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# SIMPLICIAL NERVE OF AN $\mathcal{A}_\infty$-CATEGORY

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ABSTRACT. We introduce a functor called the simplicial nerve of an $\mathcal{A}_\infty$-category defined on the category of $\mathcal{A}_\infty$-categories with values in simplicial sets. We show that the nerve of an $\mathcal{A}_\infty$-category is an $(\infty, 1)$-category in the sense of J. Lurie [Lur1]. This construction generalizes the nerve construction for differential graded categories given in [Lur2]. We prove that if a differential graded category is pretriangulated in the sense of A.I. Bondal and M. Kapranov [Bo-Ka] then its nerve is a stable $(\infty, 1)$-category in the sense of J. Lurie [Lur2].

## 1. Introduction

$\mathcal{A}_\infty$-algebras were introduced by J.D. Stasheff [Sta] in order to encode the notion of a binary operation associative up to a coherent system of homotopies. An $\mathcal{A}_\infty$-algebra is a $\mathbb{Z}$-graded vector space $A$ over some base field $\mathbb{K}$ together with degree $2-k$ morphisms

$$m_k : A^{\otimes n} \longrightarrow A, \quad k \geq 1$$

satisfying the equation

$$\sum_{n=r+t+s} (-1)^{sr+t} m_{r+t+1}(Id^{\otimes^r} \otimes m_s \otimes Id^{\otimes^t}) = 0 \tag{1}$$

for $n \geq 1$. This equation for $n = 1$ tells that $m_1$ is a differential on $A$ and for $n = 2$ provides the compatibility of the binary operation $m_2$ with the differential $m_1$ in terms of the Leibniz rule. For $n = 3$ the equation is

$$m_2(m_2 \otimes Id) - m_2(Id \otimes m_2) = m_1(m_3) + \sum_{2=r+t} m_3(Id^{\otimes^r} \otimes m_1 \otimes Id^{\otimes^t})$$

which expresses the fact that $m_2$ is an associative operation up to the data provided by $m_3$. Values of $n > 3$ in equation (1) encode all the coherences needed to be satisfied by such associativity constraints. Similarly an $\mathcal{A}_\infty$-morphism is a morphism of differential graded spaces

$$f_1 : A \longrightarrow B$$

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which preserves the binary operations only up to the data provided by

$$f_2 : A \otimes A \longrightarrow B$$

whose coherences are controlled by degree $1-k$ morphisms $f_k : A^{\otimes k} \longrightarrow B$ for $k > 3$. The notion of a dg-algebra (morphisms of dg-algebras) is recovered by considering $\mathcal{A}_\infty$-algebras with vanishing $m_k$ for $k > 2$ ( $\mathcal{A}_\infty$-morphisms with vanishing $f_k$ for $k > 1$). K. Fukaya and M. Kontsevich-Y. Soibelman later considered $\mathcal{A}_\infty$-categories and $\mathcal{A}_\infty$-functors as a natural generalization of these notions which have been essential tools in the formulation of homological mirror symmetry [Kon-Soi],[Fuk]. Namely the Fukaya category $\mathcal{F}(X)$ is an $\mathcal{A}_\infty$-category associated to a symplectic manifold $X$ and it corresponds to the $A$-side of such symmetry. Similarly to the case of algebras, differential graded categories (dg-functors) can be identified with $\mathcal{A}_\infty$-categories ($\mathcal{A}_\infty$-functors) for which the composition of morphisms is strictly associative (strictly preserving the composition of morphisms).

In the first part of this paper we define a functor (Definition 2.8) called the simplicial nerve of an $\mathcal{A}_\infty$-category

$$N_{\mathcal{A}_\infty} : \mathcal{A}_\infty Cat \longrightarrow SSet$$

defined on the category $\mathcal{A}_\infty Cat$ of $\mathcal{A}_\infty$-categories with $\mathcal{A}_\infty$-functors and values in simplicial sets. We prove (Proposition 2.15) that the simplicial nerve of an $\mathcal{A}_\infty$-category is an $(\infty, 1)$-category in the sense of J. Lurie [Lur1]. For an $\mathcal{A}_\infty$-category $\mathcal{A}$ its nerve is the simplicial set of $\mathcal{A}_\infty$-functors from a certain cosimplicial $\mathcal{A}_\infty$-category $\mathcal{A}[\Delta^-]$, generated by the standard simplices, into $\mathcal{A}$. The restriction of this functor to dg-categories provides a functorial description of the differential graded nerve $N_{dg}$ introduced in [Lur2] by J. Lurie and earlier defined in [Hin-Sch] by V.A. Hinich and V.V. Schechtman. The existence of a model category structure without limits on $\mathcal{A}_\infty Cat$ was shown in [Le-Ha] and the study of its relationships with the nerve construction presented in this paper will be subject of future work.

In the second part we establish a connection between pretriangulated differential graded categories in the sense of A.I. Bondal and M. Kapranov [Bo-Ka] and stable $(\infty, 1)$-categories in the sense of J. Lurie [Lur2]. Pretriangulated dg-categories provide a natural setting to address the lack of functoriality of the cone construction for triangulated categories following from the axioms of J.-L. Verdier [Ver]. To a dg-category with a zero object $\mathcal{D}$ it is possible to associate the dg-category of twisted complexes of $\mathcal{D}$, denoted by $PreTr(\mathcal{D})$, whose construction has to be understood as a triangulated hull of $\mathcal{D}$. $PreTr(\mathcal{D})$ has a shift functor and a functorial notion of cones inducing a triangulated structure on its homotopy category $H^0(PreTr(\mathcal{D}))$. In particular when $\mathcal{D}$ is pretriangulated the dg-embedding $\mathcal{D} \longrightarrow PreTr(\mathcal{D})$ is a quasi-equivalence of dg-categories and hence $\mathcal{D}$ inherits shift and cones from $PreTr(\mathcal{D})$ making $H^0(\mathcal{D})$ into a triangulated category [Bo-Ka]. On the $(\infty, 1)$-categorical side J. Lurie in [Lur2] introduced the notion of stable $(\infty, 1)$-category as an axiomatization of the properties of stable homotopy theory. The relevant feature for our purposes is that in a stable $(\infty, 1)$-category the notion of an exact triangle is replaced by the one of homotopy fiber of a morphism. Moreover stable $(\infty, 1)$-categories have canonically defined loop and suspension functors playing the role of the shift functor and its inverse for triangulated categories. This data induces a structure of triangulated category on the homotopy category of a stable $(\infty, 1)$-category [Lur2]. We give an explicit proof (Theorem 3.18) that if $\mathcal{D}$ is a pretriangulated dg-category the nerve $N_{dg}(\mathcal{D})$ is a stable $(\infty, 1)$-category and $H^0(\mathcal{D})$ is identified, as a triangulated category, with the homotopy category of $N_{dg}(\mathcal{D})$. The proof is based on a direct computation performed on an $(\infty, 1)$-category equivalent to $N_{dg}(\mathcal{D})$, that we call the big dg-nerve $N_{dg}^{big}(\mathcal{D})$, defined as the nerve of a certain simplicial category $\mathcal{D}_\Delta$ whose simplicial set of morphisms is obtained by applying the Dold-Kan correspondence [McL] to a truncation of the cochain complex of morphisms in $\mathcal{D}$.

1.1. Conventions and notations. From now on we fix a field $\mathbb{K}$ of characteristic 0. The category $Vect_{\mathbb{Z}}(\mathbb{K})$ is the category whose objects are $\mathbb{Z}$-graded vector spaces over $\mathbb{K}$

$$V = \bigoplus_{n\in\mathbb{Z}} V^n$$

and morphisms are degree preserving $\mathbb{K}$-linear maps. The tensor product of graded vector spaces is defined as

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q$$

and the graded space of morphisms as

$$Hom^n_{Vect_{\mathbb{Z}}(\mathbb{K})}(V, W) = \prod_{p\in\mathbb{Z}} Hom_{Vect(\mathbb{K})}(V^p, W^{p+n})$$

We say that a morphism is of degree $n$ if it belongs to such graded component. The tensor product of two morphisms is defined according to the convention

$$(f \otimes g)(x \otimes y) = (-1)^{deg(x)deg(g)} f(x) \otimes g(y) \tag{2}$$

A cochain complex is an object $V \in Ob(Vect_{\mathbb{Z}}(\mathbb{K}))$ together with a morphism $d \in Hom^1_{Vect_{\mathbb{Z}}(\mathbb{K})}(V, V)$ such that $d^2 = 0$. We call $d$ the differential of the cochain complex. A morphism of cochain complexes $f : V \longrightarrow W$ is a morphism $f \in Hom^0_{Vect_{\mathbb{Z}}(\mathbb{K})}(V, W)$ such that $d \circ f = f \circ d$. The category $Ch^\bullet(\mathbb{K})$ is the category whose objects are cochain complexes and morphisms are morphisms of cochain complex. The cohomology of a cochain complex $(V, d)$ is the $\mathbb{Z}$-graded vector space defined as

$$H^\bullet(V) = \frac{Ker(d)}{Im(d)}$$

A morphism of cochain complexes $f : V \longrightarrow W$ is called a quasi-isomorphism if the morphism induced in cohomology $H^\bullet(f) : H^\bullet(V) \longrightarrow H^\bullet(W)$ is an isomorphism. The tensor product of $\mathbb{Z}$-graded vector spaces extends to a functor on $Ch^\bullet(\mathbb{K})$ defining a symmetric monoidal structure $(Ch^\bullet(\mathbb{K}), \bigotimes, \mathbb{K})$ where, for cochain complexes $V$ and $W$, we have

$$d_{V\otimes W} = d_V \otimes Id_W + Id_V \otimes d_W$$

Similarly the category $Ch_\bullet(\mathbb{K})$ of chain complexes has objects $\mathbb{Z}$-graded vector spaces together with a morphism $d \in Hom^{-1}_{Vect_{\mathbb{Z}}(\mathbb{K})}(V,V)$ such that $d^2 = 0$ and analogue notions of homology and quasi-isomorphism. We denote by $op : Ch^\bullet(\mathbb{K}) \longrightarrow Ch_\bullet(\mathbb{K})$ the functor associating to a cochain complex the chain complex $V^{op}_p = V^{-p}$ with the same differential and by $\tau_{\geq 0} : Ch_\bullet(\mathbb{K}) \longrightarrow Ch^{\geq 0}_\bullet(\mathbb{K})$ the truncation functor

$$\tau_{\geq 0}(V)_p = \begin{cases} 0 & \text{if } p < 0 \\ Ker(d_{|V_0}) & \text{if } p = 0 \\ V_p & \text{if } p > 0. \end{cases}$$

For a category $\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$, denoted by $S(\mathcal{C})$, is the category of functors $Fun(\Delta^{op}, \mathcal{C})$ where $\Delta$ is the standard simplex category. Similarly, the category of cosimplicial objects in $\mathcal{C}$ is the category of functors $Fun(\Delta, \mathcal{C})$. We refer to [Lur1, Chap. 1-2-3] for the theory of $(\infty,1)$-categories and related constructions. The smallness assumptions necessary for the consistency of the results presented in this work are implicitly assumed.

## 2. The simplicial nerve of an $\mathcal{A}_\infty$-category

2.1. $\mathcal{A}_\infty$-CATEGORIES AND $\mathcal{A}_\infty$-FUNCTORS. We recall now the notion of $\mathcal{A}_\infty$-category and of $\mathcal{A}_\infty$-functor. We refer to [Le-Ha] for an extensive survey of the subject. For the purposes of this work we will refer to an $\mathcal{A}_\infty$-category meaning a strictly unital $\mathcal{A}_\infty$-category and to an $\mathcal{A}_\infty$-functor meaning a strictly unital $\mathcal{A}_\infty$-functor.

2.2. DEFINITION. [$\mathcal{A}_\infty$-category] *Let $\mathbb{K}$ be a field, an $\mathcal{A}_\infty$-category $\mathcal{A}$ is the data of:*

- *A set of objects $Ob(\mathcal{A})$*
- *For every pair of objects $x, y \in Ob(\mathcal{A})$ a graded space of morphisms $Hom^\bullet_{\mathcal{A}}(x,y)$*
- *For $k \geq 1$ and a sequence of objects $x_0, x_1, \ldots, x_k$, a morphism of degree $2-k$*

  $$m_k : Hom^\bullet_{\mathcal{A}}(x_{k-1}, x_k) \otimes \cdots \otimes Hom^\bullet_{\mathcal{A}}(x_0, x_1) \longrightarrow Hom^\bullet_{\mathcal{A}}(x_0, x_k)$$

  *such that, for every $n \geq 1$*

  $$\sum_{n=r+t+s} (-1)^{sr+t} m_{r+t+1}(Id^{\otimes^r} \otimes m_s \otimes Id^{\otimes^t}) = 0 \tag{3}$$

- *For every object $x \in Ob(\mathcal{A})$ a degree $0$ element $1_x \in Hom^\bullet_{\mathcal{A}}(x,x)$, called the identity at $x$, such that*

  $$m_1(1_x) = 0$$
  $$m_2(1_x \otimes a) = a = m_2(a \otimes 1_x)$$
  $$m_n(a_1 \otimes \cdots \otimes a_{j-1} \otimes 1_x \otimes a_{j+1} \otimes \ldots a_n) = 0$$

  *for $n > 2$, $1 \leq j \leq n$.*

2.3. DEFINITION. [$\mathcal{A}_\infty$-functor] *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{A}_\infty$-categories, an $\mathcal{A}_\infty$-functor*

$$f : \mathcal{A} \longrightarrow \mathcal{B}$$

*is the data of:*

- *A map of sets $f_0 : Ob(\mathcal{A}) \longrightarrow Ob(\mathcal{B})$*
- *For $k \geq 1$ and a sequence of objects $x_0, x_1, \ldots, x_k$, a morphism of degree $1-k$*

$$f_k : Hom^\bullet_{\mathcal{A}}(x_{k-1}, x_k) \otimes \cdots \otimes Hom^\bullet_{\mathcal{A}}(x_0, x_1) \longrightarrow Hom^\bullet_{\mathcal{B}}(f_0(x_0), f_0(x_k))$$

*such that, for $n \geq 1$*

$$\sum_{n=r+t+s} (-1)^{sr+t} f_{r+t+1}(Id^{\otimes^r} \otimes m_s \otimes Id^{\otimes^t}) = \sum_{\substack{1\leq r\leq n \\ i_1+\cdots+i_r=n}} (-1)^{\epsilon_r} m'_r(f_{i_1} \otimes \cdots \otimes f_{i_r}) \quad (4)$$

*where*

$$\epsilon_r = \epsilon_r(i_1, \ldots, i_r) = \sum_{2\leq k\leq r} \left( (1-i_k) \sum_{1\leq l\leq k-1} i_l \right) \quad (5)$$

*and satisfying the strict unitality conditions:*

$$f_1(1_x) = 1_{f_0(x)}$$

$$f_n(a_1 \otimes \cdots \otimes a_{j-1} \otimes 1_x \otimes a_{j+1} \otimes \ldots a_n) = 0$$

*for $n > 1$, $1 \leq j \leq n$.*

2.4. REMARK. $\mathcal{A}_\infty$-categories and $\mathcal{A}_\infty$-functors can be defined in a more canonical way using the Bar construction [Le-Ha]. The sign convention adopted in Definition 2.2 and 2.3 follows from the convention (2) defining the tensor product of graded morphisms after considering such construction.

If $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{C}$ are $\mathcal{A}_\infty$-functors, their composition $g \circ f : \mathcal{A} \longrightarrow \mathcal{C}$ has graded components given by

$$(g \circ f)_k = \sum_{r=1}^{k} \sum_{i_1+\cdots+i_r=k} (-1)^{\epsilon_r(i_1,\ldots,i_r)} g_r(f_{i_1} \otimes \cdots \otimes f_{i_r}) \quad (6)$$

This composition is strictly associative with unit given by $Id_{\mathcal{A}}$ and defines a category $\mathcal{A}_\infty Cat$ of $\mathcal{A}_\infty$-categories over $\mathbb{K}$.

A differential graded category (dg-category) over a field $\mathbb{K}$ is a category enriched over the symmetric monoidal category $(Ch^\bullet(\mathbb{K}), \bigotimes, \mathbb{K})$ of cochain complexes and a dg-functor is a functor of enriched categories (see [Ke] for the notion of enriched category). In particular dg-categories are identified with $\mathcal{A}_\infty$-categories having $m_k = 0$, for $k > 2$, and dg-functors with $\mathcal{A}_\infty$-functors having $f_k = 0$, for $k > 1$. This means that there exists a faithful functor

$$i : dgCat \longrightarrow \mathcal{A}_\infty Cat \tag{7}$$

where $dgCat$ is the category of differential graded categories over $\mathbb{K}$.

For a dg-category $\mathcal{D}$ its underlying $\mathbb{K}$-linear category $\mathcal{D}_{un}$ has the same objects of $\mathcal{D}$ and morphisms

$$Hom_{\mathcal{D}_{un}}(x, y) = Ker(d_0 : Hom^0_{\mathcal{D}}(x, y) \longrightarrow Hom^1_{\mathcal{D}}(x, y))$$

We refer to a morphism in a dg-category, without specifying the degree, meaning a morphism in $\mathcal{D}_{un}$ and we say that a dg-category $\mathcal{D}$ is a dg-enhancement of a $\mathbb{K}$-linear category $\mathcal{V}$ if there exists an isomorphism $\mathcal{V} \simeq \mathcal{D}_{un}$. The homotopy category $H^0(\mathcal{D})$ of $\mathcal{D}$ is the $\mathbb{K}$-linear category with the same objects of $\mathcal{D}$ and vector space of morphisms given by:

$$Hom_{H^0(\mathcal{D})}(x, y) = H^0(Hom^\bullet_{\mathcal{D}}(x, y))$$

A dg-functor $f : \mathcal{D} \longrightarrow \mathcal{E}$ is a quasi-equivalence of dg-categories if the induced functor on the homotopy categories $H^0(f) : H^0(\mathcal{D}) \longrightarrow H^0(\mathcal{E})$ is an equivalence and the induced morphisms of cochain complexes $f : Hom^\bullet_{\mathcal{D}}(x, y) \longrightarrow Hom^\bullet_{\mathcal{E}}(f(x), f(y))$ are quasi-isomorphisms of complexes, for every $x, y \in Ob(\mathcal{D})$.

2.5. Construction of the nerve.

2.6. Definition. *The $\mathcal{A}_\infty$-category (in fact dg-category) $\mathcal{A}_\infty[\Delta^n]$ generated by the standard $n$-simplex is the $\mathcal{A}_\infty$-category whose objects are the integers $\{0, 1, \ldots, n\}$, morphisms spaces given, for $0 \le i, j \le n$, by*

$$Hom^\bullet_{\mathcal{A}_\infty[\Delta^n]}(i, j) = \begin{cases} \mathbb{K} \cdot (i, j) & i \le j \\ \emptyset & i > j \end{cases}$$

*with deg((i, j))=0 and $\mathcal{A}_\infty$-structure determined by the maps*

$$\begin{cases} m_1 = 0 \\ m_2((j, k), (i, j)) = (i, k), & \textit{for } i \le j \le k \\ m_n = 0, & \textit{for } n > 2 \end{cases}$$

*with identities $1_i = (i, i) \in Hom^\bullet_{\mathcal{A}_\infty[\Delta^n]}(i, i)$, for $i = 0, \ldots, n$.*

2.7. Proposition. *The construction $[n] \longrightarrow \mathcal{A}_\infty[\Delta^n]$ defines a cosimplicial $\mathcal{A}_\infty$-category*

$$\mathcal{A}_\infty[\Delta^-] : \Delta \longrightarrow \mathcal{A}_\infty Cat$$

PROOF. Consider the standard cofaces and codegeneracies morphisms in $\Delta$

$$\delta_j^n : [n-1] \longrightarrow [n], \qquad 0 \le j \le n$$
$$\sigma_j^n : [n] \longrightarrow [n-1], \qquad 0 \le j \le n-1$$

with

$$\delta_j^n(k) = \begin{cases} k, & 0 \le k \le j-1 \\ k+1, & j \le k \le n-1 \end{cases}$$

$$\sigma_j^n(k) = \begin{cases} k, & 0 \le k \le j \\ k-1, & j+1 \le k \le n \end{cases}$$

The induced cofaces $\mathcal{A}_\infty$-functors

$$(\delta_j^n)_\star : \mathcal{A}_\infty[\Delta^{n-1}] \longrightarrow \mathcal{A}_\infty[\Delta^n]$$

are defined by

$$\begin{cases} (\delta_j^n)_{\star,0}(k) = \delta_j^n(k) \\ (\delta_j^n)_{\star,1}(i,k) = (\delta_j^n(i), \delta_j^n(k)) \\ (\delta_j^n)_{\star,p} = 0, & p > 2 \end{cases}$$

Similarly the codegeneracies $\mathcal{A}_\infty$-functors

$$(\sigma_j^n)_\star : \mathcal{A}_\infty[\Delta^n] \longrightarrow \mathcal{A}_\infty[\Delta^{n-1}]$$

are

$$\begin{cases} (\sigma_j^n)_{\star,0}(k) = \sigma_j^n(k) \\ (\sigma_j^n)_{\star,1}(i,k) = (\sigma_j^n(i), \sigma_j^n(k)) \\ (\sigma_j^n)_{\star,p} = 0, & p > 2 \end{cases}$$

The fact that this assignment determines a cosimplicial $\mathcal{A}_\infty$-category follows from the standard cosimplicial structure of $\Delta^-$. ∎

2.8. DEFINITION. [Simplicial Nerve of an $\mathcal{A}_\infty$-category] *For an $\mathcal{A}_\infty$-category $\mathcal{A}$ its simplicial nerve $N_{\mathcal{A}_\infty}(\mathcal{A})$ is the simplicial set whose $n$-simplices are given by*

$$N_{\mathcal{A}_\infty}(\mathcal{A})_n = Hom_{\mathcal{A}_\infty Cat}(\mathcal{A}_\infty[\Delta^n], \mathcal{A})$$

*and simplicial structure induced by applying the functor $Hom_{\mathcal{A}_\infty Cat}(-, \mathcal{A})$ to the cosimplicial $\mathcal{A}_\infty$-category $\mathcal{A}_\infty[\Delta^-]$.*

2.9. PROPOSITION. *An $n$-simplex of the simplicial nerve of an $\mathcal{A}_\infty$-category $\mathcal{A}$ is determined by $n+1$ objects*

$$x_i \in Ob(\mathcal{A}), i = 0, \ldots, n$$

*and by a collection of elements*

$$f_{i_0 \ldots i_k} \in Hom_{\mathcal{A}}^{1-k}(x_{i_0}, x_{i_k})$$

*for* $1 \leq k \leq n$ *and* $0 \leq i_0 < i_1 < \cdots < i_k \leq n$*, satisfying the conditions*

$$f_{i_0,i_0} = Id_{x_{i_0}}$$

$$f_{i_0,\ldots,i_p,i_p,\ldots,i_l} = 0, \quad \textit{for } 2 \leq l \leq n$$

$$m_1(f_{i_0\ldots i_k}) = \sum_{0<j<n} (-1)^{j-1} f_{i_0\ldots\hat{i_j}\ldots i_k} + \sum_{0<j<n} (-1)^{1+k(j-1)} m_2(f_{i_j\ldots i_k}, f_{i_0\ldots i_j}) +$$
$$+ \sum_{\substack{1\leq r\leq n \\ 0<j_1<\cdots<j_{r-1}<n}} (-1)^{1+\epsilon_r} m_r(f_{i_{j_{r-1}}\ldots i_k}, \ldots, f_{i_0\ldots i_{j_1}})$$

*where* $\epsilon_r$ *as in equation* (5).

Proof. An $n$-simplex $f \in N_{\mathcal{A}_\infty}(\mathcal{A})_n$ is by definition an $\mathcal{A}_\infty$-functor

$$f : \mathcal{A}_\infty[\Delta^n] \longrightarrow \mathcal{A}$$

This determines $n+1$ objects of

$$x_i = f_0(i) \in Ob(\mathcal{A})$$

and for $1 \leq k \leq n$ a morphism of degree $1-k$

$$f_k : Hom^\bullet_{\mathcal{A}_\infty[\Delta^n]}(x_{i_{k-1}}, x_{i_k}) \otimes \cdots \otimes Hom^\bullet_{\mathcal{A}_\infty[\Delta^n]}(x_{i_0}, x_{i_1}) \longrightarrow Hom^\bullet_{\mathcal{A}}(f_0(x_{i_0}), f_0(x_{i_k}))$$

which corresponds to the choice, for every string $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, of an element

$$f_{i_0\ldots i_k} = f_k((i_{k-1}, i_k) \otimes \cdots \otimes (i_0, i_1)) \in Hom^{1-k}_{\mathcal{A}}(x_{i_0}, x_{i_k})$$

The conditions satisfied by $f_{i_0\ldots i_k}$ follow from Definition 2.3. ∎

2.10. Corollary. *For a dg-category* $\mathcal{D}$ *its differential graded nerve [Lur2, §1.3] equals the simplicial nerve of* $i(\mathcal{D})$

$$N^{sm}_{dg}(\mathcal{D}) = N_{\mathcal{A}_\infty}(i(\mathcal{D}))$$

*where* $i$ *is defined in Remark 7.*

Proof. The proof follows from Proposition 2.9. ∎

2.11. PROPOSITION. *The simplicial structure of $N_{\mathcal{A}_\infty}(\mathcal{A})$ can be described as follows: for $f \in N_{\mathcal{A}_\infty}(\mathcal{A})_n$ the components of $j$-th face map $d^n_j(f)$ are given by*

$$d^n_j(f)_{i_0\dots i_k} = \begin{cases} f_{i_0\dots i_{p-1}(i_p+1)\dots(i_k+1)}, & j \le i_p, 0 \le p \le k \\ f_{i_0\dots i_k}, & j > i_k \end{cases}$$

*for $1 \le k \le n-1$ and a string $0 \le i_0 < i_1 < \cdots < i_k \le n-1$. The components of the $j$-th degeneracy map $s^n_j(f)$ are*

$$s^n_j(f)_{i_0 i_1} = \begin{cases} f_{(i_0-1)(i_1-1)}, & j \le i_0 - 1 \\ f_{i_0(i_1-1)}, & i_0 < j < i_1 - 1 \\ Id_{x_{i_0}}, & i_0 = j, i_1 = j+1 \\ f_{i_0 i_1}, & j \ge i_1 \end{cases} \tag{8}$$

*for $0 \le i_0 < i_1 \le n+1$ and*

$$s^n_j(f)_{i_0\dots i_k} = \begin{cases} f_{(i_0-1)\dots(i_k-1)}, & j \le i_0 - 1 \\ f_{i_0\dots i_p(i_{p+1}-1)\dots(i_k-1)}, & i_p < j < i_{p+1} - 1, 0 < p < k \\ 0, & i_p = j, i_{p+1} = j+1 \\ f_{i_0\dots i_k}, & j \ge i_k \end{cases} \tag{9}$$

*for $2 \le k \le n+1$, $0 \le i_0 < i_1 < \cdots < i_k \le n+1$.*

PROOF. The $j$-th face map

$$d^n_j : N_{\mathcal{A}_\infty}(\mathcal{A})_n \longrightarrow N_{\mathcal{A}_\infty}(\mathcal{A})_{n-1}$$

evaluated on an $n$-simplex $f \in N_{\mathcal{A}_\infty}(\mathcal{A})$ is by definition the $\mathcal{A}_\infty$-functor

$$d^n_j(f) = f \circ (\delta^n_j)_\star$$

The composition law for $\mathcal{A}_\infty$-functors (see Equation (6)) gives

$$(f \circ (\delta^n_j)_\star)_k = f_k((\delta^n_j)_{\star,1} \otimes \cdots \otimes (\delta^n_j)_{\star,1})$$

and hence, for $1 \le k \le n-1$ and a string $0 \le i_0 < i_1 < \cdots < i_k \le n-1$

$$d^n_j(f)_{i_0\dots i_k} = \begin{cases} f_{i_0\dots i_{p-1}(i_p+1)\dots(i_k+1)}, & j \le i_p, 0 \le p \le k \\ f_{i_0\dots i_k}, & j > i_k \end{cases}$$

A similar computation shows that the $j$-th degeneracy map is determined by the formulas (8) and (9). ∎

2.12. Proposition. *The simplicial nerve construction defines a functor*

$$N_{\mathcal{A}_\infty} : \mathcal{A}_\infty Cat \longrightarrow SSet$$

Proof. Any $\mathcal{A}_\infty$-functor $g : \mathcal{A} \longrightarrow \mathcal{B}$ induces a map of simplicial sets $(g)_\star : N_{\mathcal{A}_\infty}(\mathcal{A}) \longrightarrow N_{\mathcal{A}_\infty}(\mathcal{B})$ by the assignment

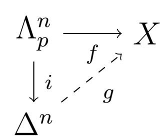
$$(g)_\star(f) = f \circ g$$

where $f \in Hom_{\mathcal{A}_\infty Cat_{\mathbb{K}}}(\mathcal{A}_\infty[\Delta^n], \mathcal{A})$ is an $n$-simplex in $N_{\mathcal{A}_\infty}(\mathcal{A})$. More explicitly, if $1 \leq k \leq n$ and $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, we have from Equation (6)

$$((g)_\star(f))_{i_0 \dots i_k} = \sum_{r=1}^{k} \sum_{j_1 + \cdots + j_r = k} (-1)^{\epsilon_r(j_1, \dots, j_r)} g_r(f_{i_{j_r + \cdots + j_2} \dots i_k}, \dots, f_{i_0 \dots i_{j_r}})$$

The functoriality of $N_{\mathcal{A}_\infty}$ hence follows from its definition. ∎

2.13. Definition. *[Lur1] An $(\infty, 1)$-category (or weak Kan complex) is a simplicial set $X$ such that, for any $0 < p < n$ and any map of simplicial sets $f : \Lambda^n_p \longrightarrow X$, there exists an extension to the full $n$-simplex $g : \Delta^n \longrightarrow X$*




*where $\Lambda^n_p$ is the $p$-th inner horn.*

2.14. Remark. A simplicial category is a category enriched over the symmetric monoidal category $(SSet, \times, pt)$ of simplicial sets where the monoidal structure is the point-wise cartesian product of simplices. Simplicial categories can be related to simplicial sets through a pair of adjoint functors

$$N_{SCat} \colon SCat \rightleftarrows SSet : \mathcal{C}[-] \tag{10}$$

where $SCat$ is the category of simplicial categories with the obvious notion of morphisms and the functor $N_{SCat}$ is called the nerve of a simplicial category. This adjunction lifts to a Quillen adjunction of model categories between the Kan model structure on $SCat$ and the Joyal model structure on $SSet$ whose fibrant objects are $(\infty, 1)$-categories. We refer to [Lur1, §2.2] for a more detailed discussion about this construction and for the definition of the homotopy category of an $(\infty, 1)$-category.

2.15. Proposition. *For an $\mathcal{A}_\infty$-category $\mathcal{A}$ its simplicial nerve $N_{\mathcal{A}_\infty}(\mathcal{A})$ is an $(\infty, 1)$-category.*

Proof. Consider a morphism of simplicial sets $f : \Lambda^n_p \longrightarrow N_{\mathcal{A}_\infty}(\mathcal{A})$, where $n > 0$ and $0 < p < n$ are fixed. Such morphism can be identified with an $n$-simplex of $N_{\mathcal{A}_\infty}(\mathcal{A})$ whose components $f_{0\dots n}$ and $f_{0\dots\hat{p}\dots n}$ are not given. The morphism $g : \Delta^n \longrightarrow N_{\mathcal{A}_\infty}(\mathcal{A})$ defined by

$$g_{0\dots n} = 0$$

$$g_{0\dots\hat{p}\dots n} = \sum_{0<j<n, j\neq p} (-1)^{j-1+p} f_{0\dots\hat{j}\dots n} + \sum_{0<j<n} (-1)^{1+n(j-1)+p} f_{j\dots n} \circ f_{0\dots j} +$$
$$+ \sum_{\substack{1\leq r\leq n \\ 0<j_1<\dots<j_{r-1}<n}} (-1)^{1+\epsilon_r(j_1,\dots,j_{r-1})+p} m_r(f_{i_{j_{r-1}}\dots i_k}, \dots, f_{i_0\dots i_{j_1}})$$

$$g_{|\Lambda^n_p} = f$$

provides an extension of $f$. ∎

## 3. Comparison between pretriangulated dg-categories and stable $(\infty, 1)$-categories

3.1. Pretriangulated dg-categories.

3.2. Definition. *A zero object of a dg-category $\mathcal{D}$ is an object $0 \in Ob(\mathcal{D})$ such that for every $X \in Ob(\mathcal{D})$*

$$Hom^\bullet_{\mathcal{D}}(X, 0) = 0^\bullet = Hom^\bullet_{\mathcal{D}}(0, X)$$

*where $0^\bullet$ is the cochain complex having 0 in each degree and zero differential.*

3.3. Definition. [dg-category of twisted complexes, [Bo-Ka]] *Let $\mathcal{D}$ be a dg-category with a zero object 0. A twisted complex of $\mathcal{D}$ is the data consisting of a pair $K = (K_i, q_{ij})_{i,j\in\mathbb{Z}}$ where:*

- *$K_i \in Ob(\mathcal{D})$ are equal to 0 for all but a finite number of indices $i \in \mathbb{Z}$*
- *$q_{ij} \in Hom^{i-j+1}_{\mathcal{D}}(K_i, K_j)$ are morphisms satisfying the Maurer-Cartan equation*

$$d(q_{ij}) + \sum_{k\in\mathbb{Z}} q_{kj} q_{ik} = 0$$

*for every $i, j \in \mathbb{Z}$.*

*The dg-category $PreTr(\mathcal{D})$ of twisted complexes of $\mathcal{D}$ is the differential category whose objects are twisted complexes of $\mathcal{D}$ and cochain complex of morphisms*

$$Hom^k_{PreTr(\mathcal{D})}(K, K') = \bigoplus_{l+j-i=k} Hom^l_{\mathcal{D}}(K_i, K'_j)$$

*with differential $d$ evaluated on $f \in Hom^l_{\mathcal{D}}(K_i, K'_j)$ given by the expression*

$$d(f) = d(f) + \sum_m (q'_{jm} f + (-1)^{l(i-m+1)} f q_{mi})$$

3.4. Remark. For a dg-category $\mathcal{D}$ the category of dg-functors $dgFun(\mathcal{D}, Ch^{\bullet}(\mathbb{K}))$ has a dg-enhancement (as described in [Bo-Ka]) in such a way that the diagram of dg-functors

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{h} & dgFun(\mathcal{D}, Ch^{\bullet}(Vect_k)) \\ & {\scriptstyle\epsilon}\searrow \quad \nearrow{\scriptstyle\alpha} & \\ & PreTr(\mathcal{D}) & \end{array}$$

is commutative. Here $h$ is the dg-Yoneda functor defined on objects by

$$h(X)(Y) = Hom^{\bullet}_{\mathcal{D}}(X, Y)$$

$\epsilon$ is the dg-functor sending an object $X \in Ob(\mathcal{D})$ to the twisted complex concentrated in degree 0 and $\alpha$ is the dg-functor that associates to a twisted complex $K = (K_i, q_{ij})_{i,j\in\mathbb{Z}}$ the dg-functor

$$\alpha(K)(Y) = \bigoplus_{i\in\mathbb{Z}} Hom^{\bullet}_{\mathcal{D}}(Y, K_i)[-i]$$

with twisted differential $d+q$. Moreover both dg-categories $PreTr(\mathcal{D})$ and $dgFun(\mathcal{D}, Ch^{\bullet}(\mathbb{K}))$ have shift functors and functorial cones which are preserved under the dg-functor $\alpha$. For a twisted complex $K$ its shift by 1 is given explicitly by the formula

$$K[1]_i = K_{i+1}$$

$$q[1]_{ij} = q_{i+1,j+1}$$

and for a morphism of twisted complexes $f : K \longrightarrow K'$ its cone is the twisted complex

$$Cone(f) = (K_{i+1} \bigoplus K'_i, q''_{ij})$$

where $q''_{ij}$ is the matrix

$$q''_{ij} = \begin{vmatrix} q_{i+1,j+1} & f_{i+1,j} \\ 0 & q'_{ij} \end{vmatrix}$$

This definition of shift functor and cone construction determines a triangulated structure on the homotopy category $H^0(PreTr(\mathcal{D}))$ [Bo-Ka] where exact triangles are given by sequences in $H^0(PreTr(\mathcal{D}))$ of the form

$$K \xrightarrow{f} K' \longrightarrow Cone(f) \longrightarrow K[1]$$

3.5. Definition. [Pretriangulated dg-category, [Bo-Ka]] *A dg-category $\mathcal{D}$ is called pretriangulated if for every twisted complex $K \in PreTr(\mathcal{D})$, the dg-functor $\alpha(K)$ is isomorphic to $h(X)$ for some object $X \in \mathcal{D}$.*

3.6. REMARK. If $\mathcal{D}$ is a pretriangulated dg-category the dg-functor $\epsilon : \mathcal{D} \longrightarrow PreTr(\mathcal{D})$ is a quasi-equivalence of dg-categories [Bo-Ka]. In this situation it is possible to transfer the shift functor and the cone construction of $PreTr(\mathcal{D})$ to $\mathcal{D}$. Namely the shift by +1 of $X \in Ob(\mathcal{D})$ is defined as

$$X[1] = T(\epsilon(X)[1]) \tag{11}$$

where $T$ is the inverse equivalence of $\epsilon$ and $T(\epsilon(X)[1])$ is an object of $\mathcal{D}$ representing the dg-functor $\alpha(T(\epsilon(X)[1]))$. Similarly the cone of a morphism $f : X \longrightarrow Y$ is

$$Cone(f) = T(Cone(\epsilon(f)))$$

Moreover there exist canonical quasi-isomorphisms of complexes

$$Hom^\bullet_{\mathcal{D}}(X[1], Y) \simeq Hom^\bullet_{\mathcal{D}}(X, Y[-1]) \simeq Hom^\bullet_{\mathcal{D}}(X, Y)[-1]$$

and for a morphism $f : X \longrightarrow Y$ one has

$$Hom^k_{\mathcal{D}}(Cone(f), Z) = Hom^k_{\mathcal{D}}(Y, Z) \oplus Hom^{k-1}_{\mathcal{D}}(X, Z) \tag{12}$$

with differential

$$d^k = \begin{vmatrix} d^k_{(Y,Z)} & 0 \\ (-1)^k(- \circ f) & d^{k-1}_{(X,Z)} \end{vmatrix}$$

where $d^k_{(X,Y)} = d_{Hom^k_{\mathcal{D}}(X,Y)}$, and

$$Hom^k_{\mathcal{D}}(Z, Cone(f)) = Hom^k_{\mathcal{D}}(Z, Y) \oplus Hom^{k+1}_{\mathcal{D}}(Z, X) \tag{13}$$

with differential

$$d^k = \begin{vmatrix} d^k_{(Z,Y)} & (- \circ f) \\ 0 & d^{k+1}_{(Z,X)} \end{vmatrix}$$

3.7. PROPOSITION. *[Bo-Ka] The homotopy category $H^0(\mathcal{D})$ of a pretriangulated dg-category $\mathcal{D}$ is triangulated in the sense of [Ver] with shift functor*

$$[1] : H^0(\mathcal{D}) \longrightarrow H^0(\mathcal{D})$$

*defined on objects by Equation (11) and class of exact triangles of the form*

$$K \xrightarrow{f} K' \longrightarrow Cone(f) \longrightarrow K[1]$$

*where $f$ is a morphism of twisted complexes. Moreover the functor*

$$H^0(T) : H^0(PreTr(\mathcal{D})) \longrightarrow H^0(\mathcal{D})$$

*is an equivalence of triangulated categories.*

3.8. Stable $(\infty,1)$-categories.

3.9. Definition. *An $(\infty,1)$-category $X$ is pointed if there exists an object $0 \in X_0$ called the zero object such that for every $A \in X_0$*

$$Map_X(A,0) \simeq * \simeq Map_X(0,A)$$

*where $Map_X(-,-)$ is the Kan complex of morphisms in $X$ (see* [Lur1, §2.2]).

3.10. Definition. *A triangle in a pointed $(\infty,1)$-category $X$ is a diagram $\Delta^1 \times \Delta^1 \longrightarrow X$ of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & C \end{array}$$

*We say that a triangle is a fiber sequence (the fiber of $g$) if it is homotopy cartesian and is a cofiber sequence (the cofiber of $f$) if it is homotopy cocartesian.*

3.11. Definition. [Stable $(\infty,1)$-category, [Lur2]] *An $(\infty,1)$-category is stable if*

- *It is pointed*
- *Every morphism admits fiber and cofiber*
- *A triangle is a fiber sequence if and only if it is a cofiber sequence*

3.12. Remark. A stable $(\infty,1)$-category $X$ has canonical constructions of the suspension and loop functors

$$\Sigma, \Omega : X \longrightarrow X$$

which are equivalences of $(\infty,1)$-categories [Lur2, Chap. 1]. Explicitly these functors are given on objects by

$$\Sigma(A) = 0 \amalg^h_A 0$$
$$\Omega(A) = 0 \times^h_A 0$$

Moreover its homotopy category $h(X)$ is an additive category and it comes equipped with a notion of distinguished triangles which are diagrams of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] = \Sigma A$$

induced by a diagram $\Delta^1 \times \Delta^2 \longrightarrow X$

$$\begin{array}{ccccc} A & \xrightarrow{\tilde{f}} & B & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0' & \longrightarrow & C & \xrightarrow[\tilde{h}]{} & D \end{array}$$

where

- 0 and 0' are both zero objects
- Both squares are pushout diagrams in $X$
- The morphisms $\tilde{f}$ an $\tilde{g}$ represent $f$ and $g$ respectively
- The map $h$ is the composition with the homotopy class of $\tilde{h}$ with an equivalence $D \simeq A[1]$.

3.13. Proposition. *[Lur2] The homotopy category $h(X)$ of a stable $(\infty,1)$-category $X$ is a triangulated category in the sense of [Ver] with shift functor induced by the suspension functor*

$$\Sigma : h(X) \longrightarrow h(X)$$

*and class of distinguished triangles as described in Remark 3.12.*

3.14. The Dold-Kan correspondence and computation of homotopy limits. Recall that the category $S(Vect_{\mathbb{K}})$ of simplicial vector spaces over $\mathbb{K}$ has a model structure called the Quillen model structure [Qui]. Weak-equivalences and fibrations of simplicial vector spaces are morphisms inducing weak-homotopy equivalences and Kan fibrations on the underlying simplicial sets. The category $Ch_\bullet^{\geq 0}(\mathbb{K})$ of non-negatively graded chain complexes has a model structure called the projective model structure [Qui]. Weak-equivalences are quasi-isomorphisms of chain complexes and fibrations are chain maps which are epimorphisms in each positive degree. Cofibrant objects in this model structure are retract of complexes of projective $\mathbb{K}$-modules. The Dold-Kan correspondence

$$DK\colon Ch_\bullet^{\geq 0}(\mathbb{K}) \rightleftarrows S(Vect_{\mathbb{K}}) :N \tag{14}$$

establishes an equivalence of categories between the category $Ch_\bullet^{\geq 0}(\mathbb{K})$ of positively graded chain complexes over $\mathbb{K}$ and the category $S(Vect_{\mathbb{K}})$ [McL]. According to S. Schwede and B. Shipley [Sch-Shi] these functors are both left and right adjoint of a Quillen adjunction between the Quillen model structure on $S(Vect_{\mathbb{K}})$ and the projective model structure on $Ch_\bullet^{\geq 0}(Vect_{\mathbb{K}})$. Moreover homology groups and homotopy groups are identified under this correspondence.

Recall that for a model category $\mathcal{C}$ and a Reedy indexing category $I$, the Reedy model structure on the category of functors $Fun(I,\mathcal{C})$ (or $I$-shaped diagrams in $\mathcal{C}$) is the model structure for which a morphism $f : X \longrightarrow Y$ is a weak-equivalence if it is an object-wise weak-equivalence in $\mathcal{C}$, a cofibration if the relative latching morphisms at every object of $I$ are cofibrations in $\mathcal{C}$ and a fibration if the relative matching morphisms at every object of $I$ are fibrations in $\mathcal{C}$. Quillen equivalences of model categories induce Quillen equivalences in the respective categories of functors with the Reedy model structures [Hir]. We say that $\mathcal{C}$ has $I$-shaped limits if the functor

$$(-)_{const.} : \mathcal{C} \longrightarrow Fun(I,\mathcal{C})$$

taking an object $x \in \mathcal{C}$ to the constant functor with value $x$, has a right adjoint

$$lim : Fun(I, \mathcal{C}) \longrightarrow \mathcal{C}$$

If $\mathcal{C}$ has $I$-shaped limits the total right derived functor of $lim$ exists [Hir] and defines the homotopy limit functor

$$holim : h(Fun(I, \mathcal{C})) \longrightarrow h(\mathcal{C})$$

given explicitly on an object $X \in h(Fun(I, \mathcal{C}))$ by

$$holim(X) = \mathbb{R}(lim)(X) = lim(P(X))$$

where $P(X)$ is the fibrant replacement for $X$ in the Reedy model structure on $Fun(I, \mathcal{C})$. These observations lead to the following lemma.

3.15. Lemma. *Let $I$ be a Reedy indexing category and consider the functor*

$$DK_* : Fun(I, Ch_\bullet^{\geq 0}(\mathbb{K})) \longrightarrow Fun(I, S(Vect_{\mathbb{K}}))$$

*defined by applying object-wise the functor $DK$ of* (14) to an $I$-shaped diagram in $Ch_\bullet^{\geq 0}(\mathbb{K})$. For $X \in Fun(I, Ch_\bullet^{\geq 0}(\mathbb{K}))$ we have

$$holim(DK_*(X)) \simeq DK(lim(P(X)))$$

where $P(X)$ is the fibrant replacement of $X$ in the Reedy model category on $Fun(I, Ch_\bullet^{\geq 0}(\mathbb{K}))$.

Proof. By definition we have

$$holim(DK_*(X)) \simeq lim(P(DK_*(X)))$$

The functor $DK$ is the right and left adjoint of a Quillen equivalence [Sch-Shi] hence

$$P(DK_*(X)) \simeq DK_*(P(X))$$

Moreover $DK$ preserves limits which implies that

$$lim(P(DK_*(X))) \simeq lim(DK_*(P(X))) \simeq DK(lim(P(X)))$$

■

3.16. Example. We give an explicit construction of the fibrant replacement in the Reedy model structure in order to compute homotopy limits when the indexing category $I$ is the category

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 2 & \longrightarrow & 0 \end{array}$$

Given $f : X \longrightarrow Y$ a morphism in $Fun(I, \mathcal{C})$ it is easy to check that relative matching morphisms are

$$M_0(f) : X_0 \longrightarrow Y_0$$

and for $i = 1, 2$

$$M_i(f) : X_i \longrightarrow Y_i \underset{Y_0}{\times} X_0$$

In particular a morphism $f$ is a fibrations if the morphisms $M_i(f)$ are fibrations in $\mathcal{C}$, for $i = 0, 1, 2$, and an object $X \in Fun(I, \mathcal{C})$ is a fibrant object if $X_0$ is a fibrant object in $\mathcal{C}$ and the morphisms $X_1 \longrightarrow X_0$, $X_2 \longrightarrow X_0$ are fibrations. Hence a fibrant replacement of a diagram

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow{\scriptstyle f_{10}} \\ X_2 & \underset{f_{20}}{\longrightarrow} & X_0 \end{array}$$

where $X_0$ fibrant is given by

$$\begin{array}{ccc} & & P(f_{10}) \\ & & \downarrow \\ P(f_{20}) & \twoheadrightarrow & X_0 \end{array}$$

where

$$\begin{array}{ccccc} & & P(f_{i0}) & & \\ & \overset{\simeq}{\nearrow} & & \searrow & \\ X_i & & \xrightarrow{f_{i1}} & & X_0 \end{array}$$

is a trivial cofibration-fibration factorization of the morphisms $f_{i0}$, $i = 1, 2$. In particular we get the following expression for the homotopy fibre product of $X$

$$X_2 \overset{h}{\underset{X_0}{\times}} X_1 := holim(X) \simeq P(f_{20}) \underset{X_0}{\times} P(f_{10})$$

When $\mathcal{C} = Ch_\bullet^{\geq 0}(\mathbb{K})$ with the projective model structure a trivial cofibration-fibration factorization of a chain map $f_\bullet : A_\bullet \longrightarrow B_\bullet$ is given by

$$\begin{array}{ccccc} & & P(f_\bullet) & & \\ & \overset{i}{\nearrow} & & \overset{p}{\searrow} & \\ A_\bullet & & \xrightarrow{f_\bullet} & & B_\bullet \end{array}$$

where $P(f_\bullet)$ is the chain complex in degree $n > 0$

$$P(f_\bullet)_n = A_n \bigoplus B_{n+1} \bigoplus B_n$$

and in degree 0

$$P(f_\bullet)_0 = A_0 \bigoplus B_1 \bigoplus D_0$$

with $D_0 \subseteq B_0$ defined by the equation $b_0 = d(b_1) + f_0(a_0)$, with $b_1 \in B_1$ and $a_0 \in A_0$. The differential is

$$d_n = \begin{vmatrix} d_{A_n} & 0 & 0 \\ -f_n & -d_{B_{n+1}} & Id_{B_n} \\ 0 & 0 & d_{B_n} \end{vmatrix}$$

and the morphisms $i$ and $p$ are

$$i_n = \begin{vmatrix} Id_{A_n} & 0 & f_n \end{vmatrix}$$

$$p_n = \begin{vmatrix} 0 \\ 0 \\ Id_{B_n} \end{vmatrix}$$

One can easily check that $H_*(P(f_\bullet)) \simeq H_*(A_\bullet)$ and that $p$ is a fibration, being degree-wise surjective. The fact that $i$ is a cofibration follows from the fact that every $\mathbb{K}$-vector space is free and hence projective.

3.17. Pretriangulated dg-categories are stable $(\infty,1)$-categories under the nerve construction.

3.18. Theorem. *For a pretriangulated dg-category $\mathcal{D}$ the dg-nerve $N_{dg}(\mathcal{D})$ is a stable $(\infty,1)$-category. Moreover $H^0(\mathcal{D})$ is identified with $h(N_{dg}(\mathcal{D}))$ as triangulated categories.*

Proof. Recall that the big dg-nerve $N_{dg}^{big}(\mathcal{D})$ is the $(\infty,1)$-category defined as the nerve of the simplicial category $\mathcal{D}_\Delta$ (see Remark 2.14) having the same objects of $\mathcal{D}$ and simplicial set of morphisms given by

$$Map_{\mathcal{D}_\Delta}(X,Y) := DK(\tau_{\geq 0}(Hom_{\mathcal{D}}(X,Y)^{op}))$$

where $DK$ is the functor of the Dold-Kan correspondence (14), $\tau_{\geq 0}$ and $(-)^{op}$ are the functors defined in Section 1.1. The big dg-nerve and the dg-nerve are equivalent $(\infty,1)$-categories [Lur2, §1.3] hence it is enough to show that the big dg-nerve is a stable $(\infty,1)$-category. Let 0 be the zero object of $\mathcal{D}$ then

$$Map_{\mathcal{D}_\Delta}(X,0) \simeq * \simeq Map_{\mathcal{D}_\Delta}(0,X)$$

for every object $X$ of $\mathcal{D}$ hence $N_{dg}^{big}(\mathcal{D})$ is pointed. For a 1-simplex $f : X \longrightarrow Y$ in $N_{dg}^{big}(\mathcal{D})$ we show that it admits fiber and cofiber. Consider the case of the cofiber first. Let

$j : Y \longrightarrow Cone(f)$ be the degree 0 morphism corresponding to $(Id_Y, 0)$ according to the equality (see Equation (13))

$$Hom^0_{\mathcal{D}}(Y, Cone(f)) = Hom^0_{\mathcal{D}}(Y, Y) \oplus Hom^1_{\mathcal{D}}(Y, X)$$

This is a closed morphism because

$$d(Id_Y, 0) = (d(Id_Y) + f \circ 0, d(0)) = (0, 0)$$

hence $j$ identifies a 1-simplex of $N^{big}_{dg}(\mathcal{D})$. The composition $j \circ f$ is null-homotopic in the sense that for

$$h = (0, Id_Y) \in Hom^{-1}_{\mathcal{D}}(Y, Cone(f)) = Hom^{-1}_{\mathcal{D}}(Y, Y) \oplus Hom^0_{\mathcal{D}}(Y, X)$$

we have

$$d(h) = (d(0) + f \circ Id_Y, d(0)) = (f, 0) = j \circ f$$

These data determines a triangle in $N^{big}_{dg}(\mathcal{D})$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle j} \\ 0 & \xrightarrow{0} & Cone(f) \end{array} \tag{15}$$

In order to show that $Cone(f)$ is the cofiber of $f$ we need to construct a weak-equivalence

$$Map_{\mathcal{D}_\Delta}(Cone(f), Z) \longrightarrow Map_{\mathcal{D}_\Delta}(Y, Z) \underset{Map_{\mathcal{D}_\Delta}(X,Z)}{\times^h} * \tag{16}$$

for every object $Z \in Ob(\mathcal{D})$. According to Lemma 3.15 this is equivalent to exhibit a quasi-isomorphism of chain complexes

$$\tau_{\geq 0}(Hom_{\mathcal{D}}(Cone(f), Z)^{op}) \longrightarrow \tau_{\geq 0}(Hom_{\mathcal{D}}(Y, Z)^{op}) \underset{\tau_{\geq 0}(Hom_{\mathcal{D}}(X,Z)^{op})}{\times^h} 0 \tag{17}$$

Following Example 3.16 the right hand side of Equation (17) can be identified with the fibre product

$$P(- \circ f) \underset{\tau_{\geq 0}(Hom^{\bullet}_{\mathcal{D}}(X,Z)^{op})}{\times} P(0)$$

where the chain complex $P(- \circ f)$ is

$$\begin{cases} P(- \circ f)_0 = Ker(d_{Hom^0_{\mathcal{D}}(Y,Z)}) \oplus Hom^{-1}_{\mathcal{D}}(X, Z) \oplus D^0, & \text{if } k = 0 \\ P(- \circ f)_k = Hom^{-k}_{\mathcal{D}}(Y, Z) \oplus Hom^{-k-1}_{\mathcal{D}}(X, Z) \oplus Hom^{-k}_{\mathcal{D}}(X, Z), & \text{if } k > 0. \end{cases}$$

with $D^0 \subseteq Hom^0_{\mathcal{D}}(X,Z)$ defined by the equation $g^0 = h^0 \circ f + d(g^{-1})$, for $h^0 \in Ker(d_{Hom^0_{\mathcal{D}}(Y,Z)})$ and $g^{-1} \in Hom^{-1}_{\mathcal{D}}(X,Z)$. The differential for $k > 0$ is

$$d_k = \begin{vmatrix} d^{-k}_{(Y,Z)} & 0 & 0 \\ -(- \circ f) & -d^{-k-1}_{(X,Z)} & Id_{(X,Z)} \\ 0 & 0 & d^{-k}_{(Y,Z)} \end{vmatrix}$$

and for $k = 0$

$$d_0 = \begin{vmatrix} d^{0}_{(Y,Z)} & 0 & 0 \\ -(- \circ f) & -d^{-1}_{(X,Z)} & Id_{(X,Z)} \\ 0 & 0 & d^{0}_{(Y,Z)} \end{vmatrix}$$

where $d^k_{(X,Y)} = d_{Hom^k_{\mathcal{D}}(X,Y)}$. The chain complex $P(0)$ is

$$\begin{cases} P(0)_0 = Hom^{-1}_{\mathcal{D}}(X,Z) \oplus Im(d_{Hom^{-1}_{\mathcal{D}}(X,Z)}), & \text{if } k = 0 \\ P(0)_k = Hom^{-k-1}_{\mathcal{D}}(X,Z) \oplus Hom^{-k}_{\mathcal{D}}(X,Z), & \text{if } k > 0. \end{cases}$$

with differential for $k > 0$

$$d_k = \begin{vmatrix} -d^{-k-1}_{(X,Z)} & Id_{(X,Z)} \\ 0 & d^{-k}_{(Y,Z)} \end{vmatrix}$$

and for $k = 0$

$$d_0 = \begin{vmatrix} -d^{-1}_{(X,Z)} & Id_{(X,Z)} \\ 0 & d^{0}_{(Y,Z)} \end{vmatrix}$$

Hence the right hand side of Equation (17) is computed by the cochain complex having degree $k > 0$ component

$$Hom^{-k}_{\mathcal{D}}(Y,Z) \oplus Hom^{-k-1}_{\mathcal{D}}(X,Z) \oplus Hom^{-k-1}_{\mathcal{D}}(X,Z) \oplus Hom^{-k}_{\mathcal{D}}(X,Z)$$

with differential

$$d_k = \begin{vmatrix} d^{-k}_{(X,Z)} & 0 & 0 & 0 \\ -(- \circ f) & -d^{-k-1}_{(X,Z)} & 0 & 0 \\ 0 & 0 & -d^{-k-1}_{(X,Z)} & Id_{(X,Z)} \\ 0 & 0 & 0 & d^{-k}_{(X,Z)} \end{vmatrix}$$

and degree 0 component the subspace of the direct sum $P(-\circ f)_0 \oplus P(0)_0$ corresponding to $D^0 \oplus Im(d_{Hom^{-1}_{\mathcal{D}}(X,Z)})$ via the inclusion in $Hom^0_{\mathcal{D}}(X,Z)$. This chain complex is quasi-isomorphic to the chain complex whose degree $k > 0$ component is

$$Hom^{-k}_{\mathcal{D}}(Y,Z) \oplus Hom^{-k-1}_{\mathcal{D}}(X,Z)$$

with differential

$$d_k = \begin{vmatrix} d^{-k}_{(X,Z)} & 0 \\ -(- \circ f) & -d^{-k-1}_{(X,Z)} \end{vmatrix}$$

and degree 0 component

$$Ker(d_{Hom^0_{\mathcal{D}}(Y,Z)}) \oplus E^{-1}$$

where $E^{-1} \subseteq Hom^{-1}_{\mathcal{D}}(X,Z)$ is defined by the equation $d(g^{-1}) + g^0 \circ f = 0$, for $g^0 \in Ker(d_{Hom^0_{\mathcal{D}}(Y,Z)})$. This is quasi-isomorphic to the truncation $\tau_{\geq 0}(Hom^{\bullet}_{\mathcal{D}}(Cone(f), Z)^{op})$ and hence it provides the weak-equivalence of equation (17). For computing the fiber of $f$, a similar computation shows that the morphism $i : Cone(f)[-1] \longrightarrow X$ corresponding to $(Id_X, 0)$ according to the equality (see Equation (12))

$$Hom^0_{\mathcal{D}}(Cone(f)[-1], X) = Hom^0_{\mathcal{D}}(X,X) \oplus Hom^1_{\mathcal{D}}(Y,X)$$

induces and homotopy cartesian triangle in $N^{big}_{dg}(\mathcal{D})$

$$\begin{array}{ccc} Cone(f)[1] & \xrightarrow{j} & X \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle f} \\ 0 & \xrightarrow{0} & Y \end{array} \tag{18}$$

We show now that the diagram (15) is also cartesian. Let $h$ and $g$ be the closed degree 0 morphisms corresponding respectively to $(0, \pi_X)$ and $(0, (-f) \oplus i_X)$ according to the equalities

$$Hom^0_{\mathcal{D}}(X, Cone(j)[-1]) = Hom^{-1}_{\mathcal{D}}(X,Y) \oplus Hom^0_{\mathcal{D}}(X, Y \oplus X)$$

$$Hom^0_{\mathcal{D}}(Cone(j)[-1], X) = Hom^1_{\mathcal{D}}(Y,X) \oplus Hom^0_{\mathcal{D}}(Y \oplus X, X)$$

where $\pi_X : Y \oplus X \longrightarrow X$ and $i_X : X \longrightarrow Y \oplus X$ are the canonical projection and inclusion morphisms. Let $\alpha$ corresponding to $(0, -i_Y, 0, 0)$ according to the identification of $Hom^{-1}_{\mathcal{D}}(Cone(j)[-1], Cone(j)[-1])$ with

$$Hom^{-1}_{\mathcal{D}}(Y,Y) \oplus Hom^0_{\mathcal{D}}(Y, Y \oplus X) \oplus Hom^{-2}_{\mathcal{D}}(Y \oplus X, Y) \oplus Hom^{-1}_{\mathcal{D}}(Y \oplus X, Y \oplus X)$$

We have that $d(\alpha) = g \circ h - Id_{Cone(j)[-1]}$ because the differential in degree $-1$ is given by (see Equations (12) and (13))

$$d_{-1}(c^{-1}, c^0, c^{-2}, d^{-1}) = \left| \begin{array}{c} d(c^{-1}) + (Id_Y \oplus f) \circ c^0 \\ d(c^0) \\ d(c^{-2}) + c^{-1} \circ (Id_Y \oplus f) + (Id_Y \oplus f) \circ d^{-1} \\ d(d^{-1}) + c^0 \circ (Id_Y \oplus f) \end{array} \right|$$

On the other hand $h \circ g = Id_X$ which implies that $X$ and $Cone(j)[-1]$ are homotopy equivalent in $N^{big}_{dg}(\mathcal{D})$. A similar argument shows that the diagram (18) is also cocartesian. The fact that $H^0(\mathcal{D})$ is equivalent to $h(N^{big}_{dg}(\mathcal{D}))$ as triangulated categories follows from the arguments in this proof. ∎

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