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Applications of the Efros Theorem to the Wright Functions of the Second Kind and Other Results

Alexander Apelblat[a] and Francesco Mainardi [b*]

[a]*Dept of Chemical Engineering, Ben Gurion University of the Negev, 84105 Beer Sheva, Israel E-mail:apelblat@bgu.ac.il*

[b]Dept of Physics and Astronomy, University of Bologna, Via Irnerio 46, 40126 Bologna, Italy E-mail:francesco.mainardi@bo.infn.it

Using a special case of the Efros theorem and operational calculus it was possible to derive many infinite integrals, finite integrals and integral identities for the Wright functions of the second kind. The integral identities derived as inverse Laplace transforms are mainly in terms of convolution integrals with the Mittag-Leffler functions. The integrands of determined integrals include elementary functions (power, exponential, logarithmic, trigonometric and hyperbolic functions) and the error functions, the Mittag-Leffler functions and the Volterra functions. Special attention was devoted to integrands with the modified Bessel function of the second kind and order one-third

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1. Introduction

In 1933 [1] and in 1940 [2] Sir Edward Maitland Wright (1906 - 2005) introduced to the mathematical literature new special functions which initially were regarded as some kind of generalization of the Bessel functions. Later on, these functions, which are named after him, started to be independent special functions and to play an important role in solution of the linear partial fractional differential equations. These differential equations describe a wide range of important physical phenomena that occur in condensed and soft matter physics, geophysics, meteorology, in fractional kinetics and diffusion processes, in statistical mechanics, in socio-economical models and in many other problems that take place in natural sciences and in engineering disciplines (see for example [3-17]).

The Wright functions $W_{\lambda,\mu}(z)$ are defined in the complex plane $\mathbb{C}$ with parameters λ and μ by the power-series:

$$W_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\ \Gamma(\lambda k + \mu)}. \tag{1}$$

*Corresponding author. Email: francesco.mainardi@bo.infn.it

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They turn out to be entire functions in $z \in \mathbb{C}$ of order $1/(1+\lambda)$ for $\lambda > -1$ and for any complex μ (here we take μ real, always $\mu \geq 0$). Thus we note that the Wright functions are of exponential order only if $\lambda \geq 0$. The case $\lambda = 0$ is trivial wince $W_{0,\mu} = \mathrm{e}^z/\Gamma(\mu)$.

Following Mainardi (see i.e. the appendix F of his 2010 book [12]), taking into account the different order, we distinguish the Wright functions of the *first kind* for $\lambda \geq 0$, and of the *second kind* for $-1 < \lambda < 0$. This distinction between the two kinds is justified for the differences arising in the asymptotic representation in the complex domain and in the expression of the corresponding Laplace transforms, as outlined by Gorenflo, Luchko and Mainardi in [8]. In this paper we restrict our attention to the Wright functions of the second kind, following the notation (formerly introduced by Mainardi) of denoting by ν the positive parameter $-\lambda$. Then for our Wright functions of the second kind we take the following definition in terms of power series

$$W_{-\nu,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(-\nu k + \mu)}, \quad 0 < \nu < 1, \quad \mu \geq 0\,. \tag{2}$$

We find worthwhile to add the main integral representation in the complex plane of these functions that reads

$$W_{-\nu,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\zeta + z\zeta^{\nu}} \frac{d\zeta}{\zeta^{\mu}}, \quad 0 < \nu < 1, \quad \mu \geq 0, \tag{3}$$

where Ha denotes the Hankel path, that is a loop which starts from $-\infty$ along the lower side of negative real axis, encircles with a small circle the axes origin and ends at $-\infty$ along the upper side of the negative real axis.

Concerning the Laplace transforms of the Wright functions of the second kind we point out the following results recently recalled in the survey by Mainardi and Consiglio [17].
For $z = -x/t^{\nu}$ with $x > 0$, $t > 0$

$$\mathcal{L}\left\{t^{\mu-1}\, W_{-\nu,\mu}(-x/t^{\nu}); t \to s\right\} = s^{-\mu} \mathrm{e}^{-x\, s^{\nu}}, \quad 0 < \nu < 1, \quad \mu \geq 0, \tag{4}$$

and for $z = t > 0$

$$\mathcal{L}\left\{W_{-\nu,\mu}(-t); t \to s\right\} = E_{\nu,\mu+\nu}\,(-s) \quad 0 < \nu < 1, \quad \mu \geq 0. \tag{5}$$

Above we have introduced the Mittag-Leffler function in two parameters $\alpha > 0$, $\beta \in \mathbb{C}$ defined by its convergent series for all $z \in \mathbb{C}$

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{6}$$

on which we will return later in Sections 2, 3. We note that in Eqs. (4) and (5) the arguments are negative so the series of the corresponding Wright functions are with sing-alternating terms.

In the class of the Wright function of the second kind particular mention is attributed to the two functions obtained when $\mu = 0$ and $\mu = 1-\nu$ for their relevance in time fractional fractional diffusion wave equations. Indeed they are frequently discussed in the literature of fractional calculus, see i.e. [6-17] and referred to as the *Mainardi auxiliary functions*. Their power series representations (depending on the single parameter ν) read

$$F_\nu(z) := W_{-\nu,0}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(-\nu k)}, \quad 0 < \nu < 1, \tag{7}$$

and

$$M_\nu(z) := W_{-\nu,1-\nu}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(-\nu(k+1)+1}, \quad 0 < \nu < 1. \tag{8}$$

The Mainardi auxiliary functions are interrelated through the following relation:

$$F_\nu(z) = \nu z M_\nu(z), \quad 0 < \nu < 1. \tag{9}$$

Using the *reflection formula* for the Gamma function

$$\Gamma(\zeta)\,\Gamma(1-\zeta) = \pi/\sin \pi\zeta,$$

we can re-write Eqs. (7) and (8) in an alternative and more convenient form:

$$F_\nu(z) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \Gamma(\nu k + 1) \sin(\pi \nu k), \tag{10}$$

and

$$M_\nu(z) := \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-z)^{k-1}}{(n-1)!} \Gamma(\nu k) \sin(\pi \nu k). \tag{11}$$

From now on we restrict our main attention to the Mainardi auxiliary functions. For them the Laplace transforms corresponding to (4) (5) take the relevant form: (i) for the functions F_ν

$$\mathcal{L}\left\{t^{-1}\, F_\nu(x/t^\nu); t \to s\right\} = \mathrm{e}^{-x\, s^\nu}, \quad 0 < \nu < 1, \tag{12}$$

$$\mathcal{L}\left\{F_\nu(t); t \to s\right\} = E_{\nu,\nu}\left(-s\right) \quad 0 < \nu < 1, \tag{13}$$

(ii) for the functions M_ν

$$\mathcal{L}\left\{t^{-\nu}\, M_\nu(x/t^\nu); t \to s\right\} = s^{\nu-1}\mathrm{e}^{-x\, s^\nu}, \quad 0 < \nu < 1, \tag{14}$$

$$\mathcal{L}\left\{M_\nu(t); t \to s\right\} = E_{\nu,1}\left(-s\right), \quad 0 < \nu < 1. \tag{15}$$

In particular, the interest in these functions comes from the fact that their Laplace transforms are in the form of exponential functions of the type $s^{-\mu}\exp(-s^{\nu})$ with $\mu \geq 0$ and $0 < \nu < 1$, whose inversion intrigued in the middle of previous century a number of well-known mathematicians. It is worthwhile to mention yearly classical papers of Humbert [18] since 1945, Pollard [19] since 1946, Wlodarski [20] since 1953, Mikusinski [21-24] in the 1951-1959 period, Wintner [25] since 1956, Ragab [26] since 1958 and finally Stankovič [27] since 1970. It was Stankovič who recognized the role of the Wright functions with first parameter negative, that we have denoted as Wright functions of the second kind.

For the Mainardi auxiliary functions we refer the reader again to [14]: in the time domain the inverse Laplace transforms were derived in the form of infinite series representations and read

$$\begin{aligned} F_\nu(t) &= \sum_{k=1}^{\infty} \frac{(-t)^k}{k!\,\Gamma(-\nu k)} = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}t^k}{k!}\,\Gamma(\nu k+1)\,\sin(\pi\nu k), \\ M_\nu(t) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!\,\Gamma\,(-\nu(k+1)+1)} = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-t)^{k-1}}{(k-1)!}\,\Gamma(\nu k)\,\sin(\pi\nu k), \\ F_\nu(t) &= \nu t M_\nu(t). \end{aligned} \tag{16}$$

Here particular attention is devoted to the Laplace inversion of $\exp(-s^{\nu})$ with $0 < \nu < 1$ because it provides the unilateral (i.e active only for $t > 0$) stable density of order ν in the probability theory of Lévy stable distributions. We denote by $f_\nu(t)$ this function for which Pollard [19] and Mikusinski [24] gave the following integral representations

$$\begin{aligned} f_\nu(t) &= \frac{1}{\pi}\int_0^\infty e^{-ut}e^{-u^\nu\cos(\pi\nu)}\sin[u^\nu\sin(\pi\nu)]\,du \\ &= \frac{2}{\pi}\int_0^\infty e^{-u^\nu\cos(\pi\nu/2)}[\cos[u^\nu\cos(\pi\nu/2)]\,\cos(ut)\,du\,. \end{aligned} \tag{17}$$

Mikusinski [24] was able to present $f_\nu(t)$ also as the finite trigonometric integral

$$\begin{aligned} f_\nu(t) &= \frac{\nu}{\pi(1-\nu)t}\int_0^\pi \xi e^{-\xi}du, \quad \text{where} \\ \xi &= \frac{1}{t^{\nu/(1-\nu)}}\left(\frac{\sin(\nu u)}{\sin u}\right)^{\nu/(1-\nu)}\frac{\sin[(1-\nu)u]}{\sin u}\,. \end{aligned} \tag{18}$$

In particular cases when $\nu = 1/2$ the inverse was established by using exponential and parabolic cylinder functions [14,26] and for $\nu = 1/3$ and $\nu = 2/3$, in terms of the Airy functions and their first derivatives [11,14]. For $\nu = 1/4$, the solution was deduced as a sum of three generalized hypergeometric functions, but it is still uncertain [15]. For rational $\nu = k/m, k < m$, the Laplace inverse of (17) can be presented in terms of the Meijer's G and the Fox H functions [15] and for $\nu = 1/m$ using MacRobert's E function [26].

The inverse operation which restores the original function in the Laplace transformation is usually the most complicated step in operational calculus because it involves integration in the complex plane. However, in many cases finding of inverse transforms is facilitated if the Laplace transform is a product of two or more transforms providing that the inverse transforms of them are known. This so-called product (convolution) theorems for product of two transforms can be written as (see e.g. [28])

$$\begin{aligned}
&\mathcal{L}\{f(t)\} = F(s), \\
&F(s) = F_1(s)\cdot F_2(s), \\
&f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}\ ; \quad f_2(t) = \mathcal{L}^{-1}\{F_2(s)\}, \\
&f(t) = f_1(t) \star f_2(t) = \int_0^t f_1(u) f_2(t-u) du = \int_0^t f_1(t-u) f_2(u) du.
\end{aligned} \tag{19}$$

In 1935 the product theorem was generalized by Efros in the following form [29]

$$\begin{aligned}
&\mathcal{L}\{f(t)\} = F(s). \\
&G(s,\lambda) = \mathcal{L}\{g(t,\lambda)\} = H(s)\, \mathrm{e}^{-\lambda q(s)}, \\
&\mathcal{L}^{-1}\{H(s)F[q(s)]\} = \int_0^\infty f(u) g(u,\lambda)\, du.
\end{aligned} \tag{20}$$

where $H(s)$ and $q(s)$ are analytic functions. If $q(s) = s$ it is possible to show that (20) reduces to (19). Wlodarski [20] in 1953 discussed the special case of the Efros theorem when $H(s)$ is $1/s^\mu$ with $\mu \geq 0$ and $q(s) = s^\nu$ with $0 < \nu < 1$. As is demonstrated in this paper, the Wlodarski formula can be applied to the Mainardi auxiliary functions M_ν and F_ν. Indeed, letting $\mu = 1 - \nu$, we obtain

$$\begin{aligned}
&\mathcal{L}\{g(t)\} = G(s)\,, \\
&\mathcal{L}^{-1}\left\{\frac{G(s^\nu)}{s^{1-\nu}}\right\} = \int_0^\infty g(u)\, \mathcal{L}^{-1}\left\{\frac{\exp\left(-u s^\nu\right)}{s^{1-\nu}}\right\} du\,; \quad 0 < \nu < 1\,.
\end{aligned} \tag{21}$$

From [12] we recall for $0 < \nu < 1$ and $\lambda > 0$,

$$\mathcal{L}\left\{\frac{1}{\nu} F_\nu\left(\frac{\lambda}{t^\nu}\right)\right\} = \mathcal{L}\left\{\frac{\lambda}{t^\nu}\, M_\nu\left(\frac{\lambda}{t^\nu}\right)\right\} = \frac{\lambda}{s^{1-\nu}}\, \exp(-\lambda\, s^\nu), \tag{22}$$

so that, by using (21) and (22), the following expressions can be derived for the auxiliary function $F_\nu(t)$

$$\nu \mathcal{L}^{-1}\left\{\frac{G(s^\nu)}{s^{1-\nu}}\right\} = \int_0^\infty g(u)\, F_\nu\left(\frac{u}{t^\nu}\right) \frac{du}{u}\,; \quad 0 < \nu < 1 \tag{23}$$

and similarly for the function $M_\nu(t)$ we have

$$t^\nu \mathcal{L}^{-1}\left\{\frac{G(s^\nu)}{s^{1-\nu}}\right\} = \int_0^\infty g(u)\, M_\nu\left(\frac{u}{t^\nu}\right) du\,; \quad 0 < \nu < 1. \tag{24}$$

Thus, if left hand side Laplace transforms in (23) and (24) can be inverted, the infinite integrals with the functions $F_\nu(t)$ and $M_\nu(t)$ can be evaluated.

The plane of the present paper is herewith illustrated. In the following four sections we devote our attention to the evaluation of infinite and convolution integrals by using expressions in (23) and (24) and operational rules of the Laplace transformation.

In section 2 we present available results for elementary functions $g(t)$ for any $\nu \in (0,1)$.

In sections 3,4 we report infinite or convolution integrals associated with the Mittag-Leffler functions, error function and with the Volterra functions, respectively.

The section 5 is dedicated to the particular case of $\nu = 1/3$, to infinite integrals with integrands having the modified Bessel functions of the second kind and order one-third.

In order to clearly illustrate the application of the Efros theorem (in the Wlodarski form given in (21)) in evaluation of infinite and convolution integrals, direct and inverse Laplace transforms are always presented in considered derivations. It should be also taken into account that all mathematical operations and manipulations with elementary and special functions, integrals and transforms are formal. Therefore, the validity of derived results is assured by considering the restrictions usually imposed on the Laplace transforms.

Finally, in section 6, we provide some concluding remarks.

2. Integrals of the Wright functions of the second kind with elementary functions

In this section we will consider a number of examples with elementary functions. We outline that all the Laplace transforms were taken from well known tables, see i.e. [30-33], always letting $0 < \nu < 1$ and $\lambda > 0$ even when not explicitly stated.

As the first example we consider the power function t^λ with $\lambda > 0$ for which

$$\begin{aligned} &g(t) = t^\lambda; \quad \lambda > 0, \\ &G(s) = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}; \quad G(s^\nu) = \frac{\Gamma(\lambda+1)}{s^{(\lambda+1)\nu}} \\ &\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{\Gamma(\lambda+1)}{s^{(\lambda+1)\nu}}\right\} = \mathcal{L}^{-1}\left\{\frac{\Gamma(\lambda+1)}{s^{\lambda\nu+1}}\right\} = \frac{\Gamma(\lambda+1)\,t^{\lambda\nu}}{\Gamma(\lambda\nu+1)}. \end{aligned} \tag{25}$$

Then, from (23), we have

$$\int_0^\infty u^{\lambda-1} F_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(\lambda)\,t^{\lambda\nu}}{\Gamma(\lambda\nu)}; \quad 0<\nu<1; \quad \lambda>0, \tag{26}$$

and from (24)

$$\int_0^\infty u^{\lambda} M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(\lambda)\,t^{(\lambda+1)\nu}}{\nu\,\Gamma(\lambda\nu)}; \quad 0<\nu<1, \quad \lambda>0. \tag{27}$$

For particular values of λ we get for $0 < \nu < 1$ and $n = 1, 2, 3, \dots$

$$\begin{aligned}
&\int_0^\infty F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u} = \nu, \quad \int_0^\infty F_\nu\left(\frac{u}{t^\nu}\right) du = \frac{t^\nu}{\Gamma(\nu)},\\
&\int_0^\infty u^{2n-1} F_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(2n) t^{2n\nu}}{\Gamma(2n\nu)},\\
&\int_0^\infty u^{2n} F_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(2n+1) t^{(2n+1)\nu}}{\Gamma[(2n+1)\nu]};
\end{aligned} \tag{28}$$

and similarly from (27)

$$\begin{aligned}
&\int_0^\infty M_\nu\left(\frac{u}{t^\nu}\right) du = t^\nu, \quad \int_0^\infty u\, M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{t^{2\nu}}{\Gamma(\nu+1)},\\
&\int_0^\infty u^{2n} M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(2n) t^{(2n+1)\nu}}{\nu\,\Gamma(2n\nu)},\\
&\int_0^\infty u^{2n+1} M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(2n+1)\, t^{2(n+1)\nu}}{\nu\,\Gamma[(2n+1)\nu]}.
\end{aligned} \tag{29}$$

If both sides of (26) are differentiated with respect to parameter λ we have

$$\int_0^\infty u^{\lambda-1} \ln u\, F_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(\lambda)\, t^{\lambda\nu}\, \{\psi(\lambda) + \nu\,[(\ln t - \psi(\lambda\nu)]\}}{\Gamma(\lambda\nu)}, \tag{30}$$

and similarly from (16) and (27)

$$\int_0^\infty u^{\lambda-1} \ln u\, F_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\Gamma(\lambda)\, t^{(\lambda+1)\nu}\, \{\psi(\lambda) + \nu\,[(\ln t - \psi(\lambda\nu)]\}}{\nu\Gamma(\lambda\nu)}. \tag{31}$$

The above results can be extended to functions $g(t)$ which are defined at finite intervals and to step functions jumping at integral values of variable t. Let us start with the simplest case

$$\begin{aligned}
&g(t) = \begin{cases} 1, & 0 < t < \lambda,\\ 0, & t > \lambda,\end{cases}\\
&G(s) = \frac{1 - \mathrm{e}^{-\lambda s}}{s}, \quad G(s^\nu) = \frac{1 - \mathrm{e}^{-\lambda s^\nu}}{s^\nu},\\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{1 - \mathrm{e}^{-\lambda s^\nu}}{s^\nu}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\mathrm{e}^{-\lambda s^\nu}}{s}\right\}
\end{aligned} \tag{32}$$

but, in view of (4) with $x = 1$,

$$\mathcal{L}\left\{W_{-\nu,1}\left(-\frac{\lambda}{t^\nu}\right)\right\} = \frac{\mathrm{e}^{-\lambda s^\nu}}{s}, \tag{33}$$

and therefore

$$\begin{aligned} &\int_0^\lambda F_\nu\left(\frac{u}{t^\nu}\right)\,\frac{du}{u} = \nu\,\left\{1 - W_{-\nu,1}\left(-\frac{\lambda}{t^\nu}\right)\right\}\,,\\ &\int_0^\lambda M_\nu\left(\frac{u}{t^\nu}\right)\,du = t^\nu\,\left\{1 - W_{-\nu,1}\left(-\frac{\lambda}{t^\nu}\right)\right\}. \end{aligned} \tag{34}$$

In next examples will appear convolution integrals.
From

$$\begin{aligned} &g(t) = \begin{cases} 1-t, & 0 < t < 1,\\ 0, & t > 1, \end{cases}\\ &G(s) = \frac{e^{-s} + s - 1}{s^2}, \quad G(s^\nu) = \frac{e^{-s^\nu} + s^\nu - 1}{s^{2\nu}};\\ &\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{e^{-s^\nu} + s^\nu - 1}{s^{2\nu}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^{\nu+1}} + \frac{1}{s^{\nu+1}}\,\frac{e^{-s^\nu}}{s}\right\} =\\ &1 - \frac{t^\nu}{\Gamma(\nu+1)} + \frac{t^{\nu-1}}{\Gamma(\nu)} \star W_{-\nu,1}\left(-\frac{1}{t^\nu}\right) \end{aligned} \tag{35}$$

we deduce

$$\begin{aligned} &\int_0^1 (1-u)F_\nu\left(\frac{u}{t^\nu}\right)\,\frac{du}{u} = \nu\,\left\{1 - \frac{t^\nu}{\Gamma(\nu+1)} + \frac{t^{\nu-1}}{\Gamma(\nu)} \star W_{-\nu,1}(-\frac{1}{t^\nu})\right\}\,,\\ &\int_0^1 (1-u)M_\nu\left(\frac{u}{t^\nu}\right)\,du = t^\nu\,\left\{1 - \frac{t^\nu}{\Gamma(\nu+1)} + \frac{t^{\nu-1}}{\Gamma(\nu)} \star W_{-\nu,1}(-\frac{1}{t^\nu}).\right\} \end{aligned} \tag{36}$$

From

$$\begin{aligned} &g(t) = \begin{cases} 0\,, & 0 < t < \lambda,\\ \dfrac{(t-\lambda)^\mu}{\Gamma(\mu+1)}\,, & t > \lambda, \end{cases}\\ &G(s) = \frac{e^{-\lambda s}}{s^{\mu+1}}\,; \quad G(s^\nu) = \frac{e^{-\lambda s^\nu}}{s^{(\mu+1)\nu}}\\ &\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{e^{-\lambda s^\nu}}{s^{(\mu+1)\nu}}\right\} = \mathcal{L}^{-1}\,\left\{\frac{1}{s^{\mu\nu}}\,\frac{e^{-\lambda s^\nu}}{s}\right\} = \frac{t^{\mu\nu-1}}{\Gamma(\mu\nu)} \star W_{-\nu,1}(-\frac{\lambda}{t^\nu}) \end{aligned} \tag{37}$$

it is a possible to derive the following integral identities with $\mu > 0$ and $0 < \nu < 1$,

$$\begin{aligned} &\textstyle\int_\lambda^\infty (t-\lambda)^\mu F_\nu\left(\frac{u}{t^\nu}\right)\,\frac{du}{u} = \frac{\nu\Gamma(\mu+1)}{\Gamma(\mu\nu)}\,t^{\mu\nu-1} \star W_{-\nu,1}\left(-\frac{\lambda}{t^\nu}\right)\,,\\ &\textstyle\int_\lambda^\infty (t-\lambda)^\mu M_\nu\left(\frac{u}{t^\nu}\right)\,du = \frac{\Gamma(\mu+1)}{\Gamma(\mu\nu)}\,t^{(\mu+1)\nu-1} \star W_{-\nu,1}\left(-\frac{\lambda}{t^\nu}\right). \end{aligned} \tag{38}$$

In the next two examples the cases of exponential functions are illustrated.

$$\begin{aligned} &g(t) = e^{-\lambda t}\,,\\ &G(s) = \frac{1}{s+\lambda}\,, \quad G(s^\nu) = \frac{1}{s^\nu+\lambda},\\ &\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{1}{s^\nu+\lambda}\right\} = \mathcal{L}^{-1}\left\{\frac{s^{\nu-1}}{s^\nu+\lambda}\right\}. \end{aligned} \tag{39}$$

Now, taking into account the Mittag-Leffler function, see e.g. [14,34,35] we recall

$$\mathcal{L}\left\{E_\nu(-\lambda t^\nu)\right\} = \frac{s^{\nu-1}}{s^\nu+\lambda}\,, \quad E_\nu(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\nu+1)}\,. \tag{40}$$

Thus, as expected, the Laplace transforms of the Mainardi functions are expressed in terms of the Mittag-Leffler functions,

$$\int_0^\infty \mathrm{e}^{-\lambda u}\, F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u} = \nu\, E_\nu(-\lambda t^\nu),\ \int_0^\infty \mathrm{e}^{-\lambda u}\, M_\nu\left(\frac{u}{t^\nu}\right) du = t^\nu\, E_\nu(-\lambda t^\nu). \tag{41}$$

The shifted increasing and decreasing exponential functions are considered in the following two examples. From

$$\begin{aligned}
&g(t) = \begin{cases} 0 & 0<t<\lambda,\\ 1-\mathrm{e}^{-(t-\lambda)} & t>\lambda; \end{cases}\\
&G(s) = \frac{e^{-\lambda s}}{s(s+1)}\,; \quad G(s^\nu) = \frac{e^{-\lambda s^\nu}}{s^\nu(s^\nu+1)}\\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{e^{-\lambda s^\nu}}{s^\nu(s^\nu+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{\nu-1}}\,\frac{s^{\nu-1}}{(s^\nu+1)}\,\frac{\mathrm{e}^{-\lambda s^\nu}}{s}\right\} = \\
&\frac{t^{\nu-2}}{\Gamma(\nu-1)}\star E_\nu(-t^\nu)\star W_{-\nu,1}\left(-\frac{\lambda}{t^\nu}\right).
\end{aligned} \tag{42}$$

we have the convolution of three functions.

$$\begin{aligned}
&\int_\lambda^\infty [1-\mathrm{e}^{-(u-\lambda)}]F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u} = \frac{\nu}{\Gamma(\nu-1)}\left\{[t^{\nu-2}\star E_\nu(-t^\nu)]\star W_{-\nu,1}(-\frac{\lambda}{t^\nu})\right\}\\
&\int_\lambda^\infty \left[1-\mathrm{e}^{-(u-\lambda)}\right] M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{t^\nu}{\Gamma(\nu-1)}\left\{[t^{\nu-2}\,\star\, E_\nu(-t^\nu)]\,\star\, W_{-\nu,1}(-\frac{\lambda}{t^\nu})\right\}
\end{aligned} \tag{43}$$

Similarly from

$$\begin{aligned}
&g(t) = \begin{cases} 0, & 0<t<\lambda\,,\\ \mathrm{e}^{-(t-\lambda)}, & t>\lambda\,; \end{cases}\\
&G(s) = \frac{e^{-\lambda s}}{(s+1)}\,; \quad G(s^\nu) = \frac{\mathrm{e}^{-\lambda s^\nu}}{(s^\nu+1)},\\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{\mathrm{e}^{-\lambda s^\nu}}{(s^\nu+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s^{\nu-1}}{(s^\nu+1)}\,\mathrm{e}^{-\lambda s^\nu}\right\} = E_\nu(-t^\nu)\,\star\, f(t,\lambda)\\
&f(t,\lambda) = \mathcal{L}^{-1}\left\{\mathrm{e}^{-\lambda s^\nu}\right\} = \frac{1}{t}\,W_{-\nu,0}(-\frac{\nu}{t^\nu})
\end{aligned} \tag{44}$$

it is possible to obtain

$$\begin{aligned}
&\int_\lambda^\infty \mathrm{e}^{-(u-\lambda)}\, F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u} = \nu E_\nu(-t^\nu)\,\star\, f(t,\lambda),\\
&\int_\lambda^\infty \mathrm{e}^{-(u-\lambda)}\, M_\nu\left(\frac{u}{t^\nu}\right) du = t^\nu E_\nu(-t^\nu)\,\star\, f(t,\lambda),
\end{aligned} \tag{45}$$

where the function $f(t,\lambda)$ is the inverse Laplace transform of the exponential function, see (17).

The next group of elementary functions are the logarithmic functions and in the simplest case from

$$
\begin{aligned}
&g(t) = \ln t\,; \quad C = e^{\gamma} \\
&G(s) = -\frac{\ln(Cs)}{s} \quad ; \quad G(s^{\nu}) = -\frac{\ln(Cs^{\nu})}{s^{\nu}} \\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{\ln(Cs^{\nu})}{s^{\nu}}\right\} = \mathcal{L}^{-1}\left\{-\frac{(1-\nu)\ln C + \nu\ln(Cs)}{s}\right\} = (\nu-1)\gamma + \nu\ln t
\end{aligned}
\tag{46}
$$

where γ is the Euler constant. it follows that

$$
\begin{aligned}
&\int_0^{\infty} \ln u\, F_{\nu}\left(\frac{u}{t^{\nu}}\right)\,\frac{du}{u} = \nu(\nu-1)\gamma + \nu\ln t^{\nu}, \\
&\int_0^{\infty} \ln u\, M_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = t^{\nu}[(\nu-1)\gamma + \ln t^{\nu}].
\end{aligned}
\tag{47}
$$

In the more complex case

$$
\begin{aligned}
&g(t) = t^{\lambda-1}\ln t \quad ; \quad \lambda > 0 \quad ; \quad 0 < \nu < 1 \\
&G(s) = \frac{\Gamma(\lambda)}{s^{\lambda}}\,[\psi(\lambda) - \ln s\,] \quad ; \qquad G(s^{\nu}) = \frac{\Gamma(\lambda)\,[\psi(\lambda) - \nu\ln s]}{s^{\lambda\nu}} \\
&\Gamma(\lambda)\,\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{[\psi(\lambda) - \nu\ln s]}{s^{\lambda\nu}}\right\} = \Gamma(\lambda)\,\mathcal{L}^{-1}\left\{\frac{\psi(\lambda)+\gamma\nu}{s^{(\lambda-1)\nu+1}} - \frac{\nu}{s^{(\lambda-1)\nu}}\,\frac{\ln(Cs)}{s}\right\} = \\
&\frac{\Gamma(\lambda)[\psi(\lambda)+\gamma\nu]\,t^{(\lambda-1)\nu}}{\Gamma[(\lambda-1)\nu+1]} + \frac{\Gamma(\lambda)\,t^{(\lambda-1)\nu-1}}{\Gamma[(\lambda-1)\nu]} \star \ln t^{\nu}
\end{aligned}
\tag{48}
$$

the final expressions for integrals of the functions $F_{\nu}(t)$ and $M_{\nu}(t)$ are

$$
\begin{aligned}
&\int_0^{\infty} u^{\lambda-1}\ln t\, F_{\nu}\left(\frac{u}{t^{\nu}}\right)\,\frac{du}{u} = \frac{\nu\Gamma(\lambda)[\psi(\lambda)+\gamma\nu]\,t^{(\lambda-1)\nu}}{\Gamma[(\lambda-1)\nu+1]} + \\
&\frac{\nu\Gamma(\lambda)}{\Gamma[(\lambda-1)\nu]}\,[t^{(\lambda-1)\nu-1} \star \ln t^{\nu}] \\
&\int_0^{\infty} u^{\lambda-1}\ln t\, M_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = \frac{\Gamma(\lambda)[\psi(\lambda)+\gamma\nu]\,t^{\lambda\nu}}{\Gamma[(\lambda-1)\nu+1]} + \\
&\frac{\Gamma(\lambda)t^{\nu}}{\Gamma[(\lambda-1)\nu]}\,[\,t^{(\lambda-1)\nu-1} \star \ln t^{\nu}]
\end{aligned}
\tag{49}
$$

The last groups of elementary functions to be considered are trigonometric and hyperbolic functions. From

$$
\begin{aligned}
&g(t) = \sin(\lambda t) \quad ; \quad 0 < \nu < 1 \\
&G(s) = \frac{\lambda}{s^2+\lambda^2} \quad ; \quad G(s^{\nu}) = \frac{\lambda}{s^{2\nu}+\lambda^2} \\
&\lambda\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{1}{s^{2\nu}+\lambda^2}\right\} = \lambda\mathcal{L}^{-1}\left\{\frac{1}{s^{\nu}}\,\frac{s^{2\nu-1}}{s^{2\nu}+\lambda^2}\right\} = \frac{\lambda}{\Gamma(\nu)}t^{\nu-1} \star E_{2\nu}(-\lambda^2 t^{2\nu})
\end{aligned}
\tag{50}
$$

the following integral identities are derived

$$\begin{aligned} \int_0^\infty \sin(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right)\, \frac{du}{u} &= \frac{\lambda\nu}{\Gamma(\nu)}[t^{\nu-1} \star E_{2\nu}(-\lambda^2 t^{2\nu})] \\ \int_0^\infty \sin(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right)\, du &= \frac{\lambda t^\nu}{\Gamma(\nu)}[t^{\nu-1} \star E_{2\nu}(-\lambda^2 t^{2\nu})] \end{aligned} \tag{51}$$

By changing variable of integration $x = t(\cos\theta)^2$, all convolution integrals can be expressed as in terms of finite trigonometric integrals. For example, the convolution integral in (51) becomes

$$\begin{aligned} & t^{\nu-1} \star E_{2\nu}(-\lambda^2 t^{2\nu}) = \int_0^t (t-x)^{\nu-1} E_{2\nu}(-\lambda^2 x^{2\nu})\, dx = \\ & \int_0^{\pi/2} \sin(2\theta)(\sin\theta)^{2(\nu-1)} E_{2\nu}[-\lambda^2 t^{2\nu}(\cos\theta)^{4\nu}]\, d\theta \end{aligned} \tag{52}$$

In an analogy with (50), for the hyperbolic cosine function, the change is only in the sign

$$\begin{aligned} & g(t) = \sinh(\lambda t) \quad ; \quad 0 < \nu < 1 \\ & G(s) = \frac{\lambda}{s^2 - \lambda^2} \quad ; \quad G(s^\nu) = \frac{\lambda}{s^{2\nu} - \lambda^2} \\ & \lambda\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{\lambda}{s^{2\nu} - \lambda^2}\right\} = \lambda\mathcal{L}^{-1}\left\{\frac{1}{s^\nu}\,\frac{s^{2\nu-1}}{s^{2\nu} - \lambda^2}\right\} = \frac{\lambda t^{\nu-1}}{\Gamma(\nu)} \star E_{2\nu}(\lambda^2 t^{2\nu}) \end{aligned} \tag{53}$$

and

$$\begin{aligned} \int_0^\infty \sinh(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right)\, \frac{du}{u} &= \frac{\lambda\nu}{\Gamma(\nu)}\,[t^{\nu-1} \star E_{2\nu}(\lambda^2 t^{2\nu})] \\ \int_0^\infty \sinh(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right)\, du &= \frac{\lambda t^\nu}{\Gamma(\nu)}[t^{\nu-1} \star E_{2\nu}(\lambda^2 t^{2\nu})] \end{aligned} \tag{54}$$

In the case of cosine function, from

$$\begin{aligned} & g(t) = \cos(\lambda t)\,; \quad 0 < \nu < 1 \\ & G(s) = \frac{s}{s^2 + \lambda^2} \quad ; \quad G(s^\nu) = \frac{s^\nu}{s^{2\nu} + \lambda^2} \\ & \mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{s^\nu}{s^{2\nu} + \lambda^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s^{2\nu-1}}{s^{2\nu} + \lambda^2}\right\} = E_{2\nu}(-\lambda^2 t^{2\nu}) \end{aligned} \tag{55}$$

we have

$$\begin{aligned} \int_0^\infty \cos(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right)\, \frac{du}{u} &= \nu E_{2\nu}(-\lambda^2 x^{2\nu}) \\ \int_0^\infty \cos(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right)\, du &= t^\nu E_{2\nu}(-\lambda^2 x^{2\nu}) \end{aligned} \tag{56}$$

Similarly as in (55) and (56)

$$\begin{aligned}
&g(t)=\cosh(\lambda t) \quad ; \quad 0<\nu<1\\
&G(s)=\frac{s}{s^2-\lambda^2} \quad ; \quad G(s^\nu)=\frac{s^\nu}{s^{2\nu}-\lambda^2}\\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{s^\nu}{s^{2\nu}-\lambda^2}\right\}=\mathcal{L}^{-1}\left\{\frac{s^{2\nu-1}}{s^{2\nu}-\lambda^2}\right\}=E_{2\nu}(\lambda^2 t^{2\nu})
\end{aligned} \tag{57}$$

for the hyperbolic cosine function we have

$$\begin{aligned}
&\int_0^\infty \cosh(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u}=\nu E_{2\nu}(\lambda^2 x^{2\nu})\\
&\int_0^\infty \cosh(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right)du=t^\nu E_{2\nu}(\lambda^2 x^{2\nu})
\end{aligned} \tag{58}$$

The direct and inverse Laplace transforms in the case of the product of trigonometric and hyperbolic sine functions are

$$\begin{aligned}
&g(t)=\sin(\lambda t)\sinh(\lambda t) \quad ; \quad 0<\nu<1\\
&G(s)=\frac{2\lambda^2 s}{s^4+4\lambda^4} \quad ; \quad G(s^\nu)=\frac{2\lambda^2 s^\nu}{s^{4\nu}+4\lambda^4}\\
&2\lambda^2\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{s^\nu}{s^{4\nu}+4\lambda^4}\right\}=2\lambda^2\mathcal{L}^{-1}\left\{\frac{s^{4\nu-1}}{s^{2\nu}(s^{4\nu}+4\lambda^4)}\right\}=\\
&\frac{2\lambda^2}{\Gamma(2\nu)}[t^{2\nu-1}\star E_{4\nu}(-4\lambda^4 t^{4\nu})]
\end{aligned} \tag{59}$$

and therefore

$$\begin{aligned}
&\int_0^\infty \sin(\lambda u)\sinh(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u}=\frac{2\lambda^2\nu t^{2\nu-1}}{\Gamma(2\nu)}\left[t^{2\nu-1}\star E_{4\nu}(-4\lambda^4 t^{4\nu})\right]\\
&\int_0^\infty \sin(\lambda u)\sinh(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right)du=\frac{2\lambda^2 t^\nu}{\Gamma(2\nu)}\left[t^{2\nu-1}\star E_{4\nu}(-4\lambda^4 t^{4\nu})\right].
\end{aligned} \tag{60}$$

For the product of trigonometric and hyperbolic cosine functions we have

$$\begin{aligned}
&g(t)=\cos(\lambda t)\cosh(\lambda t)\,; \quad 0<\nu<1,\\
&G(s)=\frac{s^3}{s^4+4\lambda^4}\,; \quad G(s^\nu)=\frac{s^{3\nu}}{s^{4\nu}+4\lambda^4}\\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{s^{3\nu}}{s^4+4\lambda^4}\right\}=\mathcal{L}^{-1}\left\{\frac{s^{4\nu-1}}{s^{4\nu}+4\lambda^4}\right\}=E_{4\nu}(-4\lambda^4 t^{4\nu})
\end{aligned} \tag{61}$$

and therefore

$$\begin{aligned}
&\int_0^\infty \cos(\lambda u)(\cosh(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u}=\nu E_{4\nu}(-4\lambda^4 x^{4\nu}),\\
&\int_0^\infty \cos(\lambda u)(\cosh(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right)du=t^\nu E_{4\nu}(-4\lambda^4 x^{4\nu}).
\end{aligned} \tag{62}$$

3. Integrals of the Wright functions of the second kind with the Mitag-Leffler functions and the error function

The Laplace transform of the two parameter Mittag-Leffler function is [14,34,35]

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}\mp\lambda}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}. \tag{63}$$

This permits to obtain from (63)

$$\begin{aligned} &g(t) = t^{\beta-1}\,E_{\alpha,\beta}(\pm\lambda t^{\alpha})\,; \quad 0<\nu<1, \\ &G(s) = \frac{s^{\alpha-\beta}}{s^{\alpha}\mp\lambda}\,; \quad G(s^{\nu}) = \frac{s^{(\alpha-\beta)\nu}}{s^{\alpha\nu}\mp\lambda}, \\ &\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{s^{(\alpha-\beta)\nu}}{s^{\alpha\nu}\mp\lambda}\right\} = \mathcal{L}^{-1}\left\{\frac{s^{\alpha\nu-[(\beta-1)\nu+1]}}{s^{\alpha\nu}\mp\lambda}\right\} \\ &= t^{(\beta-1)\nu}\,E_{\alpha\nu,(\beta-1)\nu+1}(\pm\lambda t^{\alpha}), \end{aligned} \tag{64}$$

and using (23) and (24)

$$\begin{aligned} &\int_0^{\infty} u^{\beta-1}E_{\alpha,\beta}(\pm\lambda u^{\alpha})\,F_{\nu}\left(\frac{u}{t^{\nu}}\right)\,\frac{du}{u} = \nu\,t^{(\beta-1)\nu}E_{\alpha\nu,(\beta-1)\nu+1}(\pm\lambda t^{\alpha}), \\ &\int_0^{\infty} u^{\beta-1}E_{\alpha,\beta}(\pm\lambda u^{\alpha})\,M_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = t^{\beta\nu}E_{\alpha\nu,(\beta-1)\nu+1}(\pm\lambda t^{\alpha}). \end{aligned} \tag{65}$$

Thus, in both sides of expressions (65) appear the Mittag-Leffler functions. Evidently, for $\beta=1$ they are reduced to the classical Mittag-Leffler functions

$$\begin{aligned} &\int_0^{\infty} E_{\alpha}(\pm\lambda u^{\alpha})\,F_{\nu}\left(\frac{u}{t^{\nu}}\right)\,\frac{du}{u} = \nu E_{\alpha\nu}\,(\pm\lambda t^{\alpha}), \\ &\int_0^{\infty} E_{\alpha}(\pm\lambda u^{\alpha})\,M_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = t^{\nu}E_{\alpha\nu}(\pm\lambda t^{\alpha}), \end{aligned} \tag{66}$$

and for $\beta=\alpha$

$$\begin{aligned} &\int_0^{\infty} u^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda u^{\alpha})\,F_{\nu}\left(\frac{u}{t^{\nu}}\right)\,\frac{du}{u} = \nu t^{(\alpha-1)\nu}E_{\alpha\nu,(\alpha-1)\nu+1}(\pm\lambda t^{\alpha}), \\ &\int_0^{\infty} u^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda u^{\alpha})\,M_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = t^{\alpha\nu}E_{\alpha\nu,(\alpha-1)\nu+1}(\pm\lambda t^{\alpha}). \end{aligned} \tag{67}$$

If $\beta=\alpha+1$ we have

$$\begin{aligned} &\int_0^{\infty} u^{\alpha}E_{\alpha,\alpha+1}(\pm\lambda u^{\alpha})\,F_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = \nu\,t^{\alpha\nu}E_{\alpha\nu,\alpha\nu+1}(\pm\lambda t^{\alpha}), \\ &\int_0^{\infty} u^{\alpha}E_{\alpha,\alpha+1}(\pm\lambda u^{\alpha})\,M_{\nu}\left(\frac{u}{t^{\nu}}\right)\,du = t^{(\alpha+1)\nu}E_{\alpha\nu,\alpha\nu+1}(\pm\lambda t^{\alpha}). \end{aligned} \tag{68}$$

In the case $\beta = 1/2$, in the integrands of (66) - (68), the Mittag-Leffler functions are expressed then by the error functions [35]

$$
\begin{aligned}
&E_{1/2}(\pm z) = \mathrm{e}^{z^2}[1 \pm \mathrm{erf}(z)]\,; \quad z = \lambda u^{\alpha},\\
&E_{1/2,1/2}(\pm z) = \left\{\frac{1}{\sqrt{z}} \pm z\mathrm{e}^{z^2}[1 \pm \mathrm{erf}(z)]\right\},\\
&E_{1/2,3/2}(z) = \frac{\mathrm{e}^{z}}{\sqrt{z}}\mathrm{erf}(z).
\end{aligned} \tag{69}
$$

If β is positive integer, the Mittag-Leffler functions are known to be expressed by elementary functions [36].

The Laplace transform of the error function is

$$
\begin{aligned}
&g(t) = \mathrm{erf}\left(\frac{\lambda}{2t^{1/2}}\right)\,; \quad 0 < \nu < 1,\\
&G(s) = \frac{1 - \mathrm{e}^{-\lambda s^{1/2}}}{s}\,; \quad G(s^{\nu}) = \frac{1 - \mathrm{e}^{-\lambda s^{\nu/2}}}{s^{\nu}}\\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\,\frac{1 - \mathrm{e}^{-\lambda s^{\nu/2}}}{s^{\nu}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\mathrm{e}^{-\lambda s^{\nu/2}}}{s}\right\} = 1 - W_{\nu/2,1}\left(-\frac{\lambda}{t^{\nu/2}}\right),
\end{aligned} \tag{70}
$$

which yields

$$
\begin{aligned}
&\int_0^{\infty} \mathrm{erf}\left(\frac{\lambda}{2u^{1/2}}\right)\, F_{\nu}\left(\frac{u}{t^{\nu}}\right)\, \frac{du}{u} = \nu\, \left\{1 - W_{-\nu/2,1}\left(-\frac{\lambda}{t^{\nu/2}}\right)\right\},\\
&\int_0^{\infty} \mathrm{erf}\left(\frac{\lambda}{2u^{1/2}}\right)\, M_{\nu}\left(\frac{u}{t^{\nu}}\right)\, du = t^{\nu}\, \left\{1 - W_{-\nu/2,1}\left(-\frac{\lambda}{t^{\nu/2}}\right)\right\}
\end{aligned} \tag{71}
$$

4. Integrals of the Wright functions of the second kind with the Volterra functions.

The Volterra functions are defined by the following integrals [36]

$$
\begin{aligned}
&\nu(t) = \int_0^{\infty} \frac{t^u}{\Gamma(u+1)}\, du\\
&\nu(t,\alpha) = \int_0^{\infty} \frac{t^{u+\alpha}}{\Gamma(u+\alpha+1)}\, du\\
&\mu((t,\beta,\alpha) = \int_0^{\infty} \frac{u^{\beta} t^{u+\alpha}}{\Gamma(\beta+1)\Gamma(u+\alpha+1)}\, du
\end{aligned} \tag{72}
$$

and their Laplace transforms are

$$
\begin{aligned}
&\mathcal{L}\left\{\nu(\lambda t)\right\} = \frac{1}{s\ln\left(\dfrac{s}{\lambda}\right)} \quad ; \quad \lambda > 0\\
&\mathcal{L}\left\{\nu(\lambda t,\alpha)\right\} = \frac{\lambda^{\alpha}}{s^{\alpha+1}\ln\left(\dfrac{s}{\lambda}\right)}\\
&\mathcal{L}\left\{\mu((\lambda t,\beta,\alpha)\right\} = \frac{\lambda^{\alpha}}{s^{\alpha+1}\ln\left(\dfrac{s}{\lambda}\right)^{\beta+1}}
\end{aligned} \tag{73}
$$

This form of the Laplace transforms permits to express the integrals of the Volterra functions with the Wright functions of the second kind also in terms of the Volterra functions. From

$$\begin{aligned}
&g(t) = \nu(\lambda t) \quad ; \quad \lambda > 0 \quad ; \quad 0 < \nu < 1 \\
&G(s) = \frac{1}{s \ln\left(\dfrac{s}{\lambda}\right)} \quad ; \quad G(s^\nu) = \frac{1}{s^\nu \ln\left(\dfrac{s^\nu}{\lambda}\right)} \\
&\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}} \frac{1}{s^\nu \ln[(s/\lambda^{1/\nu})^\nu]}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\nu s \ln(s/\lambda^{1/\nu})}\right\} = \frac{1}{\nu}\, \nu(\lambda^{1/\nu} t)
\end{aligned} \tag{74}$$

it follows that

$$\begin{aligned}
&\int_0^\infty \nu(\lambda u)\, F_\nu\left(\frac{u}{t^\nu}\right) \frac{du}{u} = \nu(\lambda^{1/\nu} t) \\
&\int_0^\infty \nu(\lambda u)\, M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{t^\nu}{\nu}\, \nu(\lambda^{1/\nu} t)
\end{aligned} \tag{75}$$

Similarly from

$$\begin{aligned}
&g(t) = \nu(\lambda t, \rho) \quad ; \quad \lambda, \rho > 0 \quad ; \quad 0 < \nu < 1 \\
&G(s) = \frac{\lambda^\rho}{s^{\rho+1} \ln\left(\dfrac{s}{\lambda}\right)} \quad ; \quad G(s^\nu) = \frac{\lambda^\rho}{s^{(\rho+1)\nu} \ln\left(\dfrac{s^\nu}{\lambda}\right)} \\
&\lambda^\rho \mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}} \frac{1}{s^{(\rho+1)\nu} \ln[(s/\lambda^{1/\nu})^\nu]}\right\} = \lambda^\rho \mathcal{L}^{-1}\left\{\frac{1}{\nu s^{\rho\nu+1} \ln(s/\lambda^{1/\nu})}\right\} = \\
&\frac{\lambda^\rho}{\nu}\, \nu(\lambda^{1/\nu} t, \rho\nu)
\end{aligned} \tag{76}$$

we have

$$\begin{aligned}
&\int_0^\infty \nu(\lambda u, \rho)\, F_\nu\left(\frac{u}{t^\nu}\right) \frac{du}{u} = \lambda^\rho \nu(\lambda^{1/\nu} t, \rho\nu) \\
&\int_0^\infty \nu(\lambda u, \rho)\, M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\lambda^\rho t^\nu}{\nu}\, \nu(\lambda^{1/\nu} t, \rho\nu)
\end{aligned} \tag{77}$$

Finally, the Laplace transform of the generalized Volterra function is

$$\begin{aligned}
&g(t) = \mu(\lambda t, \xi, \rho) \quad ; \quad \lambda, \rho, \xi > 0 \quad ; \quad 0 < \nu < 1 \\
&G(s) = \frac{\lambda^\rho}{s^{\rho+1} \left[\ln\left(\dfrac{s}{\lambda}\right)\right]^{\xi+1}} \quad ; \quad G(s^\nu) = \frac{\lambda^\rho}{s^{(\rho+1)\nu} \left[\ln\left(\dfrac{s^\nu}{\lambda}\right)\right]^{\xi+1}} \\
&\lambda^\rho \mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}} \frac{1}{s^{(\rho+1)\nu}\{\ln\left[(s/\lambda^{1/\nu})^\nu\right]\}^{\xi+1}}\right\} = \\
&\frac{\lambda^\rho}{\nu^{\xi+1}} \mathcal{L}^{-1}\left\{\frac{1}{s^{\mu\nu+1}\left[\ln(s/\lambda^{1/\nu})\right]^{\xi+1}}\right\} = \frac{\lambda^\rho}{\nu^{\xi+1}}\, \mu(\lambda^{1/\nu} t, \xi, \rho\nu)
\end{aligned} \tag{78}$$

and (78) yields

$$\begin{aligned} &\int_0^\infty \mu(\lambda u,\xi,\rho)\, F_\nu\left(\frac{u}{t^\nu}\right)\frac{du}{u} = \frac{\lambda^\rho}{\nu^\xi}\;\mu(\lambda^{1/\nu}t,\xi,\rho\nu) \\ &\int_0^\infty \mu(\lambda u,\xi,\rho)\, M_\nu\left(\frac{u}{t^\nu}\right) du = \frac{\lambda^\rho t^\nu}{\nu^{\xi+1}}\;\mu(\lambda^{1/\nu}t,\xi,\rho\nu) \end{aligned} \tag{79}$$

In different, but in an equivalent form, the integrals in (79) were also derived by Ansari [37].

If the Volterra functions are multiplied by t^n with $n = 1, 2, 3, \dots$, their Laplace transforms should be differentiated n times [28,36]. Only the simplest case of (76) with $n = 1$ and $\lambda = 1$ is presented here

$$\mathcal{L}\{t\nu(t,\rho)\} = -\frac{d}{ds}\left\{\frac{1}{s^{\rho+1}\ln s}\right\} = \frac{\rho+1}{s^{\rho+2}\ln s} + \frac{1}{s^{\rho+2}(\ln s)^2} \tag{80}$$

Using this and the inverse Laplace transform from (73)

$$\begin{aligned} &g(t) = t\nu(t,\rho) \quad ; \quad \rho>0 \quad ; \quad 0<\nu<1 \\ &G(s) = \frac{\rho+1}{s^{\rho+2}\ln s} + \frac{1}{s^{\rho+2}(\ln s)^2} \quad ; \quad G(s^\nu) = \frac{\rho+1}{\nu s^{(\rho+2)\nu}\ln s} + \frac{1}{\nu^2 s^{(\rho+2)\nu}(\ln s)^2} \\ &\mathcal{L}^{-1}\left\{\frac{1}{s^{1-\nu}}\left[\frac{\rho+1}{\nu s^{(\rho+2)\nu}\ln s} + \frac{1}{\nu^2 s^{(\rho+2)\nu}(\ln s)^2}\right]\right\} = \\ &\mathcal{L}^{-1}\left\{\frac{\rho+1}{\nu s^{(\rho+1)\nu+1}\ln s} + \frac{1}{\nu^2 s^{(\rho+1)\nu+1}(\ln s)^2}\right\} = \\ &\frac{\rho+1}{\nu}\;\nu[t,(\rho+1)\nu] + \frac{1}{\nu^2}\,\mu[t,1,(\rho+1)\nu] \end{aligned} \tag{81}$$

we have

$$\begin{aligned} &\int_0^\infty \nu(\lambda u,\rho)\, F_\nu\left(\frac{u}{t^\nu}\right) du = \;(\rho+1)\;\nu[t,(\rho+1)\nu] + \frac{1}{\nu}\,\mu[t,1,(\rho+1)\nu] \\ &\int_0^\infty u\nu(\lambda u,\rho)\, M_\nu\left(\frac{u}{t^\nu}\right) du = t^\nu\nu[t,(\rho+1)\nu] + \frac{1}{\nu}\,\mu[t,1,(\rho+1)\nu] \end{aligned} \tag{82}$$

5. Integrals of the modified Bessel functions of the second kind and order one-third

Derived results in previous sections are of general character for any value of $\nu \in (0,1)$. As already mentioned, in some particular cases of ν they can be expressed by elementary or by special functions, see for example [11] and [12]:

$$\begin{aligned} &M_{1/2}(t) = \frac{2}{t}\,F_{1/2}(t) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-t^2/4} \\ &M_{1/3}(t) = \frac{3}{t}\,F_{1/2}(t) = 3^{2/3}\mathrm{Ai}\left(\frac{t}{3^{1/3}}\right) \\ &M_{2/3}(t) = \frac{3}{2t}\,F_{1/2}(t) = \left\{\frac{1}{3^{1/3}}\,\mathrm{Ai}\left(\frac{t^2}{3^{4/3}}\right) - 3^{1/3}\mathrm{Ai}'\left(\frac{t^2}{3^{4/3}}\right)\right\}\mathrm{e}^{-2t^3/27} \end{aligned} \tag{83}$$

Introducing these in (83), or other known in the literature results permits to present derived above integrals in an explicit form.

This is illustrated here for $\nu = 1/3$, and only for the power functions t^{μ}. If the Airy functions in (83) are replaced by the modified Bessel functions of the second kind and order one-third, we have

$$\begin{aligned} F_{1/3}\left(\frac{u}{t^{1/3}}\right) &= \frac{u^{3/2}}{3\pi t^{1/2}}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) \\ M_{1/3}\left(\frac{u}{t^{1/3}}\right) &= \frac{u^{1/2}}{\pi t^{1/6}}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) \end{aligned} \tag{84}$$

and their Laplace transform are

$$\begin{aligned} \mathcal{L}\left\{\frac{\lambda^{3/2}}{3\pi t^{3/2}}\, K_{1/3}\left(\frac{2\lambda^{3/2}}{\sqrt{27t}}\right)\right\} &= \mathrm{e}^{-\lambda s^{1/3}} \\ \mathcal{L}\left\{\frac{\lambda^{3/2}}{\pi t^{1/2}}\, K_{1/3}\left(\frac{2\lambda^{3/2}}{\sqrt{27t}}\right)\right\} &= \frac{\lambda \mathrm{e}^{-\lambda s^{1/3}}}{s^{2/3}} \end{aligned} \tag{85}$$

Thus, equation (22) can be written as

$$\begin{aligned} \mathcal{L}\left\{\frac{1}{t}F_{1/3}(\frac{\lambda}{t^{1/3}})\right\} &= \mathcal{L}\left\{\frac{\lambda}{3t^{4/3}}M_{1/3}(\frac{\lambda}{t^{1/3}})\right\} = \mathrm{e}^{-\lambda s^{1/3}} \quad ; \quad \lambda > 0 \\ \mathcal{L}\left\{3F_{1/3}(\frac{\lambda}{t^{1/3}})\right\} &= \mathcal{L}\left\{\frac{\lambda}{t^{1/3}}M_{1/3}(\frac{\lambda}{t^{1/3}}\right\} = \frac{\lambda}{s^{2/3}}\mathrm{e}^{-\lambda s^{1/3}} \end{aligned} \tag{86}$$

If the form of integration variable of the modified Bessel functions is the same as in (84), the infinite integrals in (25) - (31) become

$$\begin{aligned} &\int_0^\infty \sqrt{u}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) du = \pi\sqrt{t} \quad ; \quad \int_0^\infty u^{3/2}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) du = \frac{3\pi t^{5/6}}{\Gamma\left(\frac{1}{3}\right)} \\ &\int_0^\infty u^{2n+1/2}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) du = \frac{3\pi\Gamma(2n)t^{2n/3+1/2}}{\Gamma\left(\frac{2n}{3}\right)} \\ &\int_0^\infty u^{2n+3/2}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) du = \frac{3\pi\Gamma(2n+1)t^{2n/3+5/6}}{\Gamma\left(\frac{2n+1}{3}\right)} \\ &\int_0^\infty u^{\lambda+1/2}\, K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right) du = \frac{3\pi\Gamma(\lambda)t^{\lambda/3+1/2}}{\Gamma\left(\frac{\lambda}{3}\right)} \end{aligned} \tag{87}$$

However, if the integration variable is changed, then these integrals take much simpler forms

$$\int_0^\infty K_{1/3}(x)\,dx = \frac{\pi}{\sqrt{3}} \quad ; \quad \int_0^\infty x^{2/3}\,K_{1/3}(x)\,dx = \frac{2^{2/3}\pi}{3^{1/2}\Gamma\left(\frac{1}{3}\right)}$$

$$\begin{aligned}
&\int_0^\infty x^{4n/3}\,K_{1/3}(x)dx = \frac{2^{4n/3}\pi\Gamma(2n)}{3^{2n-1/2}\Gamma\left(\frac{2n}{3}\right)} \\
&\int_0^\infty x^{(4n+2)/3}\,K_{1/3}(x)\,dx = \frac{2^{(4n+2)/3}\pi\Gamma(2n+1)}{3^{2n+1/2}\Gamma\left(\frac{2n+1}{3}\right)} \\
&\int_0^\infty x^{\lambda}\,K_{1/3}(x)\,dx = \frac{2^{\lambda}\,\pi\Gamma\left(\frac{3\lambda}{2}\right)}{3^{(3\lambda-1)/2}\Gamma\left(\frac{\lambda}{2}\right)}
\end{aligned} \tag{88}$$

It is possible significantly to increase a number of evaluated the modified Bessel function of second kind integrals if integrals in (87) are differentiated with respect to parameters λ and t. This can be performed by considering also properties of these functions [38,39]

$$\begin{aligned}
&K_{-\nu}(z) = K_{\nu}(z) \\
&K_{\nu}(z) = \frac{z}{2\nu}[K_{\nu+1}(z) - K_{\nu-1}(z)] \\
&\frac{dK_{\nu}(z)}{dz} = \frac{\nu}{z}\,K_{\nu}(z) - K_{\nu+1}(z) \\
&\frac{dK_{\nu}(z)}{dz} = -\,[\frac{\nu}{z}\,K_{\nu}(z) + K_{\nu-1}(z)]
\end{aligned} \tag{89}$$

Starting with (87) and by using equations in (89) with $\nu = 1/3$, the integrals with $2/3$ and $4/3$ orders are determined. If this evaluation process is continued, it is possible to obtain integrals with $n+1/3$ and $n+2/3$ with $n = 1, 2, 3, \ldots$ orders. It is worthwhile to mention that infinite integrals of the modified Bessel functions of second kind with $1/3$ and $2/3$ orders can be is determined in an alternative way by using the substitution formulas for the inverse Laplace transforms [28]

$$\begin{aligned}
&\mathcal{L}\{h(t)\} = H(s), \\
&\mathcal{L}^{-1}\left\{H(s^{1/3})\right\} = \frac{1}{3\pi}\int_0^\infty h(u)\,\left(\frac{u}{t}\right)^{3/2} K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right)\,du, \\
&\mathcal{L}^{-1}\left\{\frac{H(s^{1/3})}{s^{2/3}}\right\} = \frac{1}{\pi}\int_0^\infty h(u)\,\sqrt{\frac{u}{t}}K_{1/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right)\,du, \\
&\mathcal{L}^{-1}\left\{\frac{H(s^{1/3})}{s^{2/3}}\right\} = \frac{1}{\sqrt{3}\pi}\int_0^\infty h(u)\,\left(\frac{u}{t}\right)K_{2/3}\left(\frac{2u^{3/2}}{\sqrt{27t}}\right)\,du.
\end{aligned} \tag{90}$$

6. Conclusions

At first we have recalled the main definitions and properties for the so-called Mainardi auxiliary functions $F_\nu(t)$ and $M_\nu(t)$ with $0 < \nu < 1$, that are noteworthy examples of the Wright functions of the second kind. Then we have demonstrated that, by applying the Efros theorem in the form established by Wlodarski, it is possible to derive many infinite integrals, finite integrals and integral identities involving these functions. In evaluated integral identities, the Mittag-Leffler functions appear frequently in convolution integrals, pointing out the connection between these functions with the Wright functions of the second kind. Indeed our derived integrals of the Wright functions of second kind include in integrands elementary functions (power, exponential, logarithmic, trigonometric and hyperbolic functions) and the error functions, the Mittag-Leffler functions and the Volterra functions. Finally, our general results are illustrated in detail by presenting the particular case of integrals with the modified Bessel function of the second kind and order one-third.

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References

[1] Wright, E.M. On the coefficients of power series having exponential singularities. J. London Math. Soc., 8 (1933), 71–79.

[2] Wright, E.M. The generalized Bessel function of order greater than one. Quart. J. Math. Oxford, 11 (1940), 36–48.

[3] Montroll,E.W. Bendler, J.T. On Lévy (or stable) distributions and the Williams-Watts model of dielectric relaxation. J. Stat. Phys., 34 (1984),129–192.

[4] Zolotarev, V.M. One-dimensional Stable Distributions. Amer. Math. Soc., Providence, Rhode Island, 1986. (Translated from Russian by H.H. McFaden)

[5] Mainardi, F. On the initial value problem for the fractional diffusion-wave equation, in S. Rionero and T. Ruggeri (Editors), 7th Conference on Waves and Stability in Continuous Media (WASCOM 1993), World Scientific, Singapore, 1994, pp. 246–251. [Series on Advances in Mathematics for Applied Sciences, Vol. 23]

[6] Saichev, A.I., Zaslavsky,G.M. Fractional kinetic equations :solutions and applications. Chaos, 7 (1997), 753–764.

[7] Mainardi, F., Tomirotti, M. Seismic pulse propagation with constant Q and stable probability distributions. Ann. Geofisica, 40 (1997), 1311–1328.

[8] Gorenflo, R., Luchko, Yu., Mainardi, F. Analytical properties and applications of the Wright functions. Fract. Calc. Appl. Anal., 2 (1999), 383–414.

[9] Mainardi, F., Luchko, Yu, Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. Fract. Calc. Appl. Anal., 4 (2001), 153–192.

[10] Barkai, E. Fractional Fokker-Planck equation, solution, and application. Phys. Rev. E. 63 (2001), 046118,

[11] Hanyga, A. Multidimensional solutions of time-fractional diffusion-wave equations. Proc. Royal Soc. A., London, 458 (2002), 933–957.

[12] Mainardi. F.: Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London, 2010. [2-nd edn in preparation]

[13] Penson, K.A., Górska, K. Exact and explicit probability densities for one-sided Lévy stable distribution. Phys. Rev. Letters, 105 (2010), 210604/1–9.

[14] Górska, K., Penson, K.A. Lévy stable distributions via associated integral transform. J. Math. Phys., 53 (2012), 053302/1–10.

[15] Luchko, Yu., Mainardi, F. Cauchy and signaling problems for the time-fractional diffusion-wave equation. ASME Journal of Vibration and Acoustics, 136 No 5 (2014), 050904/1–7. DOI: 10.1115/1.4026892 [E-print arXiv:1609.05443]

[16] Luchko, Yu, Mainardi, F. Fractional diffusion-wave phenomena, In V.E Tarasov (Editor), Handbook of Fractional Calculus with Applications, Vol. 5, pp 71–98, 2019. De Gruyter, Berlin. Series editor: J.A Tenreiro. Machado

[17] Mainardi, F., Consiglio, A. The Wright functions of the second kind in mathematical physics. Mathematics, 8 No 6 (2020), 884/1–26. DOI: 103390/MATH8060884 [E-print arXiv2007.02098]

[18] Humbert, P. Nouvelle correspondances symboliques. Bull. Sci. Math., 69 (1945), 121–129.

[19] Pollard, H. The representation of $\exp(-s^{\lambda})$ as a Laplace integral. Bull. Amer. Math. Soc., 52 (1946), 908–910.

[20] Wlodarski, L. Sur une formule de Eftros. Studia Math., 13 (1953), 183–187.

[21] Mikusinski, J. Sur les fonctions exponentielles du calcul opératoire. Studia Math., 12 (1951), 208–224.

[22] Mikusinski, J. Sur la croissance de la function opérationelle $\exp(-s^{\alpha}\,\lambda)$. Bull. Acad. Polon., 4 (1956),423–425.

[23] Mikusinski, J. Sur la function dont la transformée de Laplace est $\exp(-s^{\alpha}\,\lambda)$. Bull. Acad. Polon., 6 (1958), 691–693.

[24] Mikusinski, J. On the function whose Laplace transform is $\exp(-s^{\alpha}\lambda)$. $0<\alpha<1$, Studia Math., 18 (1959), 191–198.

[25] Wintner, A. Cauchy's stable distributions and an "explicit formula" of Mellin. Amer. J. Math., 78 (1956), 819–861.

[26] Ragab, F.M. The inverse Laplace transform of a exponential function. Comm. Pure Appl. Math., 11 (1958), 115–127.

[27] Stankovič, B. On the function of E.M. Wright. Publications de L'Institute Mathématique. Nouvelle série, 10 (1970), 113–124.

[28] Apelblat, A. Laplace Transforms and Their Applications. Nova Science Publishers, Inc., New York, 2012.

[29] Efros, A.M. Some applications of operational calculus in analysis. Mat. Sbornik, 42 (1935), 699–705.

[30] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Tables of Integral Transforms. McGraw-Hill, New York, 1954.

[31] Roberts, G.E., Kaufman, H.: Tables of Laplace Transforms. W.B. Saunders Co., Philadelphia, 1966.

[32] Hladik, J.: La Transformation de Laplace a Plusieurs Variables. Masson et Cie Éditeurs. Paris, 1969.

[33] Oberhettinger, F., Badii, L.: Tables of Laplace Transforms. Springer-Verlag, Berlin, 1973.

[34] Apelblat, A. Differentiation of the Mittag-Leffler functions with respect to parameters in the Laplace transform approach. Mathematics, 8 (2020), Art. 657.

[35] Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V. Mittag-Leffler Functions, Related Topics and Applications. Springer, Heidelberg, 2020. [First Edn, 2014].

[36] Apelblat, A. Volterra Functions. Nova Science Publishers, Inc., New York, 2008.

[37] Ansari, A. On the Volterra μ functions and the M-Wright functions as kernels and eigenfunctions of Volterra type integral operators. Frac. Calc. Appl. Anal., 19 (2016), 567–572.

[38] Abramowitz, M., Stegun, I.A. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. U.S. National Bureau of Standards. Applied Mathematics Series, vol. 55, Washington, D.C. 1964.

[39] Apelblat, A. Bessel and Related Functions. Mathematical Operations with Respect to the Order. Vol 1. Theoretical Aspects. De Gruyter GmbH, Berlin, 2020.