GPS14a], the bicategorical composition is the external-canceling tensor product $\otimes_{[B]} \colon \mathscr{V}(A \times B^{\mathrm{op}}) \times \mathscr{V}(B \times C^{\mathrm{op}}) \longrightarrow \mathscr{V}(A \times C^{\mathrm{op}})$.) When $\mathscr{V}$ is the represented derivator on a complete and cocomplete closed symmetric monoidal category $\mathbf{V}$, this bicategory can be identified with the bicategory $\mathbf{Prof}(\mathbf{V})$ from §3.

As before, we define the Φ-**weighted colimit** of $X \in \mathscr{V}(A)$ by $\Phi \in \mathscr{V}(A^{\mathrm{op}})$ to be the composite $\mathrm{colim}^{\Phi}(X) = \Phi \odot X$ in $\mathbf{Prof}(\mathscr{V})$, where Φ and X are regarded as 1-cells $\mathbb{1} \twoheadrightarrow A$ and $A \twoheadrightarrow \mathbb{1}$ in $\mathbf{Prof}(\mathscr{V})$, respectively. The following observation is somewhat less obvious in the derivator case than in the classical one.

5.1. PROPOSITION. *For any $X \in \mathscr{V}(A)$, if $\Phi = (\pi_{A^{\mathrm{op}}})^*\mathbb{S}$ is constant at the unit object, then we have $\mathrm{colim}^{\Phi}(X) \cong \mathrm{colim}(X)$, where $\mathrm{colim}(X)$ denotes the usual derivator colimit $(\pi_A)_!(X)$.*

PROOF. By [GPS14a, Corollary 5.8], we have

$$\mathrm{colim}^{(\pi_{A^{\mathrm{op}}})^*\mathbb{S}}(X) = \int^A (\pi_{A^{\mathrm{op}}})^*\mathbb{S} \otimes X$$

$$\cong \int^{\mathbb{1}} \mathbb{S} \otimes (\pi_A)_! X$$

$$\cong (\pi_A)_! X. \qquad \blacksquare$$

We can now generalize the definitions and results of §3.

5.2. DEFINITION.

- A coherent diagram $X \in \mathscr{V}(A)$ is **pointwise dualizable** if it is right dualizable when regarded as a 1-cell $A \twoheadrightarrow \mathbb{1}$ in $\mathbf{Prof}(\mathscr{V})$.

- A coherent diagram $\Phi \in \mathscr{V}(A^{\mathrm{op}})$ is **absolute** if it is right dualizable when regarded as a 1-cell $\mathbb{1} \twoheadrightarrow A$ in $\mathbf{Prof}(\mathscr{V})$.

Thus, by Theorem 2.5, we have:

5.3. THEOREM. *If $X \in \mathscr{V}(A)$ is pointwise dualizable and $\Phi \in \mathscr{V}(A^{\mathrm{op}})$ is absolute, then $\mathrm{colim}^{\Phi}(X)$ is dualizable.*

We also have a version of Lemma 3.5.

5.4. LEMMA. [GPS14a, Lemma 11.5] *$X \in \mathscr{V}(A)$ is pointwise dualizable if and only if each object $X_a \in \mathscr{V}(\mathbb{1})$ is dualizable.*

Our next order of business is to construct a shadow on $\mathbf{Prof}(\mathscr{V})$.

5.5. THEOREM. *The bicategory $\mathbf{Prof}(\mathscr{V})$ has a shadow valued in $\mathscr{V}(\mathbb{1})$, defined for $H \in \mathbf{Prof}(\mathscr{V})(A, A) = \mathscr{V}(A \times A^{\mathrm{op}})$ by*

$$\langle\!\langle H \rangle\!\rangle = \int^A \mathfrak{s}^* H$$

where $\mathfrak{s} \colon A^{\mathrm{op}} \times A \xrightarrow{\sim} A \times A^{\mathrm{op}}$ is the symmetry.

PROOF. The shadow isomorphism $\langle\!\langle H \odot K \rangle\!\rangle \cong \langle\!\langle K \odot H \rangle\!\rangle$ is essentially just the Fubini theorem, [GPS14a, Lemma 5.3]. The compatibility of the shadow with associativity and unit maps [PS13, Defn. 4.1] follows from essentially the same argument that proves the associativity of composition in $\mathbf{Prof}(\mathscr{V})$, [GPS14a, Lemma 5.12]. ∎

Thus, by Theorem 2.6, we have a linearity formula.

5.6. THEOREM. *If $X \in \mathscr{V}(A)$ is pointwise dualizable and $\Phi \in \mathscr{V}(A^{\mathrm{op}})$ is absolute, then for any $f \colon X \to X$ we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \mathrm{tr}(f) \circ \mathrm{tr}(\mathrm{id}_{\Phi}).$$

As before, to make this useful, we need to analyze $\langle\!\langle A \rangle\!\rangle = \langle\!\langle \mathbb{I}_A \rangle\!\rangle$ further. By definition of $\mathbb{I}_A$ and $\langle\!\langle - \rangle\!\rangle$, this is obtained by transporting $\mathbb{S}_{\mathbb{1}}$ diagonally along the following diagram starting from the top right and ending at the bottom left. We restrict horizontally and left Kan extend vertically (ignore the top left corner for the moment):

$$
\begin{array}{ccccc}
\Lambda A & \longrightarrow & \mathrm{tw}(A) & \longrightarrow & \mathbb{1} \\
\downarrow \lrcorner & & \downarrow {\scriptstyle (t,s)} & & \\
\mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow{\;(s^{\mathrm{op}},t^{\mathrm{op}})\;} & A \times A^{\mathrm{op}} & & \\
\downarrow & & & & \\
\mathbb{1} & & & &
\end{array}
\qquad (5.7)
$$

Let ΛA be defined by the pullback in (5.7). Explicitly, the objects of ΛA are pairs of morphisms $(a \xrightarrow{\alpha} b, b \xrightarrow{\beta} a)$ in A which are composable in both orders. A morphism from (α, β) to (α', β') is a pair of morphisms $(a \xrightarrow{\xi} a', b' \xrightarrow{\zeta} b)$ such that $\alpha = \zeta \alpha' \xi$ and $\beta' = \xi \beta \zeta$. Since (t,s) is a discrete opfibration, the above pullback square is homotopy exact [Gro13, Prop. 1.24], meaning that the Beck-Chevalley condition holds for the vertical left adjoints; thus we have

$$
\begin{aligned}
\langle\!\langle A \rangle\!\rangle &\cong (\pi_{\Lambda A})_! (\pi_{\Lambda A})^* \mathbb{S}_{\mathbb{1}} \\
&= |N(\Lambda A)| \circledast \mathbb{S}.
\end{aligned}
\qquad (5.8)
$$

5.9. REMARK. In fact, the nerve $N(\Lambda A)$ is equivalent to the *cyclic nerve ZA* of A [BM12, §3]; thus $\langle\!\langle A \rangle\!\rangle$ may be regarded as the "Hochschild homology of A with respect to $\mathscr{V}$". Recall that the n-simplices of ZA are composable loops of $(n+1)$ morphisms in A:

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n \xrightarrow{\alpha_{n+1}} a_0,$$

with face and degeneracy maps defined by composition and inserting identities. There is a map of simplicial sets $N(\Lambda A) \to ZA$ which sends an object (α, β) to the composite $\beta\alpha$, and an n-simplex

$$(\alpha_0, \beta_0) \xrightarrow{(\xi_1, \zeta_1)} (\alpha_1, \beta_1) \xrightarrow{(\xi_2, \zeta_2)} \cdots \xrightarrow{(\xi_n, \zeta_n)} (\alpha_n, \beta_n)$$

of $N(\Lambda A)$ to the n-simplex

$$a_0 \xrightarrow{\xi_1} a_1 \xrightarrow{\xi_2} \cdots \xrightarrow{\xi_n} a_n \xrightarrow{\beta_0 \zeta_0 \zeta_1 \cdots \zeta_n \alpha_n} a_0$$

of ZA. This map can be shown to be a weak homotopy equivalence. However, we will need only the following weaker assertion.

5.10. LEMMA. *There is a bijection between $\pi_0(\Lambda A)$ and the set of conjugacy classes of A.*

PROOF. Each object $(\alpha, \beta) \in \Lambda A$ induces a conjugacy class $[\alpha\beta] = [\beta\alpha]$, and given a morphism $(\xi, \zeta) \colon (\alpha, \beta) \to (\alpha', \beta')$ we have

$$[\alpha'\beta'] = [\alpha'\xi\beta\zeta] = [\zeta\alpha'\xi\beta] = [\alpha\beta].$$

Thus, $\pi_0(\Lambda A)$ maps to the set of conjugacy classes, and the map is clearly surjective. For injectivity, first note that for any (α, β) we have morphisms $(\alpha, \mathrm{id}) \colon (\alpha, \beta) \to (\mathrm{id}, \alpha\beta)$ and $(\beta, \mathrm{id}) \colon (\alpha\beta, \mathrm{id}) \to (\alpha, \beta)$. Thus, it suffices to show that if $[\alpha] = [\beta]$ then (α, id) and (id, β) are in the same connected component of ΛA. But if $[\alpha] = [\beta]$, then we have $\alpha = \zeta\xi$ and $\beta = \xi\zeta$ for some ξ and ζ, and hence there is a morphism $(\xi, \zeta) \colon (\alpha, \mathrm{id}) \to (\mathrm{id}, \beta)$ in ΛA. ∎

In particular, using Lemma 5.10 we can construct from every conjugacy class $[\alpha]$ in A a uniquely determined morphism $[\alpha] \colon \mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ in $\mathscr{V}(\mathbb{1})$, in the following way. Consider the more general case of a functor $F \colon B \to A$ and a natural transformation $\nu \colon F \to F$; then we have a functor $\Lambda\nu \colon \Lambda B \to \Lambda A$ which sends $(b_1 \xrightarrow{\beta_1} b_2, b_2 \xrightarrow{\beta_2} b_1)$ to $(\nu_{b_2} \circ F(\beta_1), F(\beta_2))$ (naturality of ν makes this functorial). Thus, we have an induced map $[\nu] \colon \langle\!\langle B \rangle\!\rangle \to \langle\!\langle A \rangle\!\rangle$. In the case $B = \mathbb{1}$, where ν is just an endomorphism $a \xrightarrow{\alpha} a$ in A, this yields our desired morphism $[\alpha] \colon \mathbb{S} \to \langle\!\langle A \rangle\!\rangle$.

Now since natural transformations between functors induce homotopies between maps of nerves, if we have a transformation $\Lambda\nu \to \Lambda\rho$, then $[\nu] = [\rho] \colon \langle\!\langle B \rangle\!\rangle \to \langle\!\langle A \rangle\!\rangle$. In the case $B = \mathbb{1}$, this implies that $[\alpha]$ is determined by the connected component of (α, id) in ΛA, and hence (by Lemma 5.10) by the conjugacy class of α. (Note that the map from conjugacy classes to morphisms $\mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ is not necessarily injective, so it is a little abusive to use the same notation for both.)

There is also a component lemma.

5.11. LEMMA. [The component lemma for monoidal derivators] *If $X \in \mathscr{V}(A)$ is pointwise dualizable and $f \colon X \to X$, then for any conjugacy class $[a \xrightarrow{\alpha} a]$ in A, the composite*

$$\mathbb{S}_{\mathbb{1}} \xrightarrow{[\alpha]} \langle\!\langle A \rangle\!\rangle \xrightarrow{\mathrm{tr}(f)} \mathbb{S}_{\mathbb{1}}$$

is equal to the trace in $\mathscr{V}(\mathbb{1})$ of the composite

$$X_a \xrightarrow{X_\alpha} X_a \xrightarrow{f_a} X_a.$$

PROOF. We will prove a generalization of this result in §14 on page 684.   ∎

In contrast to the situation in §3, $\text{tr}(f)$ may not be determined by its composites with the morphisms $[\alpha]$. However, for most applications, it is sufficient to know these composites, because the coefficient vector is built out of them. This is the situation of the following definition.

We say that a derivator $\mathscr{V}$ is **semi-additive** if the category $\mathscr{V}(\mathbb{1})$ is so. This implies that all categories $\mathscr{V}(A)$ are also semi-additive. Moreover, if $\mathscr{V}$ is symmetric monoidal, then $\mathscr{V}(\mathbb{1})(\mathbb{S}, \mathbb{S})$ is a commutative semiring which acts on the homsets of all the $\mathscr{V}(A)$.

5.12. DEFINITION. If $\Phi \in \mathscr{V}(A^{\mathrm{op}})$ is absolute, then $\text{tr}(\text{id}_\Phi) \colon \mathbb{S} \to \langle\!\langle A \rangle\!\rangle$ is called its **coefficient vector**. If $\mathscr{V}$ is semi-additive and we can express the coefficient vector of Φ as a linear combination

$$\text{tr}(\text{id}_\Phi) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot [\alpha],$$

for $\phi_{[\alpha]} \in \mathscr{V}(\mathbb{1})(\mathbb{S}, \mathbb{S})$, then we refer to the $\phi_{[\alpha]}$ as the **coefficients** of Φ and say that Φ has a **coefficient decomposition**.

Often ΛA is essentially discrete (and finite), so that $\langle\!\langle A \rangle\!\rangle \cong \bigoplus_{[\alpha]} \mathbb{S}$ and thus Φ automatically has a coefficient decomposition. Whenever Φ has a coefficient decomposition, we can give a more familiar description of the formula in Theorem 5.6.

5.13. COROLLARY. *If $\mathscr{V}$ is semi-additive and $\Phi \in \mathscr{V}(A^{\mathrm{op}})$ is absolute and has a coefficient decomposition, then*

$$\text{tr}(\text{colim}^\Phi(f)) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot \text{tr}(f_a \circ X_\alpha)$$

for any endomorphism f of a pointwise dualizable $X \in \mathscr{V}(A)$.

As before, the problem is now to produce examples of absolute weights and calculate their coefficient vectors. We easily obtain derivator versions of the simple examples from §4: zero objects, direct sums, and tensor products. For these, there is not much gained by passing to derivators, since they are actual colimits in the underlying category $\mathscr{V}(\mathbb{1})$. (This follows from the fact that $\mathscr{V} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ takes coproducts to products.)

The splitting of idempotents is slightly less trivial, since idempotents in a derivator (that is, objects of $\mathscr{V}(E)$ where E is the free-living idempotent) are *coherent* idempotents, and not every idempotent in $\mathscr{V}(\mathbb{1})$ may admit a coherentification (a counterexample can be found in [Lur14, Warning 1.2.4.8]). However, coherent idempotents are, in particular, idempotents in $\mathscr{V}(\mathbb{1})$, and their coherent splitting is, in particular, a splitting in $\mathscr{V}(\mathbb{1})$. Thus, again the linearity formula follows from ordinary category theory, not requiring the derivator structure.

In the next four sections (and continuing in §11) we consider some situations where honestly new phenomena occur in the derivator context, primarily arising from *stability*.

5.14. REMARK. The component lemma for derivators is closely related to [PS12, Theorem 6.3]. Indeed, a monoidal derivator is a particular kind of indexed monoidal category where the base category **S** is $\mathcal{C}at$. It doesn't have *indexed homotopy coproducts* in the

sense of [PS12], but it does if we restrict its domain to the category $\mathcal{G}pd$ of groupoids (since comma categories for groupoids coincide with homotopy pullbacks). The bicategory constructed as in [PS12, Theorem 5.2] from this indexed monoidal category is equivalent to the sub-bicategory of $\mathbf{Prof}(\mathscr{V})$ whose objects are groupoids (using the facts that when A is a groupoid, we have $A \cong A^{\mathrm{op}}$ and $\mathrm{tw}(A) \simeq A$).

Thus, [PS12, Theorem 6.1] implies that when A is a groupoid, $X \in \mathscr{V}(A)$ is pointwise dualizable just when it is dualizable in the symmetric monoidal category $\mathscr{V}(A)$, and in that case [PS12, Theorem 6.3] says that from $\mathrm{tr}(f)\colon \langle\!\langle A \rangle\!\rangle \to \mathbb{S}$ we can extract the trace of f in $\mathscr{V}(A)$. Moreover, since for any $a \in A$ the functor $a^*\colon \mathscr{V}(A) \to \mathscr{V}(\mathbb{1})$ is strong monoidal and thus preserves traces, from the trace of f in $\mathscr{V}(A)$ we can extract the traces of each $f_a\colon X_a \to X_a$. Thus, with a little care, we can deduce the special case of the component lemma when $\alpha = \mathrm{id}_a$ from [PS12, Theorem 6.3].

The general component lemma, in the case when $\mathscr{V}$ is the homotopy derivator of chain complexes, is remarked on in [PS12, Example 6.6]. In fact, [PS12, Theorem 6.3] can also be generalized to a statement that includes the full component lemma, but we will not do that here.

6. Stable derivators and additivity

Let $\mathscr{V}$ be a closed symmetric monoidal derivator which is *pointed*, i.e. its initial and terminal objects coincide and are denoted by 0. Let 2 denote the arrow category $(a \xrightarrow{\alpha} b)$, and let $\ulcorner$ be the span category $(c \leftarrow a \to b)$. Then for any $X \in \mathscr{V}(2)$ with underlying diagram $(X_a \to X_b)$, we can right Kan extend it to an object of $\mathscr{V}(\ulcorner)$ with underlying diagram $(0 \leftarrow X_a \to X_b)$. The colimit of this coherent span is an object called the **cofiber** of X. (Sometimes this word refers to the induced map from X_b to this object, perhaps itself lifted to $\mathscr{V}(2)$; but we will care primarily about the object.)

In particular, from any object $X \in \mathscr{V}(\mathbb{1})$ we can obtain (again by right Kan extension) an object of $\mathscr{V}(2)$ with underlying diagram $(X \to 0)$. Its cofiber is called the **suspension** of X and denoted ΣX. In a represented derivator, the suspension is always 0, but in general it is nontrivial and deserves its name.

Dually, we have the **fiber** of any object of $\mathscr{V}(2)$, and the **loop space** ΩX of any object X. There is an adjunction $\Sigma \dashv \Omega$, and $\mathscr{V}$ is **stable** if and only if this adjunction is an equivalence (see [GPS14b]). A stable derivator is automatically additive.

Let us consider the possible absoluteness of these constructions. We begin by observing that cofibers in any pointed symmetric monoidal derivator are a weighted colimit. Let $\Phi \in \mathscr{V}(2^{\mathrm{op}})$ be the essentially unique diagram of the form $(0 \leftarrow \mathbb{S})$, which we can obtain as before by right Kan extension. Note that $\mathrm{tw}(2) \cong \ulcorner^{\mathrm{op}}$; and for any $X \in \mathscr{V}(2)$, the weighted colimit $\Phi \odot X$ is the pushout of the $\mathrm{tw}(2)^{\mathrm{op}}$-diagram

$$
\begin{array}{ccc}
\mathbb{S} \otimes X_a & \longrightarrow & \mathbb{S} \otimes X_b \\
\downarrow & & \\
0 \otimes X_a & &
\end{array}
\qquad \text{or equivalently} \qquad
\begin{array}{ccc}
X_a & \longrightarrow & X_b \\
\downarrow & & \\
0. & &
\end{array}
$$

Thus, $\Phi \odot X$ is exactly the cofiber of X, as defined above.

Since cofibers are a weighted colimit, we can now ask under what circumstances they are absolute. To calculate the canonical right dual $D_r\Phi$ of our weight Φ, we start with the external hom $\Phi \rhd \mathbb{I}_2 \in \mathscr{V}(2 \times 2 \times 2^{\mathrm{op}})$, restrict to $\mathrm{tw}(2) \times 2$, then right Kan extend to 2. The resulting $(\mathrm{tw}(2) \times 2)$-diagram looks like

$$
\begin{array}{ccccc}
\Phi_a \rhd (\mathbb{I}_2)_{a,a} & \longrightarrow & \Phi_b \rhd (\mathbb{I}_2)_{a,a} & \longleftarrow & \Phi_b \rhd (\mathbb{I}_2)_{b,a} \\
\downarrow & & \downarrow & & \downarrow \\
\Phi_a \rhd (\mathbb{I}_2)_{a,b} & \longrightarrow & \Phi_b \rhd (\mathbb{I}_2)_{a,b} & \longleftarrow & \Phi_b \rhd (\mathbb{I}_2)_{b,b}
\end{array}
$$

or equivalently

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{S} & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{S} & \longleftarrow & \mathbb{S}.
\end{array}
$$

Therefore, its right Kan extension to 2 looks like $(\Omega\mathbb{S} \to 0)$; this is $D_r\Phi$.

Now recall from §2 that Φ is absolute if and only if the canonical map $U \odot D_r\Phi \to \Phi \rhd U$ is an isomorphism for any U. In our case, $U \odot D_r\Phi$ is the pushout of a $\mathrm{tw}(2)^{\mathrm{op}}$-diagram that looks like $U_a \otimes \Omega\mathbb{S} \leftarrow U_b \otimes \Omega\mathbb{S} \to 0$, i.e. the cofiber of $U_b \otimes \Omega\mathbb{S} \to U_a \otimes \Omega\mathbb{S}$. On the other hand, $\Phi \rhd U$ is the limit of a $\mathrm{tw}(2)$-diagram that looks like $(U_b \to U_a \leftarrow 0)$, i.e. the *fiber* of $U_b \to U_a$. Thus, if we denote the cofiber and fiber objects of a map f by $\mathsf{C}(f)$ and $\mathsf{F}(f)$ respectively, then absoluteness of Φ requires that the canonical map

$$
\mathsf{C}(U_b \otimes \Omega\mathbb{S} \to U_a \otimes \Omega\mathbb{S}) \longrightarrow \mathsf{F}(U_b \to U_a)
$$

be an equivalence for all U; in other words, that we can calculate a cofiber as a "shifted" fiber. This is similar to Example 4.1 and Example 4.2: we are asking a colimit construction to coincide with a limit construction.

We also remarked in §2 that it suffices to require this when $U = \Phi$. The fiber of Φ is just $\mathbb{S}$, while the cofiber of $\Phi \otimes \Omega\mathbb{S}$ is the suspension $\Sigma\Omega\mathbb{S}$. Thus, absoluteness of Φ equivalently requires that the canonical map

$$
\Sigma\Omega\mathbb{S} \longrightarrow \mathbb{S}
$$

be an equivalence. This map is the counit of the adjunction $\Sigma \dashv \Omega$ at $\mathbb{S}$; thus we have proven one direction of the following result.

6.1. THEOREM. *A pointed closed symmetric monoidal derivator $\mathscr{V}$ is stable if and only if the weight Φ for cofibers is absolute. In this case, if $X \in \mathscr{V}(2)$ is pointwise dualizable, then so is its cofiber $\mathrm{colim}^\Phi(X)$, and for any $f\colon X \to X$ we have*

$$
\mathrm{tr}(\mathrm{colim}^\Phi(f)) = \mathrm{tr}(f_b) - \mathrm{tr}(f_a). \tag{6.2}
$$

PROOF. We have shown "only if". For "if", suppose that Φ is absolute. Since $\otimes$ is cocontinuous, it preserves suspensions in both arguments; thus for any $Y \in \mathscr{V}(\mathbb{1})$ we have

$$Y \otimes \Sigma\mathbb{S} \cong \Sigma(Y \otimes \mathbb{S}) \cong \Sigma Y.$$

In particular, we have

$$\Omega\mathbb{S} \otimes \Sigma\mathbb{S} \cong \Sigma\Omega\mathbb{S} \cong \mathbb{S}.$$

Thus, $\Sigma\mathbb{S}$ and $\Omega\mathbb{S}$ are invertible objects of $\mathscr{V}(\mathbb{1})$. It follows that the functor $(- \otimes \Sigma\mathbb{S})$ is an equivalence with inverse $(- \otimes \Omega\mathbb{S})$, and hence so is the functor Σ. Thus, $\mathscr{V}$ is stable.

Note that $\Lambda 2$ is the discrete category on two objects, so that $\langle\!\langle 2 \rangle\!\rangle = \mathbb{S} \oplus \mathbb{S}$. Since $\mathscr{V}$ is stable, Corollary 5.13 implies we have a linearity formula. Thus, $\mathrm{tr}(\mathrm{id}_\Phi)$ is determined by two morphisms $\phi_a, \phi_b \colon \mathbb{S} \to \mathbb{S}$, and we have

$$\mathrm{tr}(\mathrm{colim}^\Phi(f)) = \phi_a \cdot \mathrm{tr}(f_a) + \phi_b \cdot \mathrm{tr}(f_b).$$

As in Example 4.2, now that we know that ϕ_a and ϕ_b exist, we can calculate them by considering some very special cases. On one hand, if X is $\mathbb{S}_2$ (which looks like $(\mathbb{S} \to \mathbb{S})$), then its cofiber is 0, so if f is the identity, we have

$$0 = \phi_a \cdot 1 + \phi_b \cdot 1.$$

On the other hand, if X is $(0 \to \mathbb{S})$, then its cofiber is $\mathbb{S}$, so if f is again the identity, we have

$$1 = \phi_a \cdot 0 + \phi_b \cdot 1.$$

Solving these equations, we obtain $\phi_b = 1$ and $\phi_a = -1$. ∎

Thinking of distinguished triangles as incoherent images of cofiber sequences, (6.2) reproduces the additivity formula of [May01, GPS14a] for traces in closed symmetric monoidal stable derivators as a special case of the general linearity formula. In particular, it yields our motivating example of Lefschetz numbers as follows. The stable homotopy category is a stable symmetric monoidal derivator. Applying the suspension spectrum functor Σ^∞_+ to a subspace $X \hookrightarrow Y$ and a continuous map $f \colon Y \to Y$ that preserves X yields a diagram in the stable homotopy category. The trace of $\Sigma^\infty_+ f$ is then the Lefschetz number of f, and similarly for f_X and f/X; thus (6.2) becomes (1.1).

6.3. REMARK. Amusingly, we can extract from this a quick proof of the fact that closed symmetric monoidal stable derivators are not just semi-additive but additive, i.e. their homsets are not just abelian monoids but abelian groups. The proof of Theorem 6.1 requires only semi-additivity, but concludes that there is a morphism $\phi_a \colon \mathbb{S} \to \mathbb{S}$ such that $\phi_a + \mathrm{id}_\mathbb{S} = 0$. Thus, tensoring any morphism $X \to Y$ with ϕ_a yields an additive inverse. This is somewhat shorter than the proof in [Gro13, Proposition 4.12 and Corollary 4.14], but unlike that proof it only applies when $\mathscr{V}$ is closed symmetric monoidal. (In Proposition 12.6 we will see that semi-additivity in the monoidal case can be similarly extracted.)

If we combine this with the Mayer-Vietoris sequence of [GPS14b], we have a corresponding "inclusion-exclusion" result for pushouts.

6.4. COROLLARY. *Suppose $\mathscr{V}$ is stable and we have a coherent span $X \in \mathscr{V}(\ulcorner)$:*

$$X_c \longleftarrow X_a \longrightarrow X_b$$

which is pointwise dualizable. Then its pushout $\mathrm{colim}(X)$ is also dualizable. Moreover, if $f\colon X \to X$ is an endomorphism, then

$$\mathrm{tr}(\mathrm{colim}(f)) = \mathrm{tr}(f_b) + \mathrm{tr}(f_c) - \mathrm{tr}(f_a). \tag{6.5}$$

PROOF. By [GPS14b, Theorem 6.1], from X we can construct $Y \in \mathscr{V}(2)$ with $Y_a \cong X_a$ and $Y_b \cong X_b \oplus X_c$, such that the cofiber $\mathsf{C}(Y)$ is isomorphic to $\mathrm{colim}(X)$. Moreover, the construction in the proof of this is functorial, so that $f\colon X \to X$ induces $g\colon Y \to Y$ such that $\mathsf{C}(g)$ and $\mathrm{colim}(f)$ can be identified. Thus, it suffices to invoke Theorem 6.1 and Example 4.2. ∎

7. Homotopy finite categories

The additivity results of the previous section can be significantly generalized, using the fact that all "finite" colimits can be constructed from pushouts (and initial objects). The relevant notion of "finite" in the homotopical case is slightly subtle, however. We say that A is **strictly homotopy finite** if its nerve NA contains only finitely many nondegenerate simplices, which is equivalent to asking that A is finite, skeletal, and with no nonidentity endomorphisms. We call A **homotopy finite** if is equivalent to a strictly homotopy finite category; for calculating colimits this is just as good.

We begin by proving a general theorem about constructing homotopy finite colimits out of pushouts. If $\mathscr{D}$ is a derivator and A a small category, we say that a full subcategory $\mathscr{E} \subseteq \mathscr{D}(\mathbb{1})$ is **closed under A-colimits** if whenever $X \in \mathscr{D}(A)$ is such that $X_a \in \mathscr{E}$ for all $a \in A$, then also $\mathrm{colim}\, X \in \mathscr{E}$.

7.1. THEOREM. *Let A be a homotopy finite category.*

 (i) *If $\mathscr{D}$ is a derivator and $\mathscr{E} \subseteq \mathscr{D}(\mathbb{1})$ contains the initial object and is closed under pushouts, then it is closed under A-colimits.*

 (ii) *If $F\colon \mathscr{D} \to \mathscr{E}$ is a morphism of derivators that preserves the initial object and pushouts, then it preserves A-colimits.*

In the terminology of [AK88], this theorem says that in the world of derivators, homotopy finite colimits lie in the *closure* (later authors have preferred *saturation*) of pushouts and initial objects.

The idea of our proof of Theorem 7.1 is to construct colimits over A out of *geometric realizations* of (semi)simplicial *bar constructions*, which in turn can be computed using coproducts and pushouts. The latter fact is an instance of a more general theorem about colimits over Reedy categories, which we will prove first as a lemma.

Recall that a **Reedy category** is a small category C with a function deg$\colon$ ob$(C) \to \mathbb{N}$ and two subcategories $\overrightarrow{C}$ and $\overleftarrow{C}$ containing all the objects, such that every nonidentity map in $\overrightarrow{C}$ strictly raises degree, every nonidentity map in $\overleftarrow{C}$ strictly lowers degree, and every map in C factors uniquely as a map in $\overleftarrow{C}$ followed by one in $\overrightarrow{C}$. A good modern reference for Reedy categories is [RV14].

The best example to think of is the opposite $\mathbb{\Delta}^{\mathrm{op}}$ of the simplex category $\mathbb{\Delta}$, for which there is only one object $[n]$ of each degree n, while $\overrightarrow{\mathbb{\Delta}^{\mathrm{op}}}$ consists of the degeneracy maps and $\overleftarrow{\mathbb{\Delta}^{\mathrm{op}}}$ of the face maps. A $\mathbb{\Delta}^{\mathrm{op}}$-diagram in a category $\mathbf{V}$ is known as a **simplicial object** in $\mathbf{V}$, and when $\mathbf{V} = \mathbf{Top}$ is the category of topological spaces it is called a **simplicial space**. When a simplicial space X is "good" (or *proper* or *Reedy cofibrant*), its homotopy colimit is equivalent to its *geometric realization* $|X|$. The latter can be defined directly as the tensor product of functors $\Delta \otimes_{[\mathbb{\Delta}^{\mathrm{op}}]} X$, where $\Delta\colon \mathbb{\Delta} \to \mathbf{Top}$ takes $[n]$ to the standard n-simplex Δ^n. However, it is well-known in classical algebraic topology (see e.g. [EKMM97, X.2.4] or [Shu06, 23.10]) that $|X|$ can also be constructed as a sequential colimit *filtered by degree*:

$$|X| = \mathrm{colim}\left(|X|_0 \hookrightarrow |X|_1 \hookrightarrow |X|_2 \hookrightarrow \dots\right)$$

in which the spaces $|X|_n$ can be constructed inductively by a pair of pushout squares:

$$
\begin{array}{ccccc}
\partial\Delta^n \times s_n X & \longrightarrow & \partial\Delta^n \times X_n & & \\
\downarrow & & \downarrow & & \\
\Delta^n \times s_n X & \longrightarrow & P_n X & \longrightarrow & X_n \\
& & \downarrow & & \downarrow \\
& & |X|_{n-1} & \longrightarrow & |X|_n.
\end{array}
$$

Here $\partial\Delta^n$ is the boundary of Δ^n and is topologically an $(n-1)$-sphere, while $s_n X$ is the subspace of X_n consisting of the degenerate simplices (those in the image of a degeneracy map from X_k for some $k < n$). When X is "good", this sequential colimit and these pushouts are also homotopy colimits; thus we can view this as a construction of the homotopy colimit of X out of homotopy pushouts and a homotopy sequential colimit, which can thus be expressed entirely in the classical homotopy derivator $\mathscr{H}o(\mathbf{Top})$.

The lemma we prove next generalizes this construction to arbitrary derivators $\mathscr{D}$ and arbitrary Reedy categories C. For this, we need analogues of the spaces Δ^n, $\partial\Delta^n$, and $s_n X$. The space Δ^n is contractible, so when working up to homotopy we can omit it. For the other two, we introduce the following notation: if $c \in C$ with $\deg(c) = n$, $\partial(C/c)$ is the full subcategory of the slice category C/c whose objects are morphisms that factor through some object of degree strictly less than n, or equivalently which are not in $\overleftarrow{C}$. Likewise, we have the category $\partial(c/C)$ of morphisms with domain c that are not in $\overrightarrow{C}$.

Now if $X \in \mathscr{D}(C)$ and $c \in C$, let $L_c X$ denote the colimit of the restriction of X to $\partial(C/c)$. This is called the **latching object** of X at c; it generalizes the space $s_n X$ of

degenerate simplices. Similarly, we let ∂D^c denote the geometric realization of the nerve of $\partial(c/C)$; it generalizes the simplicial $(n-1)$-sphere $\partial\Delta^n$.

7.2. LEMMA. *Let C be a Reedy category, $\mathscr{D}$ a derivator, and $X \in \mathscr{D}(C)$. Then there is a coherent diagram whose shape is the poset $\mathbb{N}$ and whose colimit is $\operatorname{colim}(X)$:*

$$\emptyset = \operatorname{colim}_{-1}(X) \to \operatorname{colim}_0(X) \to \operatorname{colim}_1(X) \to \operatorname{colim}_2(X) \to \cdots . \tag{7.3}$$

Moreover, the morphisms $\operatorname{colim}_{n-1}(X) \to \operatorname{colim}_n(X)$ appear in cocartesian squares

$$
\begin{array}{ccc}
\coprod\limits_{\deg(c)=n} P_c X & \longrightarrow & \coprod\limits_{\deg(c)=n} X_c \\
\downarrow & & \downarrow \\
\operatorname{colim}_{n-1}(X) & \longrightarrow & \operatorname{colim}_n(X)
\end{array}
\tag{7.4}
$$

while the objects $P_c X$, for $c \in C$, appear in cocartesian squares

$$
\begin{array}{ccc}
\partial D^c \circledast L_c X & \longrightarrow & \partial D^c \circledast X_c \\
\downarrow & & \downarrow \\
L_c X & \longrightarrow & P_c X.
\end{array}
\tag{7.5}
$$

In the latter pushout, $\circledast$ denotes the tensoring of an arbitrary derivator $\mathscr{D}$ over $\mathscr{H}\!o(\mathbf{Top})$ which we introduced on page 620 in §5.

PROOF. In this proof we will denote $\mathscr{H}\!o(\mathbf{Top})$ by $\mathscr{S}$. Following the philosophy expressed by [RV14] as "it's all in the weights", we construct the desired data in $\mathscr{S}(C \times C^{\mathrm{op}})$ first. By [RV14, Observation 6.2], we have a sequence of monomorphisms in $\mathbf{Set}^{C \times C^{\mathrm{op}}}$ whose colimit is the hom-functor $C(-,-)$:

$$\emptyset \hookrightarrow \operatorname{sk}_0 C \hookrightarrow \operatorname{sk}_1 C \hookrightarrow \operatorname{sk}_2 C \hookrightarrow \ldots . \tag{7.6}$$

Moreover, the inclusions $\operatorname{sk}_{n-1} C \hookrightarrow \operatorname{sk}_n C$ are defined inductively (starting from $\operatorname{sk}_{-1} C = \emptyset$) by pushout diagrams

$$
\begin{array}{ccc}
\coprod\limits_{\deg(c)=n} Q_c & \hookrightarrow & \coprod\limits_{\deg(c)=n} C(c,-) \times C(-,c) \\
\downarrow & & \downarrow \\
\operatorname{sk}_{n-1} C & \hookrightarrow & \operatorname{sk}_n C
\end{array}
\tag{7.7}
$$

while the objects $Q_c \in \mathbf{Set}^{C \times C^{\mathrm{op}}}$, for $c \in C$, are defined by pushout diagrams

$$
\begin{array}{ccc}
\partial C(c,-) \times \partial C(-,c) & \hookrightarrow & \partial C(c,-) \times C(-,c) \\
\uparrow & & \uparrow \\
C(c,-) \times \partial C(-,c) & \hookrightarrow & Q_c.
\end{array}
\tag{7.8}
$$

Here $\partial C(-, c)$ denotes the subfunctor of the representable $C(-, c)$ consisting of morphisms factoring through some object of degree strictly less than $n = \deg(c)$, i.e. those morphisms that are not in $\overleftarrow{C}$. Similarly, $\partial C(c, -)$ is the subfunctor of $C(c, -)$ consisting of morphisms that are not in $\overrightarrow{C}$.

Now since (7.7) and (7.8) are pushouts along monomorphisms, they are also homotopy pushouts in **Top** (regarding sets as discrete spaces), and hence yield cocartesian squares in $\mathscr{S}$, or more precisely in the shifted derivator $\mathscr{S}^{C \times C^{\mathrm{op}}}$. Similarly, since (7.6) consists of monomorphisms, its colimit is a homotopy colimit, hence a colimit in $\mathscr{S}^{C \times C^{\mathrm{op}}}$.

We now apply the left Kan extension morphism $(\pi_C)_! \colon \mathscr{S}^{C \times C^{\mathrm{op}}} \to \mathscr{S}^{C^{\mathrm{op}}}$ to these diagrams. Since $(\pi_C)_!$ is cocontinuous, the images are again colimit diagrams in $\mathscr{S}^{C^{\mathrm{op}}}$. To compute them, notice first that $C(c, -) \in \mathscr{S}(C)$ is the left Kan extension of the unit $\mathbb{S}_{c/C}$ (which is also the terminal object) along the discrete opfibration $c/C \to C$; thus its colimit is $(\pi_{c/C})_!(\mathbb{S}_{c/C}) \cong |N(c/C)|$, which is contractible since c/C has an initial object. Similarly, $\partial C(c, -)$ is the left Kan extension of the unit along the discrete opfibration $\partial(c/C) \to C$, so its colimit is $|N\partial(c/C)|$, which we have christened ∂D^c. Thus, (7.8) becomes

$$
\begin{array}{ccc}
\partial D^c \times \partial C(-, c) & \hookrightarrow & \partial D^c \times C(-, c) \\
\uparrow & & \uparrow \\
\partial C(-, c) & \hookrightarrow & (\pi_C)_! Q_c
\end{array}
\tag{7.9}
$$

and (7.7) becomes

$$
\begin{array}{ccc}
\coprod\limits_{\deg(c)=n} (\pi_C)_! Q_c & \hookrightarrow & \coprod\limits_{\deg(c)=n} C(-, c) \\
\downarrow & & \downarrow \\
(\pi_C)_! \mathrm{sk}_{n-1} C & \hookrightarrow & (\pi_C)_! \mathrm{sk}_n C
\end{array}
\tag{7.10}
$$

while (7.6) becomes

$$
\emptyset \hookrightarrow (\pi_C)_! \mathrm{sk}_0 C \hookrightarrow (\pi_C)_! \mathrm{sk}_1 C \hookrightarrow (\pi_C)_! \mathrm{sk}_2 C \hookrightarrow \ldots
\tag{7.11}
$$

whose colimit is the terminal object $\mathbb{S}_{C^{\mathrm{op}}} \in \mathscr{S}^{C^{\mathrm{op}}}$.

Now we apply the functor $- \otimes_C X$. We define $\mathrm{colim}_n(X) = (\pi_C)_! \mathrm{sk}_n C \otimes_C X$. By essentially the same proof as Proposition 5.1, we have $\mathbb{S}_{C^{\mathrm{op}}} \otimes_C X \cong \mathrm{colim}(X)$; thus the sequence (7.11) yields (7.3) with colimit $\mathrm{colim}(X)$ as desired. It remains to extract (7.4) and (7.5) from (7.10) and (7.9).

As before, since $C(-, c) \in \mathscr{S}(C^{\mathrm{op}})$ is the left Kan extension of the unit along the discrete opfibration $(C/c)^{\mathrm{op}} \to C^{\mathrm{op}}$, by [GPS14a, Corollary 5.8] the weighted colimit $C(-, c) \otimes_C X$ is the ordinary colimit of the restriction of X to C/c. But C/c has a terminal object id_c, so this colimit is isomorphic to X_c. Thus, if we define $P_c X = (\pi_C)_! Q_c \otimes_C X$, the cocartesian square (7.10) yields (7.4).

Finally, $\partial C(-,c)$ is the left Kan extension of the unit along the discrete opfibration $(\partial(C/c))^{\mathrm{op}} \to C^{\mathrm{op}}$, so $\partial C(-,c) \circledast_C X$ is the ordinary colimit of the restriction of X to $\partial(C/c)$. In other words, it is the latching object $L_c X$. Since $\circledast$ associates with the product $\times$ in $\mathscr{S}$, from (7.9) we obtain (7.5).   ∎

PROOF OF THEOREM 7.1. Since $\mathscr{D}$ preserves equivalences of categories, we may assume A to be strictly homotopy finite. The same construction shows both parts, but we will speak only about (i); we leave it to the reader to deduce the other in a similar way. Thus, let $X \in \mathscr{D}(A)$ be such that $X_a \in \mathscr{E}$ for all $a \in A$; we want to show $\operatorname{colim} X \in \mathscr{E}$.

Let B be the opposite of the category of nondegenerate simplices in A. Its objects are strings of composable nonidentity arrows $a_0 \to a_1 \to \cdots \to a_n$ in A, and its morphisms are *face maps* generated by composing some of the arrows in a string, plus discarding some from the beginning and the end. (Composing nonidentity arrows can never produce an identity arrow, since A is strictly homotopy finite.)

There is a functor $q \colon B \to A$ sending each string to its first object. We claim it is homotopy final; recall that this means A-colimits in any derivator can equivalently be calculated as B-colimits after restriction along q. By [GPS14b, Corollary 3.13], it suffices to show that for any $a \in A$, the comma category (a/q) is homotopy contractible, i.e. has a contractible nerve. The objects of this category are strings of composable arrows $a \to a_0 \to a_1 \to \cdots \to a_n$ in which the first arrow $a \to a_0$ might be an identity. The subcategory of (a/q) consisting of those strings in which $a \to a_0$ *is* an identity is coreflective. But this subcategory has a terminal object, namely the string $a \xrightarrow{\mathrm{id}_a} a$. Thus, (a/q) is homotopy contractible, so q is homotopy final. It follows that $\operatorname{colim}(X) \cong \operatorname{colim}(q^* X)$.

Now since A is strictly homotopy finite, there is a maximum length of any string of composable nonidentity arrows in A; call that maximum n. Let $\mathbb{\Delta}'_n$ be the subcategory of the simplex category $\mathbb{\Delta}$ containing only the objects $[0]$ through $[n]$ and only the coface maps. Then there is a functor $p \colon B \to (\mathbb{\Delta}'_n)^{\mathrm{op}}$ sending each string to its length. Thus, $\operatorname{colim}(q^* X) \cong \operatorname{colim}(p_! q^* X)$. Moreover, since p is a discrete opfibration with finite fibers, each object occurring in $p_! q^* X$ is a finite coproduct of objects occurring in X. Since $\mathscr{E}$ contains the initial object and is closed under pushouts, it is also closed under finite coproducts by [GPS14b, Cor. 4.11].

The diagram $p_! q^* X$ looks like a (finite) classical semisimplicial bar construction:

$$\bigoplus_{a_0 \to \cdots \to a_n} X_{a_0} \; \underset{\vdots}{\overset{\vdots}{\rightrightarrows}} \; \cdots \; \underset{\vdots}{\overset{\vdots}{\rightrightarrows}} \; \bigoplus_{a_0 \to a_1 \to a_2} X_{a_0} \; \Rrightarrow \; \bigoplus_{a_0 \to a_1} X_{a_0} \; \rightrightarrows \; \bigoplus_a X_a$$

Therefore, we have reduced the problem to showing that $\mathscr{E}$ is closed under $(\mathbb{\Delta}'_n)^{\mathrm{op}}$-colimits. However, $(\mathbb{\Delta}'_n)^{\mathrm{op}}$ inherits a Reedy structure from $\mathbb{\Delta}^{\mathrm{op}}$, with all morphisms being in $\overleftarrow{(\mathbb{\Delta}'_n)^{\mathrm{op}}}$ and only identities in $\overrightarrow{(\mathbb{\Delta}'_n)^{\mathrm{op}}}$. Thus, by Lemma 7.2, for any $Y \in \mathscr{D}((\mathbb{\Delta}'_n)^{\mathrm{op}})$ we have a sequence

$$\emptyset \to \operatorname{colim}_0(Y) \to \operatorname{colim}_1(Y) \to \operatorname{colim}_2(Y) \to \cdots . \tag{7.12}$$

with colimit $\mathrm{colim}(Y)$. The cocartesian squares (7.4) and (7.5) reduce in the case $k \leq n$ to

$$
\begin{array}{ccc}
P_k Y & \longrightarrow & Y_k \\
\downarrow & & \downarrow \\
\mathrm{colim}_{k-1}(Y) & \longrightarrow & \mathrm{colim}_k(Y)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\partial D^k \circledast L_k Y & \longrightarrow & \partial D^k \circledast Y_k \\
\downarrow & & \downarrow \\
L_k Y & \longrightarrow & P_k Y.
\end{array}
$$

When $k > n$, then $(\Delta'_n)^{\mathrm{op}}$ has no objects of degree k, so we have $\mathrm{colim}_{k-1}(Y) \cong \mathrm{colim}_k(Y)$. Thus, (7.12) stabilizes at n, so that $\mathrm{colim}(Y) \cong \mathrm{colim}_n(Y)$.

Therefore, it will suffice to show by induction that if each $Y_j \in \mathcal{E}$, then $\mathrm{colim}_k(Y) \in \mathcal{E}$. The case $k = -1$ is the assumption that $\emptyset \in \mathcal{E}$. Since $\mathcal{E}$ is closed under pushouts, for the induction step it suffices to show that $P_k Y \in \mathcal{E}$. And since $\overrightarrow{(\Delta'_n)^{\mathrm{op}}}$ contains only identities, $L_k Y = \emptyset$, so that $P_k Y \cong \partial D^k \circledast Y_k$.

To complete the proof we show by induction on k that if $Z \in \mathcal{E}$ then $S^k \circledast Z \in \mathcal{E}$, where $S^k = \partial D^{k+1}$ is the topological k-sphere. If $k = 0$, then $S^0 \circledast Z = Z \oplus Z$, which is in $\mathcal{E}$ since $\mathcal{E}$ is closed under finite coproducts. Thus, suppose that $S^k \circledast Z \in \mathcal{E}$. Then we have a (homotopy) cocartesian square in $\mathscr{S}$:

$$
\begin{array}{ccc}
S^k & \longrightarrow & \star \\
\downarrow & & \downarrow \\
\star & \longrightarrow & S^{k+1}
\end{array}
$$

Since $\circledast$ is cocontinuous in its first variable, and the one-point space $\star$ is the unit of the monoidal structure on $\mathscr{S}$, we have an induced cocartesian square in $\mathscr{D}$:

$$
\begin{array}{ccc}
S^k \circledast Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Z & \longrightarrow & S^{k+1} \circledast Z
\end{array}
$$

Thus, $S^{k+1} \circledast Z$ is a pushout of objects in $\mathcal{E}$, hence also in $\mathcal{E}$. ∎

We note in passing that a slight modification of the proof of Theorem 7.1 shows the following. We will use this result in [PS14a].

7.13. THEOREM.

(i) If $\mathscr{D}$ is a derivator and $\mathcal{E} \subseteq \mathscr{D}(\mathbb{1})$ is closed under pushouts and coproducts, then it is closed under all colimits.

(ii) If $F \colon \mathscr{D} \to \mathcal{E}$ is a morphism of derivators that preserves pushouts and coproducts, then it is cocontinuous.

PROOF. As before, we discuss only (i). The proof of (ii) is analogous, except that we also invoke the fact that a morphism of derivators is cocontinuous (i.e. preserves left Kan extensions) as soon as it preserves colimits [Gro13, Proposition 2.3].

Given $A \in \mathcal{C}at$, we now let B be the category of *all* simplices in A, with only face maps between them. As in the previous proof $q: B \to A$ is homotopy final, and we have a discrete opfibration $p: B \to (\mathbb{\Delta}')^{\mathrm{op}}$, where $\mathbb{\Delta}'$ is the subcategory of $\mathbb{\Delta}$ containing all the objects but only the coface maps. Thus, if $X \in \mathscr{D}(A)$ then $\mathrm{colim}(X) \cong \mathrm{colim}(p_! q^* X)$, and the objects in $p_! q^* X$ are coproducts of those in X. Thus, it suffices to show that $\mathscr{E}$ is closed under $(\mathbb{\Delta}')^{\mathrm{op}}$-colimits.

Now $(\mathbb{\Delta}')^{\mathrm{op}}$ is again a Reedy category with only identities in $\overrightarrow{(\mathbb{\Delta}')^{\mathrm{op}}}$, so Lemma 7.2 applies. The same argument as in the proof of Theorem 7.1 shows that if $Y \in \mathscr{D}((\mathbb{\Delta}')^{\mathrm{op}})$ has each $Y_n \in \mathscr{E}$, then each $\mathrm{colim}_n(Y) \in \mathscr{E}$. Thus, it remains only to show $\mathscr{E}$ is closed under ω-colimits, where ω is the ordinal $(0 \leq 1 \leq 2 \leq \dots)$ regarded as a category.

The idea of this is a standard one: the sequential colimit of $Z_0 \to Z_1 \to Z_2 \to \cdots$ can equivalently be calculated as a coequalizer of two maps $\coprod_{i \geq 0} Z_i \rightrightarrows \coprod_{i \geq 0} Z_i$, one whose components are identities, and one whose components are the nonidentity maps in the diagram. To express this in derivator language, let Υ be the full sub-poset of $\omega \times \omega^{\mathrm{op}}$ whose objects have the form (n, n) or $(n, n+1)$, and let $v: \Upsilon \to \omega$ be the first projection. Then v is homotopy final, since for all $n \in \omega$ the comma category (n/v) has an initial object (namely $(n, n+1)$ with $n \xrightarrow{\mathrm{id}} v(n, n+1)$) and hence a contractible nerve. Thus, it suffices to consider Υ-colimits.

Now let P be the category $(1 \rightrightarrows 0)$, and define $u: \Upsilon \to P$ by $u(a, b) = b - a$, with morphisms of the form $(n, n+1) \to (n, n)$ going to one of the parallel arrows in P, and morphisms of the form $(n, n+1) \to (n+1, n+1)$ going to the other. Then u is a discrete opfibration, so $u_!$ computes the coproduct of each fiber; thus it suffices to consider P-colimits, i.e. coequalizers. But these are finite, so we can apply Theorem 7.1. (It is also easy to construct coequalizers explicitly out of pushouts in essentially the usual way.) ∎

Now we can prove a linearity formula for homotopy finite colimits. Unlike the proofs in the previous section, we establish this formula inductively using the formula for pushouts rather than showing the relevant weight is absolute. (We will show the weight is absolute in Example 12.5.)

Note that if A is strictly homotopy finite, then ΛA is just the discrete set A_0 of objects of A, so that $\langle\!\langle A \rangle\!\rangle = A_0 \cdot \mathbb{S}$.

7.14. THEOREM. *Let A be homotopy finite and let $\mathscr{V}$ be a stable, closed symmetric monoidal derivator. If $X \in \mathscr{V}(A)$ is pointwise dualizable, then $\mathrm{colim}(X)$ is right dualizable, and for any $f: X \to X$ we have*

$$\mathrm{tr}(\mathrm{colim}(f)) = \sum_a \mathrm{tr}(f_a) \cdot \sum_{k \geq 0} (-1)^k \cdot \# \left\{ \begin{array}{c} \textit{composable strings of nonidentity} \\ \textit{arrows of length } k \textit{ starting at } a \end{array} \right\} \quad (7.15)$$

PROOF. In Theorem 7.1(i), let $\mathscr{E} \subseteq \mathscr{V}(\mathbb{1})$ be the full subcategory of dualizable objects. This obviously contains the zero object, and is closed under pushouts by Corollary 6.4.

Thus, if $X \in \mathscr{V}(A)$ is pointwise dualizable, then $\mathrm{colim}(X)$ is right dualizable.

The inductive arguments in the proof of Theorem 7.1, together with Corollary 6.4, show that for any pointwise dualizable truncated semisimplicial diagram Y and any $g \colon Y \to Y$, we have

$$\mathrm{tr}(\mathrm{colim}_{k+1}(g)) = \mathrm{tr}(\mathrm{colim}_k(g)) + \mathrm{tr}(g_{k+1}) - \mathrm{tr}(S^k \circledast g_{k+1}).$$

Moreover, for any dualizable Z and $h \colon Z \to Z$ we have

$$\mathrm{tr}(S^{k+1} \circledast h) = 2 \, \mathrm{tr}(h) - \mathrm{tr}(S^k \circledast h).$$

Since $S^0 \circledast Z = Z \oplus Z$, we have $\mathrm{tr}(S^0 \circledast h) = 2 \, \mathrm{tr}(h)$, so by induction we have

$$\mathrm{tr}(S^k \circledast h) = \begin{cases} 2 \, \mathrm{tr}(h) & k \text{ is even} \\ 0 & k \text{ is odd.} \end{cases}$$

Thus, for Y and g as above, another induction yields

$$\mathrm{tr}(\mathrm{colim}(g)) = \sum_{0 \le k \le n+1} (-1)^k \, \mathrm{tr}(g_k). \tag{7.16}$$

In Theorem 7.1 $\mathrm{colim}(X)$ for $X \in \mathscr{V}(A)$ is constructed as the colimit of a truncated semisimplicial diagram Y where Y_k is the coproduct of the objects X_a over strings of composable nonidentity morphisms (nondegenerate simplices) in A. Since the projection from the opposite category of nondegenerate simplices to A selects the first object in a composable string, each simplex $(a_0 \to a_1 \to \cdots \to a_k)$ contributes a summand of X_{a_0}, so that we have

$$Y_k = \bigoplus_a \bigoplus_{\left\{ \substack{\text{composable strings of nonidentity} \\ \text{arrows of length } k \text{ starting at } a} \right\}} X_a.$$

Combining this with (7.16) and the simple additivity formula for coproducts, and rearranging summations, yields (7.15). ■

Note that Corollary 6.4 is also an instance of this formula, where A is the category $\{b \leftarrow a \to c\}$. Starting at each object there is one composable string of length 0, contributing $+1$, while the object a also has two strings of length 1, contributing -2; thus $\phi_b = \phi_c = 1$ while $\phi_a = -1$.

We end this section by comparing the linearity formula of Theorem 7.14 to the formula for the cardinality of a colimit in [Lei08, §3]. The latter depends on the notion of a **weighting** on a finite category A ([Lei08, Def. 1.10]), which is a function $k^\bullet$ from the objects of A to some (semi)ring such that for all $a \in A$,

$$\sum_{b \in A} \#(A(a,b)) \cdot k^b = 1.$$

Note that since $\#(A(a,b)) = \#(A(a,b'))$ whenever $b \cong b'$, the property of $k^\bullet$ being a weighting depends only on the numbers $k^C = \sum_{b \in C} k^b$, where C is an isomorphism class of objects in A. Thus, it is arguably more sensible to regard a weighting as a function defined on isomorphism classes of objects.

The linearity formula of [Lei08] is:

7.17. PROPOSITION. [Lei08, Proposition 3.1] *Let A be a finite category and $k^\bullet$ a $\mathbb{Q}$-weighting on A. If $X \colon A \to \mathbf{Set}$ is finite and a sum of representables then*

$$\#(\operatorname{colim} X) = \sum_a k^a \cdot \# X_a.$$

In fact, from this proposition, [Lei08, Corollary 1.5], and the comments after [Lei08, Definition 1.10], we obtain exactly the formula in Theorem 7.14 in the special case when X is a coproduct of representables. This restriction on X was necessary in [Lei08] essentially since $\mathbf{Set}$ is not stable, so that its colimits are not "homotopy correct". Passing to the stable case allows the formula to hold for all X.

On the other hand, Proposition 7.17 is stated for any finite category with a weighting rather than merely the homotopy finite ones. The next proposition, which was pointed out to us by Gallauer, relates our linearity formulas to weightings.

Let $\operatorname{End}(A)$ be the category whose objects are endomorphisms $h \colon a \to a$ in A, and whose morphisms from $h \colon a \to a$ to $k \colon b \to b$ are morphisms $\alpha \colon a \to b$ such that $\alpha h = k \alpha$. Note that there is a fully faithful inclusion $A \to \operatorname{End}(A)$ sending each object to its identity morphism.

Recall that an **EI-category** is a category in which every endomorphism is an isomorphism. We will study EI-categories in more detail in §9; for now we only need to observe that if A is an EI-category, then the isomorphism classes of objects in $\operatorname{End}(A)$ are precisely the conjugacy classes in automorphism groups of objects in A. Thus, if $\mathbb{S}_{A^{\mathrm{op}}}$ is absolute, its coefficient vector assigns an element of the semiring $\mathscr{V}(\mathbb{1})(\mathbb{S}, \mathbb{S})$ to each isomorphism class in $\operatorname{End}(A)$, making the following claim plausible.

7.18. PROPOSITION. *Suppose A is a finite EI-category such that $\mathbb{S}_{A^{\mathrm{op}}}$ is absolute in some semi-additive symmetric monoidal derivator $\mathscr{V}$. Then the components of the coefficient vector of $\mathbb{S}_{A^{\mathrm{op}}}$ form a weighting on $\operatorname{End}(A)$.*

PROOF. For any a, consider the profunctor $Y^a = (\operatorname{id} \times a)^* \mathbb{I}_A \in \mathbf{Prof}(\mathscr{V})(A, \mathbb{1})$. Since $(Y^a)_b \cong \bigoplus_{A(a,b)} \mathbb{S}$ and A is finite, Y^a is pointwise dualizable by Lemma 5.4. Any automorphism $\alpha \colon a \to a$ induces an automorphism Y^α of Y^a. By Lemma 5.11,

$$\operatorname{tr}(Y^\alpha) \colon \langle\!\langle A \rangle\!\rangle \cong \bigoplus_{\pi_0(\Lambda A)} \mathbb{S} \longrightarrow \mathbb{S}$$

has a component at $\beta \colon b \to b$ equal to the trace of the composed action of β and α on $\bigoplus_{A(a,b)} \mathbb{S}$. This is the number of fixed points of the composed action of β and α on the hom-set $A(a,b)$, i.e. the number of morphisms $f \colon a \to b$ such that $\beta f \alpha = f$, or equivalently such that f is a morphism from α^{-1} to β in $\operatorname{End}(A)$.

Since $\mathbb{S}_{A^{\mathrm{op}}}$ is absolute, $\mathbb{S}_{A^{\mathrm{op}}} \otimes_{[A]} Y^a$ is dualizable. Thus, the linearity formula for A-colimits allows us to express the trace of the endomorphism $\mathrm{id}_{\mathbb{S}_{A^{\mathrm{op}}}} \odot Y^\alpha$ as the sum

$$\sum_{[\beta]} \phi_\beta \cdot \#(\mathrm{End}(A)(\alpha^{-1}, \beta)).$$

The sum is over conjugacy classes of automorphisms in A, i.e. isomorphism classes of objects in $\mathrm{End}(A)$. Moreover, we have $\mathrm{colim}(Y^a) \cong a^* \mathbb{S}_{A^{\mathrm{op}}} \cong \mathbb{S}$, with $\mathrm{colim}(Y^\alpha) = \mathrm{id}_{\mathbb{S}}$. Thus,

$$1 = \sum_{[\beta]} \phi_{[\beta]} \cdot \#(\mathrm{End}(A)(\alpha^{-1}, \beta)).$$

This says exactly that the coefficients $\phi_{[\beta]}$ form a weighting on $\mathrm{End}(A)$. ∎

Of course, Proposition 7.17 is phrased in terms of a weighting on A rather than $\mathrm{End}(A)$, but in many cases we can extract the former from the latter.

7.19. COROLLARY. *Suppose A is as in Proposition 7.18 and that each automorphism group $A(b, b)$ acts freely on each hom-set $A(a, b)$. Then the components of the coefficient vector at identity endomorphisms ϕ_{id_a} form a weighting on A itself.*

PROOF. If in Proposition 7.18 we take α to be the identity automorphism id_a, then we see that

$$1 = \sum_{[\beta]} \phi_{[\beta]} \cdot \#(\mathrm{End}(A)(\mathrm{id}_{Y^a}, \beta)).$$

Moreover, the assumption ensures that

$$\#(\mathrm{End}(A)(\mathrm{id}_{Y^a}, \beta)) = \begin{cases} \#A(a, b) & \beta = \mathrm{id}_b \\ 0 & \beta \neq \mathrm{id}_b. \end{cases}$$

Thus, the previous equation reduces to

$$1 = \sum_{[b]} \phi_{[b]} \cdot \#(A(a, b))$$

which says exactly that the $\phi_{[b]}$ form a weighting on A. ∎

In particular, Corollary 7.19 applies if A contains no nonidentity endomorphisms, such as when it is homotopy finite. Thus, in such cases, our linearity formula for A-shaped colimits reduces to Proposition 7.17 when X is a coproduct of representables. However, for general X, the coefficients at nonidentity endomorphisms can matter.

For instance, consider the case when $A = \mathbf{B}G$ is a finite group G regarded as a one-object groupoid. We have seen in Theorem 4.12 that $\mathbf{B}G$-colimits are absolute in any $\#G$-divisible $\mathbf{V}$, and the coefficient at a conjugacy class $C \subseteq G$ is given by $\frac{\#C}{\#G}$. (We will generalize this result to derivators in §8.) Thus in this case, $\mathbf{B}G$ satisfies the hypotheses of Proposition 7.18 for the represented derivator $y(\mathbf{V})$, yielding the weighting consisting

of the single number $\frac{1}{\#G}$, as in [Lei08, Example 1.11(b)]. However, if X is $\mathbb{S}$ with the trivial G-action, then all components of $\chi(X)$ are equal to 1, so that our linearity formula gives

$$\chi(\mathrm{colim}(X)) = \sum_C \frac{\#C}{\#G} = 1. \tag{7.20}$$

Proposition 7.17 does not apply in this case, since this X is not a coproduct of representables. However, (7.20) is nevertheless correct even as a formula for the cardinality of the colimit of a trivial action on a one-element set.

On the other hand, if $\mathbb{S}_{A^{\mathrm{op}}}$ is absolute but the endomorphism monoids do not act freely, then our linearity formula can differ from Proposition 7.17 even for coproducts of representables. For instance, suppose A is the free-living idempotent, with one object $\star$ and one nonidentity idempotent $e \in A(\star, \star)$. Then as we have seen, $\mathbb{S}_{A^{\mathrm{op}}}$ is a retract of a representable, hence absolute, and its coefficient vector has components 0 and 1 at $\mathrm{id}_\star$ and e respectively. Thus, if $X \in \mathscr{V}(A)$ is an object equipped with a coherent idempotent, our linearity formula says that $\chi(\mathrm{colim}(X))$ is the trace of the action of this idempotent, as in §3.

However, the weighting assigned to A by [Lei08] is the single number $\frac{1}{2}$. And when $X \in \mathscr{V}(A)$ is a coproduct of n representables, its underlying object is a coproduct of $2n$ copies of $\mathbb{S}$, hence has Euler characteristic $2n$. Thus Proposition 7.17 says that the Euler characteristic of $\mathrm{colim}(X)$ is $\frac{1}{2}(2n) = n$. Our formula also gives the correct answer in this case, since the matrix of the action of the idempotent on a coproduct of n representables is block diagonal with n blocks of the form $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, hence has trace n. But it is a different formula.

8. The orbit-counting theorem

In this section we generalize the orbit-counting theorem, Theorem 4.12, to derivators. Let G be a finite group and let $\mathbf{B}G$ denote the one-object category associated to G. We will also denote the set G regarded as a discrete category by G.

The main ingredient in our proof is the following theorem, which essentially reduces the derivator case to the ordinary case. We say that a derivator $\mathscr{V}$ is n-**divisible** if $\mathscr{V}(\mathbb{1})$ is n-divisible; this implies that any morphism in any category $\mathscr{V}(A)$ can be "divided by n".

8.1. THEOREM. *If $\mathscr{V}$ is a semi-additive and $\#G$-divisible derivator, then the underlying diagram functor*

$$\mathscr{V}(\mathbf{B}G) \to \mathscr{V}(\mathbb{1})^{\mathbf{B}G}$$

is fully faithful.

We can interpret this theorem as the observation that being a coherent $\mathbf{B}G$-diagram is a mere *property* of an incoherent one. To our knowledge, this theorem first appeared ex-

plicitly in [GAdS14], although closely related statements can be found in [Coo78, Theorem 2.3] and [Opr84, Theorem 3].

To begin with, note that there is a canonical natural transformation

$$
\begin{array}{ccc}
G & \xrightarrow{\ \pi\ } & \mathbb{1} \\
{\scriptstyle \pi}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\
\mathbb{1} & \xrightarrow[u]{} & \mathbf{B}G.
\end{array}
\tag{8.2}
$$

whose component at $g \in G$ is just g regarded as a morphism in $\mathbf{B}G$. Since this is a comma square, the fourth defining property of a derivator [GPS14b, Definition 2.1] yields isomorphisms $\pi_!\pi^* \cong u^*u_!$ and $u^*u_* \cong \pi_*\pi^*$. Moreover, since G is discrete, we have $\pi_! Z \cong \sum_{g \in G} Z_g$ and $\pi_* Z \cong \prod_{g \in G} Z_g$.

The underlying diagram functor $\mathscr{V}(\mathbf{B}G) \to \mathscr{V}(\mathbb{1})^{\mathbf{B}G}$ associates to any object of $\mathscr{V}(\mathbf{B}G)$ an object in $\mathscr{V}(\mathbb{1})$ with a G-action. In particular, for any $Y \in \mathscr{V}(\mathbb{1})$, the objects $u^*u_!Y$ and u^*u_*Y of $\mathscr{V}(\mathbb{1})$ have canonical G-actions.

8.3. LEMMA. *In any derivator $\mathscr{V}$, the induced G-action on $u^*u_!Y \cong \sum_{g \in G} Y$ permutes the summands by left translation. Similarly, the G-action on $u^*u_*Y \cong \prod_{g \in G} Y$ permutes the factors by right translation.*

PROOF. For any $g \in G$, we have a natural transformation $g \colon u \to u$ with unique component g, and the action of g on u^*X for $X \in \mathscr{V}(\mathbf{B}G)$ is the image of this transformation under the 2-functor $\mathscr{V}$. Thus, for $Y \in \mathscr{V}(\mathbb{1})$, the induced isomorphism

$$
\pi_!\pi^* Y \xrightarrow{\sim} u^*u_!Y \xrightarrow{g} u^*u_!Y
$$

is the mate-transformation associated to the pasted rectangle on the left below.

$$
\begin{array}{ccccccc}
G & \xrightarrow{\pi} & \mathbb{1} & & G & \xrightarrow{\pi} & \mathbb{1} \\
{\scriptstyle \pi}\downarrow & \swarrow & \downarrow{\scriptstyle u} & & {\scriptstyle g\cdot -}\downarrow & & \| \\
\mathbb{1} & \xrightarrow{u} & \mathbf{B}G & = & G & \xrightarrow{\pi} & \mathbb{1} \\
\| & \swarrow_g & \| & & {\scriptstyle \pi}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\
\mathbb{1} & \xrightarrow{u} & \mathbf{B}G & & \mathbb{1} & \xrightarrow{u} & \mathbf{B}G
\end{array}
$$

However, this is equal to the pasted rectangle on the right. Thus, after composing again with the mate $\pi_!\pi^* \cong u^*u_!$ associated to the lower square on the right, it cancels, leaving only the left translation by g. The proof for u_* is dual. ∎

For the rest of this section, we suppose that G is finite and that $\mathscr{V}$ is a semi-additive derivator.

8.4. PROPOSITION. *When G is finite and $\mathscr{V}$ is semi-additive, there is a canonical isomorphism $\phi \colon u_! \xrightarrow{\sim} u_*$.*

PROOF. First of all, note that for any $Y \in \mathcal{V}(\mathbb{1})$ we have an isomorphism

$$
\begin{aligned}
\mathcal{V}(\mathbf{B}G)(u_!Y, u_*Y) &\cong \mathcal{V}(\mathbb{1})(u^*u_!Y, Y) \\
&\cong \mathcal{V}(\mathbb{1})(\pi_!\pi^*Y, Y) \\
&\cong \mathcal{V}(\mathbb{1})(\textstyle\sum_{g\in G} Y, Y) \\
&\cong \prod_{g\in G} \mathcal{V}(\mathbb{1})(Y, Y).
\end{aligned}
$$

Let $\phi\colon u_!Y \to u_*Y$ be the morphism that corresponds under this isomorphism to the family of morphisms $(\phi_g\colon Y \to Y)_{g\in G}$ defined by

$$
\phi_g = \begin{cases} \mathrm{id}_Y & g = e \\ 0 & g \neq e, \end{cases}
$$

where $e \in G$ is the identity element. We will show that ϕ is an isomorphism. It suffices to show that $u^*\phi$ is an isomorphism.

By a similar argument, we have

$$
\begin{aligned}
\mathcal{V}(\mathbb{1})(u^*u_!Y, u^*u_*Y) &\cong \mathcal{V}(\mathbb{1})(\pi_!\pi^*Y, \pi_*\pi^*Y) \\
&\cong \mathcal{V}(\mathbb{1})\big(\textstyle\sum_{g\in G} Y, \prod_{h\in G} Y\big) \\
&\cong \prod_{g,h\in G} \mathcal{V}(\mathbb{1})(Y, Y).
\end{aligned}
$$

We have written $\sum$ and $\prod$ for clarity in this isomorphism, but since $\mathcal{V}$ is semi-additive, finite sums and products are isomorphic and may be written as direct sums $\bigoplus$. With this identification, isomorphisms such as

$$
\mathcal{V}(\mathbb{1})\big(\textstyle\bigoplus_i Z_i, \bigoplus_j W_j\big) \cong \prod_{i,j} \mathcal{V}(\mathbb{1})(Z_i, W_j)
$$

allow us to identify composition of morphisms between finite direct sums with matrix multiplication.

We will show that $u^*\phi$ is a permutation matrix, and hence invertible. Note that for any $g \in G$, we have a 2-cell

$$
\begin{array}{ccc}
\mathbb{1} & =\!=\!= & \mathbb{1} \\
\| & \Swarrow g & \downarrow u \\
\mathbb{1} & \xrightarrow{\ u\ } & \mathbf{B}G
\end{array}
\tag{8.5}
$$

This gives rise to mates $\breve{g}\colon Y \to u^*u_!Y$ and $\widehat{g}\colon u^*u_*Y \to Y$. Moreover, since (8.5) factors through the square (8.2) by the morphism $g\colon \mathbb{1} \to G$, if we make the identifications

$$
u^*u_!Y \cong \pi_!\pi^*Y \cong \bigoplus_{g\in G} Y \qquad \text{and} \qquad u^*u_*Y \cong \pi_*\pi^*Y \cong \bigoplus_{g\in G} Y
$$

then $\breve{g}$ and $\widehat{g}$ correspond respectively to the inclusion of the gth summand and the projection onto the gth factor. Therefore, the (g, h)-component of $u^*\phi$ can be obtained as the composite

$$Y \xrightarrow{\breve{g}} u^*u_!Y \xrightarrow{u^*\phi} u^*u_*Y \xrightarrow{\widehat{h}} Y. \tag{8.6}$$

Similarly, the g-component of ϕ can be obtained as the composite

$$Y \xrightarrow{\breve{g}} u^*u_!Y \xrightarrow{\overline{\varphi}} Y$$

where $\overline{\varphi}$ is the map corresponding to ϕ under the adjunction $u^* \dashv u_*$. However, since $\overline{\varphi}$ is by definition the composite $u^*u_!Y \xrightarrow{u^*\phi} u^*u_*Y \xrightarrow{\epsilon} Y$, where ϵ is the counit of the adjunction $u^* \dashv u_*$, we can also express the g-component of ϕ as

$$Y \xrightarrow{\breve{g}} u^*u_!Y \xrightarrow{u^*\phi} u^*u_*Y \xrightarrow{\epsilon} Y. \tag{8.7}$$

Now note that ϵ is in fact $\widehat{e}$ (recall that $e \in G$ is the identity element), since when $g = e$ the square (8.5) commutes. Thus, comparing (8.6) and (8.7), we obtain an immediate identification $(u^*\phi)_{g,e} = \phi_g$.

Now observe that for any $g, h \in G$, we have

Thus, by the functoriality of mates, the composite

$$u^*u_*Y \xrightarrow{g^*} u^*u_*Y \xrightarrow{\widehat{h}} Y$$

is equal to $\widehat{hg}$, where g^* denotes the image of the 2-cell $g \colon u \to u$ under the 2-functor $\mathscr{V}$. Similarly, the composite

$$Y \xrightarrow{\breve{g}} u^*u_!Y \xrightarrow{h^*} u^*u_!Y$$

is equal to $\breve{hg}$. Now we have the commutative diagram

which shows that

$$(u^*\phi)_{g,h} = \phi_{hg} = \begin{cases} \text{id}_Y & h = g^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

This is a permutation matrix, as claimed. $\blacksquare$

8.8. LEMMA. *For any $X \in \mathscr{V}(\mathbf{B}G)$, after applying u^* the composite*

$$X \xrightarrow{\eta} u_* u^* X \xleftarrow{\sim} u_! u^* X \xrightarrow{\epsilon} X \tag{8.9}$$

is multiplication by $\#G$.

PROOF. The proof of Proposition 8.4 also showed that for any $Y \in \mathscr{V}(\mathbb{1})$, the map $\epsilon_Y \colon u^* u_* Y \to Y$ is $\widehat{e}$, which (under the identifications of $u^* u_*$ and $u^* u_!$ with $\#G$-ary direct sums) corresponds to projection onto the eth factor. In particular, this is the case for $\epsilon_{u^* X} \colon u^* u_* u^* X \to u^* X$. Since $u^* \eta_X \colon u^* X \to u^* u_* u^* X$ is a section of $\epsilon_{u^* X}$, its component onto the eth factor must be the identity. However, $u^* \eta_X$ is also an equivariant map of G-objects in $\mathscr{V}(\mathbb{1})$, and by Lemma 8.3 G acts by right translation on $u^* u_* u^* X$. Therefore, the component of $u^* \eta_X$ onto the gth factor must be the action of g.

Similarly, we can show that the component of $u^* \epsilon_X$ out of the gth summand must be the action of g. Moreover, we have seen in the proof of Proposition 8.4 that the middle factor $u^* u_! u^* X \xrightarrow{\sim} u^* u_* u^* X$ is a permutation matrix which sends the gth summand to the $(g^{-1})^{\text{th}}$ summand. Therefore, u^* of the composite (8.9) is the sum

$$\sum_{g,h \in G} h^{-1} \circ \phi_{hg} \circ g$$

with ϕ_{hg} as in Proposition 8.4. But this is equally

$$\sum_{g \in G} (g^{-1} \circ g) = (\#G) \cdot \mathrm{id}. \qquad \blacksquare$$

PROOF OF THEOREM 8.1. Since $\#G$ is invertible by assumption, Lemma 8.8 implies the composite

$$u^* X \xrightarrow{u^* \eta} u^* u_* u^* X \xleftarrow{\sim} u^* u_! u^* X \xrightarrow{u^* \epsilon} u^* X$$

is invertible. Hence so is (8.9).

In particular, every $X \in \mathscr{V}(\mathbf{B}G)$ is a natural retract of $u_! u^* X$. Therefore, if $X, X' \in \mathscr{V}(\mathbf{B}G)$, the hom-set $\mathscr{V}(\mathbf{B}G)(X, X')$ is a retract of $\mathscr{V}(\mathbf{B}G)(u_! u^* X, X')$. Moreover, this retraction is preserved by the underlying diagram functor $\mathscr{V}(\mathbf{B}G) \to \mathscr{V}(\mathbb{1})^{\mathbf{B}G}$. Thus, it will suffice to show that

$$\mathscr{V}(\mathbf{B}G)(u_! u^* X, X') \longrightarrow \mathscr{V}(\mathbb{1})^{\mathbf{B}G}(u^* u_! u^* X, u^* X') \tag{8.10}$$

is an isomorphism. However, by adjunction we have

$$\mathscr{V}(\mathbf{B}G)(u_! u^* X, X') \cong \mathscr{V}(\mathbb{1})(u^* X, u^* X'),$$

and by Lemma 8.3, $u^* u_! u^* X \cong \bigoplus_{g \in G} u^* X$ is the free G-object on $u^* X$ in $\mathscr{V}(\mathbb{1})$. Thus, the domain and codomain of (8.10) are isomorphic, and it is straightforward to check that this is indeed the desired map. $\blacksquare$

8.11. COROLLARY. *If $\mathscr{V}$ is a semi-additive derivator, then the following bicategories with shadows are equivalent:*

(i) *The full sub-bicategory of $\mathbf{Prof}(\mathscr{V})$, as defined in §5, whose objects are groupoids A such that $\mathscr{V}$ is $\#\mathrm{Aut}_A(a)$-divisible for each $a \in A$.*

(ii) *The locally full sub-bicategory of $\mathbf{Prof}(\mathscr{V}(\mathbb{1}))$, as constructed in §2, with the same objects as above, and whose 1-cells are the underlying incoherent diagrams of coherent ones.*

More precisely, since $\mathscr{V}(\mathbb{1})$ is not cocomplete (though it does have some colimits, like coproducts), the claim is that it does have the necessary colimits in order to define the bicategory (ii) by the same formulas that would be used in $\mathbf{Prof}(\mathscr{V}(\mathbb{1}))$.

PROOF. The bicategories in question have the same objects, and by Theorem 8.1 they have equivalent hom-categories. Thus, it suffices to show that their composition and identities are the same; or, more precisely, that the (homotopy) colimits and left Kan extensions used in defining (i) are also ordinary colimits and Kan extensions in $\mathscr{V}(\mathbb{1})$.

To this end, if U denotes the underlying diagram functor, then Theorem 8.1 implies that whenever $\mathscr{V}$ is $\#G$-divisible, we have

$$\mathscr{V}(\mathbb{1})(\pi_! X, Y) \cong \mathscr{V}(\mathbf{B}G)(X, \pi^*(Y)) \cong \mathscr{V}(\mathbb{1})^{\mathbf{B}G}(UX, U\pi^* Y).$$

Note that $U\pi^* Y$ is the ordinary constant $\mathbf{B}G$-shaped diagram on $Y \in \mathscr{V}(\mathbb{1})$. Therefore, this composite isomorphism exhibits $\pi_! X$ as a colimit of UX in $\mathscr{V}(\mathbb{1})$.

In other words, although $\mathscr{V}(\mathbb{1})$ is not cocomplete, it *does* have colimits of those $\mathbf{B}G$-shaped diagrams that underlie some coherent diagram, and these colimits agree with the corresponding homotopy colimits in $\mathscr{V}$. More generally, if A is a groupoid such that $\mathscr{V}$ is $\#\mathrm{Aut}_A(a)$-divisible for each $a \in A$, then $\mathscr{V}(\mathbb{1})$ has an (ordinary) colimit of any (incoherent) A-diagram that underlies some coherent one, since colimits over such an A are just colimits over the groups $\mathbf{B}\mathrm{Aut}_A(a)$ followed by a coproduct. We can extend this further to left Kan extensions along Grothendieck opfibrations whose fibers are groupoids of this sort. Moreover, any such colimit of a diagram in $\mathscr{V}(\mathbb{1})$ agrees with the corresponding homotopy colimit of any lift of that diagram to a coherent diagram in $\mathscr{V}$. Since all the colimits and Kan extensions used in defining the bicategory (ii) and its shadow are of this sort (note in particular that $\mathrm{tw}(\mathbf{B}G)$ is equivalent to $\mathbf{B}G$ itself) this completes the proof. ∎

Now we can extract the linearity formula and obtain a generalized orbit-counting theorem.

8.12. THEOREM. *Suppose G is a finite group and $\mathscr{V}$ is a stable and $\#G$-divisible symmetric monoidal derivator. Then the constant diagram $(\pi_{\mathbf{B}G^{\mathrm{op}}})^* \mathbb{S}$ is absolute, and its coefficient vector*

$$\mathrm{tr}(\mathrm{id}_{(\pi_{\mathbf{B}G^{\mathrm{op}}})^* \mathbb{S}}) \colon \mathbb{S} \longrightarrow \langle\!\langle \mathbf{B}G \rangle\!\rangle \cong \bigoplus_C \mathbb{S}$$

has components $\phi_C = \frac{\#C}{\#G}$, where C ranges over the set of conjugacy classes of G. Therefore, if $X \in \mathscr{V}(\mathbf{B}G)$ is pointwise dualizable and $f\colon X \to X$, then

$$\mathrm{tr}(\mathrm{colim}(f)) = \sum_C \frac{\#C}{\#G} \mathrm{tr}(X_C \circ f),$$

where X_C denotes the action of any element of C on X.

PROOF. As we have seen, the underlying diagram of $u_!\mathbb{S}$ is $\bigoplus_{g\in G}\mathbb{S} \in \mathscr{V}(\mathbb{1})^{\mathbf{B}G}$ with the right G-action that permutes the summands. Similarly, the underlying diagram of $(u^{\mathrm{op}})_!\mathbb{S}$ is the same coproduct with the corresponding left G-action. In Theorem 4.12 we observed that these two underlying diagrams are the representables $\mathbf{B}G(\mathrm{id}, a)$ and $\mathbf{B}G(a, \mathrm{id})$ in $\mathbf{Prof}(\mathscr{V}(\mathbb{1}))$, and hence dual to each other. Therefore, by Corollary 8.11, they are also dual in $\mathbf{Prof}(\mathscr{V})$.

Similarly, the underlying diagram of $\pi^*\mathbb{S} \in \mathscr{V}(\mathbf{B}G)$ is the constant diagram at $\mathbb{S}$. In Theorem 4.12 we showed that this diagram is a retract of $\mathbf{B}G(\mathrm{id}, a)$. Thus, by Corollary 8.11, $\pi^*\mathbb{S}$ is also a retract of $u_!\mathbb{S}$ in $\mathbf{Prof}(\mathscr{V})$, and therefore absolute.

Finally, the construction of the coefficient vector as a bicategorical trace is also preserved by the equivalence of Corollary 8.11. Thus, our calculation of the coefficients for $\mathbf{Prof}(\mathscr{V}(\mathbb{1}))$ in Theorem 4.12 yields the same result in $\mathbf{Prof}(\mathscr{V})$.   ∎

There is an easy extension to finite groupoids. Up to equivalence, a finite groupoid A is a finite disjoint union of categories $\mathbf{B}G$, where G ranges over the automorphism groups $\mathrm{Aut}_A(a)$ of isomorphism classes of objects in A. Thus, the conjugacy classes of A can be identified with pairs $([a], C)$ where $[a]$ is an isomorphism class of objects in A and C is a conjugacy class in $\mathrm{Aut}_A(a)$.

8.13. THEOREM. *Suppose $\mathscr{V}$ is a stable, closed, symmetric monoidal derivator and A is a finite groupoid such that for each object $a \in A$, $\mathscr{V}$ is $\#\mathrm{Aut}_A(a)$-divisible. If $X \in \mathscr{V}(A)$ is such that each X_a is right dualizable, then $\mathrm{colim}(X)$ is also right dualizable, and for any $f\colon X \to X$ we have*

$$\mathrm{tr}(\mathrm{colim}(f)) = \sum_{[a]} \sum_C \frac{\#C}{\#\mathrm{Aut}_A(a)} \mathrm{tr}(X_C \circ f_a). \tag{8.14}$$

PROOF. Without loss of generality assume A is skeletal, and let A_0 denote the set of objects of A. Then we have a projection $p\colon A \to A_0$ that is an opfibration, so for each $a \in A_0$ the pullback square

$$
\begin{array}{ccc}
\mathbf{B}(\mathrm{Aut}_A(a)) & \xrightarrow{\ r\ } & A \\
\downarrow & & \downarrow{\scriptstyle p} \\
\mathbb{1} & \xrightarrow{\ a\ } & A_0
\end{array}
$$

gives an isomorphism $(\pi_{\mathbf{B}(\mathrm{Aut}_A(a))})_! r^* \to a^* p_!$ as in [Gro13, Prop. 1.24]. Thus, $(p_! X)_a$ is the colimit of the restriction of X to $\mathbf{B}\mathrm{Aut}_A(a)$, which is dualizable by Theorem 8.12.

But $\mathrm{colim}(X) \cong \mathrm{colim}(p_! X)$, and the colimit over A_0 is a finite coproduct, hence also preserves dualizability. Formula (8.14) follows immediately from Theorem 8.12 and the simple additivity formula for coproducts.   ∎

9. EI categories

Finally, we combine the results of §§7–8 to prove a linearity theorem for colimits over **EI-categories**, i.e. categories in which every endomorphism is an isomorphism. Thus, every endomorphism monoid is in fact an automorphism group. Two extreme examples of EI-categories are groupoids (since all morphisms are isomorphisms) and posets (since all endomorphisms are identities). Another standard example of an EI-category is the **orbit category** $\mathscr{O}_G$ of a finite group G, whose objects are the transitive G-sets G/H and whose morphisms are G-maps.

If an EI-category is skeletal (as we may assume without loss of generality), then a morphism in it is invertible if and only if its domain and codomain are the same. There is thus a partial order induced on the objects, where $a \leq a'$ means there is an arrow $a \to a'$, and $a < a'$ if and only if there is a noninvertible such arrow. Consider finite strings of composable noninvertible arrows $a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$; we include strings of length zero, which are just objects of A. We say that two such strings are isomorphic if we have a commutative diagram

$$
\begin{array}{ccccccc}
a_0 & \xrightarrow{\alpha_1} & a_1 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_n} & a_n \\
\downarrow{\cong} & & \downarrow{\cong} & & & & \downarrow{\cong} \\
a_0 & \xrightarrow{\beta_1} & a_1 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_n} & a_n
\end{array}
$$

in which the vertical arrows are isomorphisms (hence automorphisms). In particular, we can consider the set of isomorphism classes of such strings, and the automorphism group $\mathrm{Aut}(\vec{\alpha})$ of any such string $\vec{\alpha}$.

Finally, any EI-category A has a **core** $A_\cong$, which is a groupoid obtained by discarding all noninvertible arrows. Since any cyclically composable chain of morphisms in an EI-category consists of isomorphisms, we have $\Lambda A \cong \Lambda(A_\cong)$, and thus

$$
\langle\!\langle A \rangle\!\rangle \cong \bigoplus_{[a]} \langle\!\langle \mathbf{B}\mathrm{Aut}_A(a) \rangle\!\rangle
$$

where $[a]$ ranges over isomorphism classes of objects of A. In particular, we can expect to describe a coefficient vector with components indexed by pairs $([a], C)$, where C is a conjugacy class in $\mathrm{Aut}_A(a)$.

9.1. Theorem. *Suppose that A is a finite EI-category, and that $\mathscr{V}$ is a stable, closed symmetric monoidal derivator which is $\#\mathrm{Aut}(\vec{\alpha})$-divisible for each $\vec{\alpha}$. If $X \in \mathscr{V}(A)$ is pointwise dualizable, then $\mathrm{colim}(X)$ is dualizable, and for any $f \colon X \to X$ we have*

$$
\mathrm{tr}(\mathrm{colim}(f)) = \sum_{[a]} \sum_C \mathrm{tr}(X_C \circ f_a) \cdot \sum_n (-1)^n \sum_{[\vec{\alpha}]} \sum_{\vec{C}} \frac{\#\vec{C}}{\#\mathrm{Aut}(\vec{\alpha})}.
$$

in which:

- $[a]$ *ranges over isomorphism classes of objects in* A*;*

- C *ranges over conjugacy classes in* $\mathrm{Aut}(a)$*;*

- n *ranges over natural numbers;*

- $[\vec{\alpha}]$ *ranges over isomorphism classes of strings* $\vec{\alpha}$ *of length* n *starting at* a*; and*

- $\vec{C}$ *ranges over conjugacy classes of* $\mathrm{Aut}(\vec{\alpha})$ *that restrict to* C *on* a*.*

PROOF. Without loss of generality, we may assume A is skeletal. Let B be its **preorder reflection**, i.e. the objects of B are those of A and there is a unique arrow $a \to a'$ in B precisely when there is some arrow $a \to a'$ in A. The assumptions that A is skeletal and an EI-category imply that B is also skeletal, i.e. a poset. In particular, it is homotopy finite.

Let D be the opposite of the category of nondegenerate simplices in B. The objects of D are strings $a_0 < a_1 < \cdots < a_n$, where $a < a'$ means that there is an arrow $a \to a'$ in A, but $a \neq a'$. The non-identity morphisms in D are given by deleting elements. More precisely, a morphism from $a_0 < a_1 < \cdots < a_n$ to $b_0 < b_1 < \cdots < b_m$ consists of an injection γ from $[m] = \{0, 1, \ldots, m\}$ to $[n]$ such that $a_{\gamma(i)} = b_i$ for all i.

Let $F \colon D \to \mathbf{Cat}$ be the functor sending such a string $a_0 < a_1 < \cdots < a_n$ to the category whose objects are composable strings $a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$ of (necessarily noninvertible) arrows in A, and whose morphisms are diagrams

$$
\begin{array}{ccccccc}
a_0 & \xrightarrow{\alpha_1} & a_1 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_n} & a_n \\
\downarrow{\cong} & & \downarrow{\cong} & & & & \downarrow{\cong} \\
a_0 & \xrightarrow[\beta_1]{} & a_1 & \xrightarrow[\beta_2]{} & \cdots & \xrightarrow[\beta_n]{} & a_n
\end{array}
\tag{9.2}
$$

in which the vertical arrows are automorphisms. The action of F on face maps in D is given by composing and discarding arrows.

Let E be the Grothendieck construction of F, with induced opfibration $p \colon E \to D$. Thus, the objects of E are composable strings $a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n$ of noninvertible morphisms, and a morphism from $a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n$ to $b_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_m} b_m$ consists of an injection $\gamma \colon [m] \hookrightarrow [n]$ and isomorphisms $\delta_i \colon a_{\gamma(i)} \cong b_i$ such that $\beta_i = \delta_i \alpha_{\gamma(i)} \alpha_{\gamma(i)-1} \cdots \alpha_{\gamma(i-1)+1} \delta_{i-1}^{-1}$ for all i. There is a functor $q \colon E \to A$ sending a string $a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n$ to the object a_0, and a morphism (γ, δ) as above to the composite $\alpha_{\gamma(0)} \alpha_{\gamma(0)-1} \cdots \alpha_1$. We claim that analogously to the proof of Theorem 7.1, this functor is homotopy final.

To see this, let $a \in A$. The objects of (a/q) are strings $a \xrightarrow{\alpha} a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n$ in which α *might* be invertible but the other α_i's are not. We claim first that the subcategory of such objects for which α is an identity is coreflective. Suppose given some such string; to construct its coreflection we divide into two cases according to whether α is invertible or not.

If α is not invertible, then the coreflection is $a \xrightarrow{\mathrm{id}_a} a \xrightarrow{\alpha} a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n$, with the counit discarding the first copy of a. A morphism (γ, δ) from $b_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_m} b_m$ to the given string factors through this uniquely by (γ', δ'), where $\gamma'(0) = 0$ and $\gamma'(j+1) = \gamma(j)$, while $\delta'_0 = \mathrm{id}_a$ and $\delta'_{j+1} = \delta_j$.

On the other hand, if α is invertible, so that in particular $a = a_0$, then the coreflection is $a \xrightarrow{\mathrm{id}_a} a \xrightarrow{\alpha_1 \alpha} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$, with counit $(\mathrm{id}_{[n]}, \epsilon)$ where $\epsilon_0 = \alpha$ and $\epsilon_{i+1} = \mathrm{id}_{a_{i+1}}$. Given a morphism (γ, δ) from $b_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_m} b_m$ to the given string, we necessarily have $b_0 = a = a_0$ and $\delta_0 = \alpha$. Thus (γ, δ) factors uniquely through $(\mathrm{id}_{[n]}, \epsilon)$ by (γ', δ'), where $\gamma' = \gamma$ while $\delta'_0 = \mathrm{id}_a$ and $\delta'_{j+1} = \delta_{j+1}$.

We have shown that every object of (a/q) coreflects into the subcategory of strings for which α is an identity. But this subcategory has a terminal object, namely the string $a \xrightarrow{\mathrm{id}_a} a$. Thus (a/q) is connected to $\mathbb{1}$ by a zigzag of adjoints, so it is homotopy contractible, and hence q is homotopy final. Therefore

$$\mathrm{colim}(X) \cong \mathrm{colim}(q^*X) \cong \mathrm{colim}(p_! q^* X).$$

However, since p is an opfibration, for each object $\vec{a} = (a_0 < \cdots < a_n)$ of D, the pullback square on the left of the diagram

$$
\begin{array}{ccccc}
p^{-1}(\vec{a}) & \xrightarrow{\ i\ } & E & \xrightarrow{\ q\ } & A \\
{\scriptstyle \pi}\downarrow & & {\scriptstyle p}\downarrow & & \\
\mathbb{1} & \xrightarrow[\ \vec{a}\ ]{} & D & & {\scriptstyle \pi}\downarrow \\
& & {\scriptstyle \pi}\downarrow & & \\
& & \mathbb{1} & \xrightarrow[\ \mathrm{id}\ ]{} & \mathbb{1}
\end{array}
$$

and [Gro13, Prop. 1.24] give an isomorphism $(\pi_{p^{-1}(\vec{a})})_! i^* \to (\vec{a})^* p_!$. The right half of the diagram above shows $(p_! q^* X)_{\vec{a}}$ is a colimit over $p^{-1}(\vec{a})$, which is a finite groupoid. Hence, by assumption and Theorem 8.13, $(p_! q^* X)_{\vec{a}}$ is dualizable for each $\vec{a}$. But $\mathrm{colim}(X) \cong \mathrm{colim}(p_! q^* X)$, and D is homotopy finite, so Theorem 7.14 completes the proof of dualizability.

We now extract the linearity formula from this construction. Suppose $f \colon X \to X$. The objects of E are strings $\vec{\alpha} = (a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n)$ of composable noninvertible arrows in A, and we have $(q^* X)_{\vec{\alpha}} = X_{a_0}$. Now for any object $\vec{a} = (a_0 < \cdots < a_n)$ of D, the isomorphism classes in $p^{-1}(\vec{a})$ are isomorphism classes $[\vec{\alpha}]$ connecting the objects $a_0, \dots, a_n$ in order. Thus we have

$$(p_! q^* X)_{\vec{a}} \cong \bigoplus_{[\vec{\alpha}]} (X_{a_0} / \mathrm{Aut}(\vec{\alpha}))$$

where $\mathrm{Aut}(\vec{\alpha})$ denotes the group of automorphisms of $\vec{\alpha}$ as in (9.2), which act on X_{a_0} via their first components $a_0 \xrightarrow{\sim} a_0$. Thus, by Theorem 8.13 and the simple additivity formula

for coproducts, the trace of the induced endomorphism of $(p_! q^* X)_{\vec{a}}$ is

$$\sum_{[\vec{\alpha}]} \sum_{\vec{C}} \frac{\#\vec{C}}{\#\mathrm{Aut}(\vec{\alpha})} \mathrm{tr}(X_{C_0} \circ f_{a_0})$$

where $\vec{C}$ ranges over conjugacy classes in $\mathrm{Aut}(\vec{\alpha})$, and X_{C_0} denotes the action of such a $\vec{C}$ on X_{a_0} via its first component. Now Theorem 7.14 yields the following formula for the trace of the induced endomorphism of $\mathrm{colim}(X) = \mathrm{colim}(p_! q^* X)$:

$$\mathrm{tr}(\mathrm{colim}(f)) = \sum_{\vec{a}} \left(\sum_{[\vec{\alpha}]} \sum_{\vec{C}} \frac{\#\vec{C}}{\#\mathrm{Aut}(\vec{\alpha})} \mathrm{tr}(X_{C_0} \circ f_{a_0}) \right) \cdot$$

$$\left(\sum_{k \geq 0} (-1)^k \cdot \# \left\{ \begin{array}{c} \text{composable strings of} \\ \text{nonidentity face maps of} \\ \text{length } k \text{ starting at } \vec{a} \in D \end{array} \right\} \right)$$

However, by Lemma 9.3 which we prove below, the last factor is simply equal to $(-1)^n$, where n is the length of $\vec{a}$. Therefore, rearranging the summations, we obtain

$$\mathrm{tr}(\mathrm{colim}(f)) = \sum_a \sum_C \mathrm{tr}(X_C \circ f_a) \cdot \sum_n (-1)^n \sum_{[\vec{\alpha}]} \sum_{\vec{C}} \frac{\#\vec{C}}{\#\mathrm{Aut}(\vec{\alpha})}$$

where a ranges over objects of A, C ranges over conjugacy classes of automorphisms of a, $[\vec{\alpha}]$ ranges over isomorphism classes of strings of n composable noninvertible arrows starting at a, and $\vec{C}$ ranges over conjugacy classes of automorphisms of $\vec{\alpha}$ which restrict to C on a.   ∎

9.3. LEMMA. *For any n, we have*

$$\sum_{k \geq 0} (-1)^k \cdot \# \left\{ \begin{array}{c} \text{composable strings of } k \text{ nonidentity} \\ \text{face maps starting at } [n] \in \mathbb{\Delta}^{\mathrm{op}} \end{array} \right\} = (-1)^n.$$

PROOF. This lemma has many proofs [GGT14]. Here we sketch a topological one; in §12 on page 666 we will give another. Let $g(n, k)$ be the number of composable strings of k nonidentity face maps starting at $[n]$; thus the claim is that

$$\sum_{k \geq 0} (-1)^k g(n, k) = (-1)^n.$$

Let Δ be the standard n-simplex and Δ' its barycentric subdivision. Then the k-simplices of Δ' are composable strings of $k + 1$ face maps starting at $[n] \in \mathbb{\Delta}^{\mathrm{op}}$ of which the first one might be an identity. Those for which the first map is not an identity are precisely those that do not contain the barycenter, i.e. that lie in the boundary $\partial\Delta'$. Therefore, for $k > 0$, $g(n, k)$ is the number of $(k-1)$-simplices in $\partial\Delta'$. Since the Euler

characteristic of a simplicial complex is the alternating sum of its simplices by dimension, we have

$$\chi(\partial\Delta') = \sum_{k \geq 1} (-1)^{k-1} g(n,k) = -\left(\sum_{k \geq 1} (-1)^k g(n,k)\right)$$

However, $\partial\Delta'$ is topologically an $(n-1)$-sphere, so this is also equal to $\chi(S^{n-1}) = 1 + (-1)^{n-1}$. After adding in $g(n,0) = 1$, we have the desired statement. ∎

Theorem 9.1 implies that if $\mathscr{V}$ is stable and *rational* (i.e. n-divisible for all nonzero integers n), then colimits of all finite EI-categories are absolute.[2] However, these are not the only absolute colimits in such a $\mathscr{V}$. For instance, if $p\colon B \to A$ is a homotopy final functor, then the colimit of $X \in \mathscr{V}(A)$ can be computed as the colimit of $p^*(X) \in \mathscr{V}(B)$; thus if B-colimits are absolute in $\mathscr{V}$, so are A-colimits.

As a concrete example, if $A = \mathbf{B}F_n$ is the one-object groupoid corresponding to the free group on n generators, then there is a homotopy final functor $p\colon B \to A$ where B has two objects b_1 and b_2 with $n + 1$ nonidentity arrows from b_1 to b_2. The functor p sends one of these arrows to the identity and the others to the free generators. Since B is finite, B-colimits are absolute in any stable $\mathscr{V}$, and hence so are A-colimits. The formula (7.15) gives $\mathrm{tr}(\mathrm{colim}(g)) = \mathrm{tr}(g_{b_2}) - n\,\mathrm{tr}(g_{b_1})$ for any endomorphism g of a pointwise dualizable $Y \in \mathscr{V}(B)$, whence $\mathrm{tr}(\mathrm{colim}(f)) = \mathrm{tr}(\mathrm{colim}(p^*(f))) = (1-n)\,\mathrm{tr}(f_\star)$ for any endomorphism f of a pointwise dualizable $X \in \mathscr{V}(A)$.

Thus, finiteness of the group G is not necessary for $\mathbf{B}G$-colimits to be absolute or to have a linearity formula. In fact, when $\mathscr{V}$ is the homotopy category of chain complexes, $\mathbf{B}G$-colimits are absolute whenever G is of *type FP* as defined in [Bro82, VIII.6]. See also [PS12, Example 8.7].

We might call a category **finally homotopy finite** if it admits a homotopy final functor from a homotopy finite category; these are a homotopical version of the duals of the *L-finite categories* of [Par90]. Thus, colimits over all such categories are absolute in stable derivators. More generally, we can consider categories admitting a homotopy final functor from a finite EI-category, whose colimits will be absolute in any rational stable derivator.

However, even this does not exhaust the absolute colimits in such derivators. We have seen that splitting of idempotents is absolute in *any* $\mathscr{V}$, and the free-living idempotent is not finally homotopy finite (see [Lur09, Example 4.4.5.1]). We do not know a characterization of all colimits that are absolute in any stable derivator or in any rational stable derivator.

9.4. REMARK. A related question is whether there is any $\mathscr{V}$ in which *all* colimits are absolute. We have seen in Theorem 7.14 that finite colimits are absolute in any stable $\mathscr{V}$, and in Example 4.3 that infinite coproducts are absolute in suplattices. However, these two properties are impossible to combine nontrivially, because of the "Eilenberg

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[2]To be precise, it only says that such colimits preserve dualizability, which is a *consequence* of absoluteness. However, when we generalize to bicategories, preserving dualizability becomes equivalent to absoluteness; see §12.

swindle". Specifically, if countably infinite coproducts are absolute in $\mathscr{V}$, then we can "add up countably many parallel morphisms" $\{f_i\colon X \to Y\}_{i\in\mathbb{N}}$ to get a single morphism $\sum_{i\in\mathbb{N}} f_i\colon X \to Y$, just as we can add up finitely many parallel morphisms in a semi-additive category. But if $\mathscr{V}$ is also additive, so that we can subtract morphisms, then for any $f\colon X \to Y$ we have

$$f = \left(f + \sum_{i\in\mathbb{N}} f\right) - \sum_{i\in\mathbb{N}} f = \sum_{i\in\mathbb{N}} f - \sum_{i\in\mathbb{N}} f = 0.$$

Since stable derivators are additive, countable coproducts cannot be absolute in any non-trivial stable monoidal derivator.

There are other ways to organize the terms in Theorem 9.1, one of which is the formula of [GAdS14]. As Gallauer pointed out in private communication, we can use Proposition 7.18 to see that all such expressions agree. The proof of Proposition 7.18 evidently applies to any linearity formula, however it is obtained. Now by [Lei08, Theorem 1.4], any skeletal finite EI-category has "Möbius inversion", which implies that it has a *unique* weighting. If A is a finite EI-category then so is $\mathrm{End}(A)$; thus the weighting in Proposition 7.18 is unique, and so any two linearity formulas must agree. (This would also follow from our uniqueness theorem (Theorem 12.1) if the formula of [GAdS14] can be generalized to bicategorical traces. This may be the case, but at this point we do not know.)

Part 3: Linearity in bicategories

In §3, we used bicategorical traces in the bicategory $\mathbf{Prof}(\mathbf{V})$ to prove linearity formulas for symmetric monoidal traces in a symmetric monoidal category $\mathbf{V}$. We generalized to derivators in §5. In this part, we generalize further to the case when we *start* with a bicategory (or a *derivator bicategory*) instead of a symmetric monoidal category, thereby obtaining linearity formulas for *bicategorical* traces.

The ease of this generalization is one of the primary motivations for our general approach to linearity. In particular, in §11 we will use it to generalize the additivity formula of [May01] to bicategorical traces, without having to generalize the complicated axioms of [May01] to the case of bicategorical traces. (This was done for the first four axioms (TC1)–(TC4) in [MS06, §16.7], but it is much more difficult to generalize the final axiom (TC5) to bicategories. It should be possible to generalize the version of May's proof for monoidal derivators presented in [GPS14a] to derivator bicategories, but the present approach avoids this question entirely.)

Just as the additivity formula of [May01] applies to the Lefschetz number, which is a trace in the symmetric monoidal derivator of spectra, the bicategorical version in §11 applies to an analogous invariant arising from the derivator bicategory of parametrized spectra called the *Reidemeister trace*. As remarked in §1, we postpone the details of the

application to Reidemeister trace to the companion paper [PS14a]; in §11 we will only sketch the argument.

As in Part 2, we postpone some details and proofs until Part 4.

10. Linearity in ordinary bicategories

Let $\mathbf{W}$ be a closed bicategory equipped with a shadow valued in a category $\mathbf{T}$. We assume that $\mathbf{W}$ is *locally complete and cocomplete*, i.e. its hom-categories $\mathbf{W}(R, S)$ are complete and cocomplete — since the composition functor $\odot$ has both adjoints, it automatically preserves colimits in each variable. We assume furthermore that $\mathbf{T}$ is cocomplete and the shadow functors $\langle\!\langle - \rangle\!\rangle \colon \mathbf{W}(R, R) \to \mathbf{T}$ are cocontinuous.

We now define a new closed bicategory $\mathbf{Prof}(\mathbf{W})$, with shadow also valued in $\mathbf{T}$, as follows.

- An object is a pair (A, R) where $A \in \mathcal{C}at$ and R is an object of $\mathbf{W}$.

- The hom-category from (A, R) to (B, S) is the functor-category $\mathbf{W}(R, S)^{A \times B^{\mathrm{op}}}$.

- The composite of $H \colon (A, R) \nrightarrow (B, S)$ and $K \colon (B, S) \nrightarrow (C, T)$ is defined by

$$(H \odot K)(a, c) = \int^{b \in B} H(a, b) \odot K(b, c).$$

  Note that this is a coend in the cocomplete hom-category $\mathbf{W}(R, T)$.

- The unit 1-morphism $\mathbb{I}_{(A,R)} \colon (A, R) \nrightarrow (A, R)$ is defined by the copower

$$\mathbb{I}_{(A,R)}(a, a') = A(a, a') \cdot \mathbb{I}_R.$$

- The shadow of $H \colon (A, R) \nrightarrow (A, R)$ is

$$\langle\!\langle H \rangle\!\rangle = \int^{a \in A} \langle\!\langle H(a, a) \rangle\!\rangle.$$

  Note that this is a coend in $\mathbf{T}$.

- The internal hom $\triangleright$ is defined for $H \colon (A, R) \nrightarrow (B, S)$ and $K \colon (C, T) \nrightarrow (B, S)$ by

$$(H \triangleright K)(c, a) = \int_{b \in B} H(a, b) \triangleright K(c, b),$$

  and similarly for $\triangleleft$.

The intent is to generalize as closely as possible the construction of $\mathbf{Prof}(\mathbf{V})$ from §3. Indeed, if we regard a monoidal category $\mathbf{V}$ as a bicategory with one object, the two constructions agree. A version of Remark 3.2 also applies here.

Now suppose given $X\colon A \to \mathbf{W}(R,S)$ and $\Phi\colon A^{\mathrm{op}} \to \mathbf{W}(R,R)$, which we can regard respectively as morphisms $(A,R) \nrightarrow (\mathbb{1},S)$ and $(\mathbb{1},R) \nrightarrow (A,R)$ in $\mathbf{Prof}(\mathbf{W})$. Thus, we have a composite $\Phi \odot X\colon (\mathbb{1},R) \nrightarrow (\mathbb{1},S)$, which is essentially just a morphism $R \nrightarrow S$ in $\mathbf{W}$; we call this the Φ-**weighted colimit of** X and denote it $\mathrm{colim}^{\Phi}(X)$. If Φ is furthermore constant at the unit object $\mathbb{I}_R$, then the Φ-weighted colimit of X is simply the ordinary colimit of $X\colon A \to \mathbf{W}(R,S)$ in the category $\mathbf{W}(R,S)$. Now we can apply Theorem 2.5 and Theorem 2.6.

10.1. DEFINITION.

- A functor $X\colon A \to \mathbf{W}(R,S)$ is **pointwise right dualizable** if it is right dualizable as a morphism $(A,R) \nrightarrow (\mathbb{1},S)$ in $\mathbf{Prof}(\mathbf{W})$.

- A weight $\Phi\colon A^{\mathrm{op}} \to \mathbf{W}(R,R)$ is **absolute** if it is right dualizable as a morphism $(\mathbb{1},R) \nrightarrow (A,R)$ in $\mathbf{Prof}(\mathbf{W})$.

10.2. LEMMA. *A functor $X\colon A \to \mathbf{W}(R,S)$ is pointwise right dualizable if and only if each 1-cell $X(a) \in \mathbf{W}(R,S)$ is right dualizable in $\mathbf{W}$.*

PROOF. Just like Lemma 3.5. ∎

10.3. THEOREM. *If $\Phi\colon A^{\mathrm{op}} \to \mathbf{W}(R,R)$ is absolute and $X\colon A \to \mathbf{W}(R,S)$ is pointwise right dualizable, then $\mathrm{colim}^{\Phi}(X)$ is right dualizable. In this case, for any $P\colon \mathbb{1} \to \mathbf{W}(S,S)$ and endomorphism $f\colon X \to X \odot P$, we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \mathrm{tr}(f) \circ \mathrm{tr}(\mathrm{id}_{\Phi}).$$

PROOF. By Theorem 2.5 and Theorem 2.6. ∎

Note that we are now including a twisting in the target of f; this is because the application to Reidemeister trace requires it. In this section, however, P will always be the unit $\mathbb{I}_S$. As observed in Remark 2.7, we could also include a source twisting if desired.

As before, in order to make use of this, we analyze the two factors further. The shadow of (A,R) in $\mathbf{Prof}(\mathbf{W})$ is

$$\langle\!\langle (A,R) \rangle\!\rangle = \left(\int^{a \in A} A(a,a) \right) \cdot \langle\!\langle R \rangle\!\rangle,$$

i.e. the copower of the shadow of (the identity 1-cell of) R by the set of conjugacy classes in the ordinary category A. Thus, for $X\colon A \to \mathbf{W}(R,S)$, $P\colon \mathbb{1} \to \mathbf{W}(S,S)$ and $f\colon X \to X \odot P$, the trace $\mathrm{tr}(f)\colon \langle\!\langle (A,R) \rangle\!\rangle \to \langle\!\langle P \rangle\!\rangle$ is determined by one morphism $\langle\!\langle R \rangle\!\rangle \to \langle\!\langle P \rangle\!\rangle$ for each conjugacy class of A, which we can identify using a component lemma.

10.4. LEMMA. [The component lemma for bicategories] *For any right dualizable 1-cell* $X \colon (A, R) \nrightarrow (\mathbb{1}, S)$ *with* $f \colon X \to X \odot P$, *and any morphism* $\alpha \in A(a, a)$, *the component* $\mathrm{tr}(f)_{[\alpha]}$ *is the trace in* $\mathbf{W}$ *of the composite*

$$ X_a \xrightarrow{\ X_\alpha\ } X_a \xrightarrow{\ f_a\ } X_a \odot P $$

PROOF. This lemma can be proven explicitly like Lemma 3.8, but we will deduce it from a more abstract result in §14 on page 677. ∎

Continuing with the analogy, for any absolute $\Phi \colon A^{\mathrm{op}} \to \mathbf{W}(R, R)$ we refer to

$$ \mathrm{tr}(\mathrm{id}_\Phi) \colon \langle\!\langle R \rangle\!\rangle \to \langle\!\langle (A, R) \rangle\!\rangle $$

as its **coefficient vector**. When the target category $\mathbf{T}$ of the shadow is semi-additive and A is finite, $\langle\!\langle (A, R) \rangle\!\rangle$ is a direct sum of copies of $\langle\!\langle R \rangle\!\rangle$ indexed by the conjugacy classes of A. Thus, $\mathrm{tr}(\mathrm{id}_\Phi)$ is a column vector with entries in the semiring $\mathbf{T}\big(\langle\!\langle R \rangle\!\rangle, \langle\!\langle R \rangle\!\rangle\big)$; as before, we denote these entries by $\phi_{[\alpha]}$ and call them the **coefficients** of Φ. Thus, we have a linearity formula for bicategorical traces:

10.5. COROLLARY. *If the target category* $\mathbf{T}$ *of the shadow is semi-additive,* A *is finite, and* $\Phi \colon A^{\mathrm{op}} \to \mathbf{W}(R, R)$ *is absolute, then we have*

$$ \mathrm{tr}(\mathrm{colim}^\Phi f) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot \mathrm{tr}(f_a \circ X_\alpha) $$

for any pointwise dualizable $X \colon A \to \mathbf{W}(R, S)$, *any 1-cell* $P \in \mathbf{W}(S, S)$, *and any 2-cell* $f \colon X \to X \odot P$.

This formula is syntactically identical to (3.10); the only difference is that now X is a functor $A \to \mathbf{W}(R, S)$, Φ is a functor $A^{\mathrm{op}} \to \mathbf{W}(R, R)$, f is a twisted endomorphism of X, and the equation is between morphisms $\langle\!\langle R \rangle\!\rangle \to \langle\!\langle P \rangle\!\rangle$ in $\mathbf{T}$.

We now have essentially all the same examples of absolute weights and coefficient vectors that we had in §3.

10.6. EXAMPLE. Let A be the empty category, and $\Phi \colon A \to \mathbf{W}(R, R)$ the unique $\mathbf{W}$-profunctor for some object R of $\mathbf{W}$. If $U \colon A \to \mathbf{W}(T, R)$ is the unique $\mathbf{W}$-profunctor for some other object T, then $\mu_{\Phi, U}$ from (2.3) is the unique map from an initial object to a terminal object of $\mathbf{W}(T, R)$, which is an isomorphism for all of these categories just when $\mathbf{W}$ is **locally pointed**, i.e. each category $\mathbf{W}(R, S)$ is pointed. In this case, the Φ-weighted colimit of the unique $X \colon A \nrightarrow \mathbf{W}(R, S)$ is the zero object of $\mathbf{W}(R, S)$, which is therefore pointwise dualizable. The shadow of A is the initial object of $\mathbf{T}$, so when $\mathbf{T}$ is pointed as well (as is usually the case), the trace of the identity morphism of $0 \in \mathbf{W}(R, S)$ is the zero morphism $\langle\!\langle R \rangle\!\rangle \to \langle\!\langle S \rangle\!\rangle$ in $\mathbf{T}$.

10.7. EXAMPLE. Let R be an object of $\mathbf{W}$, and let A be the discrete category with two objects. Then $X\colon A \to \mathbf{W}(R,S)$ consists of a pair of 1-cells $X_a, X_b \in \mathbf{W}(R,S)$, and is pointwise right dualizable just when X_a and X_b are right dualizable in $\mathbf{W}$. If $\Phi\colon A^{\mathrm{op}} \nrightarrow \mathbf{W}(R,R)$ is constant at $\mathbb{I}_R$, then the weighted colimit $\mathrm{colim}^{\Phi}(X) = \Phi \odot X$ is the local coproduct $X_a \sqcup X_b$. Just as in Example 4.2, we conclude that if $\mathbf{W}$ is locally pointed, then $\mu_{\Phi,U}$ is the canonical map from a binary coproduct to a binary product, which is an isomorphism if $\mathbf{W}$ is *locally semi-additive*. In this case, our Φ is absolute, so that local binary coproducts preserve right dualizability.

Now the shadow of (A, R) is the coproduct $\langle\!\langle R \rangle\!\rangle \oplus \langle\!\langle R \rangle\!\rangle$ in $\mathbf{T}$, and for a pointwise right dualizable $X\colon A \to \mathbf{W}(R,S)$ and an endomorphism $f\colon X \to X$, the trace $\mathrm{tr}(f)\colon \langle\!\langle R \rangle\!\rangle \oplus \langle\!\langle R \rangle\!\rangle \to \langle\!\langle S \rangle\!\rangle$ has components $\mathrm{tr}(f_a)$ and $\mathrm{tr}(f_b)$. If $\mathbf{T}$ is also semi-additive, then the coefficient vector of Φ is determined by two components $\phi_a, \phi_b\colon \langle\!\langle R \rangle\!\rangle \to \langle\!\langle R \rangle\!\rangle \oplus \langle\!\langle R \rangle\!\rangle$, which as before we can determine to both be 1 by a judicious choice of X. Thus, we have the linearity formula

$$\mathrm{tr}(f_a \oplus f_b) = \mathrm{tr}(f_a) + \mathrm{tr}(f_b)$$

exactly as in Example 4.2, but now for bicategorical traces. For instance, when $\mathbf{W}$ is the bicategory of rings and bimodules, this yields the additivity of the Hattori-Stallings trace under direct sums.

All the other examples from §4 generalize to bicategories in an entirely analogous way. We leave the details to the reader.

11. Linearity in derivator bicategories

We now combine the theory of linearity for monoidal derivators (§5) with that for ordinary bicategories (§10) to obtain a theory of linearity for *derivator bicategories*. This is necessary for the application to Reidemeister trace [PS14a], which is a bicategorical trace but is linear in the *stable* sense of §5. It is also necessary for the uniqueness theorem in §12.

The definition of derivator bicategory is obtained from the definition of a bicategory by simply replacing all hom-categories with derivators.

11.1. DEFINITION. A **derivator bicategory** $\mathscr{W}$ consists of the following data.

- A collection of objects R, S, T, ….

- For each pair of objects R and S a derivator $\mathscr{W}(R,S)$. We think of the category $\mathscr{W}(R,S)(A)$ as the homotopy category of A-shaped diagrams in $\mathscr{W}(R,S)$.

- For each triple of objects R, S, and T, a morphism of derivators

$$\odot\colon \mathscr{W}(R,S) \times \mathscr{W}(S,T) \to \mathscr{W}(R,T).$$

That is, we have a pseudonatural transformation between 2-functors $\mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$, which has components

$$\mathscr{W}(R,S)(A) \times \mathscr{W}(S,T)(A) \to \mathscr{W}(R,T)(A).$$

- We require the morphisms $\odot$ to be cocontinuous in each variable separately [GPS14a, Definition 3.19].

- For each object R, a morphism of derivators $\mathbb{I}_R \colon y(\mathbb{1}) \to \mathscr{W}(R,R)$ (hence an object $\mathbb{I}_{R,A} \in \mathscr{W}(R,R)(A)$, varying pseudonaturally in $A \in \mathcal{C}at$).

- Natural unit and associativity isomorphisms, i.e. invertible modifications

$$
\begin{array}{ccc}
\mathscr{W}(R,S) \times \mathscr{W}(S,T) \times \mathscr{W}(T,U) & \xrightarrow{\mathrm{id}\times\odot} & \mathscr{W}(R,S) \times \mathscr{W}(S,U) \\
\downarrow{\scriptstyle \odot\times\mathrm{id}} & \Downarrow\cong & \downarrow{\scriptstyle \odot} \\
\mathscr{W}(R,T) \times \mathscr{W}(T,U) & \xrightarrow{\quad\odot\quad} & \mathscr{W}(R,U)
\end{array}
$$

$$
\mathscr{W}(R,S) \xrightarrow{(\mathrm{id},\mathbb{I}_S)} \mathscr{W}(R,S) \times \mathscr{W}(S,S) \qquad \mathscr{W}(R,S) \xrightarrow{(\mathbb{I}_R,\mathrm{id})} \mathscr{W}(R,R) \times \mathscr{W}(R,S)
$$

with $\Downarrow\cong$ to $\mathscr{W}(R,S)$ via $\odot$, and similarly $\Downarrow\cong$ to $\mathscr{W}(R,S)$ via $\odot$.

- The usual pentagon and unit axioms for a bicategory hold.

A derivator bicategory is **closed** if the morphisms $\odot$ participate in a two-variable adjunction of derivators.

As in [GPS14a], we define the **external composition** to be the composite

$$
\mathscr{W}(R,S)(A) \times \mathscr{W}(S,T)(B) \to \mathscr{W}(R,S)(A \times B) \times \mathscr{W}(S,T)(A \times B) \to \mathscr{W}(R,T)(A \times B)
$$

where the first maps are restrictions induced by the projections. Joint cocontinuity is defined in terms of this composition rather than the original *internal* composition.

Unsurprisingly, we also need to extend the notion of shadow to the derivator case.

11.2. DEFINITION. A **shadow** on a derivator bicategory $\mathscr{W}$ consists of a derivator $\mathscr{T}$ and cocontinuous morphisms of derivators

$$
\langle\!\langle - \rangle\!\rangle \colon \mathscr{W}(R,R) \to \mathscr{T}
$$

for each object R, together with invertible modifications

$$
\begin{array}{ccc}
\mathscr{W}(R,S) \times \mathscr{W}(S,R) & \xrightarrow{\quad\cong\quad} & \mathscr{W}(S,R) \times \mathscr{W}(R,S) \\
\downarrow{\scriptstyle \odot} & \Downarrow\cong & \downarrow{\scriptstyle \odot} \\
\mathscr{W}(R,R) \xrightarrow[\langle\!\langle-\rangle\!\rangle]{} & \mathscr{T} \xleftarrow[\langle\!\langle-\rangle\!\rangle]{} & \mathscr{W}(S,S)
\end{array}
$$

satisfying the usual compatibility axioms for a shadow ([PS13, Defn. 4.1]).

Note that just as we did in §10, we require the shadow functors to be cocontinuous (in the appropriate sense).

A derivator bicategory $\mathscr{W}$ has an **underlying ordinary bicategory** with the same objects, and whose hom-category from R to S is $\mathscr{W}(R, S)(\mathbb{1})$. If $\mathscr{W}$ has a shadow, then so does its underlying ordinary bicategory.

One obvious way to construct derivator bicategories is by taking the homotopy bicategory of a *model bicategory*. Recall that from a model category $\mathscr{C}$ with weak equivalences W, we define a derivator $\mathscr{H}o(\mathscr{C})$ by

$$\mathscr{H}o(\mathscr{C})(A) := (\mathscr{C}^A)[(W^A)^{-1}].$$

Motivated by this, we say a **model bicategory** is a closed bicategory $\mathscr{B}$ with model structures on each of the hom-categories $\mathscr{B}(R, S)$ that satisfy the pushout-product and unit axioms.

- (Pushout-Product Axiom) If $X \to Y$ and $K \to L$ are cofibrations, then the map

  $$(X \odot L) +_{(X \odot K)} (Y \odot K) \to (Y \odot L)$$

  is a cofibration, which is a weak equivalence if either of the maps $X \to Y$ or $K \to L$ are.

- (Unit Axiom) If $Q\mathbb{I}_R \to \mathbb{I}_R$ is a cofibrant replacement for a bicategorical unit $\mathbb{I}_R$, then

  $$Q\mathbb{I}_R \odot Y \to \mathbb{I}_R \odot Y \cong Y \qquad \text{and} \qquad X \odot Q\mathbb{I}_R \to X \odot \mathbb{I}_R \cong X$$

  are weak equivalences for any cofibrant X and Y.

If $\mathscr{B}$ additionally has a shadow, we call it a **Quillen shadow** if it takes values in a model category $\mathscr{T}$ and each functor $\langle\!\langle - \rangle\!\rangle\colon \mathscr{B}(R, R) \to \mathscr{T}$ is left Quillen.

11.3. EXAMPLE. Any monoidal model category can be regarded as a model bicategory with one object. If it is symmetric, then its identity functor is a Quillen shadow.

11.4. EXAMPLE. There is a model bicategory whose objects are noncommutative rings, and where $\mathscr{B}(R, S)$ is the category of unbounded chain complexes of R-S-bimodules with a projective model structure. The pushout product and unit axioms can be proven by adapting the arguments of [Hov99] from the monoidal case. It also has a Quillen shadow with values in chain complexes of abelian groups, which coequalizes the left and right actions.

Now we can state the following theorem.

11.5. THEOREM. *For any model bicategory $\mathscr{B}$, there is a closed derivator bicategory $\mathscr{H}o(\mathscr{B})$ whose objects are the objects of $\mathscr{B}$, and whose hom-derivator $\mathscr{H}o(\mathscr{B})(R,S)$ is the derivator determined by the model category $\mathscr{B}(R,S)$. Moreover, if $\mathscr{B}$ has a Quillen shadow, then $\mathscr{H}o(\mathscr{B})$ has a shadow.*

This follows Cisinski's result [Cis03] that the homotopy category of any model category can be enhanced to a derivator (an easier proof for combinatorial model categories can be found in [Gro13]). One can prove Theorem 11.5 by extending results of [GPS14a] from the monoidal case. We omit the details.

11.6. REMARK. Theorem 11.5 does have its limitations, however. In particular, we cannot use it to construct the derivator bicategory $\mathcal{E}x$ of parametrized spectra, which is the example relevant for [PS14a]. In that paper we instead enhance the construction of [Shu08, PS12] to construct a derivator bicategory from an *indexed monoidal derivator*, the latter of which can be obtained from an *indexed monoidal model category*.

We now proceed to generalize §§5 and 10 by constructing a bicategory $\mathbf{Prof}(\mathscr{W})$ from a derivator bicategory $\mathscr{W}$. In fact, now that we have the notion of derivator bicategory, we can prove a stronger result: this bicategory $\mathbf{Prof}(\mathscr{W})$ is in fact the underlying ordinary bicategory of a *derivator* bicategory $\mathcal{P}rof(\mathscr{W})$ (that is, we have $\mathbf{Prof}(\mathscr{W})((A,R),(B,S)) = \mathcal{P}rof(\mathscr{W})((A,R),(B,S))(\mathbb{1})$).

11.7. THEOREM. *Given a derivator bicategory $\mathscr{W}$, we can construct a derivator bicategory $\mathcal{P}rof(\mathscr{W})$, whose underlying ordinary bicategory we denote $\mathbf{Prof}(\mathscr{W})$. The latter is described as follows:*

- *An object is a pair (A,R) where $A \in \mathcal{C}at$ and R is an object of $\mathscr{W}$.*

- *The hom-category from (A,R) to (B,S) is $\mathscr{W}(R,S)(A \times B^{\mathrm{op}})$.*

- *The composition functors are*

$$\mathscr{W}(R,S)(A \times B^{\mathrm{op}}) \times \mathscr{W}(S,T)(B \times C^{\mathrm{op}}) \xrightarrow{\odot} \mathscr{W}(R,T)(A \times B^{\mathrm{op}} \times B \times C^{\mathrm{op}})$$

$$\xrightarrow{\int^{B}} \mathscr{W}(R,T)(A \times C^{\mathrm{op}}).$$

- *The unit object of (A,R) is $\mathbb{I}_{(A,R)} = (t,s)_! \mathbb{I}_{R,\mathrm{tw}(A)} \in \mathscr{W}(R,R)(A \times A^{\mathrm{op}})$.*

For the derivator bicategory $\mathcal{P}rof(\mathscr{W})$, the hom-derivators are defined by

$$\mathcal{P}rof(\mathscr{W})((A,R),(B,S))(C) = \mathscr{W}(R,S)(A \times B^{\mathrm{op}} \times C),$$

i.e. $\mathcal{P}rof(\mathscr{W})((A,R),(B,S))$ is the shifted derivator $\mathscr{W}(R,S)^{A \times B^{\mathrm{op}}}$. The composition and units are defined analogously. If $\mathscr{W}$ is closed, then so is $\mathcal{P}rof(\mathscr{W})$ (and hence also $\mathbf{Prof}(\mathscr{W})$).

Finally, if $\mathscr{W}$ has a shadow valued in a derivator $\mathscr{T}$, then so does $\mathcal{P}rof(\mathscr{W})$, defined for $H \in \mathbf{Prof}(\mathscr{W})((A, R), (A, R)) = \mathscr{W}(R, R)(A \times A^{\mathrm{op}})$ by

$$\langle\!\langle H \rangle\!\rangle = \int^{A} \mathfrak{s}^{*} \langle\!\langle H \rangle\!\rangle_{A \times A^{\mathrm{op}}}. \tag{11.8}$$

Here $\langle\!\langle - \rangle\!\rangle_{A \times A^{\mathrm{op}}}$ denotes the shadow functor $\mathscr{W}(R, R)(A \times A^{\mathrm{op}}) \to \mathscr{T}(A \times A^{\mathrm{op}})$, and $\mathfrak{s}$ is the symmetry as before. It follows that $\mathbf{Prof}(\mathscr{W})$ also has a shadow valued in $\mathscr{T}(\mathbb{1})$.

PROOF. For the ordinary bicategory $\mathbf{Prof}(\mathscr{W})$, the proof is essentially identical to the proof of [GPS14a, Theorem 5.9].

For the derivator bicategory, we note that just as a derivator $\mathscr{D}$ induces a *shifted* derivator $\mathscr{D}^{A}$ for any $A \in \mathcal{C}at$, with $\mathscr{D}^{A}(B) = \mathscr{D}(A \times B)$, a closed derivator bicategory $\mathscr{W}$ induces a shifted version $\mathscr{W}^{A}$ with $\mathscr{W}^{A}(R, S)(B) = \mathscr{W}(R, S)(A \times B)$. This follows from [GPS14a, Example 8.14]. Thus, applying Theorem 11.7 to $\mathscr{W}^{A}$, and then letting A vary, we obtain all the data and coherence axioms of $\mathcal{P}rof(\mathscr{W})$.

For closedness of $\mathcal{P}rof(\mathscr{W})$, we must show that the composition morphisms of $\mathcal{P}rof(\mathscr{W})$ are two-variable left adjoints. However, the two-variable morphism

$$\mathcal{P}rof(\mathscr{W})((A, R), (B, S)) \times \mathcal{P}rof(\mathscr{W})((B, S), (C, T)) \xrightarrow{\odot} \mathcal{P}rof(\mathscr{W})((A, R), (C, T))$$

can be regarded as a shifted version of the composition morphism of $\mathscr{W}$:

$$\mathscr{W}(R, S)^{A \times B^{\mathrm{op}}} \times \mathscr{W}(S, T)^{B \times C^{\mathrm{op}}} \to \mathscr{W}(R, T)^{A \times C^{\mathrm{op}}}$$

in which B is canceled but A and C are treated externally. This is a two-variable left adjoint by [GPS14a, Examples 8.15 and 8.16].

Suppose $\mathscr{W}$ has a shadow valued in $\mathscr{T}$. By cocontinuity of this shadow, (11.8) is equivalent to

$$\langle\!\langle H \rangle\!\rangle = \left\langle\!\!\left\langle \int^{A} \mathfrak{s}^{*} H \right\rangle\!\!\right\rangle_{\mathbb{1}}.$$

with $\int^{A}$ now denoting the coend in the derivator $\mathscr{W}(R, R)$ rather than in $\mathscr{T}$. For $H \in \mathbf{Prof}(W)((A, R), (B, S))$ and $K \in \mathbf{Prof}(\mathscr{W})((B, S), (A, R))$, we take the shadow isomorphism $\langle\!\langle H \odot K \rangle\!\rangle \cong \langle\!\langle K \odot H \rangle\!\rangle$ to be the composite

$$\begin{aligned}
\langle\!\langle H \odot K \rangle\!\rangle &\cong \int^{A} \left\langle\!\!\left\langle \int^{B} (H \odot K) \right\rangle\!\!\right\rangle_{A \times A^{\mathrm{op}}} \\
&\cong \int^{A} \int^{B} \langle\!\langle H \odot K \rangle\!\rangle_{A \times A^{\mathrm{op}} \times B \times B^{\mathrm{op}}} \\
&\cong \int^{B} \int^{A} \langle\!\langle K \odot H \rangle\!\rangle_{A \times A^{\mathrm{op}} \times B \times B^{\mathrm{op}}} \\
&\cong \int^{B} \left\langle\!\!\left\langle \int^{A} (K \odot H) \right\rangle\!\!\right\rangle_{B \times B^{\mathrm{op}}} \\
&\cong \langle\!\langle K \odot H \rangle\!\rangle.
\end{aligned}$$

(We have omitted the symmetry isomorphisms for brevity.) Combining the argument that proves the associativity of composition in $\mathbf{Prof}(\mathscr{V})$, [GPS14a, Lemma 5.12], with the shadow axiom for $\mathscr{W}$ proves the shadow axiom for $\mathbf{Prof}(\mathscr{W})$. We extend this construction to $\mathcal{P}rof(\mathscr{W})$ by shifting, i.e. we define

$$\mathcal{P}rof(\mathscr{W})((A, R), (A, R)) \to \mathscr{T}$$

to be the composite

$$\mathscr{W}(R, R)^{A \times A^{\mathrm{op}}} \xrightarrow{\langle\!\langle - \rangle\!\rangle} \mathscr{T}^{A \times A^{\mathrm{op}}} \xrightarrow{\int^A} \mathscr{T}.$$

Since shifting preserves cocontinuity, both of these morphisms are cocontinuous. ∎

In particular, this strengthens the result of [GPS14a, Theorem 5.9] which we cited in §5: if $\mathscr{V}$ is a closed symmetric monoidal derivator, then not only do we have a bicategory $\mathbf{Prof}(\mathscr{V})$, but we have a derivator bicategory $\mathcal{P}rof(\mathscr{V})$.

We suppose from now on that $\mathscr{W}$ is a closed derivator bicategory. As before, we define the Φ-**weighted colimit** of $X \in \mathscr{W}(R, S)(A)$ by $\Phi \in \mathscr{W}(R, R)(A^{\mathrm{op}})$ to be the composite $\mathrm{colim}^{\Phi}(X) = \Phi \odot X$ in $\mathbf{Prof}(\mathscr{W})$, where Φ and X are regarded as 1-cells $(\mathbb{1}, R) \nrightarrow (A, R)$ and $(A, R) \nrightarrow (\mathbb{1}, S)$ in $\mathbf{Prof}(\mathscr{W})$, respectively. We then have the analogue of Proposition 5.1.

11.9. PROPOSITION. *For any* $X \in \mathscr{W}(R, S)(A)$, *if* $\Phi = (\pi_{A^{\mathrm{op}}})^* \mathbb{I}_R$ *is constant at the unit 1-cell of* R, *then we have* $\mathrm{colim}^{\Phi}(X) \cong \mathrm{colim}(X)$, *where* $\mathrm{colim}(X)$ *denotes the usual colimit* $(\pi_A)_!(X)$ *in the derivator* $\mathscr{W}(R, S)$.

PROOF. Just like Proposition 5.1. ∎

11.10. DEFINITION.

- A coherent diagram $X \in \mathscr{W}(R, S)(A)$ is **pointwise dualizable** if it is right dualizable when regarded as a 1-cell $(A, R) \nrightarrow (\mathbb{1}, S)$ in $\mathbf{Prof}(\mathscr{W})$.

- A coherent diagram $\Phi \in \mathscr{W}(R, R)(A^{\mathrm{op}})$ is **absolute** if it is right dualizable when regarded as a 1-cell $(\mathbb{1}, R) \nrightarrow (A, R)$ in $\mathbf{Prof}(\mathscr{W})$.

Thus, by Theorem 2.5, we have:

11.11. THEOREM. *If* $X \in \mathscr{W}(R, S)(A)$ *is pointwise dualizable and* $\Phi \in \mathscr{W}(R, R)(A^{\mathrm{op}})$ *is absolute, then* $\mathrm{colim}^{\Phi}(X)$ *is dualizable.*

We also have a version of Lemma 3.5, whose proof is essentially identical.

11.12. LEMMA. $X \in \mathscr{W}(R, R)(A)$ *is pointwise dualizable if and only if each object* $X_a \in \mathscr{W}(R, R)(\mathbb{1})$ *is right dualizable in the underlying bicategory of* $\mathscr{W}$.

Thus, by Theorem 2.6, we have a linearity formula.

11.13. THEOREM. *If $X \in \mathscr{W}(R,S)(A)$ is pointwise dualizable and $\Phi \in \mathscr{W}(R,R)(A^{\mathrm{op}})$ is absolute, then for any $P \in \mathscr{W}(S,S)(\mathbb{1})$ and $f\colon X \to X \odot P$ we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \mathrm{tr}(f) \circ \mathrm{tr}(\mathrm{id}_{\Phi}).$$

In other words, the following diagram commutes in $\mathscr{T}(\mathbb{1})$:

$$
\begin{array}{ccc}
\langle\!\langle R \rangle\!\rangle & \xrightarrow{\mathrm{tr}(\mathrm{id}_{\Phi})} & \langle\!\langle (A,R) \rangle\!\rangle \\
& {\scriptstyle \mathrm{tr}(\mathrm{colim}^{\Phi}(f))} \searrow & \downarrow {\scriptstyle \mathrm{tr}(f)} \\
& & \langle\!\langle P \rangle\!\rangle
\end{array}
$$

Here we have identified the shadow $\langle\!\langle (\mathbb{1},R) \rangle\!\rangle$ in $\mathbf{Prof}(\mathscr{W})$ with the shadow $\langle\!\langle R \rangle\!\rangle$ in $\mathscr{W}$, which as usual is the shadow of the unit 1-cell $\mathbb{I}_R$. Our analysis of the shadows of units leading to eq. (5.8) can be repeated essentially verbatim to conclude that here we have

$$
\begin{aligned}
\langle\!\langle (A,R) \rangle\!\rangle &= (\pi_{\Lambda A})_! (\pi_{\Lambda A})^* \langle\!\langle R \rangle\!\rangle \\
&= |N\Lambda A| \circledast \langle\!\langle R \rangle\!\rangle
\end{aligned}
\tag{11.14}
$$

(computed in the derivator $\mathscr{T}$). Similarly, every conjugacy class $[\alpha]$ in A yields a uniquely determined morphism $\langle\!\langle R \rangle\!\rangle \xrightarrow{[\alpha]} \langle\!\langle (A,R) \rangle\!\rangle$ in $\mathscr{T}(\mathbb{1})$.

For this theorem to be useful we need to be able to compute $\mathrm{tr}(f)$ and $\mathrm{tr}(\mathrm{id}_{\Phi})$. As before, we have a component lemma that enables us to compute $\mathrm{tr}(f)$.

11.15. LEMMA. [The component lemma for derivator bicategories] *If $X \in \mathscr{W}(R,S)(A)$ is pointwise dualizable and $f\colon X \to X \odot P$, then for any conjugacy class $[a \xrightarrow{\alpha} a]$ in A, the composite*

$$\langle\!\langle R \rangle\!\rangle \xrightarrow{[\alpha]} \langle\!\langle (A,R) \rangle\!\rangle \xrightarrow{\mathrm{tr}(f)} \langle\!\langle P \rangle\!\rangle$$

is equal to the trace in $\mathscr{W}(R,S)(\mathbb{1})$ of the composite

$$X_a \xrightarrow{X_\alpha} X_a \xrightarrow{f_a} X_a \odot P_a.$$

PROOF. We will prove a generalization of this result in §14 on page 684. ∎

Finally, in the semi-additive case we can deduce a more familiar-looking formula.

11.16. DEFINITION. *If $\Phi \in \mathscr{W}(R,R)(A^{\mathrm{op}})$ is absolute, then $\mathrm{tr}(\mathrm{id}_{\Phi})\colon \langle\!\langle R \rangle\!\rangle \to \langle\!\langle (A,R) \rangle\!\rangle$ is called its **coefficient vector**. If the target derivator $\mathscr{T}$ of the shadow is semi-additive and we can express the coefficient vector of Φ as a linear combination*

$$\mathrm{tr}(\mathrm{id}_{\Phi}) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot [\alpha],$$

*for $\phi_{[\alpha]}\colon \langle\!\langle R \rangle\!\rangle \to \langle\!\langle R \rangle\!\rangle$ in $\mathscr{T}(\mathbb{1})$, then we refer to the $\phi_{[\alpha]}$ as the **coefficients** of Φ and say that Φ has a **coefficient decomposition**.*

11.17. COROLLARY. *If $\mathscr{T}$ is semi-additive and $\Phi \in \mathscr{W}(R,R)(A^{\mathrm{op}})$ is absolute and has a coefficient decomposition, then we have*

$$\mathrm{tr}(\mathrm{colim}^{\Phi} f) = \sum_{[\alpha]} \phi_{[\alpha]} \cdot \mathrm{tr}(f_a \circ X_\alpha)$$

for any pointwise dualizable $X \in \mathscr{W}(R,S)(A)$, $P \in \mathscr{W}(S,S)(\mathbb{1})$ and $f \colon X \to X \odot P$.

All the examples from the previous sections have generalizations to derivator bicategories. To give some idea of these generalizations, and because we have a use for it in [PS14a], we will give an outline of the generalization of Theorem 6.1.

We say that a derivator bicategory $\mathscr{W}$ is **locally** semi-additive, stable, or n-divisible if each derivator $\mathscr{W}(R,S)$ has the corresponding property.

11.18. THEOREM. *If $\mathscr{W}$ is a locally-stable closed derivator bicategory with a shadow valued in a stable derivator $\mathscr{T}$, $X \in \mathscr{W}(R,S)(\mathbb{2})$ is pointwise dualizable and $f \colon X \to X$, then*

$$\mathrm{tr}(\mathsf{C}(f)) = \mathrm{tr}(f_b) - \mathrm{tr}(f_a).$$

Note that this is an equality of morphisms $\langle\!\langle R \rangle\!\rangle \to \langle\!\langle S \rangle\!\rangle$ in $\mathscr{T}(\mathbb{1})$. We also have the twisted version, which applies to traces of any $f \colon X \to X \odot P$ with $P \in \mathscr{W}(S,S)(\mathbb{1})$.

OUTLINE OF PROOF. For any object R, let $\Phi \in \mathscr{W}(R,R)(\mathbb{2}^{\mathrm{op}})$ be the essentially unique diagram of the form $(0 \leftarrow \mathbb{I}_R)$. Just as in Theorem 6.1, we can conclude that Φ-weighted colimits are **local cofibers**, i.e. cofibers in the hom-derivators of $\mathscr{W}$, and that Φ is absolute whenever $\mathscr{W}$ is locally stable.

As before, since $\Lambda\mathbb{2}$ is discrete on two objects, we have $\langle\!\langle (\mathbb{2}, R) \rangle\!\rangle = \langle\!\langle R \rangle\!\rangle \oplus \langle\!\langle R \rangle\!\rangle$. Thus, $\mathrm{tr}(\mathrm{id}_\Phi)$ is determined by two morphisms $\phi_a, \phi_b \colon \langle\!\langle R \rangle\!\rangle \to \langle\!\langle R \rangle\!\rangle$, and we have

$$\mathrm{tr}(\mathrm{colim}^{\Phi}(f)) = \phi_a \cdot \mathrm{tr}(f_a) + \phi_b \cdot \mathrm{tr}(f_b)$$

for any $f \colon X \to X$. We calculate the coefficients exactly as before: the traces of identity maps of the cofibers of $(\mathbb{I}_R \to \mathbb{I}_R)$ and $(0 \to \mathbb{I}_R)$ are 0 and 1, respectively, while the trace of the identity of $\mathbb{I}_R$ is 1, so we have $0 = \phi_a + \phi_b$ and $1 = 0 + \phi_b$, whence $\phi_b = 1$ and $\phi_a = -1$. ∎

We now sketch the promised application to the Reidemeister trace; see [PS14a] for the details.

By a **parametrized space** over a topological space B, we mean a space X, called the **total space**, together with maps $B \xrightarrow{s} X \xrightarrow{p} B$ so that $p \circ s$ is the identity. A bicategory whose 0-cells are topological spaces, whose 1-cells from A to B are parametrized spaces over $A \times B$ (or more precisely, parametrized spectra), and whose 2-cells are stable homotopy classes of maps of total spaces that commute with the maps s and p is constructed in [MS06]. The bicategorical product, denoted $\odot$, is defined in terms of the pullback. In [PS14a] we will extend this to a derivator bicategory.

Let $i\colon X \hookrightarrow Y$ be an inclusion of spaces and suppose $f\colon Y \to Y$ is a continuous map so that $f(X) \subset X$. Then we have induced maps of parametrized spaces

$$\widehat{f|_X}\colon S_X^0 \to S_X^0 \odot S_{f|_X} \text{ and } \widehat{f}\colon S_Y^0 \to S_Y^0 \odot S_f.$$

Here S_Y^0 is the parametrized space $Y \amalg Y$ over Y, and S_f is the **twisted path space** of f (the space of triples (y_1, y_2, λ) where $y_1, y_2 \in Y$ and λ is a path from y_1 to $f(y_2)$), regarded as a space over $Y \times Y$ (technically, with a disjoint section adjoined). The definitions of S_X^0 and $S_{f|_X}$ are similar.

By composing with the inclusion i, we can define a parametrized space $i_!(S_X^0)$ over Y, with an induced map $i_!(\widehat{f|_X})\colon i_!(S_X^0) \to i_!(S_X^0) \odot S_f$. We then have a map of 2-diagrams of parametrized spaces over Y:

$$\widehat{f}_i\colon \big(i_!(S_X^0) \to S_Y^0\big) \to \big(i_!(S_X^0) \to S_Y^0\big) \odot S_f$$

whose components are $i_!(\widehat{f|_X})$ and $\widehat{f}$.

It is shown in [MS06] that if X and Y are closed smooth manifolds or compact ENRs, then S_X^0 and S_Y^0 are dualizable in the derivator bicategory of parametrized spectra. In this case, Theorem 11.18 implies

$$\operatorname{tr}(\widehat{f}) - \operatorname{tr}(i_!(\widehat{f|_X})) = \operatorname{tr}(\mathsf{C}(\widehat{f}_i))$$

as maps $S^0 \to \langle\!\langle S_f \rangle\!\rangle$. Note that this is a twisted trace, with twisting object $P = S_f$. The shadow of a parametrized space over $B \times B$ is its pullback along the diagonal (technically, with the section also quotiented out), so $\langle\!\langle S_f \rangle\!\rangle$ is the **twisted loop space** of f:

$$\Lambda^f Y := \{\lambda \in Y^I \mid f(\lambda(0)) = \lambda(1)\}.$$

Thus, $\operatorname{tr}(\widehat{f})$ is an element of the zeroth stable homotopy of $\Lambda^f Y$; it can be identified with the Reidemeister trace of f. Similarly, $\operatorname{tr}(i_!(\widehat{f_X}))$ can be identified with the image of the Reidemeister trace of $f|_X$ under the map $\Lambda^{f|_X} X \to \Lambda^f Y$ induced by i. The remaining piece, $\operatorname{tr}(\mathsf{C}(\widehat{f}_i))$, is the *relative Reidemeister trace* of f from [Pon11]; it is a refinement of the Reidemeister trace of the induced map $f/X\colon Y/X \to Y/X$ on the quotient space. Thus, in the end we obtain the formula mentioned in the introduction:

$$R(f) - i(R(f|_X)) = R_{Y|X}(f).$$

11.19. REMARK. The other examples from §§6–9 also generalize directly to the bicategorical case. For instance, if $\mathcal{W}$ is locally stable and $\mathcal{T}$ is stable, and $X \in \mathcal{W}(R, S)(\ulcorner)$ is a pointwise right dualizable span, then its pushout $\operatorname{colim}(X)$ is also right dualizable, and if $f\colon X \to X$ is an endomorphism, then

$$\operatorname{tr}(\operatorname{colim}(f)) = \operatorname{tr}(f_c) + \operatorname{tr}(f_b) - \operatorname{tr}(f_a)$$

as morphisms $\langle\!\langle R \rangle\!\rangle \to \langle\!\langle S \rangle\!\rangle$.

Similarly, if A is a finite EI-category, and $\mathscr{W}$ is locally stable and locally $\#\mathrm{Aut}(\vec{\alpha})$-divisible for each $\vec{\alpha}$, with a shadow valued in a stable and $\#\mathrm{Aut}(\vec{\alpha})$-divisible derivator $\mathscr{T}$, then for any pointwise dualizable $X \in \mathscr{W}(R,S)(A)$, its colimit $\mathrm{colim}(X) \in \mathscr{W}(R,S)(\mathbb{1})$ is dualizable, and for any $f\colon X \to X$ we have

$$\mathrm{tr}(\mathrm{colim}(f)) = \sum_{[a]} \sum_{C} \mathrm{tr}(X_C \circ f_a) \cdot \sum_{n}(-1)^n \sum_{[\vec{\alpha}]} \sum_{\vec{C}} \frac{\#\vec{C}}{\#\mathrm{Aut}(\vec{\alpha})}.$$

The notation is as in Theorem 9.1.

We omit the proofs of these generalizations, since they are remarkably similar to those of Corollary 6.4 and Theorem 9.1.

12. The uniqueness of linearity formulas

As an additional application of the extension of linearity to bicategorical trace, we now prove a uniqueness theorem for linearity formulas. So far, we have shown that *if* a weight Φ is absolute, *then* Φ-weighted colimits preserve dualizability, and moreover the coefficient vector of Φ yields *a* linearity formula. Now we will reverse these implications: we show that *if* Φ-weighted colimits preserve dualizability, *then* Φ is absolute; and that moreover in this case, the coefficient vector of Φ is the *only* linearity formula.

The main idea involved in the proof of the latter statement should not be surprising since we have already seen it in many examples. Namely, once we know that a linearity formula exists, we can deduce what its coefficients *must* be, by considering some simple examples. There are even a canonical set of examples to consider, namely the representable diagrams. This naive method doesn't quite work in all cases (for instance, we weren't able to use it to calculate the coefficients for the orbit-counting theorem), but a refinement of it does: we must consider not the individual representable diagrams, but their totality, as a 1-cell in the bicategory of profunctors.

For this we need linearity formulas that apply to bicategorical traces. It follows that the generality of linearity for bicategorical trace is necessary even to *state* the uniqueness theorem.

First we introduce a bit of terminology, to make precise the location in which our "coefficients" live when comparing linearity formulas in different categories. Let Z be a commutative semiring; a **Z-module** is a commutative monoid equipped with an associative, unital, bilinear action of Z. We say that a derivator $\mathscr{D}$ is **Z-linear** if each category $\mathscr{D}(A)$ is Z-linear (i.e. enriched over Z-modules) and each functor $u^*\colon \mathscr{D}(B) \to \mathscr{D}(A)$ is likewise Z-linear. This implies that the functors $u_!$ and u_* are also Z-linear.

We say that a derivator bicategory $\mathscr{W}$ is **locally Z-linear** if each derivator $\mathscr{W}(R,S)$ is Z-linear and the composition morphisms are Z-bilinear. In particular, for any object R of $\mathscr{W}$, we have a semiring homomorphism $Z \to \mathscr{W}(R,R)(\mathbb{1})(\mathbb{I}_R, \mathbb{I}_R)$, defined by multiplying by the identity. If $\mathscr{W}$ has a shadow valued in $\mathscr{T}$, we say that the shadow is **Z-linear** if $\mathscr{T}$ is Z-linear and the shadow morphisms $\mathscr{W}(R,R) \to \mathscr{T}$ are Z-linear.

For example, any locally semi-additive $\mathscr{W}$ is $\mathbb{N}$-linear. If it is locally additive (such as if it is locally stable), then it is $\mathbb{Z}$-linear. And it is n-divisible, as defined in §8, exactly when it is $\mathbb{N}[\frac{1}{n}]$-linear. (In these cases linearity is a mere property rather than a structure, since the unique map $\mathbb{N} \to Z$ from the initial semiring $\mathbb{N}$ is an epimorphism.)

Now we can state the uniqueness theorem. For simplicity, we consider only the case of constant weights ("conical colimits"); for the general case we would need a way to say in what sense two weights in different bicategories are "the same".

12.1. THEOREM. *Let Z be a commutative semiring and $\mathfrak{B}$ be a class of closed, locally semi-additive, Z-linear derivator bicategories with Z-linear shadows. Assume that $\mathfrak{B}$ is closed under $\mathcal{P}rof$, i.e. if $\mathscr{W} \in \mathfrak{B}$ then $\mathcal{P}rof(\mathscr{W}) \in \mathfrak{B}$. Let A be a finite category, and assume the following.*

(i) If $\mathscr{W} \in \mathfrak{B}$ and $X \in \mathscr{W}(R,S)(A)$ is pointwise right dualizable, then $\mathrm{colim}(X)$ is right dualizable.

(ii) There are elements $\phi_{[\alpha]} \in Z$ indexed by the conjugacy classes of A such that for any $\mathscr{W}$ and X as in (i) and any $f\colon X \to X$, we have

$$\mathrm{tr}(f) = \sum_{[\alpha]} \phi_{[\alpha]} \, \mathrm{tr}(f_a \circ X_\alpha).$$

Then for any $\mathscr{W} \in \mathfrak{B}$ and any object R of $\mathscr{W}$, the constant diagram

$$(\pi_{A^{\mathrm{op}}})^* \mathbb{I}_R \in \mathscr{W}(R,R)(A^{\mathrm{op}})$$

is absolute and has a coefficient decomposition, and its coefficients are (the images in $\mathscr{T}$ of) the $\phi_{[\alpha]}$.

PROOF. See page 685 in §15.   ∎

12.2. REMARK. As stated, this theorem does not imply that the elements $\phi_{[\alpha]} \in Z$ are themselves uniquely determined. It could be, for instance, that all the Z-linearity in $\mathfrak{B}$ factors through some quotient of Z. However, as soon as $\mathfrak{B}$ contains one derivator bicategory for which the map from Z to the center of $\mathscr{T}$ is injective, then the coefficients $\phi_{[\alpha]} \in Z$ are uniquely determined. Moreover, if Z is a commutative *ring*, then such a derivator bicategory always exists; for instance, we can generalize Example 11.4 to replace rings by Z-algebras. We are indebted to the referee for this remark.

To explain the role of Z and $\mathfrak{B}$, we show how this theorem applies to a few of our previous examples.

12.3. EXAMPLE. Consider the case $A = \{a,b\}$ of direct sums, as in Example 4.2. Let $Z = \mathbb{N}$ and let $\mathfrak{B}$ be the class of *all* closed, locally semi-additive derivator bicategories with semi additive shadows. This is clearly closed under $\mathcal{P}rof$. As remarked in Example 4.2 (in the monoidal case), it is not hard to prove by explicit calculation that if X and Y are right

dualizable in such a derivator bicategory, then so is $X \oplus Y$, and that for any $f \colon X \to X$ and $g \colon Y \to Y$ we have $\mathrm{tr}(f \oplus g) = \mathrm{tr}(f) + \mathrm{tr}(g)$. Therefore, Theorem 12.1 implies that the weights $(\pi_{A^{\mathrm{op}}})^* \mathbb{I}_R$ for coproducts are right dualizable in any locally semi-additive derivator bicategory, and their coefficients are 1 and 1.

12.4. EXAMPLE. Let $A = \ulcorner$ be the diagram for pushouts, as in Corollary 6.4 and Remark 11.19. Let $Z = \mathbb{Z}$ and let $\mathfrak{B}$ be the class of *locally stable* closed derivator bicategories with shadows valued in a stable derivator; this is also closed under $\mathcal{P}rof$. Combining the Mayer-Vietoris sequence of [GPS14b] with [GPS14a, Lemma 11.6], it follows that the pushout of right dualizable 1-cells in such a bicategory is again right dualizable. Now [GPS14a] showed, by generalizing the method of [May01], that the additivity formula (6.2) holds in any stable closed symmetric monoidal derivator. It might be possible to generalize this method to apply to derivator bicategories as well. If so, then in Corollary 6.4, from this and the additivity of direct sums we would obtain the additivity formula (6.5) for pushouts. Therefore, Theorem 12.1 would imply that the weights for pushouts are right dualizable in any locally stable derivator bicategory, and their coefficients are 1, 1, and -1 as in Corollary 6.4.

Note that in both of these cases, the uniqueness theorem is calculating the coefficient vector in essentially the same way that we did before: namely, once we know that a linearity formula exists, we can look at simple examples to identify its coefficients. Examples 12.3 and 12.4 do this in a more redundant way, deriving the general linearity formula first and then restricting it to the simple examples to conclude that its coefficients coincide with those of the coefficient vector.

There are other examples, however, where it does seem to be easier to proceed in this way. Specifically, if A-colimits can be constructed out of smaller colimits in some concrete way, then it is sometimes easier to derive the linearity formula for A-colimits directly from those for the smaller colimits, rather than to explicitly analyze the coefficient vector of A. Theorem 12.1 tells us that in this case, the coefficients in the resulting linearity formula are automatically the components of the coefficient vector itself.

12.5. EXAMPLE. In Theorem 7.14 we proved a linearity formula for homotopy finite colimits that holds in any stable, closed symmetric monoidal derivator, and in Remark 11.19 we observed that essentially the same proofs apply to any locally-stable closed derivator bicategory with a shadow that takes values in a stable derivator. The proof involved constructing such colimits out of pushouts, rather than directly analyzing the weight for absoluteness and calculating its coefficient vector. However, if we let $Z = \mathbb{Z}$ and $\mathfrak{B}$ be the class of locally-stable closed derivator bicategories, then Theorem 12.1 implies that in fact, for any homotopy finite A, the weight $(\pi_{A^{\mathrm{op}}})^* \mathbb{I}_R$ is absolute in any $\mathscr{W} \in \mathfrak{B}$, and its coefficients are precisely those appearing in Theorem 7.14.

Similarly, let A be a finite EI-category, let $Z = \mathbb{Z}[S^{-1}]$ where S is the set of cardinalities of automorphism groups of objects of A, and let $\mathfrak{B}$ be the class of S-divisible locally-stable closed derivator bicategories with S-divisible shadows. Then from Theorem 9.1 and

Theorem 12.1, we conclude that $(\pi_{A^{\mathrm{op}}})^*\mathbb{I}_R$ is absolute in any $\mathscr{W} \in \mathfrak{B}$, and its coefficients are those appearing in Theorem 9.1.

To be honest, in concrete applications the uniqueness theorem is not very important: usually what we care about is *having* a linearity formula, not about whether it comes from a coefficient vector. However, it does reassure us that our theory is "complete", in the sense that it includes all sufficiently general linearity formulas.

We end this section with two amusing applications. First, we give a quick proof that stable implies semi-additive in the monoidal or bicategorical case. (Compare to the proof of [Gro13, Proposition 4.7], which applies in any stable derivator.)

12.6. PROPOSITION. *Any locally-stable closed derivator bicategory is locally semi-additive (and hence locally additive, by Remark 6.3).*

PROOF. Suppose $\mathscr{W}$ is locally stable. Then for any 1-cells $X, Y \in \mathscr{W}(R, S)(\mathbb{1})$, we have two cocartesian squares

$$
\begin{array}{ccc}
\Omega X \longrightarrow 0 & & 0 \longrightarrow Y \\
\downarrow \qquad \downarrow & \text{and} & \downarrow \qquad \downarrow \\
0 \longrightarrow X & & 0 \longrightarrow Y
\end{array}
$$

(the first being cocartesian in addition to cartesian because of stability). Taking their coproduct, we have a cocartesian square

$$
\begin{array}{ccc}
\Omega X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X + Y,
\end{array}
$$

so $X + Y$ is the cofiber of a coherent morphism $\Omega X \to Y$. Now if X and Y are right dualizable, so is $\Omega X \cong \Omega \mathbb{I}_R \odot X$ by stability; thus by Theorem 6.1 $X + Y$ is also right dualizable.

We have shown that in any locally-stable closed derivator bicategory, the coproduct of two right dualizable 1-cells is right dualizable. Therefore, the first part of the uniqueness theorem implies that coproducts are absolute in any such $\mathscr{W}$, which is to say that it is locally semi-additive. ∎

Second, we give another proof of Lemma 9.3.

RESTATEMENT OF LEMMA 9.3. *For any n, we have*

$$
\sum_{k \geq 0} (-1)^k \cdot \# \left\{ \begin{array}{c} \text{composable strings of } k \text{ nonidentity} \\ \text{face maps starting at } [n] \in \Delta^{\mathrm{op}} \end{array} \right\} = (-1)^n.
$$

PROOF. Applying formula (7.15) to the strictly homotopy finite category $(\mathbb{\Delta}'_m)^{\mathrm{op}}$ for some $m \geq n$, we conclude that for any truncated semisimplicial diagram Y and $g \colon Y \to Y$ we have

$$\mathrm{tr}(\mathrm{colim}(g)) = \sum_{n \leq m} \mathrm{tr}(g_n) \cdot \sum_{k \geq 0} (-1)^k \cdot \# \left\{ \begin{array}{l} \text{composable strings of nonidentity} \\ \text{arrows of length } k \text{ starting at } [n] \end{array} \right\}$$

However, by (7.16) we also have

$$\mathrm{tr}(\mathrm{colim}(g)) = \sum_{n \leq m} (-1)^k \, \mathrm{tr}(g_n).$$

The claim now follows from the Uniqueness Theorem 12.1. ∎

Karol Szumiło has pointed out that this proof is basically the same as that given in §9, since the construction of $(\mathbb{\Delta}'_m)^{\mathrm{op}}$-colimits by the method of Theorem 7.1 amounts to barycentrically subdividing it.

Part 4: Base change objects and component lemmas

In this fourth and final part of the paper, we complete the proofs of the component lemmas. This consists mainly of calculations with derivators using the tools of [Gro13, GPS14b, GPS14a]. However, to simplify some of these calculations, we first introduce a slightly more abstract framework for bicategorical trace.

13. The abstract theory of base change objects

In this section we introduce an abstract framework that will enable us to give a high-level proof of Lemma 5.11, Lemma 10.4, Lemma 11.15, and other similar statements. It is based on the notion of *framed bicategory* [Shu08] (also known as a *proarrow equipment* [Woo82]), which is an enhancement of a bicategory that includes *base change objects* such as the representable profunctors $B(\mathrm{id}, f)$ and $B(f, \mathrm{id})$. This will allow us to establish a very general form of the component lemmas in Theorem 13.9. In the next two sections we will apply this to bicategories and derivators.

We start by recalling the definitions.

13.1. DEFINITION. A **(pseudo) double category** $\mathbb{W}$ consists of:

- A category $\mathbb{W}_0$, whose objects we call objects of $\mathbb{W}$ and whose morphisms we call vertical arrows.

- A category $\mathbb{W}_1$, whose objects we call horizontal arrows and whose morphisms we call 2-cells.

- Functors $S, T \colon \mathbb{W}_1 \to \mathbb{W}_0$ called horizontal source and horizontal target.

- A composition functor $\odot\colon \mathbb{W}_1 \times_{\mathbb{W}_0} \mathbb{W}_1 \to \mathbb{W}_0$.

- A unit functor $\mathbb{I}\colon \mathbb{W}_0 \to \mathbb{W}_1$.

- Associativity and unit isomorphisms for composition, satisfying appropriate axioms.

We draw a 2-cell $\phi\colon M \to N$ as

$$
\begin{array}{ccc}
S(M) & \xrightarrow{\;M\;} & T(M) \\
{\scriptstyle S(\phi)}\big\downarrow & \Downarrow_{\phi} & \big\downarrow{\scriptstyle T(\phi)} \\
S(N) & \xrightarrow[N]{} & T(N)
\end{array}
$$

Composition in $\mathbb{W}_1$ is vertical pasting of such squares, while the composition functor $\odot$ yields a horizontal pasting.

Every double category has a **horizontal bicategory** obtained by neglecting the vertical arrows. If $\mathbb{W}$ is a double category, we denote its horizontal bicategory by $\mathbf{W}$. Conversely, any bicategory can be regarded as a double category whose only vertical arrows are identities; thus any monoidal category can similarly be regarded as a double category with one object and one vertical arrow.

Just as any bicategory is equivalent to a strict 2-category, any pseudo double category is equivalent to a strict one (see [GP99]). Thus, we will write as if our double categories were strict, even though the examples we care about are not.

13.2. DEFINITION. [Shu08, Theorem 4.1] The following three conditions on a double category $\mathbb{W}$ are equivalent; when they hold we call it a **framed bicategory**.

- $(S,T)\colon \mathbb{W}_1 \to \mathbb{W}_0 \times \mathbb{W}_0$ is a categorical fibration.

- $(S,T)\colon \mathbb{W}_1 \to \mathbb{W}_0 \times \mathbb{W}_0$ is a categorical opfibration.

- For every vertical arrow $f\colon A \to B$, there exist horizontal arrows $B(\mathrm{id}, f)\colon A \nrightarrow B$ and $B(f, \mathrm{id})\colon B \nrightarrow A$ and 2-cells

$$
\begin{array}{cccc}
\begin{array}{ccc}
A & \xrightarrow{\mathbb{I}_A} & A \\
\| & \Downarrow_\alpha & \big\downarrow f \\
A & \xrightarrow[B(\mathrm{id},f)]{} & B
\end{array}
&
\begin{array}{ccc}
A & \xrightarrow{B(\mathrm{id},f)} & B \\
{\scriptstyle f}\big\downarrow & \Downarrow_\beta & \| \\
B & \xrightarrow[\mathbb{I}_B]{} & B
\end{array}
&
\begin{array}{ccc}
A & \xrightarrow{\mathbb{I}_A} & A \\
{\scriptstyle f}\big\downarrow & \Downarrow_\eta & \| \\
B & \xrightarrow[B(f,\mathrm{id})]{} & A
\end{array}
&
\begin{array}{ccc}
B & \xrightarrow{B(f,\mathrm{id})} & A \\
\| & \Downarrow_\epsilon & \big\downarrow f \\
B & \xrightarrow[\mathbb{I}_B]{} & B
\end{array}
\end{array}
$$

such that the following four composites are identities.

Any bicategory (or monoidal category), regarded as a double category, is a framed bicategory. The other primary examples we have in mind are extensions of the bicategories **Prof**(**V**) and **Prof**(**W**) from previous sections, whose vertical arrows come from functors between small categories. These can be obtained by applying the following two general constructions.

13.3. THEOREM. [Shu08, Prop. 11.10] *If* $\mathbb{W}$ *is a framed bicategory with local coequalizers (i.e. coequalizers in each category* $\mathbf{W}(A, B)$ *that are preserved by* $\odot$ *in each variable), then there is a framed bicategory* $\mathbb{M}\mathrm{od}(\mathbb{W})$ *described as follows.*

- *Its objects are monads in* **W***: endo-1-morphisms* $M \colon A \nrightarrow A$ *together with a multiplication* $M \odot M \to M$ *and unit* $\mathbb{I}_A \to M$ *satisfying associativity and unit axioms.*

- *Its horizontal arrows from* $M \colon A \nrightarrow A$ *to* $N \colon B \nrightarrow B$ *are modules in* **W***: horizontal arrows* $H \colon A \nrightarrow B$ *together with actions* $M \odot H \to H$ *and* $H \odot N \to H$ *that are associative and unital and commute with each other.*

- *Its vertical arrows from* $M \colon A \nrightarrow A$ *to* $N \colon B \nrightarrow B$ *are pairs* (f, ϕ) *of a vertical arrow* $f \colon A \to B$ *and a 2-cell*

$$
\begin{array}{ccc}
R & \xrightarrow{\;M\;} & R \\
{\scriptstyle f}\big\downarrow & \Swarrow{\scriptstyle \phi} & \big\downarrow{\scriptstyle f} \\
S & \xrightarrow[\;N\;]{} & S
\end{array}
$$

   in $\mathbb{W}$*, which commute with the monad structures of* M *and* N*.*

- *Its 2-cells are 2-cells in* $\mathbb{W}$ *which commute with the module actions of their domain and codomain.*

- *The composite of modules* $H \colon M \nrightarrow N$ *and* $K \colon N \nrightarrow P$ *is the local coequalizer of the two actions*

$$
H \odot N \odot K \rightrightarrows K \odot K.
$$

- *The horizontal unit* $\mathbb{I}_M$ *of a monad* $M \colon A \nrightarrow A$ *is the horizontal arrow* M *itself, regarded as a module.*

PROOF. See [Shu08, Prop. 11.10]. ∎

13.4. THEOREM. *If* $\mathbb{W}$ *is a framed bicategory with local coproducts, then there is a framed bicategory* $\mathbb{M}\mathrm{at}(\mathbb{W})$ *described as follows.*

- *Its objects are set-indexed families* $(A_i)_{i \in I}$ *of objects of* $\mathbb{W}$*.*

- *Its horizontal arrows from* $(A_i)_{i \in I}$ *to* $(B_j)_{j \in J}$ *are matrices in* **W***: families of horizontal arrows* $(H_{ij} \colon A_i \nrightarrow B_j)_{i \in I, j \in J}$*.*

- *Its vertical arrows from* $(A_i)_{i \in I}$ *to* $(B_j)_{j \in J}$ *are pairs* (f, ϕ) *of a function* $f \colon I \to J$ *and a family* $(\phi_i \colon A_i \to B_{f(i)})_{i \in I}$ *of vertical arrows in* $\mathbb{W}$*.*

- *Its 2-cells are families of 2-cells in $\mathbb{W}$.*

- *The composite of matrices $(H_{ij}: A_i \nrightarrow B_j)_{i \in I, j \in J}$ and $(K_{jl}: B_j \nrightarrow C_l)_{j \in J, l \in L}$ is the family of local coproducts*

$$\Big(\coprod_{j \in J} H_{ij} \odot K_{jl} \Big)_{i \in I, l \in L}.$$

- *The horizontal unit $\mathbb{I}_A$ of a family $(A_i)_{i \in I}$ is the identity matrix defined by*

$$(\mathbb{I}_A)_{ii'} = \begin{cases} \mathbb{I}_{A_i} & i = i' \\ \emptyset & i \neq i'. \end{cases}$$

PROOF. A straightforward modification of the proof of [Shu08, Prop. 11.10]. ■

Any local colimits possessed by $\mathbb{W}$ are inherited by $\mathbb{M}\text{od}(\mathbb{W})$ and $\mathbb{M}\text{at}(\mathbb{W})$. In particular, if $\mathbb{W}$ is locally cocomplete, we can construct $\mathbb{M}\text{od}(\mathbb{M}\text{at}(\mathbb{W}))$.

If we regard a cocomplete closed symmetric monoidal category $\mathbf{V}$ as a framed bicategory, then the objects, vertical arrows, and horizontal arrows of $\mathbb{M}\text{od}(\mathbb{M}\text{at}(\mathbf{V}))$ are exactly $\mathbf{V}$-enriched categories, functors, and profunctors respectively. Recall that a $\mathbf{V}$-profunctor between $\mathbf{V}$-categories $A \nrightarrow B$ is equivalently a $\mathbf{V}$-enriched functor $A \otimes B^{\text{op}} \to \mathbf{V}$. In §3 we defined $\mathbf{V}$-profunctors between unenriched categories, but a $\mathbf{V}$-profunctor $A \nrightarrow B$ in this sense can be identified with a $\mathbf{V}$-profunctor in the enriched sense from $\mathbf{V}[A]$ to $\mathbf{V}[B]$, where $\mathbf{V}[A]$ is the $\mathbf{V}$-category "freely generated" by the ordinary category A, with $\mathbf{V}[A](a, a') = A(a, a') \cdot \mathbb{S}$. Thus, the bicategory $\mathbf{Prof}(\mathbf{V})$ from §3 is equivalent to the full sub-bicategory of the horizontal bicategory of $\mathbb{M}\text{od}(\mathbb{M}\text{at}(\mathbf{V}))$ determined by the objects $\mathbf{V}[A]$. In this case, the representable horizontal arrows $B(\text{id}, f)$ and $B(f, \text{id})$ defined in the abstract framed-bicategory context agree with the representable profunctors from §3.

With these examples in mind, for a general locally cocomplete framed bicategory $\mathbb{W}$ we denote $\mathbb{M}\text{od}(\mathbb{M}\text{at}(\mathbb{W}))$ by $\mathbb{P}\text{rof}(\mathbb{W})$ and refer to its objects, vertical arrows, and horizontal arrows as $\mathbb{W}$**-enriched categories**, **functors**, and **profunctors** respectively. The more classical case is when $\mathbb{W}$ is a locally cocomplete bicategory $\mathbf{W}$; see [Wal81, Str81, CKW87]. In this case, the bicategory $\mathbf{Prof}(\mathbf{W})$ constructed in §10 sits inside $\mathbb{P}\text{rof}(\mathbf{W})$ similarly to the monoidal case, where the pair (A, R) corresponds to the $\mathbf{W}$-category whose object family is $(R)_{a \in \text{ob}(A)}$ and whose matrix monad is $(A(a, a') \cdot \mathbb{I}_R)_{a, a' \in \text{ob}(A)}$.

We now recall some basic properties of framed bicategories.

13.5. LEMMA. *For any vertical arrow $f: A \to B$ in a framed bicategory, $B(f, \text{id})$ is right dual to $B(\text{id}, f)$ in the horizontal bicategory.*

PROOF. The evaluation and coevaluation are the composites

The required identities follow from those for α, β, η, and ϵ. ■

13.6. LEMMA. *In a framed bicategory, there is a natural bijection between 2-cells*

$$
\begin{array}{ccc}
R & \xrightarrow{H} & R' \\
{\scriptstyle g}\downarrow & \Swarrow & \downarrow{\scriptstyle f} \\
S & \xrightarrow{K} & S'
\end{array}
$$

and 2-cells $H \odot B(\mathrm{id}, f) \to B(\mathrm{id}, g) \odot K$ *in* $\mathbf{W}(R, S')$.

PROOF. The bijection is defined via pasting

$$
\begin{array}{ccc}
R & \xrightarrow{H} & R' \\
{\scriptstyle g}\downarrow & \Swarrow & \downarrow{\scriptstyle f} \\
S & \xrightarrow{K} & S'
\end{array}
\quad \mapsto \quad
\begin{array}{ccccccc}
R & \xrightarrow{\mathbb{I}_R} & R & \xrightarrow{H} & R' & \xrightarrow{B(\mathrm{id},f)} & S' \\
\| & \Swarrow_{\alpha}\ {\scriptstyle g} & \downarrow & \Swarrow & \downarrow{\scriptstyle f} & \Swarrow_{\beta} & \| \\
R & \xrightarrow[B(\mathrm{id},g)]{} & S & \xrightarrow[K]{} & S' & \xrightarrow[\mathbb{I}_{S'}]{} & S'
\end{array}
\qquad \blacksquare
$$

Next we extend the definition of a shadow from bicategories to framed bicategories. Given a double category $\mathbb{W}$, let $\mathbb{W}^{\circlearrowleft}$ denote the **category of endo-1-cells** in $\mathbb{W}$, defined as follows:

- Its objects are pairs (R, H) where R is an object of $\mathbb{W}$ and $H \colon R \nrightarrow R$ is a horizontal arrow.

- Its morphisms from (R, H) to (S, K) are pairs (f, ϕ) where $f \colon R \to S$ is a vertical arrow and ϕ is a 2-cell

$$
\begin{array}{ccc}
R & \xrightarrow{H} & R \\
{\scriptstyle f}\downarrow & \Swarrow_{\phi} & \downarrow{\scriptstyle f} \\
S & \xrightarrow{K} & S.
\end{array}
$$

Note that for any object R, the horizontal hom-category $\mathbf{W}(R, R)$ is a non-full subcategory of $\mathbb{W}^{\circlearrowleft}$.

13.7. DEFINITION. A **shadow** on a double category $\mathbb{W}$ is a functor

$$
\langle\!\langle - \rangle\!\rangle \colon \mathbb{W}^{\circlearrowleft} \to \mathbf{T}
$$

to some other category $\mathbf{T}$, such that

- the composite functors $\mathbf{W}(R, R) \to \mathbb{W}^{\circlearrowleft} \to \mathbf{T}$ equip the bicategory $\mathbf{W}$ with a shadow, and

- the isomorphisms $\langle\!\langle H \odot K \rangle\!\rangle \cong \langle\!\langle K \odot H \rangle\!\rangle$ of this shadow are natural with respect to all morphisms in $\mathbb{W}^{\circlearrowleft}$. By this we mean that given 2-cells

$$
\begin{array}{ccc}
R \xrightarrow{H} S & & S \xrightarrow{K} R \\
f \downarrow \quad \Swarrow_{\phi} \quad \downarrow g & \text{and} & g \downarrow \quad \Swarrow_{\psi} \quad \downarrow f \\
R' \xrightarrow{H'} S' & & S' \xrightarrow{K'} R'
\end{array}
$$

in $\mathbb{W}$, so that

$$(f, \phi \odot \psi) \colon (R, H \odot K) \to (R', H' \odot K') \qquad \text{and}$$
$$(g, \psi \odot \phi) \colon (S, K \odot H) \to (S', K' \odot H')$$

are morphisms in $\mathbb{W}^{\circlearrowleft}$, the following square commutes in $\mathbf{T}$:

$$
\begin{array}{ccc}
\langle\!\langle H \odot K \rangle\!\rangle & \xrightarrow{\cong} & \langle\!\langle K \odot H \rangle\!\rangle \\
\langle\!\langle (f,\phi \odot \psi) \rangle\!\rangle \downarrow & & \downarrow \langle\!\langle (g,\psi \odot \phi) \rangle\!\rangle \\
\langle\!\langle H' \odot K' \rangle\!\rangle & \xrightarrow{\cong} & \langle\!\langle K' \odot H' \rangle\!\rangle.
\end{array}
$$

Our central examples inherit this structure.

13.8. THEOREM. *If $\mathbb{W}$ is a locally cocomplete framed bicategory with a cocontinuous shadow valued in a cocomplete category $\mathbf{T}$, then $\mathbb{M}\mathsf{od}(\mathbb{W})$ and $\mathbb{M}\mathsf{at}(\mathbb{W})$ (and also $\mathbb{P}\mathsf{rof}(\mathbb{W})$) also have cocontinuous shadows valued in $\mathbf{T}$.*

This generalizes the construction in [Pon10, §9.3].

PROOF. For $\mathbb{M}\mathsf{od}(\mathbb{W})$, we define the shadow of a module $H \colon M \nrightarrow M$ to be the coequalizer of the two maps

$$\langle\!\langle M \odot H \rangle\!\rangle \rightrightarrows \langle\!\langle H \rangle\!\rangle$$

in $\mathbf{T}$. The first of these two maps is the left action of M on H, while the second is the right action composed with the shadow isomorphism. The shadow of a module 2-cell is induced using the universal property of coequalizers and the action of the shadow of $\mathbb{W}$ on the corresponding 2-cells in $\mathbb{W}$. The axioms are straightforward to verify.

Similarly, for $\mathbb{M}\mathsf{at}(\mathbb{W})$, we define the shadow of a matrix $(H_{ii'} \colon A_i \nrightarrow A_{i'})_{i,i' \in I}$ to be the coproduct

$$\coprod_{i \in I} \langle\!\langle H_{ii} \rangle\!\rangle.$$

The rest of the structure is analogous. ∎

The following is the central abstract result we will use to deduce all the variants of the component lemma.

13.9. THEOREM. *Suppose given a 2-cell in a framed bicategory with a shadow:*

$$
\begin{array}{ccc}
R & \xrightarrow{H} & R \\
f \downarrow & \Downarrow_{\phi} & \downarrow f \\
S & \xrightarrow{K} & S
\end{array}
$$

and let $B(\mathrm{id}, \phi)\colon H \odot B(\mathrm{id}, f) \to B(\mathrm{id}, f) \odot K$ denote the corresponding 2-cell obtained from Lemma 13.6. Since $B(\mathrm{id}, f)$ is right dualizable, $B(\mathrm{id}, \phi)$ has a trace in the horizontal bicategory $\mathbf{W}$. On the other hand, we can directly apply the shadow $\langle\!\langle - \rangle\!\rangle\colon \mathbb{W}^{\circlearrowleft} \to \mathbf{T}$ to (f, ϕ); then

$$
\mathrm{tr}(B(\mathrm{id}, \phi)) = \langle\!\langle (f, \phi) \rangle\!\rangle
$$

as morphisms $\langle\!\langle H \rangle\!\rangle \to \langle\!\langle K \rangle\!\rangle$ in $\mathbf{T}$.

PROOF. Invoking the definitions of $B(\mathrm{id}, \phi)$ and of the evaluation and coevaluation in terms of the framed bicategory structure, we find that $\mathrm{tr}(B(\mathrm{id}, \phi))$ is the following composite. We have drawn pasting composites of 2-cells in $\mathbb{W}$ with $\langle\!\langle - \rangle\!\rangle$ around them to indicate application of $\langle\!\langle - \rangle\!\rangle\colon \mathbb{W}^{\circlearrowleft} \to \mathbf{T}$; thus the picture below denotes a morphism in $\mathbf{T}$.
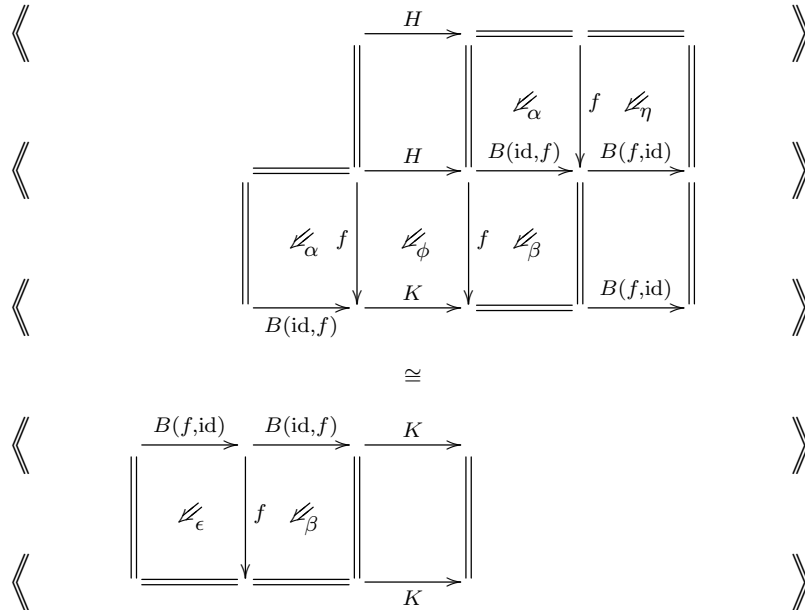
The unit isomorphism and one of the defining equations of $B(\mathrm{id}, f)$ enables us to simplify this to

$$
\left\langle\!\!\left\langle
\begin{array}{c}
\xrightarrow{\ H\ } \\
\Downarrow_\alpha\ f \quad \Downarrow_\phi \quad f\ \Downarrow_\eta \\
\xrightarrow[B(\mathrm{id},f)]{}\ \xrightarrow{\ K\ }\ \xrightarrow[B(f,\mathrm{id})]{}
\end{array}
\right\rangle\!\!\right\rangle
$$

$$
\cong
$$

$$
\left\langle\!\!\left\langle
\begin{array}{c}
\xrightarrow[B(f,\mathrm{id})]{}\ \xrightarrow[B(\mathrm{id},f)]{}\ \xrightarrow{\ K\ } \\
\Downarrow_\epsilon \quad f\ \Downarrow_\beta \\
\xrightarrow[\qquad K\qquad]{}
\end{array}
\right\rangle\!\!\right\rangle
$$

Now the naturality of the shadow isomorphism on $\mathbb{W}^{\circlearrowleft}$ implies that this is equal to

$$
\left\langle\!\!\left\langle
\begin{array}{c}
\xrightarrow{\ H\ } \\
\Downarrow_\alpha\ f \quad \Downarrow_\phi \quad f\ \Downarrow_\eta \\
\xrightarrow[B(\mathrm{id},f)]{}\ \xrightarrow{\ K\ }\ \xrightarrow[B(f,\mathrm{id})]{} \\
f\ \Downarrow_\beta \qquad\qquad \Downarrow_\epsilon \\
\xrightarrow[\qquad K\qquad]{}
\end{array}
\right\rangle\!\!\right\rangle
$$

followed by the shadow isomorphism $\langle\!\langle \mathbb{I}_S \odot K \rangle\!\rangle \cong \langle\!\langle K \odot \mathbb{I}_S \rangle\!\rangle$. However, the latter is the identity by one of the axioms of a shadow. Finally, two of the defining laws of $B(\mathrm{id}, f)$ and $B(f, \mathrm{id})$ reduce this composite to simply $\langle\!\langle (f, \phi) \rangle\!\rangle$. ∎

Theorem 13.9 implies in particular that a shadow on a framed bicategory is uniquely determined by the underlying shadow on its horizontal bicategory. However, at present our interest is in applying it in the other direction: we will construct a shadow on a framed bicategory and use Theorem 13.9 to yield a more explicit characterization of traces for endomorphisms of representable proarrows.

13.10. REMARK. There is also a sort of converse to Theorem 13.9. If $\mathbf{W}$ is any bicategory, assumed for simplicity to be a strict 2-category, then there is a framed bicategory $\mathbb{W}$ whose horizontal bicategory is $\mathbf{W}$, whose vertical arrows are right dualizable 1-cells in $\mathbf{W}$, and whose 2-cells

$$
\begin{array}{ccc}
A & \xrightarrow{\ Q\ } & B \\
N\downarrow & \Downarrow_\phi & \downarrow M \\
C & \xrightarrow[\ P\ ]{} & D
\end{array}
$$

are 2-cells $\phi\colon Q \odot M \to N \odot P$ in $\mathbf{W}$. The converse of Theorem 13.9 says that any shadow on $\mathbf{W}$ can be (uniquely) extended to a shadow on this $\mathbb{W}$, where the shadow of ϕ above is the trace of the corresponding 2-cell in $\mathbf{W}$. The axioms required of this shadow are basically the properties of bicategorical traces from [PS13, §7]. In particular, its functoriality on 2-cells in $\mathbb{W}^{\circlearrowleft}$ is precisely Theorem 2.6, the composition theorem for traces that plays such a major role in this paper.

14. Base change objects for bicategories

In this section, we will use Theorem 13.9 to identify the components of bicategorical traces in the non-homotopical examples (Lemma 10.4). We start by describing base change objects in terms of the bicategorical structure used in §10.

From the categorical fibration $(S, T)\colon \mathbb{W}_1 \to \mathbb{W}_0 \times \mathbb{W}_0$ of a framed bicategory $\mathbb{W}$, we obtain a pseudofunctor $(\mathbb{W}_0 \times \mathbb{W}_0)^{\mathrm{op}} \to \mathcal{CAT}$ sending (A, B) to the category $\mathbf{W}(A, B)$. We write $f^* M g^*$ for the action of a pair of vertical arrows $(f, g)\colon (A, B) \to (A', B')$ on $M\colon A \nrightarrow B$ under this pseudofunctor; it is called the **restriction** of M along f and g.

We can further extend this to a pseudofunctor whose domain is a 2-category. The **vertical 2-category** $\mathcal{V}\mathbb{W}$ of a double category has underlying category $\mathbb{W}_0$, and a 2-cell from $f\colon A \to B$ to $g\colon A \to B$ is a 2-cell

$$
\begin{array}{ccc}
A & \overset{\mathbb{I}_A}{\longrightarrow} & A \\
g\downarrow & \Swarrow & \downarrow f \\
B & \underset{\mathbb{I}_B}{\longrightarrow} & B
\end{array}
$$

in $\mathbb{W}$. Pasting of 2-cells in $\mathbb{W}$ yields the required compositional structure making $\mathcal{V}\mathbb{W}$ a 2-category.

14.1. LEMMA. *The pseudofunctor* $\mathbf{W}\colon (\mathbb{W}_0 \times \mathbb{W}_0)^{\mathrm{op}} \to \mathcal{CAT}$ *can be extended to a pseudofunctor* $\mathcal{V}\mathbb{W}^{\mathrm{op}} \times \mathcal{V}\mathbb{W}^{\mathrm{coop}} \to \mathcal{CAT}$.

As usual, $\mathcal{V}\mathbb{W}^{\mathrm{op}}$ denotes reversal of 1-morphisms but not 2-morphisms, while $\mathcal{V}\mathbb{W}^{\mathrm{coop}}$ denotes reversal of both.

PROOF. Suppose given 2-cells

$$
\begin{array}{ccc}
A & \overset{\mathbb{I}_A}{\longrightarrow} & A \\
g\downarrow & \Swarrow_{\phi} & \downarrow f \\
B & \underset{\mathbb{I}_B}{\longrightarrow} & B
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
C & \overset{\mathbb{I}_C}{\longrightarrow} & C \\
k\downarrow & \Swarrow_{\psi} & \downarrow h \\
D & \underset{\mathbb{I}_D}{\longrightarrow} & D.
\end{array}
$$

Their image under the desired pseudofunctor must be a natural transformation whose component at $M\colon B \nrightarrow D$ is a map $f^* M k^* \to g^* M h^*$. The fibrational structure of (S, T)

yields cartesian 2-cells

$$
\begin{array}{ccc}
A & \xrightarrow{f^*Mk^*} & C \\
f\downarrow & \Downarrow\mathrm{cart}_1 & \downarrow k \\
B & \xrightarrow[M]{} & D
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{g^*Mh^*} & C \\
g\downarrow & \Downarrow\mathrm{cart}_2 & \downarrow h \\
B & \xrightarrow[M]{} & D.
\end{array}
$$

Pasting cart$_1$ of these with ϕ and ψ we have

$$
\begin{array}{ccccccc}
A & \mapsto & A & \mapsto & C & \mapsto & C \\
g\downarrow & \Downarrow_\phi & \downarrow f & \Downarrow\mathrm{cart}_1 k\downarrow & \Downarrow_\psi & \downarrow h \\
B & \mapsto & B & \mapsto & D & \mapsto & D
\end{array}
$$

whose unique factorization through cart$_2$ yields the desired map $f^*Mk^* \to g^*Mh^*$. The verification of functoriality is straightforward. ∎

Given a 2-cell $\nu\colon f \to g$ in $\mathcal{V}\mathbb{W}$, we have an induced map $B(\mathrm{id},\nu)\colon B(\mathrm{id},f) \to B(\mathrm{id},g)$ in $\mathbf{W}(A,B)$, defined as the following composite:

$$
\begin{array}{ccccccc}
& \xrightarrow{\mathbb{I}_A} & & \xrightarrow{\mathbb{I}_A} & & \xrightarrow{B(\mathrm{id},f)} & \\
\| & \Downarrow_\beta & g\downarrow & \Downarrow_\nu & f\downarrow & \Downarrow_\alpha & \| \\
& \xrightarrow[B(\mathrm{id},g)]{} & & \xrightarrow[\mathbb{I}_B]{} & & \xrightarrow[\mathbb{I}_B]{} &
\end{array}
$$

Similarly, we have an induced map $B(\nu,\mathrm{id})\colon B(g,\mathrm{id}) \to B(f,\mathrm{id})$. These constructions yield pseudofunctors $\mathcal{V}\mathbb{W} \to \mathbf{W}$ and $\mathcal{V}\mathbb{W}^{\mathrm{coop}} \to \mathbf{W}$; see [Shu08, Appendix C].

14.2. LEMMA. *For f, g, M, ϕ, and ψ as in the proof of Lemma 14.1, we have isomorphisms*

$$
f^*Mk^* \cong B(\mathrm{id},f) \odot M \odot D(k,\mathrm{id})
$$
$$
g^*Mh^* \cong B(\mathrm{id},g) \odot M \odot D(h,\mathrm{id})
$$

which identify the map $f^*Mk^* \to g^*Mh^*$ from Lemma 14.1 with the map

$$
B(\mathrm{id},f) \odot M \odot D(k,\mathrm{id}) \xrightarrow{B(\mathrm{id},\phi)\odot\mathrm{id}_M\odot D(\psi,\mathrm{id})} B(\mathrm{id},g) \odot M \odot D(h,\mathrm{id}).
$$

PROOF. The isomorphisms follow from the proof of equivalence of the three definitions of framed bicategory in [Shu08, Theorem 4.1]. The identification of maps is a straightforward calculation using the proof of [Shu08, Theorem 4.1]. ∎

We can now prove the Component Lemma for bicategories.

RESTATEMENT OF LEMMA 10.4. [The component lemma for bicategories]. *Let* $\mathbf{W}$ *be a locally cocomplete bicategory with a cocontinuous shadow. For any right dualizable* $X\colon (A, R) \nrightarrow (\mathbb{1}, S)$ *in* $\mathbf{Prof}(\mathbf{W})$ *with* $f\colon X \to X \odot P$, *and any morphism* $\alpha \in A(a, a)$, *the component* $\mathrm{tr}(f)_{[\alpha]}$ *is the trace in* $\mathbf{W}$ *of the composite*

$$X(a) \xrightarrow{\ X_\alpha\ } X(a) \xrightarrow{\ f_a\ } X(a) \odot P. \tag{14.3}$$

PROOF. In outline, the proof is identical to that of Lemma 3.8. With $a\colon (\mathbb{1}, R) \to (A, R)$ the functor picking out $a \in A$, the representable profunctor $(A, R)(\mathrm{id}, a)$ is right dualizable. Moreover, we have a 2-cell in $\mathbb{P}\mathsf{rof}(\mathbf{W})$:

$$
\begin{array}{ccc}
(\mathbb{1}, R) & \xrightarrow{\ \mathbb{I}_{(\mathbb{1}, R)}\ } & (\mathbb{1}, R) \\
{\scriptstyle a}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle a} \\
(A, R) & \xrightarrow[\ \mathbb{I}_{(A, R)}\ ]{} & (A, R)
\end{array}
$$

whose unique component is the coprojection $\mathbb{I}_R \to A(a, a) \cdot \mathbb{I}_R$ indexed by α. Thus, there is an induced endomorphism $(A, R)(\mathrm{id}, \alpha)\colon (A, R)(\mathrm{id}, a) \to (A, R)(\mathrm{id}, a)$, and by Theorem 2.6, the trace of the composite

$$(A, R)(\mathrm{id}, a) \odot X \xrightarrow{\ (A, R)(\mathrm{id}, \alpha) \odot \mathrm{id}\ } (A, R)(\mathrm{id}, a) \odot X \xrightarrow{\ \mathrm{id} \odot f\ } (A, R)(\mathrm{id}, a) \odot X \odot P \tag{14.4}$$

is equal to the composite

$$\langle\!\langle R \rangle\!\rangle \xrightarrow{\ \mathrm{tr}((A, R)(\mathrm{id}, \alpha))\ } \langle\!\langle (A, R) \rangle\!\rangle \xrightarrow{\ \mathrm{tr}(f)\ } \langle\!\langle P \rangle\!\rangle.$$

Now note that $(A, R)(\mathrm{id}, a) \odot X \cong a^* X \cong X(a)$. Moreover, Lemma 14.2 informs us that under this isomorphism, (14.4) is identified with (14.3). The proof is now completed by Theorem 13.9, which identifies $\mathrm{tr}((A, R)(\mathrm{id}, \alpha))$ with $\langle\!\langle \alpha \rangle\!\rangle$, where by construction the latter picks out the component of $\langle\!\langle (A, R) \rangle\!\rangle$ indexed by $[\alpha]$. ∎

14.5. REMARK. Since the framed bicategories of profunctors constructed in this section have general enriched categories as their objects, we can obtain versions of the linearity formulas from §3 and §10 that apply to the more general case of weighted colimits of *enriched* diagrams. However, we do not have any interesting examples, so we leave the details to the reader.

15. Base change objects for derivators

We now apply the theory of §13 to prove the component lemmas for derivators (Lemma 5.11 and Lemma 11.15). The structure is similar to §14, and we build on the results there.

15.1. THEOREM. *Let $\mathscr{W}$ be a closed derivator bicategory with a shadow. Then the bicategory* $\mathbf{Prof}(\mathscr{W})$ *and its shadow constructed in* §11 *extend to a framed bicategory* $\mathbb{Prof}(\mathscr{W})$ *with a shadow, whose vertical arrows are of the form* $(f, R) \colon (A, R) \to (B, R)$ *for a functor* $f \colon A \to B$.

We could state this theorem more generally replacing $\mathscr{W}$ by a *framed derivator bicategory* $\mathbb{W}$, thereby allowing the vertical arrows in $\mathbb{Prof}(\mathbb{W})$ to incorporate a vertical arrow in $\mathbb{W}$ as well as a functor in $\mathcal{C}at$. Moreover, most motivating examples of derivator bicategories are in fact framed, including the bicategory of rings and complexes of bimodules (its vertical arrows are ring homomorphisms) and the bicategory of parametrized spectra that we will use in [PS14a] (its vertical arrows are maps of base spaces). However, here we have no need for these more general vertical arrows, so we can avoid giving a definition of framed derivator bicategories.

PROOF. We define a 2-cell

$$
\begin{array}{ccc}
(A, R) & \overset{M}{\longrightarrow} & (B, S) \\
{\scriptstyle (f,R)}\big\downarrow & {\Downarrow_\phi} & \big\downarrow{\scriptstyle (g,S)} \\
(A', R) & \underset{N}{\longrightarrow} & (B', S)
\end{array}
$$

in $\mathbb{Prof}(\mathscr{W})$ to be a morphism $M \to (f \times g^{\mathrm{op}})^* N$ in $\mathscr{W}(R, S)(A \times B^{\mathrm{op}})$. Vertical composites and identities are obvious. The horizontal identity 2-cell of $(f, R) \colon (A, R) \to (B, R)$ is the mate-transformation $\mathbb{I}_{(A,R)} \to (f \times f^{\mathrm{op}})^* \mathbb{I}_{(B,R)}$ induced by the following commutative square:

$$
\begin{array}{ccc}
\mathrm{tw}(A) & \overset{\mathrm{tw}(f)}{\longrightarrow} \mathrm{tw}(B) \longrightarrow & \mathbb{1} \\
{\scriptstyle (t,s)}\big\downarrow & \big\downarrow{\scriptstyle (t,s)} & \\
A \times A^{\mathrm{op}} & \underset{f \times f^{\mathrm{op}}}{\longrightarrow} B \times B^{\mathrm{op}} &
\end{array}
$$

The horizontal composite of

$$
\begin{array}{ccccc}
(A, R) & \overset{M}{\longrightarrow} & (B, S) & \overset{P}{\longrightarrow} & (C, T) \\
{\scriptstyle (f,R)}\big\downarrow & {\Downarrow_\phi} & \big\downarrow{\scriptstyle (g,S)} \; {\Downarrow_\psi} & & \big\downarrow{\scriptstyle (h,T)} \\
(A', R) & \underset{N}{\longrightarrow} & (B', S) & \underset{Q}{\longrightarrow} & (C', T)
\end{array}
$$

is the composite

$$
\begin{aligned}
(\pi_{\mathrm{tw}(B)^{\mathrm{op}}})_! (t^{\mathrm{op}}, s^{\mathrm{op}})^* (M \odot P) & \overset{\phi \odot \psi}{\longrightarrow} (\pi_{\mathrm{tw}(B)^{\mathrm{op}}})_! (t^{\mathrm{op}}, s^{\mathrm{op}})^* ((f \times g^{\mathrm{op}})^* N \odot (g \times h^{\mathrm{op}})^* Q) \\
& \cong (\pi_{\mathrm{tw}(B)^{\mathrm{op}}})_! (t^{\mathrm{op}}, s^{\mathrm{op}})^* (f \times g^{\mathrm{op}} \times g \times h^{\mathrm{op}})^* (N \odot Q) \\
& \cong (\pi_{\mathrm{tw}(B)^{\mathrm{op}}})_! (f \times \mathrm{tw}(g)^{\mathrm{op}} \times h^{\mathrm{op}})^* (t^{\mathrm{op}}, s^{\mathrm{op}})^* (N \odot Q) \\
& \longrightarrow (f \times h^{\mathrm{op}})^* (\pi_{\mathrm{tw}(B')^{\mathrm{op}}})_! (t^{\mathrm{op}}, s^{\mathrm{op}})^* (N \odot Q).
\end{aligned}
$$

Here $\odot$ denotes the *external* version of the two-variable derivator morphism

$$\odot \colon \mathscr{W}(R,S) \times \mathscr{W}(S,T) \to \mathscr{W}(R,T),$$

and the final map in the composite is the mate-transformation induced by the following commutative square:

$$
\begin{array}{ccc}
A \times \mathrm{tw}(B)^{\mathrm{op}} \times C^{\mathrm{op}} & \xrightarrow{\ f \times \mathrm{tw}(g)^{\mathrm{op}} \times h^{\mathrm{op}}\ } & A' \times \mathrm{tw}(B')^{\mathrm{op}} \times (C')^{\mathrm{op}} \\
{\scriptstyle \pi_{\mathrm{tw}(B)^{\mathrm{op}}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{tw}(B')^{\mathrm{op}}}} \\
A \times C^{\mathrm{op}} & \xrightarrow{\quad f \times h^{\mathrm{op}} \quad} & A' \times (C')^{\mathrm{op}}
\end{array}
$$

Associativity and unitality of the horizontal composition of 2-cells are automatic because the construction of the associativity and unit isomorphisms for $\mathbf{Prof}(\mathscr{W})$ in [GPS14a] uses homotopy exact squares that are natural with respect to the small categories appearing therein, and mates are functorial under pasting of squares. Thus, we have a double category $\mathbb{Prof}(\mathscr{W})$. Moreover, the definition of the 2-cells implies immediately that it has restrictions, hence is a framed bicategory.

Similarly, the shadow of a 2-cell $\phi \colon M \to (f \times f^{\mathrm{op}})^* N$, for $M \colon (A,R) \nrightarrow (A,R)$ and $N \colon (B,R) \nrightarrow (B,R)$ with $f \colon A \to B$, is the map

$$
\begin{aligned}
(\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^* M & \xrightarrow{\ \phi\ } (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*(f \times f^{\mathrm{op}})^* N \\
& \cong (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(\mathrm{tw}(f)^{\mathrm{op}})^*(t^{\mathrm{op}}, s^{\mathrm{op}})^* N \\
& \longrightarrow (\pi_{\mathrm{tw}(B)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^* N
\end{aligned}
$$

where the final morphism is induced by the commutative square

$$
\begin{array}{ccc}
\mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow{\ \mathrm{tw}(f)^{\mathrm{op}}\ } & \mathrm{tw}(B)^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\mathbb{1} & = \!\!=\!\!=\!\!=\!\!= & \mathbb{1}.
\end{array}
$$

Finally, the naturality of the shadow isomorphism with respect to 2-cells also follows from the naturality with respect to small categories of the homotopy exact squares appearing in its definition. ∎

The proof of Theorem 15.1 implies that for $M \in \mathscr{W}(R,S)(B,D)$ and $f \colon A \to B$ and $g \colon C \to D$, the restriction $(f,R)^* M(g,S)^*$ in the framed-bicategory sense may be identified with the restriction $(f \times g^{\mathrm{op}})^* M \in \mathscr{W}(R,S)(A,C)$ in the derivator sense. Thus, by Lemma 14.2, we have

$$(f \times g^{\mathrm{op}})^* M \cong (B,R)(\mathrm{id}, f) \odot M \odot (D,S)(g, \mathrm{id}). \tag{15.2}$$

Now, any natural transformation $\nu\colon f \to g\colon A \to B$ induces a 2-cell

$$
\begin{array}{ccc}
(A,R) & \xrightarrow{\mathbb{I}_{(A,R)}} & (A,R) \\
{\scriptstyle (g,R)}\downarrow & \Downarrow_{(\nu,R)} & \downarrow{\scriptstyle (f,R)} \\
(B,R) & \xrightarrow[\mathbb{I}_{(B,R)}]{} & (B,R)
\end{array}
$$

as the following composite:

$$
(t,s)_! \pi^*_{\mathrm{tw}(A)} \mathbb{I}_R = (t,s)_! \tilde{\nu}^* \pi^*_{\mathrm{tw}(B)} \mathbb{I}_R
$$
$$
\longrightarrow (g \times f^{\mathrm{op}})^*(t,s)_! \pi^*_{\mathrm{tw}(B)} \mathbb{I}_R.
$$

Here $\tilde{\nu}\colon \mathrm{tw}(A) \to \mathrm{tw}(B)$ is the functor sending a morphism $\alpha\colon a \to a'$ of A (regarded as an object of $\mathrm{tw}(A)$) to the morphism $\nu_{a'} \circ f(\alpha) = g(\alpha) \circ \nu_a$ of B, and the final morphism above is induced by the following commutative square:

$$
\begin{array}{ccc}
\mathrm{tw}(A) & \xrightarrow{\tilde{\nu}} & \mathrm{tw}(B) \\
{\scriptstyle (t,s)}\downarrow & & \downarrow{\scriptstyle (t,s)} \\
A \times A^{\mathrm{op}} & \xrightarrow[g \times f^{\mathrm{op}}]{} & B \times B^{\mathrm{op}}.
\end{array}
$$

In fact, for any R, this construction defines a 2-functor $(-,R)\colon \mathcal{C}at \to \mathcal{V}(\mathbb{P}\mathsf{rof}(\mathscr{W}))$. However, we will not need its functoriality, so we leave the proof to the reader. What we do need is that the induced action of ν on horizontal arrows of $\mathbb{P}\mathsf{rof}(\mathscr{W})$ agrees with that arising from the derivator structure of $\mathscr{W}$. For simplicity, we state and prove only the one-sided version.

15.3. LEMMA. *For $\nu\colon f \to g\colon A \to B$ and $M \in \mathscr{W}(R,S)(B,C)$, the derivator restriction map*

$$
(\nu \times \mathrm{id})^* M\colon (f \times \mathrm{id})^* M \to (g \times \mathrm{id})^* M
$$

is equal to the framed-bicategory restriction map

$$
(\nu,R)^* M\colon (f,R)^* M \to (g,R)^* M
$$

defined as in Lemma 14.1. Therefore, by Lemma 14.2, the isomorphism (15.2) identifies $\nu^* M$ with $(B,R)(\mathrm{id},\nu) \odot \mathrm{id}_M$.

PROOF. By definition, $(\nu,R)^* M$ is obtained by factoring the horizontal pasting

$$
\begin{array}{ccccc}
(A,R) & \xrightarrow{\mathbb{I}_{(A,R)}} & (A,R) & \xrightarrow{f^* M} & (C,S) \\
{\scriptstyle (g,R)}\downarrow & \Downarrow_{(\nu,R)} & \downarrow{\scriptstyle (f,R)} & \Downarrow_{\mathrm{cart}} & \| \\
(B,R) & \xrightarrow[\mathbb{I}_{(B,R)}]{} & (B,R) & \xrightarrow[M]{} & (C,S)
\end{array}
$$

through the cartesian 2-cell defining $(g, R)^* M$. However, in the case of $\mathbb{P}\mathsf{rof}(\mathscr{W})$, these cartesian 2-cells are simply identities $(f \times \mathrm{id})^* M = (f \times \mathrm{id})^* M$ and $(g \times \mathrm{id})^* M = (g \times \mathrm{id})^* M$. Thus, by definition of horizontal composition of 2-cells in $\mathbb{P}\mathsf{rof}(\mathscr{W})$ and of (ν, R), we must show that the composite

$$
\begin{aligned}
(f \times \mathrm{id})^* M &\cong (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*(\mathbb{I}_{(A,R)} \odot (f \times \mathrm{id})^* M) \\
&\cong (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*((t, s)_! \pi^*_{\mathrm{tw}(A)} \mathbb{I}_R \odot (f \times \mathrm{id})^* M) \\
&\cong (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*((t, s)_! \tilde{\nu}^* \pi^*_{\mathrm{tw}(B)} \mathbb{I}_R \odot (f \times \mathrm{id})^* M) \\
&\longrightarrow (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*((g \times f^{\mathrm{op}})^*(t, s)_! \pi^*_{\mathrm{tw}(B)} \mathbb{I}_R \odot (f \times \mathrm{id})^* M) \\
&= (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*((g \times f^{\mathrm{op}})^* \mathbb{I}_{(B,R)} \odot (f \times \mathrm{id})^* M) \\
&\cong (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*(g \times f^{\mathrm{op}} \times f \times \mathrm{id})^*(\mathbb{I}_{(B,R)} \odot M) \\
&\cong (\pi_{\mathrm{tw}(A)^{\mathrm{op}}})_!(g \times \mathrm{tw}(f)^{\mathrm{op}} \times \mathrm{id})^*(t^{\mathrm{op}}, s^{\mathrm{op}})^*(\mathbb{I}_{(B,R)} \odot M) \\
&\longrightarrow (g \times \mathrm{id})^*(\pi_{\mathrm{tw}(B)^{\mathrm{op}}})_!(t^{\mathrm{op}}, s^{\mathrm{op}})^*(\mathbb{I}_{(B,R)} \odot M) \\
&\cong (g \times \mathrm{id})^* M.
\end{aligned}
$$

is equal to $(\nu \times \mathrm{id})^* M$.

Note that the first and last isomorphisms are unit isomorphisms in $\mathbf{Prof}(\mathscr{W})$. If we omit these two isomorphisms for the moment, we may see the remaining composite as a sequence of natural transformations applied to $\mathbb{I}_R \odot M$. From the definitions of the two noninvertible factors, it is the composite mate-transformation relating the two outer paths from the top-right to the bottom-left in the following diagram of functors (restricting to the left and left Kan extending downwards). (For brevity, we omit the factor C^{op} from the notation from now on, as it plays no role in the calculation.)

$$
\begin{array}{ccccc}
 & \mathrm{tw}(A) \times A & \xrightarrow{\tilde{\nu} \times f} & \mathrm{tw}(B) \times B & \longrightarrow B \\
 & \downarrow & & \downarrow & \\
A \times \mathrm{tw}(A)^{\mathrm{op}} & \longrightarrow A \times A^{\mathrm{op}} \times A & \xrightarrow{g \times f^{\mathrm{op}} \times f} & B \times B^{\mathrm{op}} \times B & \\
\| & & & \| & \\
A \times \mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow{g \times \mathrm{tw}(f)^{\mathrm{op}}} B \times \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B \times B^{\mathrm{op}} \times B & \\
\downarrow & & \downarrow & & \\
A & \xrightarrow{g} & B & &
\end{array}
\tag{15.4}
$$

Now in order to obtain the actual composite of interest, we must also compose with the unit isomorphisms at the beginning and the end: one for B and one for A. As constructed in [GPS14a], the unit isomorphism of $\mathbf{Prof}(\mathscr{V})$ for B is composed of the mate-isomorphisms associated to two homotopy exact squares: one pullback

$$
\begin{array}{ccc}
\mathrm{tw}(B) \times_{B^{\mathrm{op}}} \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & \mathrm{tw}(B) \times B \\
\downarrow & & \downarrow \\
B \times \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B \times B^{\mathrm{op}} \times B
\end{array}
\tag{15.5}
$$

and one transformation that composes up pairs of morphisms:

$$
\begin{array}{ccc}
\mathrm{tw}(B) \times_{B^{\mathrm{op}}} \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B \\
\downarrow & \Downarrow_{\sigma} & \| \\
B & =\!\!=\!\!=\!\!=\!\!= & B.
\end{array}
\tag{15.6}
$$

(The $(-)^{\mathrm{op}}$'s are switched around from the proof in [GPS14a] since here we are considering the *left* unit isomorphism.) Thus, we need to compose with both of these squares for both A and B.

If we paste (15.5) for A onto the upper-left corner of (15.4) and forget about some intermediate objects, we get

$$
\begin{array}{ccccc}
\mathrm{tw}(A) \times_{A^{\mathrm{op}}} \mathrm{tw}(A)^{\mathrm{op}} & \longrightarrow & \mathrm{tw}(B) \times B & \longrightarrow & B \\
\downarrow & & \downarrow & & \\
A \times \mathrm{tw}(A)^{\mathrm{op}} \xrightarrow{\ g \times \mathrm{tw}(f)^{\mathrm{op}}\ } B \times \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B \times B^{\mathrm{op}} \times B & & \\
\downarrow & & \downarrow & & \\
A \xrightarrow{\quad g \quad} B. & & & &
\end{array}
$$

If we then factor this through the pullback square (15.5) for B, using the universal property of pullbacks, we get

$$
\begin{array}{ccccc}
\mathrm{tw}(A) \times_{A^{\mathrm{op}}} \mathrm{tw}(A)^{\mathrm{op}} \xrightarrow{\ \tilde{\nu} \times_f \mathrm{tw}(f)^{\mathrm{op}}\ } \mathrm{tw}(B) \times_{B^{\mathrm{op}}} \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & \mathrm{tw}(B) \times B & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
A \times \mathrm{tw}(A)^{\mathrm{op}} \xrightarrow{\ g \times \mathrm{tw}(f)^{\mathrm{op}}\ } & B \times \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B \times B^{\mathrm{op}} \times B & \\
\downarrow & & \downarrow & & \\
A \xrightarrow{\quad g \quad} & B. & & &
\end{array}
$$

Now we paste (15.6) for B onto the lower-right corner and forget about more intermediate objects to get

$$
\begin{array}{ccc}
\mathrm{tw}(A) \times_{A^{\mathrm{op}}} \mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow{\ ft^{\mathrm{op}}\ } & B \\
s \downarrow & \Downarrow_{\overline{\nu}} & \| \\
A & \xrightarrow{\quad g \quad} & B.
\end{array}
\tag{15.7}
$$

It remains only to compose with (15.6) for A. However, inspecting the definitions of all these squares, we see that the transformation $\overline{\nu}$ in (15.7) is defined by

$$
\begin{aligned}
\overline{\nu}(a \xrightarrow{\alpha} a', a' \xrightarrow{\alpha'} a) &= \sigma_B((\tilde{\nu} \times_f \mathrm{tw}(f)^{\mathrm{op}})(\alpha, \alpha')) \qquad (\sigma \text{ is } (15.6)) \\
&= \tilde{\nu}(\alpha) \circ f(\alpha') \\
&= \nu_{a'} \circ f(\alpha) \circ f(\alpha') \\
&= \nu_{a'} \circ f(\alpha \circ \alpha') \\
&= \tilde{\nu}(\sigma_A(\alpha, \alpha')).
\end{aligned}
$$

Thus, (15.7) factors into

$$
\begin{array}{ccccc}
\mathrm{tw}(A) \times_{A^{\mathrm{op}}} \mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow{t^{\mathrm{op}}} & A & \xrightarrow{f} & B \\
{\scriptstyle s}\downarrow & \Downarrow \sigma_A & \| & \Swarrow_\nu & \| \\
A & =\!=\!=\!= & A & \xrightarrow{g} & B
\end{array}
$$

in which the left-hand square is exactly (15.6) for A. Thus, after composing with both parts of both unit isomorphisms, we obtain simply ν^*. ∎

We will also need to know about the shadows of such natural transformations. As we observed in eq. (11.14), the argument for (5.8) applies essentially verbatim to conclude

$$\langle\!\langle (A, R) \rangle\!\rangle \cong |N(\Lambda A)| \circledast \langle\!\langle R \rangle\!\rangle. \tag{15.8}$$

15.9. LEMMA. *For a natural transformation $\nu\colon f \to f\colon A \to B$ and any R, the isomorphism (15.8) identifies the shadow $\langle\!\langle (\nu, R) \rangle\!\rangle\colon \langle\!\langle (A, R) \rangle\!\rangle \to \langle\!\langle (B, R) \rangle\!\rangle$ in $\mathbb{P}\mathsf{rof}(\mathscr{W})$ with the map induced by the functor $\Lambda\nu\colon \Lambda A \to \Lambda B$ from §5.*

PROOF. Putting together the definitions of (ν, R) and the shadow of $\mathbb{P}\mathsf{rof}(\mathscr{W})$, we see that $\langle\!\langle (\nu, R) \rangle\!\rangle$ is the composite mate-transformation comparing the two extreme paths from the top right to the bottom left below (restricting to the left and left Kan extending downwards):

$$
\begin{array}{ccccc}
\mathrm{tw}(A) & \xrightarrow{\tilde{\nu}} & \mathrm{tw}(B) & \longrightarrow & \mathbb{1} \\
\downarrow & & \downarrow & & \\
\mathrm{tw}(A)^{\mathrm{op}} \longrightarrow A \times A^{\mathrm{op}} & \xrightarrow{f \times f^{\mathrm{op}}} & B \times B^{\mathrm{op}} & & \\
\| & & \| & & \\
\mathrm{tw}(A)^{\mathrm{op}} \xrightarrow{\mathrm{tw}(f)^{\mathrm{op}}} \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B \times B^{\mathrm{op}} & & \\
\downarrow & & \downarrow & & \\
\mathbb{1} & =\!=\!=\!= & \mathbb{1} & &
\end{array}
\tag{15.10}
$$

In §5 we introduced ΛA as a pullback

$$
\begin{array}{ccc}
\Lambda A & \longrightarrow & \mathrm{tw}(A) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (t,s)} \\
\mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow[(s^{\mathrm{op}}, t^{\mathrm{op}})]{} & A \times A^{\mathrm{op}}
\end{array}
\tag{15.11}
$$

and the isomorphism (15.8) is essentially the mate associated to this homotopy exact square. Thus, we need to compose the composite mate of (15.10) with the mates of (15.11) for both A and B. To do this we paste (15.11) for A onto the top-left corner, and then

factor through (15.11) for B using the universal property of the pullback; thus we obtain

$$
\begin{array}{ccccccc}
\Lambda A & \xrightarrow{\ \Lambda\nu\ } & \Lambda B & \longrightarrow & \mathrm{tw}(B) & \longrightarrow & \mathbb{1} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{tw}(A)^{\mathrm{op}} & \xrightarrow{\ \mathrm{tw}(f)^{\mathrm{op}}\ } & \mathrm{tw}(B)^{\mathrm{op}} & \longrightarrow & B\times B^{\mathrm{op}} & & \\
\downarrow & & \downarrow & & & & \\
\mathbb{1} & =\!=\!=\!=\!=\!= & \mathbb{1}. & & & &
\end{array}
$$

It is easy to see that the induced map is indeed $\Lambda\nu$, as shown. ∎

We can now prove a generalization of Lemma 11.15 (which includes Lemma 5.11 as a special case). Recall that any $\alpha\colon a\to a$ in A induces a map $[\alpha]\colon \langle\!\langle R\rangle\!\rangle \to \langle\!\langle (A,R)\rangle\!\rangle$ by way of the functor $\Lambda\alpha\colon \mathbb{1}\to \Lambda A$.

RESTATEMENT OF LEMMA 11.15. [The component lemma for derivator bicategories]. *If $X\in\mathscr{W}(R,S)(A)$ is pointwise dualizable and $f\colon X\to X\odot P$, then for any conjugacy class $[a\xrightarrow{\alpha}a]$ in A, the composite*

$$
\langle\!\langle R\rangle\!\rangle \xrightarrow{\ [\alpha]\ } \langle\!\langle (A,R)\rangle\!\rangle \xrightarrow{\ \mathrm{tr}(f)\ } \langle\!\langle P\rangle\!\rangle
$$

is equal to the trace in $\mathscr{W}(R,S)(\mathbb{1})$ of the composite

$$
X_a \xrightarrow{\ X_\alpha\ } X_a \xrightarrow{\ f_a\ } X_a\odot P_a. \tag{15.12}
$$

PROOF. As before, let $a\colon \mathbb{1}\to A$ be the functor picking out $a\in A$. Then the representable profunctor $(A,R)(\mathrm{id},a)\colon (\mathbb{1},R)\nrightarrow (A,R)$ is right dualizable, and we have the 2-cell

$$
\begin{array}{ccc}
(\mathbb{1},R) & \xrightarrow{\ \mathbb{I}_{(\mathbb{1},R)}\ } & (\mathbb{1},R) \\
{\scriptstyle (a,R)}\downarrow & \Swarrow{\scriptstyle (\alpha,R)} & \downarrow{\scriptstyle (a,R)} \\
(A,R) & \xrightarrow[\ \mathbb{I}_{(A,R)}\ ]{} & (A,R)
\end{array}
$$

defined as above, with induced endomorphism $(A,R)(\mathrm{id},\alpha)\colon (A,R)(\mathrm{id},a)\to (A,R)(\mathrm{id},a)$. Thus, by Theorem 2.6, the trace of the composite

$$
(A,R)(\mathrm{id},a)\odot X \xrightarrow{\ (A,R)(\mathrm{id},\alpha)\odot\mathrm{id}\ } (A,R)(\mathrm{id},a)\odot X \xrightarrow{\ \mathrm{id}\odot f\ } (A,R)(\mathrm{id},a)\odot X\odot P \tag{15.13}
$$

is equal to the composite

$$
\langle\!\langle R\rangle\!\rangle \xrightarrow{\ \mathrm{tr}((A,R)(\mathrm{id},\alpha))\ } \langle\!\langle (A,R)\rangle\!\rangle \xrightarrow{\ \mathrm{tr}(f)\ } \langle\!\langle P\rangle\!\rangle.
$$

But by the proof in Theorem 15.1 that $\mathbb{P}\mathsf{rof}(\mathscr{W})$ is framed, we have

$$
(A,R)(\mathrm{id},a)\odot (X\odot P)\cong a^*(X\odot P)\cong (X\odot P)_a,
$$

while Lemma 15.3 and Lemma 14.2 inform us that under this isomorphism, (15.13) is identified with (15.12). Finally, Theorem 13.9 identifies $\mathrm{tr}((A,R)(\mathrm{id},\alpha))$ with $\langle\!\langle (\alpha,R)\rangle\!\rangle$, while Lemma 15.9 identifies this with the map $[\alpha]$. ∎

Finally, we can establish the uniqueness of our linearity formula using Lemma 15.9.

PROOF OF THEOREM 12.1. For any $\mathscr{W} \in \mathfrak{B}$ and any object R of $\mathscr{W}$, let $(A, R)(\mathrm{id}, -)$ denote the unit object $\mathbb{I}_{(A,R)} \in \mathcal{P}rof(\mathscr{W})((A, R), (A, R)) = \mathscr{W}(R, R)(A^{\mathrm{op}} \times A)$ regarded as an object of $\mathcal{P}rof(\mathscr{W})((\mathbb{1}, R), (A, R))(A)$. Then by Proposition 11.9, we have

$$\mathrm{colim}((A, R)(\mathrm{id}, -)) \cong (\pi_{A^{\mathrm{op}}})^{*}\mathbb{I}_R \otimes_{[A]} \mathbb{I}_{(A,R)} \cong (\pi_{A^{\mathrm{op}}})^{*}\mathbb{I}_R.$$

In other words, the colimit of $(A, R)(\mathrm{id}, -)$ is the constant weight $(\pi_{A^{\mathrm{op}}})^{*}\mathbb{I}_R$, regarded as a morphism from $(\mathbb{1}, R)$ to (A, R) in $\mathcal{P}rof(\mathscr{W})$. Thus, by (i) applied in $\mathcal{P}rof(\mathscr{W})$, to show that $(\pi_{A^{\mathrm{op}}})^{*}\mathbb{I}_R$ is absolute, it will suffice to show that $(A, R)(\mathrm{id}, -)$ is pointwise right dualizable. But its value at $a \in A$ is just the representable profunctor $(A, R)(\mathrm{id}, a)$, which as we have seen is always right dualizable.

Similarly, applying (ii) to the identity map of $(A, R)(\mathrm{id}, -)$, we find that the coefficient vector of the constant weight $(\pi_{A^{\mathrm{op}}})^{*}\mathbb{I}_R$ is the sum

$$\sum_{[\alpha]} \phi_{[\alpha]} \mathrm{tr}((A, R)(\mathrm{id}, \alpha))$$

where for $\alpha \in A(a, a)$, the map $(A, R)(\mathrm{id}, \alpha)$ is the induced endomorphism of $(A, R)(\mathrm{id}, a)$. However, by Theorem 13.9 and Lemma 15.9, $\mathrm{tr}((A, R)(\mathrm{id}, \alpha))$ is just the map $\langle\!\langle R \rangle\!\rangle \to \langle\!\langle (A, R) \rangle\!\rangle$ induced by $[\alpha]$, so to say that the coefficient vector is the above sum is exactly to say that the $\phi_{[\alpha]}$ are its components.   ∎

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