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On a Problem of Non-Homogeneous Piezoelectric Elastic Rod with Variable Constitutive Coefficients

Archili Sakevarashvili [a*] and Natalia Chinchaladze [a]

[a] *I. Vekua Institute of Applied Mathematics of Iv. Javakhishvili Tbilisi State University 2 University St., 0186 Tbilisi, Georgia*

In the present work a problem for non-homogeneous piezoelectric elastic rod is studied in the case when constitutive coefficients vary from zero as power functions of spatial variable x_3, i.e. equal to $const. \times x_3^{\kappa}$, $\kappa = const. \in [0, 1)$. It is assumed that all other functions depend on time t and spatial variable x_3, with prescribed charge density (f_e) and volume force component (Φ_3). The well-posedness of initial-boundary value problem is studied. The displacement vector (u_3) as well as electric (χ) and magnetic (η) potentials that arise during the deformation are represented as absolutely and uniformly convergent series. The conditions on the volume force components Φ_1 and Φ_2, which guarantee the strain state under consideration, are established.

Introduction

The development of science, industry and technologies on the one hand made the possibility of constructing such new composite materials with different physical properties (piezoelectric, piezomagnetic, multi-component mixtures, bio-materials, meta-materials etc.) that are not found naturally on Earth. On the other hand these new materials can be used for future development of the same fields. Several examples include piezoelectric sensors for vibration control ([39]), high precision actuators ([1]), materials with higher strength and stiffness ([33]) or ones that lower energy consumption ([37], [13]), production cost and size of sensors or actuators ([1], [39]).

"Piezoelectric materials did not come into widespread use until the World War I, when quartz was used as resonators for ultrasound sources in SONAR to detect submarines through echolocation. Although nowadays such materials can be seen in daily life even in devices such as speakers, headphones or microphones" see [16].

The increasing demand on developing new types of materials makes it necessary to describe mathematically how do they behave under the influence of various physical fields.

Direct piezoelectric effect was discovered by the brothers Jacques Curie (1856-1941) and Pierre Curie (1859-1906) ([14], [15]).

The Magnetoelectric effect was first predicted by Landay and Lifshitz in 1957 ([29]) and was later confirmed in an antiferromagnetic single crystal Cr_2O_3 ([2], [18]).

*Corresponding author. Email: asakevarashvili@gmail.com

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The electromagnetic effects in solid bodies was studied by V. Nowacki ([36]), P. Denieva at.al. ([16]). Other examples of studies can be found in [17], [34], [40], [30], [5], [6], [7], [35].

The governing equations for thermo-piezo-electro-magneto-elastic material with voids are given e.g. in G. Jaiani [23]. The governing equations consist of: 1. motion equations; 2. kinematic relations; 3. constitutive equations. Constitutive equations and constants (e.g. piezoelectric and piezomagnetic coefficients, dielectric and magnetic permittivity constant, etc.) are determined by experimentally.

In 1955 I. Vekua published his models of elastic prismatic shells ([42]). In 1965 he offered analogous models for standard shells ([43]). Works of I. Babuška, D. Gordeziani, V. Guliaev, I. Khoma, A. Khvoles, T. Meunargia, C. Schwab, T. Vashakmadze, V. Zhgenti, and others (see, e.g., [3], [4], [19], [20], [28], [32], [38], [41], [45]) are devoted to further analysis of I.Vekuas models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (to non-shallow shells, to the anisotropic case, etc.). Solving boundary and/or initial value problems for differential equation systems related to body deformations can be challenging for example when cusped plates are considered, i.e. such ones whose thickness on the part of the plate boundary or on the whole one vanishes (see [8]-[12], [21]-[26], and the references there).

In the present work the problem is studied for longitudinal oscillation of non-homogeneous piezoelectric elastic rod when constitutive coefficients are power functions of the spatial variable x_3. The elastic rod can be thought as a rectangular prism with constant height, length and width (generally, width and height of a rod can be variable, but in the present work they are considered constants). We consider spacial 1D particular case of 3D model, all functions where depended only on x_3 spatial variable and on time t. The main problem is to find the displacement (u_3), electric potential (χ) and magnetic potential (η) when charge density (f_e) and the projection of volume force on x_3 (Φ_3) are given. The top and the bottom ends of the rod are fixed. The conditions on the volume force components Φ_1 and Φ_2 which guarantee the strain state under consideration are established.

The work is organized as follows: in Section 1 some preliminary materials are provided: in Section 1.1 the system of differential equations is given for spacial 1D case; In Section 2 the problem is discussed when the constitutive coefficients are considered as power functions of spatial variable x_3, i.e. these coefficients equal to $const. \times x_3^\kappa$, $\kappa = cosnt. \in [0, 1)$. All the mechanical quantities are calculated by means of $u_3(x_3, t)$. For $u_3(x_3, t)$ we get Fredholm type linear integro-differential equation of the second kind. The solutions are represented as series, absolute and uniform convergence of the series are proved.

1. Preliminary Materials

1.1. *System of Differential Equations*

We consider a piezoelectric elastic rod ([23], [24]):

$$\bar{V} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_3 \leq L, 0 \leq x_1 \leq d, 0 \leq x_2 \leq h\}, \tag{1}$$

where $L, h = const.$

The governing equations for piezoelectric Kelvin-Voigt materials with voids has the following form (see e.g. [23]):

Motion Equations

$$X_{ji,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \ \ t > t_0; i, j = \overline{1,3}, \tag{2}$$

$$D_{j,j} = f_e, \qquad B_{j,j} = 0, \qquad \Omega \times]0, T[, \qquad j = \overline{1,3}, \tag{3}$$

where $X_{ij} \in C^1(\Omega)$ is the stress tensor; Φ_i are the volume force components; ρ is the mass density; $u_i \in C^2(\Omega)$ are the displacements; $f_e : \Omega \times]0, T[\to \mathbb{R}^1$ is the electric charge density; $\mathbf{D} := (D_1, D_2, D_3) : \Omega \times]0, T[\to \mathbb{R}^3$ is the electrical displacement vector; $\mathbf{B} := (B_1, B_2, B_3) : \Omega \times]0, T[\to \mathbb{R}^3$ is the magnetic induction vector. Here and in the future Einstein summation convention is used.

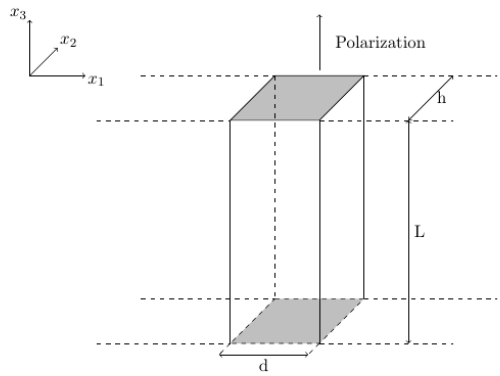

Figure 1. Rod given by region $\bar{V}$

Kinematic Relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i, j = \overline{1,3}, \tag{4}$$

where $e_{ij} \in C^1(\Omega)$ is the strain tensor.

Constitutive Equations

$$X_{ji} = X_{ij} = E_{ijkl} e_{kl} + p_{kij} \chi_{,k} + q_{kij} \eta_{,k}, \qquad i, j, k, l = \overline{1,3}, \tag{5}$$

$$D_j = p_{jkl} e_{kl} - \varsigma_{jl} \chi_{,l} - \tilde{a}_{jl} \eta_{,l}, \qquad i, j, k, l = \overline{1,3}, \tag{6}$$

$$B_j = q_{jkl} e_{kl} - \tilde{a}_{jl} \chi_{,l} - \xi_{jl} \eta_{,l}, \qquad i, j, k, l = \overline{1,3}, \tag{7}$$

where E_{ijkl} are elastic constants (measured at constant electric and magnetic fields), $\chi : \Omega \times]0, T[\to \mathbb{R}^1$ and $\eta : \Omega \times]0, T[\to \mathbb{R}^1$ are electric and magnetic potentials, respectively; p_{kij} are piezoelectric coefficients (measured at constant magnetic field), and q_{kij} are piezomagnetic coefficients (measured at constant electric field); ς_{jl} and ξ_{jl} are dielectric permittivity coefficients (measured at constant strain and magnetic filed) and magnetic permeability coefficients (measured at constant strain and electric field), respectively; $\tilde{a}_{jl}$ are the coupling coefficients (so called magnetoelectric coefficients) connecting electric and magnetic fields (measured at constant strain) ([23], [44]). The constitutive coefficients $E_{ijkl}, p_{kij}, q_{kij}, \varsigma_{jl}, \tilde{a}_{jl}, \xi_{jl}$

satisfy the following symmetry relations ([36]):

$$E_{ijkl} = E_{jikl} = E_{jilk} = E_{klij}, \quad \xi_{jl} = \xi_{lj}, \quad \tilde{a}_{jl} = \tilde{a}_{lj},$$
$$p_{kij} = p_{kji}, \quad q_{kij} = q_{kji}, \quad \varsigma_{jl} = \varsigma_{lj}, \quad i,j,k,l = \overline{1,3}. \tag{8}$$

Let us consider the case when $u_1 = u_2 \equiv 0$ and $u_3 \not\equiv 0$, and polarization is parallel to x_3 axis. Under these consideration, if we insert (4) into (5)-(7) and the result into equations (2) and (3), and use (8) relations it will lead us to the following system of equations:

$$(E_{\alpha 333}u_{3,3} + p_{3\alpha 3}\chi_{,3} + q_{3\alpha 3}\eta_{,3})_{,3} + \Phi_\alpha = 0, \alpha = 1,2, \tag{9}$$

$$(E_{3333}u_{3,3} + p_{333}\chi_{,3} + q_{333}\eta_{,3})_{,3} + \Phi_3 = \rho\ddot{u}_3, \tag{10}$$

$$(p_{333}u_{3,3} - \varsigma_{33}\chi_{,3} - \tilde{a}_{33}\eta_{,3})_{,3} = f_e, \tag{11}$$

$$(q_{333}u_{3,3} - \tilde{a}_{33}\chi_{,3} - \xi_{33}\eta_{,3})_{,3} = 0. \tag{12}$$

Here, Φ_3, ρ and f_e are known functions and we have to solve the system of equations for u_3, χ and η. The general idea of solving this system of equations is to find functions (x_3, χ, η) from the system (10)-(12). Then from (9) can be find conditions for the volume force components Φ_1 and Φ_2 which guarantee the deformation under consideration.

Note that equations (9) and (10) are obtained from motion equation (2), whereas equations (11) and (12) from (3).

2. Oscillation of Piezoelectric Elastic Rod with Variable Constitutive Coefficients

In the following section we consider the case when constitutive coefficients are power functions of spatial variable x_3:

$$\begin{aligned} E_{i333} = E^0_{i333}x_3^\kappa, \quad p_{3i3} = p^0_{3i3}x_3^\kappa, \quad q_{3i3} = q^0_{3i3}x_3^\kappa, \quad \varsigma_{33} = \varsigma^0_{33}x_3^\kappa, \\ \xi_{33} = \xi^0_{33}x_3^\kappa, \quad \tilde{a}_{33} = \tilde{a}^0_{33}x_3^\kappa, \quad x_3 \in [0, L], \qquad i = \overline{1,3}, \end{aligned} \tag{13}$$

where $E^0_{i333}, p^0_{3i3}, q^0_{3i3}, \varsigma^0_{33}, \xi^0_{33}, \tilde{a}^0_{33}, \kappa = const, i = \overline{1,3}$ and $\kappa \geq 0$.

From (13) the system of equations (10)-(12) becomes as follows

$$(E^0_{3333}x_3^\kappa u_{3,3} + p^0_{333}x_3^\kappa\chi_{,3} + q^0_{333}x_3^\kappa\eta_{,3})_{,3} + \Phi_3 = \rho\ddot{u}_3, \tag{14}$$

$$(p^0_{333}x_3^\kappa u_{3,3} - \varsigma^0_{33}x_3^\kappa\chi_{,3} - \tilde{a}^0_{33}x_3^\kappa\eta_{,3})_{,3} = f_e, \tag{15}$$

$$(q^0_{333}x_3^\kappa u_{3,3} - \tilde{a}^0_{33}x_3^\kappa\chi_{,3} - \xi^0_{33}x_3^\kappa\eta_{,3})_{,3} = 0. \tag{16}$$

In the case $\kappa = 0$ from physical considerations it follows that ([35], [36])

$$\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2 > 0.$$

Let farther $E^0_{3333} > 0$. Under these conditions it can be proved that

$$D := \begin{vmatrix} E^0_{3333} & p^0_{333} & q^0_{333} \\ p^0_{333} & -\varsigma^0_{33} & -\tilde{a}^0_{33} \\ q^0_{333} & -\tilde{a}^0_{33} & -\xi^0_{33} \end{vmatrix} > 0. \tag{17}$$

If we consider equations (14)-(16) as algebraic system for $(x_3^\kappa u_{3,3})_{,3}$, $(x_3^\kappa \chi_{,3})_{,3}$ and $(x_3^\kappa \eta_{,3})_{,3}$, then from (17) it is evident, that the system of equations (14)-(16) has a unique solution.

2.1. *Solution of The System of Differential Equations (14)-(16)*

Let the following conditions be fulfilled:

$$u_3(\cdot, t) \in C^2(]0, L[) \cap C([0, L]),$$

$$u_3(x_3, \cdot) \in C^2(t > 0) \cap C^1(t \geq 0), u_3(x_3, t) \in C(0 \leq x_3 \leq L, t \geq 0).$$

Furthermore let $\kappa < 1$ and consider the following homogeneous boundary conditions:

$$u_3(0, t) = u_3(L, t) = \xi(0, t) = \xi(L, t) = \eta(0, t) = \eta(L, t) = 0 \tag{18}$$

and non-homogeneous initial conditions:

$$u_3(x_3, 0) = \varphi_1(x_3), \tag{19}$$

$$\dot{u}_3(x_3, 0) = \varphi_2(x_3). \tag{20}$$

Integration of (14) from L to x_3, dividing both sides of the resulted equation by x_3^κ and integration of the result a second time from L to x_3 gives us the following general equation:

$$\begin{aligned} E^0_{3333} u_3 + p^0_{333}\chi + q^0_{333}\eta - \frac{\rho}{1-\kappa}\int_L^{x_3} (x_3^{1-\kappa} - y^{1-\kappa})\ddot{u}_3(y, t)dy \\ = -\frac{1}{1-\kappa}\int_L^{x_3} (x_3^{1-\kappa} - y^{1-\kappa})\Phi_3(y)dy + \frac{c_{11}}{1-\kappa}(x_3^{1-\kappa} - L^{1-\kappa}) + c_{12}. \end{aligned} \tag{21}$$

Then using boundary conditions (18) we have

$$\begin{gathered} c_{11} = \frac{1}{L^{1-\kappa}}\left[\rho\int_0^L y^{1-\kappa}\ddot{u}_3(y, t)dy - \int_0^L y^{1-\kappa}\Phi_3(y)dy\right], \\ c_{12} = 0. \end{gathered} \tag{22}$$

Substituting (22) into (21) we get

$$\begin{aligned}
&E^0_{3333}u_3(x_3,t)+p^0_{333}\chi(x_3,t)+q^0_{3333}\eta(x_3,t)\\
&\qquad-\frac{\rho}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa}-y^{1-\kappa})\ddot{u}_3(y,t)dy\\
&\qquad=-\frac{1}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa}-y^{1-\kappa})\Phi_3(y,t)dy\\
&\qquad+\frac{x_3^{1-\kappa}-L^{1-\kappa}}{(1-\kappa)L^{1-\kappa}}\left(\rho\int_0^L y^{1-\kappa}\ddot{u}_3(y,t)dy-\int_0^L y^{1-\kappa}\Phi_3(y,t)dy\right).
\end{aligned}\tag{23}$$

Similarly, from (15) and (16) we can express χ and η by u_3 as follows

$$\begin{aligned}
\chi(x_3,t)&=\frac{\xi^0_{33}p^0_{333}-\tilde{a}^0_{33}q^0_{333}}{\xi^0_{33}\varsigma^0_{33}-(\tilde{a}^0_{33})^2}u_3(x_3,t)\\
&-\frac{1}{1-\kappa}\frac{\xi^0_{33}}{\xi^0_{33}\varsigma^0_{33}-(\tilde{a}^0_{33})^2}\int_L^{x_3}(x_3^{1-\kappa}-y^{1-\kappa})f_e(y)dy\\
&-\frac{1}{\xi^0_{33}\varsigma^0_{33}-(\tilde{a}^0_{33})^2}\frac{c_{21}}{1-\kappa}(x_3^{1-\kappa}-L^{1-\kappa})-\frac{c_{22}}{\xi^0_{33}\varsigma^0_{33}-(\tilde{a}^0_{33})^2}.
\end{aligned}\tag{24}$$

$$\begin{aligned}
\eta(x_3,t)&=\frac{\tilde{a}^0_{33}p^0_{333}-\varsigma^0_{33}q^0_{333}}{(\tilde{a}^0_{33})^2-\xi^0_{33}\varsigma^0_{33}}u_3(x_3,t)\\
&-\frac{1}{1-\kappa}\frac{\tilde{a}^0_{33}}{(\tilde{a}^0_{33})^2-\xi^0_{33}\varsigma^0_{33}}\int_L^{x_3}(x_3^{1-\kappa}-y^{1-\kappa})f_e(y)dy\\
&-\frac{1}{(\tilde{a}^0_{33})^2-\xi^0_{33}\varsigma^0_{33}}\frac{c_{31}}{1-\kappa}(x_3^{1-\kappa}-L^{1-\kappa})-\frac{c_{32}}{\xi^0_{33}\varsigma^0_{33}-(\tilde{a}^0_{33})^2}.
\end{aligned}\tag{25}$$

Using boundary conditions (18) from (24) and (25) we get

$$\begin{aligned}
c_{21}&=\frac{\xi^0_{33}}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y)dy,\quad c_{22}=0,\\
c_{31}&=\frac{\tilde{a}^0_{33}}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y)dy,\quad c_{32}=0.
\end{aligned}\tag{26}$$

Finally, using (26) we have

$$\begin{aligned}
&\chi(x_3,t)=\frac{1}{\xi^0_{33}\varsigma^0_{33}-(\tilde{a}^0_{33})^2}\times\\
&\Bigg[(\xi^0_{33}p^0_{333}-\tilde{a}^0_{33}q^0_{333})u_3(x_3,t)-\frac{\xi^0_{33}}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa}-y^{1-\kappa})f_e(y,t)dy\\
&-\frac{\xi^0_{33}(x_3^{1-\kappa}-L^{1-\kappa})}{(1-\kappa)L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y,t)dy\Bigg],
\end{aligned}\tag{27}$$

$$\eta(x_3,t)=\frac{-1}{\xi^0_{33}\varsigma^0_{33}-(\tilde a^0_{33})^2}\times$$
$$\left[(\tilde a^0_{33}p^0_{3333}-\varsigma^0_{33}q^0_{3333})u_3(x_3,t)-\frac{\tilde a^0_{33}}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa}-y^{1-\kappa})f_e(y,t)dy\right. \tag{28}$$
$$\left.-\frac{\tilde a^0_{33}(x_3^{1-\kappa}-L^{1-\kappa})}{(1-\kappa)L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y,t)dy\right].$$

If we substitute (27) and (28) into (23) we obtain

$$\begin{aligned}u_3(x_3,t)+\rho\int_0^L K(x_3,y)\ddot u_3(y,t)dy=-A_2\int_0^L K(x_3,y)f_e(y,t)dy\\ +\int_0^L K(x_3,y)\Phi_3(y,t)dy,\end{aligned} \tag{29}$$

where

$$K(x_3,y)=\frac{1}{(1-\kappa)A_1L^{1-\kappa}}\times\begin{cases}y^{1-\kappa}(L^{1-\kappa}-x_3^{1-\kappa}), & 0\le y\le x_3,\\ x_3^{1-\kappa}(L^{1-\kappa}-y^{1-\kappa}), & x_3\le y\le L,\end{cases} \tag{30}$$

$$A_1=E^0_{3333}+\frac{(p^0_{333})^2\xi^0_{33}-2p^0_{333}q^0_{333}\tilde a^0_{33}+(q^0_{333})^2\varsigma^0_{33}}{\xi^0_{33}\varsigma^0_{33}-(\tilde a^0_{33})^2}, \tag{31}$$

$$A_2=\frac{p^0_{333}\xi^0_{33}-q^0_{333}\tilde a^0_{33}}{\xi^0_{33}\varsigma^0_{33}-(\tilde a^0_{33})^2}. \tag{32}$$

It can be easily proof, that $K(x_3,y)$ is a symmetric kernel (see [9]).
Furthermore, all of the eigenvalues of $K(x,t)$ are real
Using (27) and (28) we obtain

$$\begin{aligned}\chi_{,3}(x_3,t)=\frac{1}{(\xi^0_{33}\varsigma^0_{33}-(\tilde a^0_{33})^2)x_3^\kappa}\left[(p^0_{3333}\xi^0_{33}-q^0_{3333}\tilde a^0_{33})u_{3,3}(x_3,t)x_3^\kappa\right.\\ \left.+\xi^0_{33}\int_{x_3}^L f_e(y,t)dy-\frac{\xi^0_{33}}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y,t)dy\right],\end{aligned} \tag{33}$$

$$\begin{aligned}\eta_{,3}(x_3,t)=-\frac{1}{(\xi^0_{33}\varsigma^0_{33}-(\tilde a^0_{33})^2)x_3^\kappa}\left[(\tilde a^0_{33}p^0_{3333}-\varsigma^0_{33}q^0_{3333})u_{3,3}(x_3,t)x_3^\kappa\right.\\ \left.+\tilde a^0_{33}\int_{x_3}^L f_e(y,t)dy-\frac{\tilde a^0_{33}}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y,t)dy\right].\end{aligned} \tag{34}$$

Substitution of (33)-(34) into (14) gives us the following equation:

$$[A_1u_{3,3}x_3^\kappa]_{,3}-\rho\ddot u_3=F(x_3,t), \tag{35}$$

where

$$F(x_3,t) := A_2 f_e(x_3,t) - \Phi_3(x_3,t). \tag{36}$$

Let us firstly assume that $f_e(x_3,t) \equiv 0$ and $\Phi_3(x_3,t) \equiv 0$ for all $x_3 \in [0,L]$ and $t > 0$. Thus from (36):

$$F(x_3,t) \equiv 0. \tag{37}$$

If we look for $u_3(x_3,t)$ in the following form:

$$u_3(x_3,t) = X(x_3)T(t) \tag{38}$$

then from (35), (37) and (38) we get

$$\frac{\ddot{T}(t)}{T(t)} = \frac{A_1(X_{,3}(x_3)x_3^{\kappa})_{,3}}{\rho X(x_3)} = -\lambda^2 = const. \tag{39}$$

In view of boundary conditions (18) from (38) we have

$$X(0) = X(L) = 0. \tag{40}$$

Therefore, from (29), (38) and (39) we obtain

$$\frac{\ddot{T}(t)}{T(t)} = -\lambda^2 = -\frac{X(x_3)}{\rho \int_0^L K(x_3,y)X(y)dy} \tag{41}$$

and

$$X(x_3) = \lambda^2 \rho \int_0^L K(x_3,y)X(y)dy. \tag{42}$$

Let us prove the following two lemmas:

Lemma 2.1: *Number of λ_n^2 eigenvalues of the equation (42) is not finite.*

Proof: Assume, for the sake of contradiction, that the number of λ_n^2 is finite, and $n = \overline{1,m}$. Then $K(x_3,y)$ can be written as (see, e.g., [9], [27], [31]):

$$K(x_3,y) = \sum_{n=1}^{m} \frac{X_n(x_3)X_n(y)}{\lambda_n^2},$$

where $X_n(x_3) \in C^2(]0,L[)$. Thus

$$K(x_3,y) \in C^2(]0,L[). \tag{43}$$

Then

$$K'(x_3,y)\big|_{y\to x-} - K'(x_3,y)\big|_{y\to x_3+} = -\frac{x_3^{-\kappa}}{A_1},$$

i.e., $K(x_3, y) \notin C^2(]0, L[)$ that contradicts (43). □

Lemma 2.2: *The solution of the problem is oscillatory.*

Proof: From (39) we have

$$X(x_3) = -\frac{A_1}{\lambda^2 \rho}(X_{,3}(x_3) x_3^{\kappa})_{,3}. \tag{44}$$

Without loss of generality $X_n(x_3)$ be orthonormalized eigenfunctions of (44) (see, e.g., [9], [27], [31]), then

$$\lambda_n^2 X_n(x_3) = -\frac{A_1}{\rho}(X_{n,3}(x_3) x_3^{\kappa})_{,3}.$$

If we multiply both sides of the last expression by $X_n(x_3)$ and integrate from 0 to L, we get

$$\lambda_n^2 = -\frac{A_1}{\rho}\int_0^L X_n(x_3)(X_{n,3}(x_3) x_3^{\kappa})_{,3} dx_3 = \frac{A_1}{\rho}\int_0^L (X_{n,3} x_3^{\kappa/2})^2 dx_3.$$

On the other hand, from (17) and (31) we have

$$A_1 = D(\xi_{33}^0 \varsigma_{33}^0 - (\tilde{a}_{33}^0)^2) > 0.$$

Thus, $\lambda_n^2 > 0$. □

Using the result of Lemma 2.2 the solutions of (41) for functions $T_n(t)$ with corresponding eigenvalues λ_n^2 are:

$$T_n(t) = b_1^n sin(\lambda_n^2 t) + b_2^n cos(\lambda_n^2 t).$$

Together with (38) this gives us a formal expression for $u_3(x_3, t)$:

$$u_3(x_3, t) = \sum_{n=1}^{\infty} X_n(x_3)\left(b_1^n sin(\lambda_n^2 t) + b_2^n cos(\lambda_n^2 t)\right). \tag{45}$$

If we formally take the derivative of (45) with respect to time t we obtain

$$\frac{du_3(x_3, t)}{dt} = \sum_{n=1}^{\infty} \lambda_n^2 X_n(x_3)\left(b_1^n cos(\lambda_n^2 t) - b_2^n sin(\lambda_n^2 t)\right). \tag{46}$$

In view of initial conditions (19)-(20) from (45) and (46) we formally have

$$\varphi_1(x_3) = \sum_{n=1}^{\infty} X_n(x_3) b_2^n, \tag{47}$$

$$\varphi_2(x_3) = \sum_{n=1}^{\infty} \lambda_n^2 X_n(x_3) b_1^n. \tag{48}$$

To find expressions for b_1^n and b_2^n let us assume

$$\Psi_\alpha(x_3) := \frac{A_1}{\rho}(\varphi_{\alpha,3}(x_3)x_3^\kappa)_{,3} \in C([0,L]), \quad \alpha = 1,2. \tag{49}$$

If we integrate (49) from L to x_3, divide both sides of the obtained equation by x_3 and integrate the result a second time from L to x_3, under the boundary conditions (18) we get

$$\varphi_\alpha(x_3) = -\rho \int_0^L K(x_3, y)\Psi_\alpha(y)dy, \ \ \alpha = 1,2, \tag{50}$$

where $K(x_3, y)$ is defined by (30).

Since $\Psi_i(\xi) \in C([0,L])$ and $K(x_3,\xi) \in C([0,L] \times [0,L])$ is symmetric, $\varphi_\alpha(x_3)$ can be represented as the following absolutely and uniformly convergent series on the interval $[0,L]$ (see, e.g., [9], [27], [31]):

$$\varphi_\alpha(x_3) = \sum_{n=1}^{\infty} \left(\int_0^L \varphi_\alpha(y) X_n(y) dy \right) X_n(x_3), \qquad \alpha = 1,2. \tag{51}$$

Finally, (51) together with (47) and (48) gives us:

$$b_1^n = \frac{1}{\lambda_n^2} \int_0^L \varphi_2(y) X_n(y) dy, \tag{52}$$

$$b_2^n = \int_0^L \varphi_1(y) X_n(y) dy, \tag{53}$$

Absolute and uniform convergence of the series in the right-hand side (RHS) of (45) and (46), as well as of the series for $x^\kappa u_{3,3}(x_3,t)$ and $(x^\kappa u_{3,3}(x_3,t))_{,3}$ in case of homogeneous problem (see eq. (37)) is proved in Section 2.2.

Now, let us consider the case when $f_e(x_3,t) \not\equiv 0$ and $\Phi_3 \not\equiv 0$. Additionally, let us firstly consider the problem when $\varphi_i(x_3)$ given by initial conditions (19)-(20) are equivalently zero on the interval $x_3 \in [0,L]$.

Let $F(x_3,t) \in L_2([0,L])$. Then $F(x_3,t)$ can be represented as:

$$F(x_3,t) = \sum_{n=1}^{\infty} c_n \phi_n,$$

where ϕ_n form an orthogonal family in $L_2([0,L])$. Then, $F(x_3,t)$ can be represented as a uniformly convergent series:

$$
\begin{aligned}
F(x_3,t) &= \sum_{n=1}^{\infty}(F(x_3,t),X_n(x_3))X_n(x_3) \\
&= \sum_{n=1}^{\infty}\left(\int_0^L F(x_3,t)X_n(x_3)dx_3\right)X_n(x_3) \\
&= \sum_{n=1}^{\infty}F_n(t)X_n(x_3),
\end{aligned}
\tag{54}
$$

where

$$
F_n(t) = \int_0^L F(x_3,t)X_n(x_3)dx_3. \tag{55}
$$

We look for the solution in the form:

$$
u_3(x_3,t) = \sum_{n=1}^{\infty}u_n(x_3,t), \tag{56}
$$

where $u_n(x_3,t)$ is a solution of the problem with $F(x_3,t)$ replaced by $X_n(x_3)F_n(t)$. Using the method of separation of variables we can write:

$$
u_n(x_3,t) = X_n(x_3)T_{1n}(t). \tag{57}
$$

Then from equation (35) we have

$$
\frac{(A_1X_{n,3}(x_3)x_3^{\kappa})_{,3}}{X_n(x_3)} = \frac{\rho\ddot{T}_{1n}(t)+F_n(t)}{T_{1n}(t)} = -\lambda_n^2, \tag{58}
$$

where $X_n(x_3)$ satisfies (42).

If we solve (58) for $T_{in}(t)$ using the method of variation of parameters, then from (56), (57) and initial-boundary conditions, T_{1n} can be written as:

$$
T_{1n} = \frac{\sqrt{\rho}}{\lambda_n^2}\int_0^t F_n(\tau)sin\left(\frac{\lambda_n^2}{\sqrt{\rho}}(t-\tau)\right)d\tau. \tag{59}
$$

Furthermore, from (57) and (59) we get the following series for $u_3(x_3,t)$:

$$
u_3(x_3,t) = \sum_{n=1}^{\infty}\frac{\sqrt{\rho}}{\lambda_n^2}X_n(x_3)\int_0^t\left[\int_0^L F(\xi,\tau)X_n(\xi)d\xi\right]sin\left(\frac{\lambda_n^2}{\sqrt{\rho}}(t-\tau)\right)d\tau. \tag{60}
$$

If $F(.,t)\in C([0,L])$ and $F(x_3,.)\in C(t>0)\cap C^1(t>0)\cap C^2(t>0)$, the proofs of absolute and uniform convergence of the series in the right-hand side of (60), of its first and second order derivatives with respect to time, as well as of the series for $x^{\kappa}u_{3,3}(x_3,t)$ and $(x^{\kappa}u_{3,3}(x_3,t))_{,3}$ are given in Section 2.2.2.

Finally, if $\varphi_i(x_3) \not\equiv 0$ then the solution can be expressed as:

$$u_3(x_3,t) = \sum_{n=1}^{\infty} u_n(x_3,t), \tag{61}$$

where

$$u_n(x_3,t) = X_n(x_3)(T_n + T_{1n}),$$

$X_n(x_3)T_n$ is here given by (45) and $X_n(x_3)T_{1n}$ is given by (60).

The solutions for χ and η can be found by direct substitution of (61) into (24) and (25), correspondingly. The conditions for Φ_1 and Φ_2 can be found from (9):

$$\Phi_\alpha = -(E_{\alpha 333}u_{3,3} + p_{3\alpha 3}\chi_{,3} + q_{3\alpha 3}\eta_{,3})_{,3}, \alpha = 1, 2, \tag{62}$$

where $\chi_{,3}$ and $\eta_{,3}$ are given by (33) and (34), correspondingly. The expression for $u_{3,3}$ can be obtained from (61):

$$u_{3,3}(x_3,t) = \sum_{n=1}^{\infty} X_{n,3}(T_n + T_{1n}),$$

where

$$X_{n,3}(x_3) = -\frac{1}{x_3^\kappa}\frac{\rho\lambda_n^2}{A_1}\int_0^L K_1(x_3,\xi)X_n(\xi)d\xi \tag{63}$$

and

$$K_1(x_3,\xi) = \begin{cases} \dfrac{\xi^{1-\kappa}}{L^{1-\kappa}}, & 0 \le \xi < x_3, \\ \dfrac{\xi^{1-\kappa}}{L^{1-\kappa}} - 1, & x_3 \le \xi \le L. \end{cases}$$

We get expression (63) from (39) using boundary conditions (18).

2.2. *Absolute Uniform Convergence of the Solution*

Remark 1: For simplicity, throughout the following proofs, functions in LHS of inequalities mean the corresponding series.

2.2.1. Convergence of the solution of homogeneous differential equation

Theorem 2.3: *The series in RHS of (47) and (48) are absolutely and uniformly convergent on $x_3 \in [0, L]$.*

Proof: From (39) and (52) we have

$$
\begin{aligned}
b_1^n &= -\frac{A_1}{\lambda_n^4\rho}\int_0^L \left(X_{n,3}(x_3)x_3^\kappa\right)_{,3}\varphi_2(x_3)dx_3 \\
&= \frac{A_1}{\lambda_n^4\rho}\int_0^L X_{n,3}(x_3)x_3^\kappa\varphi_{2,3}(x_3)dx_3 = -\frac{A_1}{\lambda_n^4\rho}\int_0^L X_n(x_3)\varphi_2(x_3)dx_3.
\end{aligned} \tag{64}
$$

Analogously:

$$
b_2^n = -\frac{A_1}{\lambda_n^2\rho}\int_0^L X_n(x_3)\varphi_1(x_3)dx_3. \tag{65}
$$

As the series given in RHS of (51) is absolutely and uniformly convergent on $[0,L]$, and $K(x_3,\xi)\in C([0,L]\times[0,L])$, from (42) and (65) we have

$$
\begin{aligned}
|\varphi_1| &\le \sum_{n=1}^{\infty}|X_n(x_3)b_2^n| = \sum_{n=1}^{\infty}\left|\lambda_n^2\rho\int_0^L K(x_3,y)X_n(y)b_2^n dy\right| \\
&\le |A_1|\sum_{n=1}^{\infty}\left|\int_0^L K(x_3,y)\left[\int_0^L X_n(\xi)\varphi_1(\xi)X_n(y)d\xi\right]dy\right| \\
&\le |A_1|\int_0^L |K(x_3,y)|\sum_{n=1}^{\infty}\left|\int_0^L X_n(\xi)\varphi_1(\xi)X_n(y)d\xi\right|dy \\
&\le |A_1|\int_0^L |K(x_3,y)|M(y)dy \le |A_1|M\int_0^L |K(x_3,y)|dy < \infty,
\end{aligned}
$$

where

$$
M(y) = \sum_{n=1}^{\infty}\left|\int_0^L X_n(\xi)\varphi_1(\xi)X_n(y)d\xi\right|.
$$

RHS of last expression is absolutely and uniformly convergent on $[0,L]$ and $M = \underset{0\le y\le L}{max}\, M(y)$. Analogously using (64) we have

$$
|\varphi_2| \le \sum_{n=1}^{\infty}\left|\lambda_n^2 X_n(x_3)b_1^n\right| < \infty.
$$

□

Theorem 2.4: *The series in RHS of (45), as well as its first and second order derivatives with respect to time t is absolutely and uniformly convergent on $x_3\in[0,L]$.*

Proof: From (45), using results from Theorem 2.3, we have

$$|u_3(x_3,t)| \leq \sum_{n=1}^{\infty} |X_n(x_3)||b_1^n| \left|sin\left(\lambda_n^2 t\right)\right| + \sum_{n=1}^{\infty} |X_n(x_3)||b_2^n| \left|cos\left(\lambda_n^2 t\right)\right|$$
$$\leq \frac{1}{\lambda_0} \sum_{n=1}^{\infty} \left|\lambda_n^2 X_n(x_3) b_1^n\right| + \sum_{n=1}^{\infty} |X_n(x_3) b_2^n| < \infty.$$

From (46) and absolute uniform convergence of the series in RHS of (51) we have

$$|\dot{u}_3(x_3,t)| \leq \sum_{n=1}^{\infty} \lambda_n^2 |X_n(x_3)||b_1^n cos(\lambda_n^2 t)| + \sum_{n=1}^{\infty} \lambda_n^2 |X_n(x_3)||b_2^n sin(\lambda_n^2 t)|$$
$$\leq \frac{A_1}{\lambda_0^2 \rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_2(\xi) d\xi \right| + \frac{A_1}{\lambda_0 \rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_1(\xi) d\xi \right|$$
$$\leq \frac{A_1}{\lambda_0^2 \rho} M_2(x_3) + \frac{A_1}{\lambda_0 \rho} M_1(x_3) < \infty,$$

where $\lambda_0^2 := \min_n \lambda_n^2$ and

$$M_\alpha(x_3) := \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_\alpha(\xi) d\xi \right|. \tag{66}$$

Analogously,

$$|\ddot{u}_3(x_3,t)| \leq \frac{A_1}{\lambda_0 \rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_2(\xi) d\xi \right|$$
$$+ \frac{A_1}{\rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_1(\xi) d\xi \right| < \infty.$$

□

Remark 2: As it was proved in Lemma 2.1, eigenvalues λ_n^2 are infinite in number. Furthermore, it was stated, that the kernel $K(x_3, y)$ given by (30) is symmetric. In ([31]) it is shown, that in case of symmetric kernel, to each eigenvalue belongs the normalized orthogonal system of eigenfunctions and there exists at least one eigenvalue. Additionally, if they are infinite in number, they form a denumerable set and they may be arranged in the order of magnitude of their absolute values:

$$|\lambda_1^2| \leq |\lambda_2^2| \leq ... \leq |\lambda_n^2| \leq ....$$

Consequently, we can chose λ_0^2 such that $\lambda_0^2 := \min_n \lambda_n^2$.

Theorem 2.5: *The corresponding series of $x_3^\kappa u_{3,3}(x_3,t)$ is absolutely and uniformly convergent on $x_3 \in [0, L]$.*

Proof: Together with (45) and (64)-(65):

$$
\begin{aligned}
|x_3^\kappa u_{3,3}\ (x_3,t)| &= \left| x_3^\kappa \sum_{n=1}^{\infty} X_{n,3}(x_3)\left(b_1^n sin\left(\lambda_n^2 t\right) + b_2^n cos\left(\lambda_n^2 t\right)\right)\right| \\
&= \left| \frac{\rho}{A_1}\sum_{n=1}^{\infty}\lambda_n^2 \int_0^L K_1(\xi)X_n(\xi)\left(b_1^n sin\left(\lambda_n^2 t\right) + b_2^n cos\left(\lambda_n^2 t\right)\right) d\xi\right| \\
&\leq M\left[\frac{1}{\lambda_0^2}\sum_{n=1}^{\infty}\int_0^L \left|X_n(\xi)\int_0^L X_n(\eta)\varphi_2(\eta)d\eta\right| d\xi\right. \\
&\left. + \sum_{n=1}^{\infty}\int_0^L \left|X_n(\xi)\int_0^L X_n(\eta)\varphi_1(\eta)d\eta\right| d\xi\right] \\
&\leq M\left[\frac{M_2}{\lambda_0} + M_1\right] < \infty,
\end{aligned}
$$

where $M := \underset{\xi}{max} K_1(\xi)$, M_α, $\alpha = 1, 2$ is defined by (66). □

Theorem 2.6: *The corresponding series of* $(x_3^\kappa u_{3,3}(x_3,t))_{,3}$ *is absolutely and uniformly convergent on* $x_3 \in (0, L]$.

Proof: Using the result of Theorem 2.5 and proceeding in the same way, we get

$$
\begin{aligned}
\left|\left(x_3^\kappa u_{3,3}(x_3,t)\right)_{,3}\right| &= \left|\kappa x_3^{\kappa-1} u_{3,3} + x_3^\kappa u_{3,33}\right| \\
&\leq \frac{2\kappa\rho}{x_3 A_1}\left|\sum_{n=1}^{\infty}\lambda_n^2 \int_0^L K_1(\xi)X_n(\xi)\left(b_1^n sin\left(\lambda_n t\right)\right.\right. \\
&\left.\left. + b_2^n cos\left(\lambda_n t\right)\right) d\xi\right| \leq \frac{2\kappa C^*}{x_3},
\end{aligned}
$$

where C^* is a constant such that $|x^\kappa u_{3,3}(x_3,t)| \leq C^*$ (see Theorem 2.5). □

2.2.2. Convergence of the solution of non-homogeneous differential equation

Remark 3: Note, that from (42) and (54) $F(x_3,t)$ can be written in the form:

$$
F(x_3,t) = \int_0^L K(x_3,y)g(x_3,y,t)dy,
$$

where $g(x_3,y,t) \in C([0,L],[0,L],t > 0)$.

Theorem 2.7: *If* $F(x_2,t) \in C([0,L],t > 0)$, *the series in RHS of (60) is absolutely and uniformly convergent on* $x_3 \in [0, L]$.

Proof:

$$
|u_3(x_3,t)| \leq \left|\frac{1}{\rho}\int_0^t \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n^2}\int_0^L F(\xi,\tau)X_n(x_3)(\xi)d\xi\right) X_n(x_3)\right|.
$$

If conditions of the theorem hold for $F(x_3,t)$ then by virtue of Remark **3** (see, e.g., [9], [27], [31])

$$\sum_{n=1}^{\infty}\left(\int_0^L F(\xi,\tau)X_n(x_3)(\xi)d\xi\right)X_n(x_3)$$

is absolutely and uniformly convergent on $[0,L]$, thus

$$\sum_{n=1}^{\infty}\left(\int_0^L F(\xi,\tau)X_n(x_3)(\xi)d\xi\right)X_n(x_3)\le c(\tau)$$

and

$$|u_3(x_3,t)|\le\left|\frac{1}{\rho\lambda_0^2}\int_0^t c(\tau)d\tau\right|<\infty,\quad \lambda_0^2:=\min_n\lambda_n^2.$$

□

Theorem 2.8: *If $F(.,t)\in C([0,L])$ and $F(x_3,.)\in C(t>0)\cap C^1(t>0)\cap C^2(t>0)$ then first and second order derivatives of the series given in RHS of (60) with respect to time is absolutely and uniformly convergent on $x_3\in[0,L]$.*

Proof: Similarly to the proof of Theorem (2.7) we have

$$|\dot{u}_3(x_3,t)|\le\left|\frac{1}{\rho}\int_0^t\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n^2}\int_0^L \dot{F}(\xi,\tau)X_n(x_3)(\xi)d\xi\right)X_n(x_3)\right|$$
$$+\left|\frac{1}{\rho}\int_0^t\sum_{n=1}^{\infty}\left(\int_0^L F(\xi,\tau)X_n(x_3)(\xi)d\xi\right)X_n(x_3)\right|<\infty.$$

□

Theorem 2.9: *If $F(x_2,t)\in C([0,L],t>0)$, the corresponding series of $x_3^{\kappa}u_3(x_3,t)$ is absolutely and uniformly convergent on $x_3\in[0,L]$.*

Proof: Similarly to the proof of Theorem 2.7:

$$|x_3^{\kappa}u_3(x_3,t)|\le\left|\frac{x_3^{\kappa}}{\rho\lambda_0^2}\int_0^t c(\tau)d\tau\right|<\infty.$$

□

Theorem 2.10: *If $F(x_2,t)\in C([0,L],t>0)$, the corresponding series of $(x_3^{\kappa}u_3(x_3,t))_{,3}$ is absolutely and uniformly convergent on $x_3\in(0,L]$.*

Proof: Similarly to the proofs of Theorem 2.7 and Theorem 2.9 we can show that

$$|(x_3^{\kappa}u_3(x_3,t))_{,3}|\le\left|\frac{x_3^{\kappa}}{\rho\lambda_0^2}\int_0^t c(\tau)d\tau\right|+\left|\frac{\kappa x_3^{\kappa-1}}{\rho\lambda_0^2}\int_0^t c(\tau)d\tau\right|<\infty.$$

□

3. Conclusions

A problem for non-homogeneous piezoelectric elastic rod is studied in the case when constitutive coefficients vary from zero as power function of spatial variable x_3, i.e. equal to $const. \times x_3^\kappa$, $\kappa = const. \in [0,1)$. It is assumed that all other functions depend on time t and/or spatial variable x_3, with prescribed charge density (f_e) and volume force component (Φ_3). The well-posedness of initial-boundary value problem is studied. The conditions on the volume force components Φ_1 and Φ_2 which guarantee the deformation under consideration are established.

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