Theory and Applications of Categories, Vol. 32, No. 42, 2017, pp. 1485–1500.

MAL'TSEV OBJECTS, R_1-SPACES AND ULTRAMETRIC SPACES

THOMAS WEIGHILL

ABSTRACT. In this paper we introduce a notion of Mal'tsev object, and the dual notion of co-Mal'tsev object, in a general category. In particular, a category $\mathbb{C}$ is a Mal'tsev category if and only if every object in $\mathbb{C}$ is a Mal'tsev object. We show that for a well-powered regular category $\mathbb{C}$ which admits coproducts, the full subcategory of Mal'tsev objects is coreflective in $\mathbb{C}$. We show that the co-Mal'tsev objects in the category of topological spaces and continuous maps are precisely the R_1-spaces, and that the co-Mal'tsev objects in the category of metric spaces and short maps are precisely the ultrametric spaces.

1. Introduction

A variety $\mathbb{X}$ of universal algebras is called a *Mal'tsev variety* [14] if it satisfies the following condition:

(M_1) the algebraic theory of $\mathbb{X}$ contains a ternary term μ satisfying the term equations

$$\mu(x,y,y) = x = \mu(y,y,x).$$

A famous theorem of Mal'tsev states that these varieties are precisely those in which the composition of congruences on any object is commutative [13]. The notion of *Mal'tsev category* is a generalisation of the notion of Mal'tsev variety. Recall that a Mal'tsev category was originally defined in [4] to be a category $\mathbb{C}$ which is exact in the sense of Barr [1] and which satisfies the following condition:

(M_2) every reflexive internal relation in $\mathbb{C}$ is an equivalence relation.

In the present paper, by a Mal'tsev category, we mean (as in [2]) a category $\mathbb{C}$ which satisfies the following relational reformulation of (M_1) due to Lambek [12]:

(M_3) every internal relation R in $\mathbb{C}$ is *difunctional*, i.e. it satisfies

$$(x_1 R y_2 \wedge x_2 R y_2 \wedge x_2 R y_1) \Rightarrow x_1 R y_1.$$

Received by the editors 2016-03-19 and, in final form, 2017-11-17.
Transmitted by Stephen Lack. Published on 2017-11-20.
2010 Mathematics Subject Classification: 18A05, 18A32, 18B30, 54D10, 54E35.
Key words and phrases: Mal'tsev object, Mal'tsev category, R_1-space, ultrametric space.

Note that conditions (M_2) and (M_3) can both be formulated in a general category. Recall that an internal relation from an object X to an object Y in a category $\mathbb{C}$ is a triple (R, r_1, r_2) with R an object of $\mathbb{C}$ and $r_1 : R \to X$ and $r_2 : R \to Y$ morphisms of $\mathbb{C}$ such that r_1 and r_2 are jointly monomorphic. Note that if $\mathbb{C}$ admits binary products, then an internal relation from X to Y can also be viewed as a monomorphism $(r_1, r_2) : R \to X \times Y$. We say that a relation (R, r_1, r_2) from X to Y is reflexive, symmetric, transitive or difunctional when for every object S of $\mathbb{C}$, the relation

$$\hom(S,X) \xleftarrow{\hom(S,r_1)} \hom(S,R) \xrightarrow{\hom(S,r_2)} \hom(S,Y)$$

between sets is reflexive, symmetric, transitive or difunctional in the usual sense. For a category $\mathbb{C}$ with finite limits, (M_2) is equivalent to (M_3) (see [5]).

A category $\mathbb{C}$ is thus a Mal'tsev category if and only if every object S in $\mathbb{C}$ satisfies the following condition:

(D) for any internal relation (R, r_1, r_2) from an object X to an object Y, the following relation is difunctional:

$$\hom(S,X) \xleftarrow{\hom(S,r_1)} \hom(S,R) \xrightarrow{\hom(S,r_2)} \hom(S,Y)$$

For a general category $\mathbb{C}$, we will call an object S satisfying (D) above a *Mal'tsev object*. Note that the Mal'tsev objects in $\mathbb{C}$ are precisely those objects S for which the functor $\hom(S, -)$ is M-closed in the sense of [10], where M is the matrix

$$M = \left(\begin{array}{ccc|c} y & x & x & y \\ u & u & v & v \end{array}\right).$$

We call an object S in $\mathbb{C}$ a *co-Mal'tsev object* if it is a Mal'tsev object as an object of the dual category $\mathbb{C}^{\mathsf{op}}$. We denote the full subcategory of Mal'tsev objects in $\mathbb{C}$ by $\mathsf{Mal}(\mathbb{C})$.

In this paper, we first give a characterisation of Mal'tsev objects in the case when $\mathbb{C}$ is a category satisfying certain conditions. This characterisation is based on recent work by Bourn and Z. Janelidze [2]. We then show that, for a regular category $\mathbb{C}$ with binary coproducts, $\mathsf{Mal}(\mathbb{C})$ contains every full subcategory of $\mathbb{C}$ which is a Mal'tsev category and which is closed under binary coproducts and regular quotients in $\mathbb{C}$. In Section 3, we show that the co-Mal'tsev objects in **Top** (the category of topological spaces and continuous maps) are precisely the R_1 spaces [6], i.e. topological spaces satisfying the following "separation axiom":

(R_1) for all $x, y \in X$, if there exists an open set A such that $x \in A$ and $y \notin A$, then there exist disjoint open sets B and C such that $x \in B$ and $y \in C$.

In Section 4, we consider the category **Met** of metric spaces and short maps, and show that the co-Mal'tsev objects in this category are precisely the ultrametric spaces, i.e. metric spaces X satisfying

$$d(x,z) \leqslant \mathsf{max}\{(d(x,y), d(y,z)\}$$

for any x, y, z in X. A classical example of an ultrametric space is the set of rationals $\mathbb{Q}$ equipped with the metric arising from the p-adic norm for some prime p.

2. General properties of Mal'tsev objects

By a *(regular) quotient/subobject* of an object X in a category $\mathbb{C}$ we mean a (regular) epi/mono with domain/codomain X. We say that a subcategory $\mathbb{D}$ in $\mathbb{C}$ is *closed under regular quotients/subobjects* in $\mathbb{C}$ if the codomain/domain of every regular quotient/subobject of an object in $\mathbb{D}$ is in $\mathbb{D}$.

2.1. Proposition. *For any category $\mathbb{C}$, $\mathsf{Mal}(\mathbb{C})$ is closed under colimits and regular quotients in $\mathbb{C}$.*

Proof. This can be deduced from general considerations via the Yoneda embedding, but it is also easy to prove directly, as we now show. Let $D : \mathbb{D} \to \mathbb{C}$ be a diagram whose image is contained in $\mathsf{Mal}(\mathbb{C})$ and which has a colimit C in $\mathbb{C}$, and let (R, r_1, r_2) be an internal relation from X to Y. Let $x_1, x_2 : C \to X$ and $y_1, y_2 : C \to Y$ be morphisms such that $x_1 R y_2$, $x_2 R y_2$ and $x_2 R y_1$. Then for each object A in D, $x_1\iota_A R y_2\iota_A$, $x_2\iota_A R y_2\iota_A$ and $x_2\iota_A R y_1\iota_A$, where ι_A is the colimit injection. Since each $D(A)$ is a Mal'tsev object, we have $x_1\iota_A R y_1\iota_A$ for each object A in $\mathbb{D}$. Thus there is a family of morphisms $h_A : D(A) \to R$ such that $r_1 h_A = x_1\iota_A$ and $r_2 h_A = y_1\iota_A$ for every object A in $\mathbb{D}$. Using the fact that r_1 and r_2 are jointly monic, it follows that the morphisms h_A induce a morphism $h : C \to R$, and it is easy to check that $r_1 h = x_1$ and $r_2 h = y_1$, so $x_1 R y_1$ as required. Suppose now that S is a Mal'tsev object and that $q : S \to T$ is a regular epimorphism, which is the coequalizer of $a, b : Q \to S$. Let $u_1, u_2 : T \to X$ and $v_1, v_2 : T \to Y$ be morphisms such that $u_1 R v_2$, $u_2 R v_2$ and $u_2 R v_1$. Then $u_1 q R v_2 q$, $u_2 q R v_2 q$ and $u_2 q R v_1 q$. Since S is a Mal'tsev object, we have that $u_1 q R v_1 q$. In other words, there is a map $f : S \to R$ such that $r_1 f = u_1 q$ and $r_2 f = v_1 q$. But then $r_1 f a = u_1 q a = u_1 q b = r_1 f b$ and $r_2 f a = v_1 q a = v_1 q b = r_2 f b$, so since r_1 and r_2 are jointly monic, we have $fb = fa$. Thus there is a morphism $g : T \to R$ such that $gq = f$, and one checks that $r_1 g = u_1$ and $r_2 g = v_1$, which gives $u_1 R v_1$ as required. ∎

Recall that a category $\mathbb{C}$ is *well-powered* if for every object X of $\mathbb{C}$, the collection of all isomorphism classes of subobjects of X may be labelled by a set.

2.2. Corollary. *Let $\mathbb{C}$ be a well-powered regular category which admits coproducts. Then $\mathsf{Mal}(\mathbb{C})$ is a coreflective subcategory of $\mathbb{C}$. In particular, $\mathsf{Mal}(\mathbb{C})$ will be (finitely) complete if $\mathbb{C}$ is (finitely) complete.*

Proof. This follows from the following general fact: if $\mathbb{B}$ is a full subcategory of a well-powered regular category $\mathbb{X}$ which admits coproducts and $\mathbb{B}$ is closed under coproducts and regular quotients, then $\mathbb{B}$ is a coreflective subcategory. Indeed, if X is any object in $\mathbb{X}$, take a set of representatives M of all subobjects of X which lie in $\mathbb{B}$, and let $\coprod M$ be the coproduct of their domains. The coreflection of X into the subcategory $\mathbb{B}$ is then given by the domain of the mono part of the factorisation of the canonical morphism from $\coprod M$ to X. ∎

Proposition 2.3 below follows straightforwardly from the proofs of Proposition 4.1 and Theorem 4.2 in [2], but we present a direct proof here for the sake of completeness. For convenience, given an internal relation (R, r_1, r_2) from X to Y and two morphisms $f : S \to X$ and $g : S \to Y$, we write fRg to mean that f and g are related by the image of (R, r_1, r_2) under $\hom(S, -)$. Dually, given an internal co-relation (R, r_1, r_2) from X to Y (i.e a pair of jointly epimorphic morphisms $r_1 : X \to R$ and $r_2 : Y \to R$) and two morphisms $f : X \to S$ and $g : Y \to S$, we write fRg to mean that f and g are related by the relation

$$\hom(X, S) \xleftarrow{\hom(r_1,S)} \hom(R, S) \xrightarrow{\hom(r_2,S)} \hom(Y, S)$$

2.3. Proposition. *Let $\mathbb{C}$ be a regular category which admits binary coproducts. Then for any object S in $\mathbb{C}$, the following are equivalent:*

(a) S is a Mal'tsev object;

(b) $\iota_1 R' \iota_1$, where $\iota_1 : S \to 2S$ is the first coproduct injection and (R', r'_1, r'_2) is the internal relation from $2S$ to $2S$ appearing in the (regular epi, mono)-factorisation of the vertical morphism in the following diagram:

$$\begin{array}{ccc} 3S & \xrightarrow{\;e\;} & R' \\ {\scriptsize\begin{pmatrix}\iota_1 & \iota_2\\ \iota_2 & \iota_2\\ \iota_2 & \iota_1\end{pmatrix}}\Big\downarrow & \swarrow{\scriptstyle r'=(r'_1,r'_2)} & \\ 2S \times 2S & & \end{array} \tag{1}$$

Proof. (a)⇒(b): For the internal relation R' in diagram (1) we have $\iota_1 R' \iota_2$, $\iota_2 R' \iota_2$ and $\iota_2 R' \iota_1$, so $\iota_1 R' \iota_1$ by difunctionality of $\hom(S, R')$.

(b)⇒(a): Suppose (R, r_1, r_2) is an internal relation from X to Y and $x_1, x_2 : S \to A$ and $y_1, y_2 : S \to B$ are morphisms such that $x_1 R y_2$, $x_2 R y_2$ and $x_2 R y_1$. Consider the diagram of solid arrows:

$$\begin{array}{ccc} 3S & \overset{p}{\dashrightarrow} & R \\ {\scriptsize\begin{pmatrix}\iota_1 & \iota_2\\ \iota_2 & \iota_2\\ \iota_2 & \iota_1\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle r=(r_1,r_2)} \\ 2S \times 2S & \xrightarrow[\binom{x_1}{x_2}\times\binom{y_1}{y_2}]{} & X \times Y \end{array}$$

By the assumptions on R, the morphisms (x_1, y_2), (x_2, y_2) and (x_2, y_1) from S to $X \times Y$ all factor through r. It follows that there is a morphism p as shown which makes the diagram commute. By the property of the factorisation, since r is a monomorphism, there is a morphism $f : R' \to R$ which makes the following diagram commute:

$$\begin{array}{ccc} R' & \xrightarrow{\quad f \quad} & R \\ {\scriptstyle r'=(r'_1,r'_2)}\downarrow & & \downarrow{\scriptstyle r=(r_1,r_2)} \\ 2S \times 2S & \xrightarrow[\binom{x_1}{x_2}\times\binom{y_1}{y_2}]{} & X \times Y \end{array}$$

By hypothesis, ι_1 and ι_1 are related by R', so that the map $(\iota_1, \iota_1) : S \to 2S \times 2S$ factors through r'. By commutativity of the above diagram, the map $(x_1, y_1) : S \to X \times Y$ must then factor through r, as required. ■

Note that Proposition 2.3 holds more generally for any category with (strong epi, mono)-factorizations and binary products and coproducts, where one replaces the (regular epi, mono)-factorization of the vertical morphism in (1) with its (strong epi, mono)-factorization.

2.4. Corollary. *Let $\mathbb{C}$ be a regular category admitting binary coproducts. Let $\mathbb{D}$ be a full subcategory of $\mathbb{C}$ which is a Mal'tsev category and which is closed under regular quotients and binary coproducts in $\mathbb{C}$. Then $\mathbb{D}$ is contained in* $\mathsf{Mal}(\mathbb{C})$.

Proof. Suppose that $\mathbb{D}$ is a full subcategory of $\mathbb{C}$ which is Mal'tsev and which is closed under binary coproducts and regular quotients in $\mathbb{C}$. Then for every object S in $\mathbb{D}$, the objects $2S$ and R' from diagram (1), and hence also the morphisms r'_1 and r'_2, are contained in $\mathbb{D}$. Since the morphisms r'_1 and r'_2 are jointly monic in $\mathbb{C}$, they are also jointly monic in $\mathbb{D}$ and thus represent a internal relation in $\mathbb{D}$. Since $\mathbb{D}$ is assumed to be Mal'tsev, the relation $\hom(S, R')$ between sets must be difunctional. But then, since $\iota_1 R' \iota_2$, $\iota_2 R' \iota_2$ and $\iota_2 R' \iota_1$, we have that $\iota_1 R' \iota_1$, so S is a Mal'tsev object by Proposition 2.3. ■

It is not clear in general if the full subcategory $\mathsf{Mal}(\mathbb{C})$ is itself a Mal'tsev category, since jointly monic pairs in $\mathsf{Mal}(\mathbb{C})$ may not be jointly monic as morphisms in $\mathbb{C}$. The following corollary gives a condition under which $\mathsf{Mal}(\mathbb{C})$ is indeed a Mal'tsev category.

2.5. Corollary. *Let $\mathbb{C}$ be regular category with binary coproducts. Consider the following conditions on $\mathbb{C}$:*

(1) every morphism in $\mathsf{Mal}(\mathbb{C})$ which is a regular epimorphism in $\mathbb{C}$ is also a regular epimorphism as a morphism in $\mathsf{Mal}(\mathbb{C})$;

(2) if a pair of morphisms $r_1 : R \to X$ and $r_2 : R \to Y$ are jointly monic (that is, an internal relation) in $\mathsf{Mal}(\mathbb{C})$ then they are also jointly monic in $\mathbb{C}$;

(3) $\mathsf{Mal}(\mathbb{C})$ is the largest full subcategory of $\mathbb{C}$ which is a Mal'tsev category and which is closed under regular quotients and binary coproducts in $\mathbb{C}$.

Then (1) ⇒ (2) and (2) ⇒ (3).

Proof. (1) $\Rightarrow$ (2): Suppose $r_1 : R \to X$ and $r_2 : R \to Y$ are jointly monic in $\mathsf{Mal}(\mathbb{C})$. Consider the map $(r_1, r_2) : R \to X \times Y$ in $\mathbb{C}$ and its (regular epi, mono)-factorization $(r_1, r_2) = me$ in $\mathbb{C}$. Since e is a regular epi in $\mathbb{C}$, it exists as a morphism in $\mathsf{Mal}(\mathbb{C})$ where it is also a regular epi. Since r_1 and r_2 are jointly monic in $\mathsf{Mal}(\mathbb{C})$, it is easy to check that e must be a monomorphism in $\mathsf{Mal}(\mathbb{C})$, so that in fact e is an isomorphism. It follows that (r_1, r_2) is a monomorphism in $\mathbb{C}$ as required.

(2) $\Rightarrow$ (3): Since internal relations in $\mathsf{Mal}(\mathbb{C})$ are internal relations in $\mathbb{C}$, $\mathsf{Mal}(\mathbb{C})$ is a Mal'tsev category, and the result follows from Corollary 2.4. ■

Conditions (1) and (2) in Corollary 2.5 will turn out to hold for the categories we are interested in in the next two sections ($\mathbf{Top}^{\mathrm{op}}$ and $\mathbf{Met}_\infty^{\mathrm{op}}$). However, they are not very natural to require of a general category $\mathbb{C}$.

2.6. Question. *Are there natural conditions on a general category $\mathbb{C}$ such that $\mathsf{Mal}(\mathbb{C})$ is a Mal'tsev category?*

It follows from the work in [2] that, for a finitely cocomplete regular category $\mathbb{C}$ and an object S in $\mathbb{C}$, conditions (a) and (b) in Proposition 2.3 are further equivalent to the following:

(c) S admits an *approximate Mal'tsev co-operation μ with approximation α* a regular epimorphism, i.e. there exists an object A and morphisms $\mu : A \to 3S$ and $\alpha : A \to S$, with α a regular epimorphism, such that the following diagram commutes:

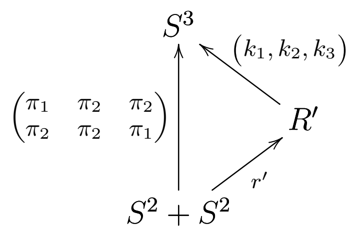
$$\begin{array}{ccc} A & \xrightarrow{\mu_S} & 3S \\ {\scriptstyle \alpha_S}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} \iota_1 & \iota_2 \\ \iota_2 & \iota_2 \\ \iota_2 & \iota_1 \end{pmatrix}} \\ S & \xrightarrow[(\iota_1,\iota_1)]{} & 2S \times 2S \end{array} \tag{2}$$

Indeed, if such a diagram (2) exists with α_S a regular epimorphism, then by the universal property of the (regular epi, mono)-factorization system, the morphism (ι_1, ι_1) factors through the monomorphism (r_1', r_2') in (1), which implies (b). Conversely, if (b) holds, there is a map $g : S \to R'$ such that $(r_1', r_2') \circ g = (\iota_1, \iota_1)$, and one can take α_S to be the pullback of the map e in (1) along the map g. Since $\mathbb{C}$ was assumed to be regular, α_S is a regular epimorphism.

3. Co-Mal'tsev objects in **Top**

It is easy to check that the regular monomorphisms in **Top** are precisely the embeddings of spaces. In particular, an embedding $f : A \to X$ is the coequalizer of the continuous maps a and b to $J = \{0, 1\}$, the two element indiscrete space, where a sends $f(A)$ to 0 and its complement to 1 and b sends all of X to 0. It is easy to show, moreover, that topological embeddings are closed under pushouts in **Top**; it follows that the category $\mathbf{Top}^{\mathsf{op}}$ is regular and finitely complete.

3.1. Theorem. *Let S be an object in* **Top**, *and let the following diagram in* **Top** *represent the (epi, regular mono)-factorization of the vertical morphism:*



Then the following are equivalent:

(a) S is a co-Mal'tsev object;

(b) there is a (unique) morphism $f : R' \to S$ such that

$$f \circ r' = \begin{pmatrix} \pi_1 \\ \pi_1 \end{pmatrix};$$

(c) for every open set A in S, there is an open set A' in S^3 such that

$$x \in A \Leftrightarrow (x, y, y) \in A' \Leftrightarrow (y, y, x) \in A'$$

for all $(x, y) \in S^2$;

(d) S is an R_1-space.

Proof. (a) $\Leftrightarrow$ (b) follows from the dual of Proposition 2.3.

(b) $\Leftrightarrow$ (c): Since (k_1, k_2, k_3) is a regular monomorphism, i.e. an embedding of spaces, R' has underlying set

$$\{(x, y, y) \mid (x, y) \in S^2\} \cup \{(y, y, x) \mid (x, y) \in S^2\}$$

with the subspace topology induced by S^3. Let f be the function from the underlying set of R' to the underlying set of S defined by $f(x, y, y) = x$ and $f(y, y, x) = x$. Condition (b) is then equivalent to f being a continuous map from R' to S, which is clearly equivalent to (c).

(c) $\Rightarrow$ (d): Let x, y be two points in S and let A be an open set such that $x \in A$ and $y \notin A$. Then take A' as in (c). Now $(x, y, y) \in A'$, so there exist open sets U, V, and W in S such that $(x, y, y) \in U \times V \times W \subseteq A'$. Moreover, $(x, y, y) \in U \times (V \cap W) \times (V \cap W) \subseteq A'$. Now suppose $z \in U \cap V \cap W$. Then (z, z, y) is in A' and thus y must be in A, a contradiction. So $U \cap (V \cap W) = \varnothing$ and thus U and $V \cap W$ are disjoint open sets such that $x \in U$ and $y \in V \cap W$.

(d) $\Rightarrow$ (c): Let A be an open set in S. Let (x, y) be a pair of points with $x \in A, y \notin A$. Then since S is an R_1-space, there exist disjoint open sets $B_{(x,y)}, C_{(x,y)}$ such that $x \in B_{(x,y)} \subseteq A$ and $y \in C_{(x,y)}$. Now consider the family of all such pairs $(B_{(x,y)}, C_{(x,y)})$ indexed by pairs of points (x, y) with $x \in A$ and $y \notin A$. Now it is easy to see that the desired set A' may be chosen to be :

$$A' = A^3 \cup \left(\bigcup_{x \in A, y \notin A} B_{(x,y)} \times C_{(x,y)} \times C_{(x,y)} \right) \cup \left(\bigcup_{x \in A, y \notin A} C_{(x,y)} \times C_{(x,y)} \times B_{(x,y)} \right)$$

■

We thus have the following corollaries of Corollary 2.2, Corollary 2.5 and the remark on approximate Mal'tsev co-operations at the end of the previous section.

3.2. Corollary. *Let* $\mathbf{R}_1$ *be the full subcategory of* **Top** *whose objects are the* R_1*-spaces. Then the dual of* $\mathbf{R}_1$ *is a finitely complete Mal'tsev category. Moreover,* $\mathbf{R}_1$ *is reflective in* **Top** *and is the largest full subcategory of* **Top** *whose dual is Mal'tsev and which is closed under binary products and regular subobjects (i.e. subspaces) in* **Top**.

Proof. The only part which needs proving is that if a morphism f in $\mathbf{R}_1$ is a regular monomorphism in **Top** then it is a regular monomorphism in $\mathbf{R}_1$, after which one can apply Corollary 2.5. This is easy to check given that the two element indiscrete space is in $\mathbf{R}_1$. ■

The notion of *approximate Mal'tsev operation* is dual to that of an approximate Mal'tsev co-operation.

3.3. Corollary. *Let* X *be an object of* **Top**. *Then* X *is an* R_1*-space if and only if* X *admits an approximate Mal'tsev operation with approximation* α *a regular monomorphism (i.e. an embedding of spaces).*

4. Co-Mal'tsev objects in **Met**

Let **Met** be the category whose objects are metric spaces and whose morphisms are all *short maps* between metric spaces, i.e. maps $f : X \to Y$ such that for any x_1, $x_2 \in X$,

$$d(f(x_1), f(x_2)) \leqslant d(x_1, x_2).$$

This is, for example, the category of metric spaces implicit in Isbell's definition of injective metric space in [8]. Note that short maps are always continuous with respect to the topology induced by the metric on each space, and that the isomorphisms in **Met** are precisely the global isometries. In this section we will prove that the co-Mal'tsev objects in this category are the ultrametric spaces.

The category **Met** does not admit coproducts, so we will also want to consider the category $\mathbf{Met}_\infty$ whose objects are *extended metric spaces* and whose maps are short maps between extended metric spaces. Recall that an extended metric space is a set equipped with a distance function which takes values in the extended reals $\mathbb{R} \cup \{\infty\}$ and which satisfies the axioms for a metric. In particular, every metric space can be viewed as an extended metric space. We now collect some elementary facts about $\mathbf{Met}_\infty$, leading eventually to Proposition 4.2 below. The results are straightforward to prove, but we include proofs for the sake of completeness.

To construct colimits in $\mathbf{Met}_\infty$ we need to be able to take quotients of metric spaces by equivalence relations. This topic is classical (see for example [3, 7]). Let A be a metric space and E an equivalence relation on A. Let A_E be the set of equivalence classes under E and define a distance function d' on A_E as follows:

$$d'([a]_E, [b]_E) = \mathsf{inf}\{\sum_{i=1}^{n} d(a_i, b_i) \mid a_1Ea,\ b_nEb,\ b_iEa_{i+1},\ n \in \mathbb{Z}_+\}$$

where $\mathbb{Z}_+$ is the set of positive integers. A sequence of pairs $(a_i, b_i)_{1\leq i\leq n}$ satisfying a_1Ea, b_nEb, b_iEa_{i+1} will usually be referred to as a *chain* from a to b. In general this defines a pseudometric on A_E, which may not be a metric (some distinct points may be distance 0 apart). If d' is a metric, then we define $\overline{A}_E$ to be A_E with the metric d'; in such a case, we will call the equivalence relation E *well-behaved*. If d' is not a metric, consider the equivalence relation $x \sim y \Leftrightarrow d'(x, y) = 0$ on A_E, and define $\overline{A}_E$ to be the set of equivalence classes under $\sim$ with the metric

$$d_E([x]_\sim, [y]_\sim) = d'(x, y)$$

(note that this is well-defined). It is an easy exercise to check that $\overline{A}_E$ with the obvious quotient map is universal amongst all short maps with domain A which are constant on equivalence classes under E.

Using this construction it is easy to define coequalizers in $\mathbf{Met}$ and $\mathbf{Met}_\infty$: for two maps $f, g : X \to Y$ simply take the quotient of Y by the equivalence relation generated by the pairs $(f(x), g(x))_{x\in X}$. Coproducts in $\mathbf{Met}_\infty$ are easy to construct (but don't exist in $\mathbf{Met}$): to form $X + Y$ simply take the disjoint union of the two spaces and declare the distance between any point in X and any point in Y to be infinite. It follows that $\mathbf{Met}_\infty$ is finitely cocomplete. It is easy to check that $\mathbf{Met}$ and $\mathbf{Met}_\infty$ also admit equalizers. Given a pair of objects X and Y in either $\mathbf{Met}$ or $\mathbf{Met}_\infty$, their product is given by the set $X \times Y$ equipped with the metric

$$d((x_1, y_1), (x_2, y_2)) = \mathsf{max}(d(x_1, x_2), d(y_1, y_2)).$$

4.1. Lemma. *A morphism $f : X \to Y$ in* $\mathbf{Met}$ *or* $\mathbf{Met}_\infty$ *is a regular monomorphism if and only if it is an isometric embedding with closed image.*

Proof. The same proof will work for both categories. Suppose f is the equalizer of a pair $a, b : Y \to Z$. Since a and b agree on $f(X)$, they also agree on the closure $\overline{f(X)}$. It follows that $f(X) = \overline{f(X)}$ so that the image of f is closed, and it is easy to check that f must be an isometric embedding.

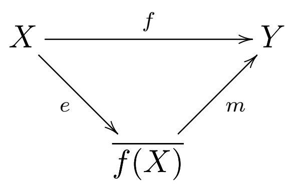
Conversely, suppose $f : X \to Y$ is an isometric embedding with $f(X)$ closed. The case when X is empty is easy to check, so assume $x_0 \in X$. Consider the quotient Z of Y by the equivalence relation

$$y \sim y' \Leftrightarrow \{y, y'\} \subseteq f(X) \text{ or } y = y',$$

which one checks is well-behaved because $f(X)$ is closed. It is now easy to check that f is the equalizer of the quotient map $a : Y \to Z$ and the map b which sends all of Y to $[f(x_0)]$. ■

4.2. Proposition. *The dual of the category* $\mathbf{Met}_\infty$ *is a finitely complete and finitely cocomplete regular category.*

Proof. Given a morphism f in $\mathbf{Met}_\infty$, we have an (epi, regular mono)-factorization $f = me$ given by



where the metric on $\overline{f(X)}$ is inherited from Y. From this it is easy to show that $\mathbf{Met}_\infty$ admits a (regular epi, mono)-factorization system, and we have already noted that it is finitely complete and cocomplete.

It remains to show that regular monos are closed under pushouts. Let $m : X \to Y$ be a regular monomorphism, which we may suppose is an inclusion of a closed subspace $X \subseteq Y$, and let $f : X \to Z$ an arbitrary morphism. The pushout of m along f is given by the quotient of $Y + Z$ by the equivalence relation $\sim$ generated by the pairs $(x, f(x))_{x \in X}$. Denote this space by Q and let $m' : Z \to Q$, $f' : Y \to Q$ be the maps induced by inclusions. We claim that m' is an isometric embedding. Let $z, z' \in Z$ and let $(a_i, b_i)_{1 \le i \le n}$ be a chain in $Y + Z$ from z to z' with respect to $\sim$. We want to show that

$$\sum_{i=1}^{n} d_{Y+Z}(a_i, b_i) \ge d_Z(z, z').$$

We may assume that none of the distances $d_{Y+Z}(a_i, b_i)$ is infinite. If some a_i or b_i is in $Y \setminus X \subseteq Y + Z$, then there is a subchain $(a_i, b_i)_{k \le i \le k'}$ with $a_k \in X$, $b_{k'} \in X$ and all other elements in $Y \setminus X$. Since $\sim$ is trivial on $Y \setminus X$, we have $b_i = a_{i+1}$ for $k \le i \le k' - 1$, so by the triangle inequality

$$\sum_{i=k}^{k'} d_{Y+Z}(a_i, b_i) \ge d_Y(a_k, b_{k'}) = d_X(a_k, b_{k'}).$$

Thus we may eliminate subchains in $Y \setminus X$ to reduce to the case when all the a_i and b_i are in $X \sqcup Z \subseteq Y + Z$.

If all the a_i and b_i are in $X \sqcup Z \subseteq Y + Z$, then applying the short map $f'' : X + Z \to Z$ induced by f and 1_Z, we obtain a chain $(f''(a_i), f''(b_i))_{1 \le i \le n}$ with $f''(a_1) = z$, $f''(b_n) = z'$ and $f''(b_i) = f''(a_{i+1})$, and by the triangle inequality

$$\sum_{i=1}^{n} d_{Y+Z}(a_i, b_i) \geq \sum_{i=1}^{n} d_Z(f''(a_i), f''(b_i)) \geq d_Z(z, z').$$

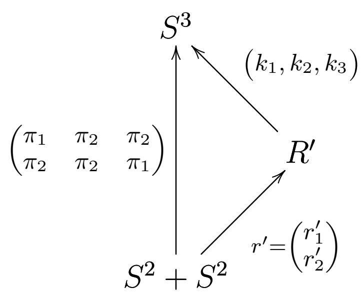
This concludes the proof that $d_Z(z, z') = d_Q(m'(z), m'(z'))$ for all $z, z' \in Z$. It remains to show that $m'(Z)$ is closed. To see this, let $y \in Y \setminus X \subseteq Y + Z$. Since $\sim$ is trivial on $Y \setminus X$, any chain $(a_i, b_i)_{1 \le i \le n}$ from y to $z \in Z \subseteq Y + Z$ must have some minimal j with $a_j \in Y \setminus X$ and $b_j \in X$. It follows that

$$\sum_{i=1}^{n} d_{Y+Z}(a_i, b_i) \geq \sum_{i=1}^{j} d_{Y+Z}(a_i, b_i) \geq d_Y(y, b_j)$$

so that $d_Q(f'(y), m'(z))$ is bounded below by $d_Y(y, X)$, which is positive since X is closed. Thus $f'(y)$ has a neighbourhood which does not intersect $m'(Z)$ as required. ■

4.3. Theorem. *Let S be an object of* **Met** *or* $\mathbf{Met}_\infty$*. Then S is a co-Mal'tsev object in the respective category if and only if it is an ultrametric space.*

Proof. Consider, in $\mathbf{Met}_\infty$, the dual picture to diagram (1), namely the diagram



where $(k_1, k_2, k_3) \circ r'$ represents the factorisation of the vertical morphism into an epi followed by a regular mono. It follows from the proof of Proposition 4.2 that the object R' is the closure of the subspace

$$T = \{(x, y, y) \mid (x, y) \in S^2\} \cup \{(x, x, y) \mid (x, y) \in S^2\} \subseteq S^3.$$

One can easily check, however, that T is itself closed, so that R' is just the subspace T. Thus the internal co-relation (R', r'_1, r'_2) is the co-relation

$$S^2 \xrightarrow{r'_1} R' \xleftarrow{r'_2} S^2 \qquad (3)$$

with $R' = T$ and where the maps r'_1 and r'_2 send (x, y) to (x, y, y) and (x, x, y) respectively. We are now ready to prove the theorem.

(⇒) If S is a metric space, then the space R' in (3) is a metric space, as is S^2, so we can form the co-relation R' in diagram (3) in both **Met** and $\mathbf{Met}_\infty$. We have $\pi_1 R' \pi_2$, $\pi_2 R' \pi_2$ and $\pi_2 R' \pi_1$, so if S is a co-Mal'tsev object then there must exist a morphism $f : R' \to S$ such that $f \circ r'_1 = f \circ r'_2 = \pi_1$. The map f is uniquely defined: it sends (x, y, y) to x and (x, x, y) to y. Let x, y, z be points in S. Then since f is a short map, we have

$$\begin{aligned} d(x, z) &= d(f(x, y, y), f(y, y, z)) \\ &\leqslant d((x, y, y), (y, y, z)) \\ &= \mathsf{max}\{d(x, y), d(y, y), d(y, z)\} \\ &= \mathsf{max}\{d(x, y), d(y, z)\}. \end{aligned}$$

(⇐) Let S an object of $\mathbf{Met}_\infty$ which is an ultrametric space, and let the internal co-relation R' be as above. By the above results, the dual of $\mathbf{Met}_\infty$ satisfies the assumptions of Proposition 2.3, so it is enough to show the existence of a map $f : R' \to S$ such that $f \circ r'_1 = f \circ r'_2 = \pi_1$. Define f to send (x, y, y) to x and (x, x, y) to y. It remains to show that f is a short map. We have

$$\begin{aligned} d(f(u, v, v), f(x, x, y)) &= d(u, y) \\ &\leqslant \mathsf{max}\{d(u, v), d(v, y)\} \\ &\leqslant \mathsf{max}\{d(u, x), d(x, v), d(v, y)\} \\ &= d((u, v, v), (x, x, y)), \end{aligned}$$

and it follows easily that f is a short map. Thus S is a co-Mal'tsev object of $\mathbf{Met}_\infty$.

Suppose now that S is actually an object of **Met**. Since S is a co-Mal'tsev object in $\mathbf{Met}_\infty$, it is enough to check that a pair of jointly epic morphisms (that is, an internal co-relation) $a : A \to X$, $b : B \to X$ in **Met** remains jointly epic in $\mathbf{Met}_\infty$. Suppose $f, g : X \to Y$ are short maps such that Y is an ∞-metric space, $fa = fb$ and $ga = gb$. If $A = B = \varnothing$, then clearly $X = \varnothing$, so we may exclude this case. Since X is a metric space, the images of f and g each lie in a subspace of Y which is a metric space. In fact, since f and g agree on at least one point, we may choose the subspaces to be the same. It follows that f and g each factor through a monomorphism $Y' \to Y$ with Y' a metric space, and so $f = g$ as required. ■

Let **UMet** and $\mathbf{UMet}_\infty$ be the full subcategories of **Met** and $\mathbf{Met}_\infty$ respectively whose objects are the ultrametric spaces.

4.4. Lemma. *Let $f : X \to Y$ be a regular monomorphism in* **Met** *or* $\mathbf{Met}_\infty$ *where X and Y are ultrametric spaces. Then f is a regular monomorphism in* **UMet** *or* $\mathbf{UMet}_\infty$ *respectively.*

PROOF. The same proof works for both categories. By Lemma 4.1, f is an isometric embedding with closed image. We may assume that X is non-empty, with $x_0 \in X$. Consider the space Z whose underlying set is the quotient of Y by the equivalence relation

$$y \sim y' \Leftrightarrow \{y, y'\} \subseteq f(X) \text{ or } y = y',$$

and whose metric is given by

$$d_Z([a]_\sim, [b]_\sim) = \mathsf{inf}\{\max\{d(a_i, b_i) \mid 1 \leq i \leq n\} \mid a_1 E a,\ b_n E b,\ b_i E a_{i+1},\ n \in \mathbb{Z}_+\}.$$

It is easy to see that d_Z satisfies $d_Z(z, z'') \leq \max(d_Z(z, z'), d_Z(z', z''))$, so we want to check that $d_Z(z, z') = 0 \implies z = z'$. Let $y \neq y'$ be in Y such that $[y] \neq [y']$, and let $(a_i, b_i)_{1 \leq i \leq n}$ be a chain from y to y' in Y. If all the a_i and b_i are in $Y \setminus f(X)$, then

$$\max\{d_Y(a_i, b_i) \mid 1 \leq i \leq n\} \geq d_Y(y, y') > 0$$

because $\sim$ is trivial on $Y \setminus X$ and d_Y is an ultrametric. Thus we may restrict to chains where some b_i is in $f(X)$. Let j be the minimal index for which b_j is in $f(X)$. We have

$$\max\{d_Y(a_i, b_i) \mid 1 \leq i \leq j\} \geq d_Y(y, b_j) \geq d(y, f(X)) > 0$$

since $f(X)$ is closed. Thus $d_Z([y], [y']) > 0$ as required. Finally, it is easy to check that f is the coequalizer of the maps a and b where $a : Y \to Z$ is the quotient map and b sends all of Y to $[f(x_0)]$. ∎

4.5. COROLLARY. $\mathbf{UMet}_\infty$ *(resp.* $\mathbf{UMet}$*) is the largest full subcategory of* $\mathbf{Met}_\infty$ *(resp.* $\mathbf{Met}$*) whose dual is a Mal'tsev category and which is closed under products and regular subobjects (i.e. isometric embeddings of closed subspaces) in* $\mathbf{Met}_\infty$ *(resp.* $\mathbf{Met}$*) .*

PROOF. The result about $\mathbf{UMet}_\infty$ follows from Corollary 2.5 and Lemma 4.4. Since $\mathbf{Met}^{\mathrm{op}}$ is not finitely complete, we need to make some elementary arguments to prove the result for $\mathbf{Met}$. If $\mathbb{D}$ is a full subcategory of $\mathbf{Met}^{\mathrm{op}}$ which is Mal'tsev and closed under coproducts and regular quotients in $\mathbf{Met}^{\mathrm{op}}$, then it is also closed under coproducts and regular quotients in $\mathbf{Met}^{\mathrm{op}}_\infty$, so by the above assertion, it is contained in $\mathbf{UMet}^{\mathrm{op}}_\infty$, and hence in $\mathbf{UMet}^{\mathrm{op}}_\infty \cap \mathbf{Met}^{\mathrm{op}} = \mathbf{UMet}^{\mathrm{op}}$. It remains to show that $\mathbf{UMet}^{\mathrm{op}}$ is itself a Mal'tsev category. If a, b is a jointly monic pair in $\mathbf{UMet}^{\mathrm{op}}$, then it is also jointly monic in $\mathbf{UMet}^{\mathrm{op}}_\infty$ by similar arguments to the end of the proof of Theorem 4.3. It is thus also jointly monic in $\mathbf{Met}^{\mathrm{op}}_\infty$ by Lemmas 2.5 and 4.4. Thus since every object in $\mathbf{UMet}^{\mathrm{op}}$ is a Mal'tsev object in $\mathbf{Met}^{\mathrm{op}}_\infty$ and every internal relation in $\mathbf{UMet}^{\mathrm{op}}$ is an internal relation in $\mathbf{Met}^{\mathrm{op}}_\infty$, $\mathbf{UMet}^{\mathrm{op}}$ is a Mal'tsev category. ∎

5. Other term conditions

It would be interesting to see if there are other connections of the form of Theorem 3.1 and Theorem 4.3 between well-known conditions from universal algebra and well-known conditions from topology and geometry. As remarked by the authors of [2], it is straightforward to adapt Proposition 2.3 by replacing the underlying notion of Mal'tsev category with another *category with closed relations* in the sense of [10]. One of the examples mentioned in [2] is that of a *subtractive category* [11]. A variety of universal algebras is *subtractive* in the sense of Ursini [15] if its theory contains a constant 0 and a binary term s satisfying the term equations $s(x,x)=0$ and $s(x,0)=x$. A subtractive category can be defined as a pointed category $\mathbb{C}$ with finite limits such that every internal relation R satisfies the following condition (see [10]):

$$xRx \wedge xR0 \Rightarrow 0Rx \tag{4}$$

Note that 0 denotes the zero morphism in the above condition. Consider now the following condition on an object S in a pointed category $\mathbb{C}$:

(S) for any relation (R, r_1, r_2) from an object X to X, the following relation satisfies condition (4) above (where 0 is the zero morphism from S to X):

$$\hom(S,X) \xleftarrow{\hom(S,r_1)} \hom(S,R) \xrightarrow{\hom(S,r_2)} \hom(S,X)$$

Theorem 5.1 was originally proved by Z. Janelidze [9], in a form involving the analogue of approximate Mal'tsev operations for the subtractive case, and it served as the original inspiration for this paper. A sketch of the proof is given here.

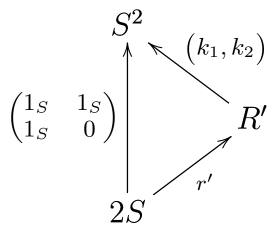
5.1. Theorem. *Let S be an object of the category $\mathbf{Top}_*$ of pointed topological spaces. Then S satisfies* (S) *as an object of the dual category $\mathbf{Top}_*^{\mathsf{op}}$ if and only if it satisfies the following condition, where 0 is the base point of S:*

(S′) *if A is any open set and x a point such that either $x \in A \wedge 0 \notin A$ or $x \notin A \wedge 0 \in A$, then there exists disjoint open sets B and C in S such that $0 \in B$ and $x \in C$.*

Proof. It follows from arguments similar to the proof of Proposition 2.3 that for a pointed category $\mathbb{C}$ with binary products and coproducts and a (strong epi, mono)-factorization system, an object S in $\mathbb{C}$ satisfies (S) if and only if $0R'1_S$, where (R', r'_1, r'_2) is the internal relation from S to S appearing in the (strong epi, mono)-factorisation of the vertical morphism in the following diagram:

$$\begin{array}{ccc} 2S & & \\ \left\downarrow \begin{pmatrix} 1_S & 1_S \\ 1_S & 0 \end{pmatrix} \right. & \searrow^{e} & \\ & & R' \\ & \swarrow_{r'=(r'_1, r'_2)} & \\ S \times S & & \end{array} \tag{5}$$

Consider the dual diagram (5) in the category of pointed topological spaces (which has products and which admits (epi, regular mono)-factorizations):



We see that the space R' is the subspace of S^2 given by

$$R' = \{(x, x) \mid x \in S\} \cup \{(x, 0) \mid x \in S\}$$

where 0 is the base point of S. We conclude that the pointed topological space S satisfies (S) as an object of the dual category $\mathbf{Top}_*^{\mathsf{op}}$ if and only if the set map $g : R' \to S$ defined by $g(x, x) = 0$ and $g(x, 0) = x$ is continuous. This is to say that for every open set $A \subseteq S$, there is an open set A' in S^2 such that $A \times \{0\} = A' \cap R'$ if $0 \notin A$ and $A \times \{0\} \cup \Delta = A' \cap R'$ if $0 \in A$. It remains to check that this is equivalent to (S'), which is not hard to show. ∎

Acknowledgements

This paper is based on research which the author carried out while he was a Masters student under the supervision of Z. Janelidze. The author would like to thank the anonymous referee for their valuable suggestions and for identifying an important mistake in the first draft of this paper.

References

[1] M. Barr, P. A. Grillet and D. H. van Osdol, Exact categories and categories of sheaves, Lecture Notes in Mathematics 236, Springer-Verlag, 1971.

[2] D. Bourn and Z. Janelidze, Approximate Mal'tsev operations, Theory and Applications of Categories 21, 2008, 152–171.

[3] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, Graduate Studies in Mathematics Volume 33, American Mathematical Society, Providence, 2001.

[4] A. Carboni, J. Lambek, and M. C. Pedicchio, Diagram chasing in Mal'cev categories, Journal of Pure and Applied Algebra 69, 1990, 271–284.

[5] A. Carboni, M. C. Pedicchio, and N. Pirovano, Internal graphs and internal groupoids in Mal'tsev categories, Canadian Mathematical Society Conference Proceedings 13, 1992, 97–109

[6] A. S. Davis, Indexed Systems of Neighborhoods for General Topological Spaces, The American Mathematical Monthly 68, 1961, 886–893.

and its Applications 346, Kluwer Academic Publishers, 1995.

[7] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Springer Science & Business Media, 2007.

[8] J. R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helv. 39 (1): 6576, 1964.

[9] Z. Janelidze, Cosubtractive objects in the category of pointed topological spaces, 2006 (unpublished).

[10] Z. Janelidze, Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories, Theory and Applications of Categories 16, 2006, 236–261.

[11] Z. Janelidze, Subtractive categories, Applied Categorical Structures 13, 2005, 343-350.

[12] J. Lambek, Goursat's theorem and the Zassenhaus lemma, Canadian Journal of Mathematics 10, 1957, 45–56.

[13] A. I. Maltsev, On the general theory of algebraic systems, Mat. Sbornik, N.S. 35 (77), 1954, 3–20 (in Russian); English translation: American Mathematical Society Translations (2) 27, 1963, 125–341.

[14] J. D. H. Smith, Mal'cev varieties, Lecture Notes in Mathematics 554, Springer, 1976.

[15] A. Ursini, On subtractive varieties, I, Algebra Universalis 31, 1994, 204–222.

Centre for Mathematical Structures
Department of Mathematical Sciences
Stellenbosch University
7602 Matieland
South Africa

Department of Mathematics
The University of Tennessee
Knoxville, TN 37996-1320
United States of America

Email: tweighil@vols.utk.edu

This article may be accessed at http://www.tac.mta.ca/tac/

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available from the journal's server at `http://www.tac.mta.ca/tac/`. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS The typesetting language of the journal is TEX, and LATEX2e is required. Articles in PDF format may be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.