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A CHARACTERIZATION OF THE HUQ COMMUTATOR

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Abstract. We study a categorical commutator, introduced by Huq, defined for a pair of coterminal morphisms. We show that in a normal unital category $\mathbb{C}$ with finite colimits, the normal closure of the regular image of the Huq commutator of a pair of subobjects under an arbitrary morphism is the same as the Huq commutator of their respective regular images. Then we use this property to characterize the Huq commutator as the largest commutator satisfying certain properties

1. Introduction

The concept of the commutator of subgroups of a given group has been generalized in various ways from groups to various categorical contexts. In the present paper, we study the concept of commutator in the sense of [Huq, 1986], which is derived from the notion of commuting morphisms introduced by [Huq, 1986] and developed further by several other mathematicians working in categorical algebra.

The commutator introduced by [Smith, 1976] for congruences in a Mal'tsev variety and later on defined categorically for internal equivalence relations by [Pedicchio, 1995], had been characterized by [Hagemann, Herrmann, 1979] as the largest commutator defined on congruence lattices satisfying certain properties. In the present paper, we characterize the Huq commutator as the largest commutator defined for subobjects satisfying certain properties; a result that could be thought of as a Huq commutator version of Herrmann and Hagemann characterization of Smith commutator.

In Section 2 we recall the notion of commuting morphisms, as well as the construction of the Huq commutator of a pair of coterminal morphisms in a normal unital category with finite colimits. Furthermore, we recall (in a slightly weaker context) the categorical definition of the Higgins commutator as formulated by [Mantovani, Metere, 2010] for a pair of subobjects in an ideal-determined category [G. Janelidze et al., 2010]. Relating these two commutators (Huq and Higgins), we prove (in Proposition 2.2) that in a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator is the normal closure of the Higgins commutator. This result was initially proven for an ideal-determined unital category in [Mantovani, Metere, 2010, Proposition 5.7].

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In Section 3 we show (in Theorem 3.2) that in a normal unital category $\mathbb{C}$ with finite colimits, the normal closure of the regular image of the Huq commutator of a pair of subobjects under an arbitrary morphism is the same as the Huq commutator of their respective regular images.

In the last section we prove (in Theorem 4.5) that in a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator is the largest commutator defined for subobjects satisfying certain properties.

2. Preliminaries

Recall that a unital category $\mathbb{C}$ [Bourn, 1996] is a pointed finitely complete category with the property that for each pair of objects X and Y, the morphisms $\langle 1, 0\rangle : X \longrightarrow X \times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X \times Y$ are jointly strongly-epimorphic. Following [Z. Janelidze, 2010], we define a normal category to be a pointed regular category where every regular epimorphism is a normal epimorphism. In this paper we will call a subobject $h : H \rightarrowtail X$ normal if h is the kernel of some morphism.

Commuting morphisms [Huq, 1986] A pair of coterminal morphisms $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ in a unital category $\mathbb{C}$ **commutes** when there exists a (necessarily unique) morphism $\varphi : X \times Z \to Y$ making the diagram

$$\begin{tikzcd} & X \arrow[dl, "{\langle 1,0\rangle}"'] \arrow[dr, "f"] & \\ X \times Z \arrow[rr, dashed, "\varphi"] & & Y \\ & Z \arrow[ul, "{\langle 0,1\rangle}"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute. Following [Bourn, 2002], we will call φ the **cooperator** of f and g.

Note that in the category of groups, a pair of inclusions of subgroups $H \hookrightarrow G$ and $K \hookrightarrow G$ commutes in the above sense if and only if the subgroups commute element-wise in G.

In a regular unital category $\mathbb{C}$, given a diagram as

$$X' \xrightarrow{f'} \!\!\!\!\rightarrow X \xrightarrow{f} Z \xleftarrow{g} Y \twoheadleftarrow\!\!\!\xleftarrow{g'} Y',$$

where f' and g' are regular epimorphisms, $f \circ f'$ and $g \circ g'$ commute if and only if f and g commute (see [Borceux, Bourn, 2004, Proposition 1.6.4]). For this reason, we will only define the Huq commutator for subobjects.

In this paper, for a morphism $t : V \longrightarrow W$ in a regular category, by "regular epimorphism-monomorphism factorization of t" we will mean the factorization of t as a regular epimorphism followed by a monomorphism.

For a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in a normal unital category $\mathbb{C}$ with finite colimits, we will denote by $\kappa_{H,K} : H \diamond K \rightarrowtail H + K$ the kernel of the canonical morphism

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : H + K \longrightarrow H \times K$$

(which is a normal epimorphism in a normal unital category with finite colimits). The object $H \diamond K$ is the co-smash product of H and K.

Huq commutator [Huq, 1986] For a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator $[H,K]_Q$ of $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ is the smallest normal subobject of X for which the composites $q \circ k$ and $q \circ h$ in the diagram

$$\begin{tikzcd}
 & & K \arrow[d, tail, "k"] \\
 & & X \arrow[d, two heads, "q"] \\
H \arrow[r, tail, "h"'] & X \arrow[r, two heads, "q"'] & X/[H,K]_Q
\end{tikzcd}$$

where q is the cokernel of $[H,K]_Q \rightarrowtail X$, commute.

The Huq commutator of a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in a normal unital category $\mathbb{C}$ with finite colimits always exists, and can be constructed as the kernel of q in the diagram

$$\begin{tikzcd}
 & H \arrow[dl, "\langle 1,0 \rangle"'] \arrow[d, dotted] \arrow[dr, tail, "h"] & & \\
H \times K \arrow[r, dotted, "\varphi"] & Q & X \arrow[l, dotted, two heads, "q"'] & {[H,K]_Q} \arrow[l, dotted, "\ker(q)"'] \\
 & K \arrow[ul, "\langle 0,1 \rangle"] \arrow[u, dotted] \arrow[ur, tail, "k"'] & &
\end{tikzcd}$$

where Q is the colimit of the solid arrows (see [Bourn, 2004]). Or equivalently (see e.g [Mantovani, Metere, 2010, Proposition 5.5]) as the kernel of q in the following pushout

$$\begin{tikzcd}
H + K \arrow[r, "{[h,k]}"] \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & X \arrow[d, two heads, "q"] & {[H,K]_Q} \arrow[l, dotted, "\ker(q)"'] \\
H \times K \arrow[r, "\varphi"'] \arrow[ur, phantom, very near end, "\ulcorner"] & Q. &
\end{tikzcd}$$

The morphism q is the universal arrow for which the composites $q \circ h$ and $q \circ k$ commute, and is clearly a normal epimorphism (being the pushout of a normal epimorphism) in a normal unital category with finite colimits.

Higgins commutator [Higgins, 1956] [Mantovani, Metere, 2010] Let $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ be a pair of subobjects of X in a normal unital category $\mathbb{C}$ with finite colimits. The Higgins commutator $[H, K]_H$ of $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ is obtained as the regular image of the kernel $\kappa_{H,K} : H \diamond K \rightarrowtail H + K$ under the morphism $[h, k] : H + K \longrightarrow X$ (see the diagram below)

$$\begin{tikzcd}
H \diamond K \arrow[r, two heads] \arrow[d, tail, "\kappa_{H,K}"'] & {[H,K]_H} \arrow[d, tail] \\
H + K \arrow[r, "{[h,k]}"'] & X.
\end{tikzcd}$$

Ideal-determined categories [G. Janelidze et al., 2010] A category $\mathbb{C}$ with finite colimits is called ideal-determined if it is normal and regular images of kernels under regular epimorphisms are kernels.
For a subobject $h : H \rightarrowtail X$ of X in a normal category $\mathbb{C}$ with finite colimits, the normal closure of h in X is given by the kernel of the cokernel of h. In an ideal-determined unital category, the Huq commutator is the normal closure of the Higgins commutator [Mantovani, Metere, 2010, Proposition 5.7]. We will prove this result in a normal unital category with finite colimits.

Let's first recall the following:

2.1. Lemma. *Let $\mathbb{C}$ be a pointed category with finite colimits. For each commutative diagram as below*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "e"'] & Y \arrow[r, two heads, "\mathrm{coker}(f)"] \arrow[d, "s"'] \arrow[dr, phantom, "(1)"] & Z \arrow[d, "r"] \\
A \arrow[r, "k"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

where e is an epimorphism, the diagram (1) is a pushout if g is the cokernel of k.

2.2. Proposition. *In a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator is the normal closure of the Higgins commutator.*

PROOF. Let $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ be a pair of subobjects of X in a normal unital category $\mathbb{C}$ with finite colimits. Consider the commutative diagram

$$\begin{tikzcd} H \diamond K \arrow[r, tail, "\kappa_{H,K}"] \arrow[d, two heads, "e"'] & H+K \arrow[r, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[d, "{[h,k]}"'] \arrow[dr, phantom, "(1)"] & H \times K \arrow[d, dashed, "\varphi"] \\ {[H,K]_H} \arrow[r, tail, "m"'] & X \arrow[r, two heads, "\mathrm{coker}(m)"'] & Q \end{tikzcd}$$

in which the morphism φ is given by the universal property of cokernels. According to Lemma 2.1 the diagram (1) is a pushout. Therefore the kernel of $\mathrm{coker}(m)$, which, as explained above is the Huq commutator $[H, K]_Q$, is the normal closure of the Higgins commutator $[H, K]_H$. ■

3. Preservation of the Huq commutator by the normal closure of the regular image.

In a unital category $\mathbb{C}$ with finite coproducts it can be seen that a pair of coterminal morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ commutes if and only if there exists a (necessarily unique) morphism φ making the diagram

$$\begin{tikzcd} X+Y \arrow[r, "{[f,g]}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & Z \\ X \times Y \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

commute.

3.1. LEMMA. *For a pair of regular epimorphisms $r : R \twoheadrightarrow X$ and $s : S \twoheadrightarrow Y$ in a regular unital category $\mathbb{C}$ with finite coproducts, the square*

$$\begin{tikzcd} R+S \arrow[r, two heads, "r+s"] \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & X+Y \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \\ R \times S \arrow[r, two heads, "r \times s"'] & X \times Y \end{tikzcd}$$

is a pushout.

Proof. If $u : R \times S \longrightarrow T$ and $v : X + Y \longrightarrow T$ are a pair of morphisms making the outer diagram

$$\begin{tikzcd}
R+S \arrow[r, two heads, "r+s"] \arrow[d, two heads, "{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}"'] & X+Y \arrow[d, two heads, "{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}"] \arrow[ddr, bend left, "{v=[v_1,v_2]}"] & \\
R \times S \arrow[r, two heads, "r \times s"'] \arrow[drr, bend right, "u"'] & X \times Y \arrow[dr, dashed, "\alpha"] & \\
& & T
\end{tikzcd}$$

commute, then by precomposing with the coproduct inclusions into $R+S$, it can be seen that the composites $v_1 \circ r$ and $v_2 \circ s$ commute with cooperator u. Since r and s are regular epimorphisms and the composites $v_1 \circ r$ and $v_2 \circ s$ commute, it follows (see e.g [Borceux, Bourn, 2004, Proposition 1.6.4]) that the morphisms v_1 and v_2 also commute. Therefore, as said above, there is a unique morphism $\alpha : X \times Y \longrightarrow T$ such that

$$\alpha \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [v_1, v_2].$$

Moreover, since the canonical morphism from $R+S$ to $R \times S$ is a (regular) epimorphism it can be seen that the lower triangle commutes, and clearly α is the unique morphism making the two triangles commute. ■

Before we state the main result of this section, let's observe the following:

Given a morphism $f : X \longrightarrow Y$ and a normal epimorphism $q : X \twoheadrightarrow Q$ in a normal category $\mathbb{C}$ with finite colimits, if $t \circ r$ is the regular epimorphism-monomorphism factorization of $f \circ \ker(q)$ where $\ker(q)$ is the kernel of q, then, applying Lemma 2.1 to the commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(q) \arrow[r, tail, "\ker(q)"] \arrow[d, two heads, "r"'] & X \arrow[r, two heads, "q"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & Q \arrow[d, dashed, "\alpha"] \\
f(\mathrm{Ker}(q)) \arrow[r, tail, "t"'] & Y \arrow[r, two heads, "\mathrm{coker}(t)"'] & \mathrm{Coker}(t)
\end{tikzcd}$$

in which the morphism α is given by the universal property of cokernels, it follows that the diagram (1) is a pushout.

3.2. Theorem. *Let $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ be a pair of subobjects of X and $f : X \longrightarrow Y$ a morphism in a normal unital category $\mathbb{C}$ with finite colimits. Then,*

$$[f(H), f(K)]_Q = \overline{f([H,K]_Q)}.$$

Proof. Let $h' \circ u$ and $k' \circ v$ be the regular epimorphism-monomorphism factorizations of $f \circ h$ and $f \circ k$ respectively. Consider the diagram

$$\begin{tikzcd}
 & f(H)+f(K) \arrow[rr, "{[h',k']}"] \arrow[dd, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & & Y \arrow[dd, two heads, "\mathrm{coker}(t)"] \\
H+K \arrow[ur, "u+v"] \arrow[rr, "{[h,k]}"] \arrow[dd, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & & X \arrow[ur, "f"] \arrow[dd, two heads, "q"'] & \\
 & f(H)\times f(K) \arrow[rr, dashed, "\beta"'] & & \mathrm{Coker}(t) \\
H\times K \arrow[ur, "u\times v"] \arrow[rr, "\varphi"'] & & Q \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

in which the front face is a pushout. As explained above, the right hand face of the cube in which t is the regular image of $[H,K]_Q \rightarrowtail X$ (the kernel of q) under f and α is the unique morphism given by the universal property of cokernels, is a pushout. Furthermore, since the left hand face of the cube is a pushout by Lemma 3.1, and the whole diagram commutes, there exists a unique morphism β making the back face of the cube commute. The composite of a pushout with another is a pushout, therefore the outer rectangle of the diagram

$$\begin{tikzcd}
H+K \arrow[r, two heads, "u+v"] \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & f(H)+f(K) \arrow[r, "{[h',k']}"] \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & Y \arrow[d, two heads, "\mathrm{coker}(t)"] \\
H\times K \arrow[r, two heads, "u\times v"'] & f(H)\times f(K) \arrow[r, "\beta"'] & \mathrm{Coker}(t)
\end{tikzcd}$$

is a pushout, and hence the square

$$\begin{tikzcd}
f(H)+f(K) \arrow[r, "{[h',k']}"] \arrow[d, two heads, "\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]"'] & Y \arrow[d, two heads, "\mathrm{coker}(t)"] \\
f(H)\times f(K) \arrow[r, "\beta"'] & \mathrm{Coker}(t)
\end{tikzcd}$$

is a pushout. Since the subobject $\overline{f([H,K]_Q)} \rightarrowtail Y$ is the kernel of coker(t), it follows that $[f(H),f(K)]_Q = \overline{f([H,K]_Q)}$. ■

Recall that each morphism $f : X \longrightarrow Y$ in a regular category $\mathbb{C}$ induces a Galois connection between the posets of subobjects $Sub(X)$ and $Sub(Y)$ of X and Y respectively, i.e. a pair of maps $f_* : Sub(X) \longrightarrow Sub(Y)$ and $f^* : Sub(Y) \longrightarrow Sub(X)$ defined (respectively) by taking regular images of subobjects of X under f and pulling back subobjects of Y along f, such that $f_*f^*(L) \le L$ and $H \le f^*f_*(H)$ for all subobjects $h : H \rightarrowtail X$ and $l : L \rightarrowtail Y$ of X and Y respectively.

3.3. Lemma. *For a subobject $h : H \rightarrowtail X$ of X and a morphism $f : X \longrightarrow Y$ in a normal category $\mathbb{C}$ with finite colimits, one has*

$$\overline{f(H)} = \overline{f(\overline{H})}.$$

Proof. It is clear that $\overline{f(H)} \le \overline{f(\overline{H})}$. On the other hand, $\overline{f(\overline{H})} \le \overline{f(H)}$ if $f(\overline{H}) \le \overline{f(H)}$ or equivalently $f_*(\overline{H}) \le \overline{f_*(H)}$, since $\overline{f(H)}$ is a normal subobject of Y and $\overline{f(\overline{H})}$ is the smallest normal subobject of Y such that $f(\overline{H}) \le \overline{f(\overline{H})}$. By the Galois connection above, $f_*(H) \le \overline{f_*(H)}$ implies $H \le f^*(\overline{f_*(H)})$. Hence, $\overline{H} \le f^*(\overline{f_*(H)})$ since $f^*(\overline{f_*(H)})$ is normal in X, and thus $f_*(\overline{H}) \le \overline{f_*(H)}$ follows. ■

For a morphism $f : X \longrightarrow Y$ and a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in a normal unital category $\mathbb{C}$ with finite colimits, since the Huq commutator is the normal closure of the Higgins commutator, by the previous lemma it immediately follows that $\overline{f([H,K]_H)} = \overline{f([H,K]_Q)}$. We further observe that

$$\begin{aligned}
[f(H),f(K)]_H &\le \overline{[f(H),f(K)]_H} \\
&= [f(H),f(K)]_Q \\
&= \overline{f([H,K]_Q)} \\
&= \overline{f(\overline{[H,K]_H})} \\
&= \overline{f([H,K]_H)}.
\end{aligned} \tag{1}$$

The following result seems to be known and can be deduced from [Mantovani, Metere, 2010, Lemma 5.11], however we insert a proof to make the paper more self-contained.

3.4. Lemma. *For a pair of subobjects* $h : H \rightarrowtail X$ *and* $k : K \rightarrowtail X$ *of* X *and a morphism* $f : X \longrightarrow Y$ *in an ideal-determined unital category* $\mathbb{C}$, $f([H,K]_H) = [f(H), f(K)]_H$.

Proof. Let $h' \circ u$ and $k' \circ v$ be the regular epimorphism-monomorphism factorizations of $f \circ h$ and $f \circ k$ respectively. Consider the diagram

$$\begin{tikzcd} H \diamond K \arrow[r, tail, "\kappa_{H,K}"] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "(i)"] & H + K \arrow[r, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[d, two heads, "u+v"] \arrow[dr, phantom, "(ii)"] & H \times K \arrow[d, two heads, "u \times v"] \\ f(H) \diamond f(K) \arrow[r, tail, "m"'] & f(H) + f(K) \arrow[r, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & f(H) \times f(K) \end{tikzcd}$$

where $m \circ e$ is the regular epimorphism-monomorphism factorization of $(u + v) \circ \kappa_{H,K}$. Using Lemma 3.1 the square (ii) is a pushout. Therefore, since $u + v$ is a regular epimorphism and $\mathbb{C}$ is an ideal-determined unital category, it follows (see for instance [Mantovani, Metere, 2010, Lemma 4.1]) that the regular image m of the kernel $\kappa_{H,K}$ under $u + v$ is the kernel of the canonical morphism from $f(H) + f(K)$ to $f(H) \times f(K)$. Hence $f([H,K]_H) = [f(H), f(K)]_H$ follows by the uniqueness of regular images. ∎

In the next remark we will see that Theorem 3.2 has a much simpler proof in an ideal-determined unital category.

3.5. Remark. From the previous lemma, $f([H,K]_H) = [f(H), f(K)]_H$ for a morphism $f : X \longrightarrow Y$ and a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in an ideal-determined unital category $\mathbb{C}$. Now since the Huq commutator is the normal closure of the Higgins commutator and $\overline{f([H,K]_H)} = \overline{f(\overline{[H,K]}_H)}$, then

$$\begin{aligned} [f(H), f(K)]_Q &= \overline{[f(H), f(K)]}_H \\ &= \overline{f([H,K]_H)} \\ &= \overline{f(\overline{[H,K]}_H)} \\ &= \overline{f([H,K]_Q)}. \end{aligned} \tag{2}$$

4. A characterization of the Huq commutator.

Let $\mathbb{C}$ be a normal unital category with finite colimits. For a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in $\mathbb{C}$, it is easy to check that $[H,K]_Q \leq \overline{H} \wedge \overline{K}$. Furthermore, for a morphism $f : X \longrightarrow Y$ in $\mathbb{C}$, $\overline{f([H,K]_Q)} = [f(H), f(K)]_Q$ (see Theorem 3.2); in particular, $[f(H), f(K)]_Q \leq \overline{f([H,K]_Q)}$. One can also check that the Huq commutator satisfies the monotonicity condition i.e. if $H' \leq H$ and $K' \leq K$ for subobjects $h' : H' \rightarrowtail X$, $h : H \rightarrowtail X$, $k' : K' \rightarrowtail X$ and $k : K \rightarrowtail X$ of X, then $[H', K']_Q \leq [H,K]_Q$. In this section we will characterize the Huq commutator as the largest commutator defined for subobjects in a normal unital category $\mathbb{C}$ with finite colimits, satisfying the monotonicity condition as well as the following two conditions:

- A1 $[H,K] \leq \overline{H} \wedge \overline{K}$;
- A2 $[f(H), f(K)] \leq \overline{f([H,K])}$.

In contrast with the Huq commutator, we will provide an example of a commutator of subobjects in a normal unital category $\mathbb{C}$ with finite colimits satisfying all the conditions above, in which the condition $A2$ does not necessarily imply $[f(H), f(K)] = \overline{f([H,K])}$.

4.1. Example. The Higgins commutator is monotone and satisfies $A1$. Moreover, as explained in equation (1) above, the Higgins commutator satisfies $A2$ in a normal unital category with finite colimits. Since in general the Higgins commutator is not always normal (see e.g [Cigoli, 2009, Example 5.3.9]), then $[f(H), f(K)]_H \leq \overline{f([H,K]_H)}$ does not necessarily imply $[f(H), f(K)]_H = \overline{f([H,K]_H)}$. For example if f is the inclusion of the alternating group A_5 into S_5 then for the two subgroups $H = \langle (1\ 2)(3\ 4) \rangle$ and $K = \langle (1\ 2)(4\ 5) \rangle$ of A_5, we have $[f(H), f(K)]_H = \langle (345) \rangle$ and $\overline{f([H,K]_H)} = A_5$.

For a pair of objects H and K in a normal unital category $\mathbb{C}$ with finite colimits, we will denote the Huq commutator of the coproduct inclusions $i_1 : H \longrightarrow H + K$ and $i_2 : K \longrightarrow H + K$ by $[i_1(H), i_2(K)]_Q$. The morphisms $\ker[1,0] : H \flat K \rightarrowtail H + K$ and $\ker[0,1] : K \flat H \rightarrowtail H + K$ will be the kernels of the induced morphisms $[1,0] : H + K \longrightarrow H$ and $[0,1] : H + K \longrightarrow K$ respectively. It is easy to check that the co-smash product $H \diamond K$ of H and K is the meet of $K \flat H$ and $H \flat K$ (see for instance [Mantovani, Metere, 2010]). In the next lemma we will observe that the co-smash product of H and K coincides with the Huq commutator of the coproduct inclusions $i_1 : H \longrightarrow H + K$ and $i_2 : K \longrightarrow H + K$.

4.2. Lemma. *Let H and K be objects in a normal unital category $\mathbb{C}$ with finite colimits. Then $H \diamond K = [i_1(H), i_2(K)]_Q$.*

Proof. Since the square

$$\begin{tikzcd}
H+K \arrow[r, equal] \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & H+K \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \\
H \times K \arrow[r, equal] & H \times K
\end{tikzcd}$$

is a pushout, it follows that the kernel $\kappa_{H,K} : H \diamond K \rightarrowtail H + K$ of the canonical morphism from $H+K$ to $H \times K$, is the Huq commutator of the coproduct inclusions $i_1 : H \longrightarrow H+K$ and $i_2 : K \longrightarrow H + K$. ∎

4.3. Lemma. *Let $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ be a pair of subobjects of X in a normal unital category $\mathbb{C}$ with finite colimits. Then*

$$[K\flat H, H\flat K]_Q = [i_1(H), i_2(K)]_Q.$$

Proof. We see that $[K\flat H, H\flat K]_Q \leq (K\flat H) \wedge (H\flat K) = H \diamond K = [i_1(H), i_2(K)]_Q$. On the other hand, since $H \leq K\flat H$, $K \leq H\flat K$ and the Huq commutator is monotone, it follows that $[i_1(H), i_2(K)]_Q \leq [K\flat H, H\flat K]_Q$. Hence, $[K\flat H, H\flat K]_Q = [i_1(H), i_2(K)]_Q$. ∎

4.4. Proposition. *For each pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in a normal unital category $\mathbb{C}$ with finite colimits,*

$$[[h,k](K\flat H), [h,k](H\flat K)]_Q = [H, K]_Q.$$

Proof. Applying Theorem 3.2 and then the previous lemma, we get

$$\begin{aligned}
[[h,k](K\flat H), [h,k](H\flat K)]_Q &= \overline{[h,k]([K\flat H, H\flat K]_Q)} \\
&= \overline{[h,k]([i_1(H), i_2(K)]_Q)} \\
&= [H, K]_Q.
\end{aligned} \tag{3}$$

Note that the last equality holds since $[h,k]([i_1(H), i_2(K)]_Q)$ is the Higgins commutator $[H,K]_H$, and in a normal unital category with finite colimits the Huq commutator is the normal closure of the Higgins commutator (see Proposition 2.2). ∎

For a pair of subobjects $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ of X in a normal category $\mathbb{C}$ with finite coproducts, since in particular $H \leq K\flat H$ in $H + K$, we see that the diagram

$$\begin{tikzcd}
 & K\flat H \arrow[r, two heads] \arrow[d, tail] & {[h,k](K\flat H)} \arrow[d, tail] \\
H \arrow[ur] \arrow[r, "i_1"] \arrow[rr, tail, bend right, "h"'] & H + K \arrow[r, "{[h,k]}"] & X
\end{tikzcd}$$

commutes and this implies that $H \leq [h,k](K\flat H)$ in X. Similarly, $K \leq [h,k](H\flat K)$ in X.

Now we are ready to state our main result.

4.5. Theorem. *In a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator is the largest monotone commutator defined for subobjects satisfying the conditions $A1$ and $A2$.*

Proof. Suppose there is a monotone commutator $[|-,-|]$ defined for subobjects, satisfying the conditions $A1$ and $A2$. Let $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$ be a pair of subobjects of X. Since $[|K\flat H, H\flat K|] \leq K\flat H \wedge H\flat K$ and $K\flat H \wedge H\flat K = H \diamond K = [K\flat H, H\flat K]_Q$, then $\overline{[h,k]([|K\flat H, H\flat K|])} \leq \overline{[h,k]([K\flat H, H\flat K]_Q)}$. Since $H \leq [h,k](K\flat H)$ and $K \leq [h,k](H\flat K)$, applying the monotonicity condition, the condition $A2$, Theorem 3.2 and Proposition 4.4, we get

$$\begin{aligned} [|H,K|] &\leq [|[h,k](K\flat H), [h,k](H\flat K)|] \\ &\leq \overline{[h,k]([|K\flat H, H\flat K|])} \\ &\leq \overline{[h,k]([K\flat H, H\flat K]_Q)} \\ &= [[h,k](K\flat H), [h,k](H\flat K)]_Q \\ &= [H,K]_Q. \end{aligned} \tag{4}$$

■

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