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PROJECTIVITY, CONTINUITY, AND ADJOINTNESS: QUANTALES, Q-POSETS, AND Q-MODULES

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ABSTRACT. In this paper, projective modules over a quantale are characterized by distributivity, continuity, and adjointness conditions. It is then show that a morphism $Q \longrightarrow A$ of commutative quantales is coexponentiable if and only if the corresponding Q-module is projective, and hence, satisfies these equivalent conditions.

1. Introduction

In [Niefield, 1978/1982a], we characterized exponentiable affine schemes over Spec(R), for a commutative ring R, by showing that a commutative R-algebra A is coexponentiable if and only if the corresponding R-module is finitely generated projective, or equivalently, finitely presented flat. Note that this characterization holds without the commutativity assumption on A when coexponentiability is replaced by the condition that $- \otimes_R A$ has a left adjoint. Of course, this is equivalent to coexponentiability in the commutative case, since $\otimes_R$ is the coproduct of algebras there. These results were subsequently generalized to graded and differential graded R-algebras in [Niefield, 1986]. In both cases, the algebras are objects of the category of commutative monoids for an appropriate monoidal category.

The question of coexponentiability of commutative quantales arose in preparation for a talk including an extension of Proposition 4.3 of [Niefield, 2012] which applied to a double category of quantales not considered therein. Since quantales are monoids in the monoidal category **Sup** of sup-lattices (i.e., complete lattices and sup-preserving maps), following an approach similar to that of rings is plausible for quantales, or more generally for algebras over a commutative quantale Q. If M is a Q-module, then the functor $- \otimes_Q M : Q\mathbf{Mod} \longrightarrow Q\mathbf{Mod}$ has a left adjoint if and only if M is projective if and only if M is flat (cf. [Joyal/Tierney, 1984]). It turns out that essentially the same proof as in [Niefield, 1978/1982a] yields a characterization of coexponentiable morphisms $Q \longrightarrow A$ of commutative quantales as those for which A is projective (or equivalently, flat) when considered as modules over Q.

More can be said about projective Q-modules. In [Niefield, 1982b], we showed that X is projective in **Sup** if and only if X is a completely distributive lattice if and only if X is a totally continuous lattice (i.e., a strong version of a continuous lattice, in the sense of [Scott, 1972]). A constructive version of complete distributivity (CCD) was introduced

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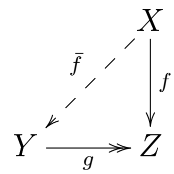
in [Fawcett/Wood, 1990] by requiring that the sup map $\bigvee: \mathcal{D}(X) \longrightarrow X$ has left adjoint (which is necessarily a right inverse in **Sup**), where $\mathcal{D}(X)$ is the lattice of downward closed subsets of X. To generalize these concepts to Q-modules, consider how the CCD condition is related to projectivity.

Recall that an object X of a category $\mathcal{A}$ is projective if and only for every $f: X \longrightarrow Z$ and every epimorphism $g: Y \longrightarrow Z$, there exists $\bar{f}: X \longrightarrow Y$ such that the following diagram commutes



When $\mathcal{A} = \mathbf{Sup}$, since g preserves suprema, it has a right adjoint $g_*: Z \longrightarrow Y$, and so the order-preserving map $g_* f: X \longrightarrow Y$ induces a sup-preserving map $\mathcal{D}(X) \longrightarrow Y$, and hence, a sup-preserving map $\bar{f}: X \longrightarrow Y$ such that $g\bar{f} = f$, when X is a retract of $\mathcal{D}(X)$ in **Sup**. Conversely, if X is projective, then one can show that the right inverse to $\bigvee: \mathcal{D}(X) \longrightarrow X$ is a left adjoint as well. If $\mathcal{A} = Q\mathbf{Mod}$, then $g_* f$ also satisfies $ag_* f(x) \leq g_* f(ax)$, since $g(ag_* f(x)) = agg_* f(x) \leq af(x) = f(ax)$. Thus, we consider a category of Q-posets X (i.e., posets on which Q acts) and morphisms which are order-preserving maps $f: X \longrightarrow Y$ satisfying $af(x) \leq f(ax)$. We then use their free Q-modules $\mathcal{D}_Q(X)$ to define a general notion of CCD_Q (and hence, totally Q-continuous) lattices to characterize projectivity in $Q\mathbf{Mod}$. Moreover, the latter holds without the assumption that Q is commutative.

In [Stubbe, 2007], the CCD and continuity characterizations of projectivity were generalized to cocontinuous $\mathcal{Q}$-categories for a quantaloid $\mathcal{Q}$. Since quantales are one-object quantaloids, $\mathcal{Q}$-modules are cocontinuous $\mathcal{Q}$-categories, and our Q-posets are necessarily Q-categories, it should be noted that our characterization is implicit in Stubbe's results.

The paper proceeds as follows. In Section 2, we generalize the Joyal/Tierney characterization of projective/flat Q-modules to the case where Q is not necessarily commutative. The category of left Q-posets X with their free left Q-modules $\mathcal{D}_Q(X)$ is introduced in Section 3. This is followed, in Section 4, by the notions of CCD_Q and totally Q-continuous objects of $Q\mathbf{Mod}$ which characterize projective left Q-modules and lead to some concrete examples. Restricting to the case where Q is commutative, in Section 5, we show that the coexponentiable quantale morphisms $Q \longrightarrow A$ are precisely those whose corresponding Q-modules satisfy these equivalent conditions.

2. Projectivity and Adjoints for Left Q-Modules

Let **Sup** denote the category of sup-lattices, i.e., complete lattices and suprema preserving maps. Monoids in **Sup** are *(unital) quantales*, in the sense of [Mulvey, 1986]. Thus, a quantale is a complete lattice together with a monoid structure such that the multiplication preserves suprema in both variables. If Q is a quantale, we write ab for the product, e for the unit, and $\bigvee a_\alpha$ for the suprema of $\{a_\alpha\}$. A *(left) Q-module* is a complete lattice M

together with a suprema preserving action of Q, i.e., satisfying $a(bm) = (ab)m$, $em = m$, $a(\bigvee m_\alpha) = \bigvee am_\alpha$, and $(\bigvee a_\alpha)m = \bigvee(a_\alpha m)$. Morphisms of quantales and Q-modules are defined in the usual way.

Recall [Joyal/Tierney, 1984] that **Sup** is self dual via $f\colon X \longrightarrow Y \;\mapsto\; f^\circ\colon Y^\circ \longrightarrow X^\circ$, where X° is the opposite poset and f° corresponds to the right adjoint f_* to f, which exists since f preserves suprema. Limits in **Sup** are formed in **Set** and equipped with point-wise suprema. Thus, limits and colimits agree, and so **Sup** is a bicomplete category. Now, $\mathbf{Sup}(X, Y)$ is a sup-lattice with point-wise suprema and the functor $\mathbf{Sup}(X, -)$ has a left adjoint, by Freyd's Special Adjoint Functor Theorem [Freyd, 1964], giving **Sup** the structure of a symmetric monoidal closed category with unit $\mathbb{2}$. Moreover, one can show that $X \otimes Y \cong \mathbf{Sup}(X, Y^\circ)^\circ$, and it follows that X is flat if and only if X is projective in **Sup**. Similarly, if Q is a commutative quantale, then the category $Q\mathbf{Mod}$ of left Q-modules in **Sup** is also a self-dual bicomplete symmetric monoidal category with unit Q. Moreover, $M \otimes_Q N \cong Q\mathbf{Mod}(M, N^\circ)^\circ$, and so projectivity and flatness agree in $Q\mathbf{Mod}$.

Dropping the commutativity assumption on Q, the duality on $Q\mathbf{Mod}$ is replaced by one between $Q\mathbf{Mod}$ and the category $\mathbf{Mod}Q$ of right Q-modules as follows. Recall that if M is a left Q-module, then M° is a right Q-module via $m \otimes a \mapsto [a, m]$, where $a \cdot - \dashv [a, -]$. Similarly, if M is a right Q-module, then M° is a left Q-module via $a \otimes m \mapsto \{a, m\}$, where $- \cdot a \dashv \{a, -\}$. Limits and colimits in $Q\mathbf{Mod}$ and $\mathbf{Mod}Q$ are computed in **Sup**, and so both categories are bicomplete. Although $Q\mathbf{Mod}$ and $\mathbf{Mod}Q$ are no longer monoidal, we can still define sup-lattices $L \otimes_Q M$ and $Q\mathbf{Mod}(M, N)$ via the following coequalizer and equalizer in **Sup**

$$L \otimes Q \otimes M \underset{L \otimes \cdot}{\overset{\cdot \otimes M}{\rightrightarrows}} L \otimes M \twoheadrightarrow L \otimes_Q M$$

$$Q\mathbf{Mod}(M, N) \rightarrowtail \mathbf{Sup}(M, N) \underset{\varphi_N}{\overset{\varphi_M}{\rightrightarrows}} \mathbf{Sup}(Q \otimes M, N)$$

where φ_M and φ_N are induced by actions of Q on M and N, respectively (cf. [Barr, 1996]).

Although we lose the adjointness between $- \otimes_Q M$ and $Q\mathbf{Mod}(M, -)$ in this case, each of these **Sup**-valued functors has an adjoint, namely,

$$\mathbf{Sup} \underset{\mathbf{Sup}(M,-)}{\overset{-\otimes_Q M}{\leftrightarrows}} \mathbf{Mod}Q \qquad \text{and} \qquad \mathbf{Sup} \underset{Q\mathbf{Mod}(M,-)}{\overset{M \otimes -}{\rightleftarrows}} Q\mathbf{Mod}$$

Similarly, $\mathbf{Mod}Q(M, N)$ can be defined for right Q-modules and we obtain analogous adjunctions. Then, as in the commutative case, one can show that

$$L \otimes_Q M \cong Q\mathbf{Mod}(M, L^\circ)^\circ \cong \mathbf{Mod}Q(L, M^\circ)^\circ$$

Thus, we will say that a left Q-module M is *projective* if $Q\mathbf{Mod}(M, -)\colon Q\mathbf{Mod} \longrightarrow \mathbf{Sup}$ preserves epimorphisms, and *flat* if $- \otimes_Q M\colon \mathbf{Mod}Q \longrightarrow \mathbf{Sup}$ preserves monomorphisms.

Now, if R is also a quantale and N is an QR-bimodule, then $Q\mathbf{Mod}(M, N)$ is a right R-module. In particular, $M^* = Q\mathbf{Mod}(M, Q)$ is a right Q-module and the counit $M \otimes M^* \longrightarrow Q$ is a morphism of QQ-bimodules. Thus, we get a morphism $L \otimes_Q M \otimes M^* \longrightarrow L$ in $\mathbf{Mod}Q$, and hence, a sup-preserving map $L \otimes_Q M \longrightarrow \mathbf{Mod}Q(M^*, L)$.

2.1. Theorem. *The following are equivalent for a left Q-module M:*

(a) $L \otimes_Q M \longrightarrow \mathbf{Mod}Q(M^*, L)$ *is an isomorphism in* $\mathbf{Sup}$*, for all right Q-modules L*

(b) $- \otimes_Q M \colon \mathbf{Mod}Q \longrightarrow \mathbf{Sup}$ *has a left adjoint*

(c) M is a flat

(d) M is a projective

Proof. First, (a) implies (b), since $\mathbf{Mod}Q(M^*, -) \colon \mathbf{Mod}Q \longrightarrow \mathbf{Sup}$ has a left adjoint, and (b) implies (c), since right adjoints preserve monomorphisms. To see that (c) implies (d), suppose $N \twoheadrightarrow P$ is a epimorphism in $Q\mathbf{Mod}$. Then $P^\circ \rightarrowtail N^\circ$ is a monomorphism in $\mathbf{Mod}Q$, and so $P^\circ \otimes_Q M \rightarrowtail N^\circ \otimes_Q M$ is a monomorphism in $\mathbf{Sup}$, since M is flat. Thus,

$$Q\mathbf{Mod}(M, N) \twoheadrightarrow Q\mathbf{Mod}(M, P)$$

is an epimorphism in $\mathbf{Sup}$, as desired. For (d) implies (a), suppose M is projective. Then M is a retract of a free left Q-module F, i.e, a biproduct of copies of Q, and so we have the commutative diagram

$$\begin{array}{ccc} L \otimes_Q F & \xrightarrow{\psi_F} & \mathbf{Mod}Q(F^*, L) \\ \downarrow\uparrow & & \uparrow\downarrow \\ L \otimes_Q M & \xrightarrow[\psi_M]{} & \mathbf{Mod}Q(M^*, Y) \end{array}$$

where the vertical down arrows are retractions. Since $L \otimes_Q -$ and $\mathbf{Mod}Q(-, L)$ preserve biproducts, one can show that ψ_F is an isomorphism, and it follows that so is ψ_M. ∎

Note that the equivalence of (a), (c), and (d) was established in [Joyal/Tierney, 1984], in the case where Q is commutative and $\mathbf{Sup}$ is replaced by $Q\mathbf{Mod}$ in (a). Since every left Q-module becomes a right Q-module with the same action, we get:

2.2. Corollary. *The following are equivalent for a left Q-module M, where Q is a commutative quantale:*

(a) $L \otimes_Q M \longrightarrow Q\mathbf{Mod}(M^*, L)$ *is an isomorphism in* $Q\mathbf{Mod}$

(b) $- \otimes_Q M \colon Q\mathbf{Mod} \longrightarrow Q\mathbf{Mod}$ *has a left adjoint*

(c) $- \otimes_Q M \colon Q\mathbf{Mod} \longrightarrow \mathbf{Sup}$ *has a left adjoint*

(d) M is a flat

(e) M is a projective

Proof. The proof that $(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a) \Rightarrow (b)$ follows as in Theorem 2.1, where the penultimate implication uses the fact that every bijection in $Q\mathbf{Mod}$ is an isomorphism; and (b) implies (c) holds, since the forgetful functor $Q\mathbf{Mod} \longrightarrow \mathbf{Sup}$ is right adjoint to $Q \otimes -$. ∎

3. Q-Posets and Free Q-Modules

In this section, fixing a quantale Q, we introduce the notion of a left Q-poset X and give a construction of the free left Q-module $\mathcal{D}_Q(X)$ on X. The latter will be used to generalize CCD and completely continuous to left Q-modules, and thus to obtain continuity and distributivity characterizations of projectivity in $Q\mathbf{Mod}$.

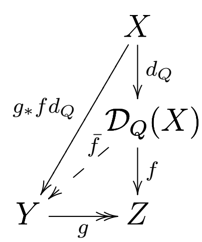
A *left Q-poset* is a poset X together with an order-preserving map $Q \times X \stackrel{\cdot}{\longrightarrow} X$ satisfying $a(bx) = (ab)x$ and $ex = x$, where e is the unit of Q. Let $Q\mathbf{Pos}$ denote the category of left Q-posets and order-preserving maps $f\colon X \longrightarrow Y$ satisfying $af(x) \leq f(ax)$. Note that if f has a right adjoint f_* which is also in $Q\mathbf{Pos}$, then $f(ax) \leq af(x)$, since $ax \leq af_*f(x) \leq f_*(af(x))$, and it follows that f is equivariant. Thus, we get:

3.1. Proposition. *Suppose M and N are left Q-modules and $f\colon M \longrightarrow N$ is a morphism of the underlying left Q-posets. Then f is a morphism of left Q-modules if and only if f has a right adjoint in $Q\mathbf{Pos}$.* ■

If e is the top element of Q, then every poset X becomes a left Q-poset $\hat{X}$ via the projection $Q \times \hat{X} \longrightarrow \hat{X}$, thus providing a left adjoint to the forgetful functor $Q\mathbf{Pos} \longrightarrow \mathbf{Pos}$. One can also consider the forgetful functor $Q\mathbf{Mod} \longrightarrow Q\mathbf{Pos}$ which has a left adjoint, in any case, by Freyd's Special Adjoint Functor Theorem, since it is easily seen to preserve limits. We denote this adjoint by $\mathcal{D}_Q$ and the unit by $d_Q\colon X \longrightarrow \mathcal{D}_Q(X)$, since $\mathcal{D}$ generalizes the down-set lattice functor $\mathbf{Pos} \longrightarrow \mathbf{Sup}$.

3.2. Proposition. *If X is a left Q-poset, then $\mathcal{D}_Q(X)$ is projective in $Q\mathbf{Mod}$.*

Proof. Consider the diagram



where f and g are morphisms of left Q-modules and g_* is a right inverse right adjoint to g. Since g_*fd_Q is in $Q\mathbf{Pos}$, there exists a unique $\bar{f}\colon \mathcal{D}_Q(X) \longrightarrow Y$ in $Q\mathbf{Mod}$ such that $\bar{f}d_Q = g_*fd_Q$. Thus, $g\bar{f}d_Q = gg_*fd_Q = fd_Q$, and so $g\bar{f} = f$, since d_Q is the unit for the adjunction. ■

We will see that a left Q-poset X is a left Q-module if and only if $d_Q\colon X \longrightarrow \mathcal{D}_Q(X)$ has a left adjoint left inverse in $Q\mathbf{Pos}$, which we will denote by s_Q, and that X is projective in $Q\mathbf{Mod}$ if and only if s_Q has a right inverse left adjoint in $Q\mathbf{Mod}$. To do so, we will use the following construction of $\mathcal{D}_Q(X)$.

3.3. Proposition. *$Q\mathbf{Pos}(X, Q^\circ)^\circ$ is the free left Q-module $\mathcal{D}_Q(X)$ on the Q-poset X.*

Proof. Since $\mathcal{D}_Q(X)^\circ \cong \mathbf{Mod}Q(Q, \mathcal{D}_Q(X)^\circ) \cong Q\mathbf{Mod}(\mathcal{D}_Q(X), Q^\circ) \cong Q\mathbf{Pos}(X, Q^\circ)$, the desired result follows. ■

A detailed description of $\mathcal{D}_Q(X)$ and its universal property will be given below. First, we present some general properties which will be used in this description as well as in subsequent sections.

We know that if M is a left Q-module, then M° is a right Q-module via $m \otimes a \mapsto [a, m]$, where $a \cdot - \dashv [a, -]$. Now, $- \cdot m\colon Q \longrightarrow M$ is also sup-preserving, and we denote its right adjoint by $M(m, -)$.

Of course, these adjoints do not exist, in general, for a left Q-poset X, but one can define $X(x, -)\colon X \longrightarrow Q$ by $X(x, x') = \bigvee\{a | ax \leq x'\}$. Note that $ax \leq x'$ implies $a \leq X(x, x')$, but not conversely. For example, if X is the left $\mathbb{2}$-poset $\hat{\mathbb{2}}$, then $0 \leq X(1, 0)$, but $0 \cdot 1 \not\leq 0$, since $0 \cdot 1 = 1$ in $\hat{\mathbb{2}}$.

Now, $ex \leq x$ implies $e \leq X(x, x)$. Since Q is a quantale one can show that

$$X(x', x'')X(x, x') \leq X(x, x'')$$

and so X becomes a Q-category, in the sense of [Lawvere, 1973]. Thus, a Q-poset is a Q-category with an order-preserving action of Q. This notion is weaker than that of a tensored Q-category, in the sense of [Kelly, 1982], i.e., one satisfying $X(ax, x') = Q(a, X(x, x'))$, since, in that case, it easily follows that $a \leq X(x, x')$ implies $ax \leq x'$.

If $f\colon X \longrightarrow Y$ is a morphism of left Q-posets, then f is a Q-functor (i.e., $X(x, x') \leq Y(f(x), f(x'))$, since $ax \leq x'$ implies $af(x) \leq f(ax) \leq f(x')$), but the converse need not hold, for consider the identity function $f\colon \mathbb{2} \longrightarrow \hat{\mathbb{2}}$. Then $\hat{\mathbb{2}}(x, x') = \mathbb{2}(x, x')$, for all x, x', but f is not a morphism of $\mathbb{2}$-posets, since $0 \cdot f(1) = 0 \cdot 1 = 1$, but $f(0 \cdot 1) = f(0) = 0$ so $0 \cdot f(1) \not\leq f(0 \cdot 1)$. However, when the codomain of f is a left Q-module, we have:

3.4. PROPOSITION. *The following are equivalent for a function $f\colon X \longrightarrow M$, where X is a left Q-poset and M is a left Q-module:*

(a) f is a morphism of left Q-posets

(b) $X(x, x') \leq M(f(x), f(x'))$ in Q, for all $x, x' \in X$

(c) $X(x, x')f(x) \leq f(x')$ in M, for all $x, x' \in X$

PROOF. We know (a) implies (b), as noted above, and (b) is equivalent to (c), since M is a left Q-module. To show (b) implies (a), suppose $X(x, x') \leq M(f(x), f(x'))$, for all $x, x' \in X$. Note that $a \leq M(f(x), m)$ implies $af(x) \leq m$, since M is a left Q-module. Thus, f is order preserving, since

$$x \leq x' \Rightarrow e \leq X(x, x') \leq M(f(x), f(x')) \Rightarrow f(x) \leq f(x')$$

and $af(x) \leq f(ax)$, since $a \leq X(x, ax) \leq M(f(x), f(ax))$. ∎

Identifying $Q\mathbf{Pos}(X, Q^\circ)^\circ$ with $\mathcal{D}_Q(X)$ and applying Proposition 3.4, we can take elements of $\mathcal{D}_Q(X)$ to be functions $\sigma\colon X \longrightarrow Q$ such that $\sigma(x')X(x,x') \le \sigma(x)$ in Q, since $X(x,x') \cdot \sigma(x) \le \sigma(x')$ in Q° precisely when the following holds in Q

$$\{X(x,x'), \sigma(x)\} \ge \sigma(x') \iff \sigma(x') \le \{X(x,x'), \sigma(x)\} \iff \sigma(x')X(x,x') \le \sigma(x)$$

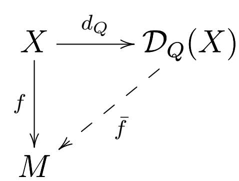
Moreover, $\mathcal{D}_Q(X)$ becomes a left Q-module with action, suprema, and infima defined point-wise in Q.

Given $x \in X$, since $X(x'', x)X(x', x'') \le X(x', x)$, it follows that the function $X(-,x)$ is an element of $\mathcal{D}_Q(X)$, and these elements generate $\mathcal{D}_Q(X)$ as a left Q-module since:

3.5. Proposition. *If $\sigma \in \mathcal{D}_Q(X)$, then $a \le \sigma(x) \iff aX(-,x) \le \sigma$, for all $a \in Q$, $x \in X$, and $\sigma = \bigvee\{aX(-,x) | a \le \sigma(x)\}$ in $\mathcal{D}_Q(X)$.*

Proof. First, $a \le \sigma(x)$ if and only if $aX(-,x) \le \sigma$, since $a \le \sigma(x)$ implies $aX(x',x) \le \sigma(x)X(x',x) \le \sigma(x')$, for all x', and $aX(-,x) \le \sigma$ implies $a = ae \le aX(x,x) \le \sigma(x)$. Thus, $\bigvee\{aX(-,x)|a \le \sigma(x)\} \le \sigma$. The reverse inequality holds since suprema in $\mathcal{D}_Q(X)$ are computed point-wise in Q and $\sigma(x) \le (aX(-,x))(x)$, for $a = \sigma(x)$. ■

The universal property of $\mathcal{D}_Q(X)$ as a free left Q-module on the left Q-poset X



can be described directly as follows. Define

$$d_Q\colon X \longrightarrow \mathcal{D}_Q(X)$$

by $d_Q(x) = X(-,x)$. Then d_Q is clearly order preserving; and $ad_Q(x) \le d_Q(ax)$, or equivalently, $aX(x',x) \le X(x',ax)$, for all x', since

$$bx' \le x \Rightarrow abx' \le ax \Rightarrow ab \le X(x', ax)$$

Given a left Q-module M and a morphism $f\colon X \longrightarrow M$ of left Q-posets, define $\bar{f}\colon \mathcal{D}_Q(X) \longrightarrow M$ by $\bar{f}(\sigma) = \bigvee\{\sigma(x)f(x) \mid x \in X\}$. Then $\bar{f}$ is clearly order preserving and

$$a\bar{f}(\sigma) = a\bigvee_x \sigma(x)f(x) = \bigvee_x a\sigma(x)f(x) = \bar{f}(a\sigma)$$

To see that $\bar{f}$ is a left Q-module morphism, by Proposition 3.1, it suffices to show that $\bar{f}$ has a right adjoint in $Q\mathbf{Pos}$. Consider $\bar{f}_*\colon M \longrightarrow \mathcal{D}_Q(X)$ defined by $\bar{f}_*(m) = M(f(-), m)$, which is clearly in $Q\mathbf{Pos}$, and $\bar{f}(\bar{f}_*(m)) \le m$, since

$$\bar{f}(\bar{f}_*(m)) = \bigvee_x M(f(x), m)f(x) \le m$$

To see that $\sigma \leq \bar{f}_*(\bar{f}(\sigma)) = M(f(-), \bar{f}(\sigma))$, by Proposition 3.5, it suffices to show that $a \leq \sigma(x)$ implies $aX(-,x) \leq M(f(-), \bar{f}(\sigma))$. But, $a \leq \sigma(x)$ implies $af(x) \leq \bar{f}(\sigma)$ by definition of $\bar{f}$ and $aX(x',x)f(x') \leq af(x)$ by Proposition 3.4, and it follows that $aX(x',x) \leq M(f(x'), \bar{f}(\sigma))$, for all x'. Therefore, $\bar{f}$ is a morphism of left Q-modules, and $\bar{f}d_Q = f$, since

$$\bar{f}(X(-,x)) = \bigvee_{x'} X(x',x)f(x') = f(x)$$

Uniqueness of $\bar{f}$ holds since the morphisms $X(-,x)$ generate $\mathcal{D}_Q(X)$.

We conclude this section with a characterization of left Q-modules as the 'cocontinuous' Q-posets.

3.6. Proposition. *Suppose X is a left Q-poset. Then X is a left Q-module if and only if $d_Q \colon X \longrightarrow \mathcal{D}_Q(X)$ has a left adjoint left inverse in Q***Pos**.

Proof. Suppose X is a left Q-module. Then the identity $id \colon X \longrightarrow X$ induces a morphism $s_Q \colon \mathcal{D}_Q(X) \longrightarrow X$ in Q**Mod** such that $s_Q d_Q = id$ and

$$s_Q(\sigma) = \overline{id}(\sigma) = \bigvee_x \sigma(x)x$$

Since $\sigma(x)x \leq s_Q(\sigma)$, we know that $\sigma(x) \leq X(x, s_Q(\sigma))$, for all x, and so $\sigma \leq d_Q s_Q(\sigma)$

Conversely, suppose d_Q has a left adjoint left inverse in Q**Pos**, call it s_Q. Then one can show X is complete via $\sup x_\alpha = s_Q(\bigvee d_Q(x_\alpha))$ and $a \sup x_\alpha = a s_Q(\bigvee d_Q(x_\alpha)) \leq s_Q(a \bigvee d_Q(x_\alpha)) = s_Q(\bigvee a d_Q(x_\alpha)) \leq s_Q(\bigvee d_Q(a x_\alpha)) = \bigvee a x_\alpha$. Since $\bigvee a x_\alpha \leq a \sup x_\alpha$, in any case, the desired result follows. ■

Note that the left adjoint s_Q above is necessarily in Q**Mod** by Proposition 3.1.

4. Projectivity, Complete Distributivity, and Total Continuity

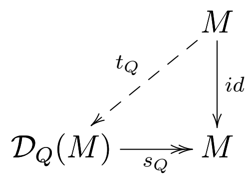
In this section, we generalize the notions of completely distributive and totally continuous sup lattices, and use these concepts to characterize projectivity in the category of left modules over a quantale Q. As noted in the introduction, this is a special case of Stubbe's characterization of projective quantaloid modules, presented here in a more elementary setting.

A left Q-module M is called CCD_Q (*constructively completely distributive over* Q) if the morphism $s_Q \colon \mathcal{D}_Q(M) \longrightarrow M$ has a left adjoint right inverse in Q**Mod**.

4.1. Proposition. *M is projective in Q***Mod** *if and only if M is CCD_Q.*

Proof. Suppose M is CCD_Q. Since $\mathcal{D}_Q(M)$ is projective (by Proposition 3.2) and M is a retract of $\mathcal{D}_Q(M)$, it follows that M is projective in Q**Mod**.

Conversely, suppose M is projective in Q**Mod**. Consider the diagram



where t_Q is the right inverse to s_Q in Q**Mod** induced by projectivity of M. To see that $t_Q \dashv s_Q$, it suffices to show that $t_Q s_Q \leq id$, or equivalently, $t_Q s_Q d_Q \leq d_Q$, since $\{M(-,m)|m \in M\}$ generates $\mathcal{D}_Q(M)$ by Proposition 3.5. But, s_Q is left adjoint left inverse to d_Q in Q**Pos**, and so $s_Q t_Q \leq id \Rightarrow t_Q \leq d_Q \Rightarrow t_Q s_Q d_Q \leq d_Q$, as desired. ∎

Suppose M is a left Q-module and $m, m' \in M$. Then m' *is totally below* m *relative* a, written $m' \lhd_a m$, if, for all $a' \in Q$ and $\sigma \in \mathcal{D}_Q(M)$,

$$a'm \leq s_Q(\sigma) \Rightarrow a'a \leq \sigma(m')$$

and M is called *totally Q-continuous* if it satisfies

$$m = \bigvee \{am' | m' \lhd_a m\}$$

Note that $m' \lhd_a m$ is order preserving in the second variable and order reversing in the first. Also, we can replace "$a'a \leq \sigma(m')$" by "$a'aM(-,m') \leq \sigma$" in the definition of $m' \lhd_a m$, since they are equivalent by Proposition 3.5.

4.2. Lemma. *Suppose $m' \lhd_a m$ and $b \in Q$. Then*

(a) $am' \leq m$

(b) $m' \lhd_{ba} bm$

(c) $[b, m'] \lhd_{ab} m$*, where* $b \cdot - \dashv [b, -]$

Proof. Suppose $m' \lhd_a m$. Then $am' \leq m$, since $m = s_Q(M(-,m))$ implies $a \leq M(m', m)$, and so (a) holds. To prove (b), suppose $a'bm \leq s_Q(\sigma)$. Then $a'ba \leq \sigma(m')$, since $m' \lhd_a m$, and it follows that $m' \lhd_{ba} bm$. For (c), suppose $a'm \leq s_Q(\sigma)$. Then $a'a \leq \sigma(m')$, and so

$$a'ab \leq \sigma(m')b \leq \sigma(b[b, m'])b \leq \sigma([b, m'])$$

where the second inequality holds since $b[b, m'] \leq m'$ and σ is order reversing, and the third holds since $\{b, \sigma(n)\} \geq \sigma(bn)$ implies $\sigma(bn)b \leq \sigma(n)$, for all $n \in M$. Thus, $[b, m'] \lhd_{ab} m$. ∎

4.3. Proposition. *M is totally Q-continuous if and only if M is CCD$_Q$.*

Proof. Suppose M is totally Q-continuous and define $t_Q\colon M \longrightarrow \mathcal{D}_Q(M)$ by

$$[t_Q(m)](m') = \bigvee\{a | m' \lhd_a m\}$$

To show $t_Q(m) \in \mathcal{D}_Q(M)$, i.e., $[t_Q(m)](m')M(m'', m') \le [t_Q(m)](m'')$, suppose $m' \lhd_a m$ and $b \le M(m'', m')$. Then $[b, m'] \lhd_{ab} m$, by Lemma 4.2, and $m'' \le [b, m']$, since $bm'' \le m'$, and so $m'' \lhd_{ab} m$. Thus, $[t_Q(m)](m')M(m'', m') \le [t_Q(m)](m'')$.

Now, $t_Q s_Q(\sigma) \le \sigma$, since $m' \lhd_a s_Q(\sigma)$ implies $a \le \sigma(m')$, and

$$s_Q t_Q(m) = \bigvee_{m'} \left(\bigvee\{a \mid m' \lhd_a m\}\right) m' = \bigvee\{am' \mid m' \lhd_a m\} = m$$

since M is totally B-continuous. Thus, t_Q is a left adjoint right inverse of s_Q in **Sup**, since t_Q is clearly order preserving, so it remains to show that t_Q is equivariant. By Proposition 3.1, it suffices to show that t_Q is a left Q-poset morphism. But,

$$b[t_Q(m)](m') = b\bigvee\{a|m' \lhd_a m\} \le \bigvee\{ba|m' \lhd_{ba} bm\} \le [t_Q(bm)](m')$$

Conversely, suppose M is CCD$_Q$, and let t_Q denote the left adjoint right inverse of s_Q in Q**Mod**. To prove M is totally Q-continuous, it suffices to show that $aM(-, m') \le t_Q(m)$ implies $m' \lhd_a m$, for then

$$m = s_Q t_Q(m) = s_Q(\bigvee\{aM(-, m')|aM(-, m') \le t_Q(m)\}) \le \bigvee\{am'|m' \lhd_a m\} \le m$$

where the second equality follows from Proposition 3.5. Given $aM(-, m') \le t_Q(m)$, to show $m' \lhd_a m$, suppose $a'm \le s_Q(\sigma)$. Then $a'a \le a'aM(m', m') \le a'[t_Q(m)](m') \le [t_Q(a'm)](m') \le [t_Q(s_Q(\sigma))](m') \le \sigma(m')$, and so $m' \lhd_a m$, as desired. ∎

Combining Theorem 2.1 with Propositions 4.1 and 4.3, we get:

4.4. Theorem. *The following are equivalent for a left Q-module M:*

(a) $- \otimes_Q M\colon \mathbf{Mod}Q \longrightarrow \mathbf{Sup}$ *has a left adjoint*

(b) M is flat

(c) M is projective

(d) M is CCD$_Q$

(e) M is totally Q-continuous ∎

Note that as in Theorem 2.1, we can replace (a) by "$-\otimes_Q M\colon Q\mathbf{Mod} \to Q\mathbf{Mod}$ has a left adjoint" when Q is commutative.

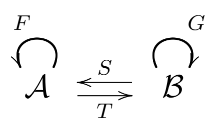
We conclude this section with examples of projective Q-modules. By Proposition 3.2, we know $\mathcal{D}_Q(X)$ is projective, and hence, CCD_Q. Also, the unit interval $[0,1]$ is a CCD suplattice (see [Fawcett/Wood, 1990]), and hence, projective in **Sup**. Now, if P is any projective suplattice, then one easily shows that the free Q-module $Q \otimes P$ is projective in $\mathbf{Mod}Q$. Thus, $Q \otimes \mathcal{D}(X)$ and $Q \otimes [0,1]$ are CCD_Q. For an example of a projective Q-module which is not of this form, let u be an idempotent element of Q, i.e., $u^2 = u$. Then, one can show that uQ is projective in $\mathbf{Mod}Q$ and not, in general, isomorphic to $Q \otimes P$, for any P.

5. Coexponentiable Morphisms of Commutative Quantales

In this section, we show that a morphism $Q \to A$ is coexponentiable in the category $\mathbf{Quant}_c$ of commutative (unital) quantales if and only if the corresponding Q-module is projective, and hence, satisfies the equivalent conditions of Theorem 4.4. As in the case of rings, this characterization holds when A is not necessarily commutative provided that coexponentiability is replaced by the existence of a left adjoint to $-\otimes_Q A$.

We begin with a general lemma from [Niefield, 1978/1982a]

5.1. Lemma. *Suppose $\mathcal{A}$ is a category with coequalizers, and consider the diagram*



where $TF = GT$ and $S \dashv T$ such that the counit $\varepsilon_A\colon STA \to A$ is a regular epimorphism, for all A. Then, if G has a left adjoint, so does F. ∎

Suppose Q is a commutative quantale and X is a Q-module. Then one can show that $Q\times X$ becomes a commutative quantale with $(a,x)(a',x') = (aa', ax' \vee a'x)$ and unit $(e,0)$, where 0 is the bottom element of X. Furthermore, every Q-module morphism $f\colon X \to Y$ induces a quantale morphism $Q \times f\colon Q\times X \to Q \times Y$, and it is not difficult to show that if f is a monomorphism, then so is $Q \times f$.

5.2. Lemma. *If $Q \to A$ is coexponentiable in $\mathbf{Quant}_c$, then A is flat in $Q\mathbf{Mod}$.*

Proof. To show that A is flat, suppose $f\colon X \to Y$ is a monomorphism in $Q\mathbf{Mod}$. Then $Q \times f\colon Q\times X \to Q\times Y$ is a monomorphism in $\mathbf{Quant}_c$. Thus, we get a commutative diagram in $Q\mathbf{Mod}$.

$$\begin{array}{ccccc}
X\otimes_Q A & \rightarrowtail & A\oplus(X\otimes_Q A)\cong(Q\oplus X)\otimes_Q A & \cong & (Q\times X)\otimes_Q A \\
{\scriptstyle f\otimes_Q A}\downarrow & & \downarrow & & \downarrow{\scriptstyle (Q\times f)\otimes_Q A} \\
Y\otimes_Q A & \rightarrowtail & A\oplus(Y\otimes_Q A)\cong(Q\oplus Y)\otimes_Q A & \cong & (Q\times Y)\otimes_Q A
\end{array}$$

where $(Q \times f) \otimes_Q A$ is a monomorphism in $Q\mathbf{Mod}$, since $- \otimes_Q A$ preserves monomorphisms (being a right adjoint) and the forgetful functor preserves monomorphisms, as well. Therefore, $f \otimes_Q A$ is a monomorphism, and it follows that A is flat in $Q\mathbf{Mod}$. ■

5.3. Theorem. *The following are equivalent for a morphism $f\colon Q \longrightarrow A$ of commutative quantales:*

(a) f is coexponentiable, i.e., $- \otimes_Q A\colon \mathbf{Quant}_c\backslash Q \longrightarrow \mathbf{Quant}_c\backslash Q$ has a left adjoint

(b) A is flat in $Q\mathbf{Mod}$

(c) A is projective in $Q\mathbf{Mod}$

(d) A is CCD_Q in $Q\mathbf{Mod}$

(e) A is totally Q-continuous in $Q\mathbf{Mod}$

(f) $- \otimes_Q A\colon Q\mathbf{Mod} \longrightarrow Q\mathbf{Mod}$ has a left adjoint

Proof. First, (a) implies (b) by Lemma 5.2, and the equivalence of (b) through (f) follows from the remark following Theorem 4.4. For (f) implies (a), apply Lemma 5.1 to

$$\mathbf{Quant}_c\backslash Q \circlearrowleft^{-\otimes_Q A} \; \underset{T}{\overset{S}{\leftrightarrows}} \; Q\mathbf{Mod} \circlearrowright^{-\otimes_Q A}$$

where T is the forgetful functor and S is its left adjoint the symmetric algebra functor. ■

Note that, as in the case of rings, this characterization holds without the commutativity assumption on A when the coexponentiability part of (a) is omitted. Finally, one can also show that $(f) \Rightarrow (a) \Rightarrow (b)$ holds for monoids in any symmetric monoidal closed category with colimits and finite biproducts, and the proof is essentially the same as that of Theorem 5.3. This would provide a single proof for quantales and rings, but one must add the additional finiteness conditions in the latter case to obtain the equivalence of (a), (b), (c), and (f).

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