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On the Roots of Complex r × r Matrices

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The pseudo-Chebyshev functions, recently introduced, have been used in a preceding article in order to compute the roots of 2×2 and 3×3 non-singular complex matrices. In this article another method is shown, which works, in some case, even for singular matrices (provided that such roots exist), based on the $F_{k,n}$ functions, that is the functions which constitute a basis for the solutions of linear recurrence relations. A representation formula of these functions by means of a contour integral encircling the eigenvalues and the use of Cauchy's residue theorem gives the possibility to derive a closed form for roots of $r \times r$ matrices based on the knowledge of the matrix invariants.

Keywords: Basic solution of linear recurrences, Matrix powers, Matrix roots.

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1. Introduction

The problem of finding roots of a $r \times r$ matrix [8] is in general hard to be solved in his generality, since there exist matrices without roots (for instance the Jordan blocks, see e.g. [30]), and other which have infinite many roots (for instance , see e.g. [15]). Some papers on this subject, mainly referred to 2×2 matrices, can be found in the Mathematical Gazette [11, 28, 30] or in Linear Algebra journals [6, 10, 18]. N.J. Higham [9] proposed a numerical technique for computing matrix square root applying Newton-Rhapson's method.
The Cayley-Hamilton Theorem was applied to compute roots of a 2×2 non-singular matrices by I.A. al-Tamimi [1], and by S.S. Rao et al. [20] for $n \times n$ matrices with non-negative distinct eigenvalues, since this subject appears while solving certain differential equations related to Markov models of finance, and related topics.
P.J. Psarrakos [18] gave a necessary and sufficient condition for the existence of mth roots of a singular complex matrix $\mathcal{A}$, in terms of the dimensions of the null spaces of the powers $\mathcal{A}^k$, $(k = 0, 1, 2, \dots)$.
In a recent article [26] a method for finding matrix roots for non-singular 2×2 or 3×3 has been proposed. This method uses the Cayley-Hamilton Theorem in order to derive matrix powers [21, 22] and the pseudo-Chebyshev functions introduced in [23–25]. However the technique considered there can hardly be extended to higher order matrices, since it would be necessary to use a difficult inductive procedure.
But there is another approach used in past time to construct matrix powers which

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makes use of the $F_{k,n}$ functions (see [3–5, 19]). These functions, which will be denoted for shortness by the acronym FKN, are related to the Lucas polynomials of the second kind and to the multivariate Chebyshev polynomials [5].
It is worth to note thate the FKN have been used even for solution of linear dynamical systems [16, 17].
In an old paper [3], largely unnoticed because it was written in Italian, a Cauchy type integral representation formula for the FKN, was proven. This representation makes possible to define the FKN even for rational values of their indexes, and gives the key to extend the matrix powers formulas given in [5] to the case of matrix roots.
This technique is used in what follows, and gives the possibility to construct the n-th roots of matrices based on the knowledge of the matrix invariants, which are the elementary symmetric functions of the eigenvalues.
Since matrix properties depend on the hidden numbers, i.e. by the eigenvalues of $\mathcal{A}$, the proposed technique could be defined of "canonical" type. In fact it ignores other methods, based, for instance, on indeterminate entries, by means of which the roots of identity matrices are derived.
By using this procedure, we can find at most a finite number of roots, and more precisely, since the obtained equations depend on the r complex roots of eigenvalues, we find n^r possible values for the n-th root of $\mathcal{A}$. Of course, sometimes it is sufficient to derive a less number of roots, and then to change the determinations for deriving the other ones.
The paper is presented as follows: first we recall the FKN and their connection with powers of matrices. Then, by using a contour integral representation for the FKN, which makes sense even for rational values of powers, the representation of matrix powers in terms of the FKN is extended to the case of matrix roots. By using Cauchy's residue theorem an explicit expression for matrix powers is derived.
Some developed examples are shown in Sect. 4, and a list of previous papers is collected in the Bibliography section, with particular reference to recent articles, which generally show particular techniques and do not frame the problem in its generality.

2. Recalling $\mathbf{F_{k,n}}$ functions and Lucas polynomials of the second kind

A basis for the r-dimensional vectorial space $\mathcal{V}_r$ of solutions of the $(r+1)$-terms homogeneous linear bilateral recursion with complex coefficients u_k $(k = 1, 2, \ldots, r)$ (with $u_r \neq 0$):

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \cdots + (-1)^{r-1} u_r X_{n-r}\,, \qquad (n \in \mathbf{Z})\,. \tag{2.1}$$

is given by the functions $F_{k,n} = F_{k,n}(u_1, u_2, \ldots, u_r)$, $(k = 1, 2, \ldots, r\,,\ n \geq -1)$, defined by the initial conditions:

$$\begin{array}{cccc}
F_{1,-1} = 0 & F_{1,0} = 0 & \ldots & F_{1,r-2} = 1, \\
F_{2,-1} = 0 & F_{2,0} = 1 & \ldots & F_{2,r-2} = 0, \\
\ldots\ \ \ldots & \ldots\ \ \ldots & \ldots & \ldots\ \ \ldots \\
F_{r,-1} = 1 & F_{r,0} = 0 & \ldots & F_{r,r-2} = 0.
\end{array} \tag{2.2}$$

Since we have assumed $u_r \neq 0$, the FKN are defined even when $n < -1$, putting:

$$F_{k,n}(u_1, \dots, u_r) = F_{r-k+1,-n+r-3}\left(\frac{u_{r-1}}{u_r}, \dots, \frac{u_1}{u_r}, \frac{1}{u_r}\right), \qquad (k = 1, \dots, r;\ n \in \mathbf{Z})\,. \tag{2.3}$$

Therefore, any solution of the recursion (2.1) is a linear combination of the FKN.

Remark 2.1 – It is worth to recall that another basis for the $\mathcal{V}_r$ space is usually obtained by using the roots of the characteristic equation

$$\lambda^r - u_1\lambda^{r-1} + \dots + (-1)^{r-1}u_{r-1}\lambda + (-1)^r u_r = 0\,, \qquad (r \in \mathbf{Z})\,, \tag{2.4}$$

however, this implies the knowledge of roots, whereas the use of the FKN is independent of that. Furthermore, the solution based on the FKN does not depend of the multiplicity of roots.

An important result, originally stated by É Lucas [14] in the case $r = 2$, is given by the equations

$$\begin{cases} F_{1,n} = u_1 F_{1,n-1} + F_{2,n-1} \\ F_{2,n} = -u_2 F_{1,n-1} + F_{3,n-1} \\ \dots \\ F_{r-1,n} = (-1)^{r-2}u_{r-1}F_{1,n-1} + F_{r,n-1} \\ F_{r,n} = (-1)^{r-1}u_r F_{1,n-1} \end{cases} \tag{2.5}$$

showing that all the FKN are expressed by the sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$.
Therefore, we assume the following

Definition 2.1 – The bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$, that is the solution of *(2.1)* corresponding to the initial conditions:

$$F_{1,-1} = 0\,, \quad F_{1,0} = 0\,, \quad \dots\,, \quad F_{1,r-3} = 0\,, \quad F_{1,r-2} = 1\,, \tag{2.6}$$

is called the *fundamental solution* of *(2.1)*, that is the “fonction fondamentale” by É. Lucas *[14]*.

Putting

$$F_{1,n}(u_1, \dots, u_r) =: \Phi_n(u_1, \dots, u_r) = \Phi_n\,, \qquad (n \in \mathbf{Z})\,. \tag{2.7}$$

For $n \geq -1$, the $\Phi_n(u_1, \dots, u_r)$ functions are called in literature [19] *Lucas polynomials of the second kind in r variables.*

Remark 2.2 – Note that, for $r = 2$, $u_2 = 1$, putting $u_1 = x$, we find

$$\Phi_n(u_1, 1) = \Phi_n(x, 1) \equiv U_n\left(\frac{x}{2}\right), \qquad (n \in \mathbf{N}_0)\,,$$

where $\{U_n(x)\}_{n\in\mathbf{N}_0}$ are the second kind Chebyshev polynomials.
Therefore, for $r \geq 3$, putting $u_r = 1$, the $(r-1)$-variable Chebyshev polynomials of the second kind have been introduced, putting:

$$\Phi_n(u_1, \dots, u_{r-1}, 1) =: U_n^{(r-1)}(u_1, \dots, u_{r-1}), \qquad (n \in \mathbf{N}_0)\,,$$

see, for instance: R. Lidl, C. Wells [12], R. Lidl [13], M. Bruschi, P.E. Ricci [5], K.B. Dunn, R. Lidl [7], R.J. Beerends [2].
Note that the choice of indexes in equations (2.2) and (2.3) was made in such a way as to find the Chebyshev polynomials with the same indexes in case $r = 2$.

3. Matrix powers representation

In preceding articles [4, 21], the following result was proved:

Theorem 3.1 – Given an $r \times r$ matrix $\mathcal{A}$, putting by definition $u_0 := 1$, and denoting by

$$P(\lambda) := \det(\lambda\mathcal{I} - \mathcal{A}) = \sum_{j=0}^{r}(-1)^j u_j \lambda^{r-j} \tag{3.1}$$

its characteristic polynomial, the matrix powers $\mathcal{A}^n$, with integer exponent n, are represented by the equation:

$$\begin{aligned}\mathcal{A}^n = F_{1,n-1}(u_1, \dots, u_r)\, \mathcal{A}^{r-1} + F_{2,n-1}(u_1, \dots, u_r)\, \mathcal{A}^{r-2} + \\ + \dots + F_{r,n-1}(u_1, \dots, u_r)\, \mathcal{I}\,,\end{aligned} \tag{3.2}$$

where the *FKN* are defined in Section *2*.
Moreover, if $\mathcal{A}$ is not singular, i.e. $u_r \neq 0$, equation (3.2) still works for negative integers n, assuming the definition (2.3) for the *FKN*.

Of course, if the degree of the minimal polynomial of $\mathcal{A}$ is $q < r$, the linear combination of powers in equation (3.2) is reduced to the degree $q - 1$.
It is worth to recall that the knowledge of eigenvalues is equivalent to that of invariants, since the second ones are the elementary symmetric functions of the first ones.

Remark 3.1 – Note that, as a consequence of the above result, the higher powers of an $r \times r$ matrix $\mathcal{A}$ are always expressible in terms of the lowers ones (at most up to the power $q - 1$, if q is the degree of the minimal polynomial of $\mathcal{A}$).

In a preceding article [3], the following representation formulas for the FKN have been established:

$$F_{k,n-1}(u_1, \dots, u_r) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^n \sum_{h=0}^{k-1}(-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\,, \tag{3.3}$$

where γ denotes a closed positively oriented (i.e. traversed counterclockwise) contour encircling all the zeros of $P(\lambda)$, for example a circle centered at the origin

whose radius is greater then the spectral radius of $\mathcal{A}$.
Therefore, equation (3.2) can be written as

$$\mathcal{A}^n = \frac{1}{2\,\pi\,\mathrm{i}}\left[\sum_{k=1}^{r}\oint_\gamma \frac{\lambda^n\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)}\;d\lambda\;\mathcal{A}^{r-k}\right]\;. \tag{3.4}$$

Noting that equation (3.3) makes sense even if n is a fractional number, we can put:

$$F_{k,\frac{1}{n}-1}(u_1,\ldots,u_r) = \frac{1}{2\,\pi\,\mathrm{i}}\oint_\gamma \frac{\lambda^{1/n}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)}\;d\lambda\;. \tag{3.5}$$

Therefore, by equations (3.2)-(3.5), we find:

$$\begin{aligned}\mathcal{A}^{1/n} &= \sum_{k=1}^{r} F_{k,\frac{1}{n}-1}(u_1,\ldots,u_r)\,\mathcal{A}^{r-k} = \\ &= \frac{1}{2\,\pi\,\mathrm{i}}\left[\sum_{k=1}^{r}\oint_\gamma \frac{\lambda^{1/n}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)}\;d\lambda\;\mathcal{A}^{r-k}\right]\;.\end{aligned} \tag{3.6}$$

Recalling Cauchy's residue theorem [27], and denoting by $\lambda_1,\lambda_2,\ldots,\lambda_r$ the eigenvalues of $\mathcal{A}$, and by $f = f(\lambda)$ the integrand in equation (3.6), the contour integral is given by:

$$\oint_\gamma \frac{\lambda^{1/n}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)}\;d\lambda = (2\,\pi\,\mathrm{i})\sum_{\ell=1}^{r} Res_{f_k}(\lambda_\ell)\,. \tag{3.7}$$

Supposing, for simplicity, the eigenvalues are all distinct, and putting

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_r),$$

we find:

$$\begin{aligned}\sum_{\ell=1}^{r} Res_{f_k}(\lambda_\ell) &= \sum_{\ell=1}^{r}\lim_{\lambda\to\lambda_\ell}(\lambda-\lambda_\ell)\;\frac{\lambda^{1/n}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda^{k-h-1}}{P(\lambda)} = \\ &= \sum_{\ell=1}^{r}\;\frac{\lambda_\ell^{1/n}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda_\ell^{k-h-1}}{(\lambda_\ell-\lambda_1)\cdots(\lambda_\ell-\lambda_{\ell-1})(\lambda_\ell-\lambda_{\ell+1})\cdots(\lambda_\ell-\lambda_r)}\;,\end{aligned} \tag{3.8}$$

where we have put, by definition: $(\lambda-\lambda_0) = (\lambda-\lambda_{r+1}) := 1$.
Then, equation (3.6) becomes:

$$\mathcal{A}^{1/n} = \sum_{k=1}^{r}\sum_{\ell=1}^{r}\;\frac{\lambda_\ell^{1/n}\,\sum_{h=0}^{k-1}(-1)^h u_h\,\lambda_\ell^{k-h-1}}{(\lambda_\ell-\lambda_1)\cdots(\lambda_\ell-\lambda_{\ell-1})(\lambda_\ell-\lambda_{\ell+1})\cdots(\lambda_\ell-\lambda_r)}\,\mathcal{A}^{r-k}\,. \tag{3.9}$$

A similar result can be found in case of multiple roots of the characteristic polynomial, by using the more general equation, which holds for a pole of order m at the point λ_ℓ:

$$Res_{f_k}(\lambda_\ell) = \frac{1}{(m-1)!} \lim_{\lambda \to \lambda_\ell} \frac{d^{m-1}}{d\lambda^{m-1}} \left[(\lambda - \lambda_\ell)^m f(\lambda) \right] . \tag{3.10}$$

Therefore, we can proclaim the following resut:

Theorem 3.2 Given a $r \times r$ complex matrix $\mathcal{A}$, and denoting by

$$u_1 = tr\mathcal{A},\ u_2,\ u_3,\ \ldots,\ u_r = \det \mathcal{A}$$

its invariants, and by $\lambda_1, \lambda_2, \ldots, \lambda_r$ its eigenvalues, a n-th root of $\mathcal{A}$ is given by equation *(3.6)*.
If the eigenvalues are all distinct, equation *(3.6)*, by Cauchy's residue theorem reduces to equation *(3.9)*. By using equation *(3.10)* the result can be extended to the case of multiple eigenvalues.
Since the $(1/n)$-powers appearing in equation *(3.9)* have n determinations, by using this method we can find at most n^r roots of $\mathcal{A}$.

Remark 3.2 Note that the knowledge of eigenvalues is not strictly necessary. It is mandatory if we compute the integral in equation (3.6) by Cauchy's residue theorem, but actually only the knowledge of the invariants is necessary, since we could compute the contour integral by choosing as γ a circle centered at the origin with radius greater then the spectral radius of $\mathcal{A}$.

4. Examples

4.1. *A square root for a 3×3 matrix*

Let $\mathcal{A}$ be a 3×3 matrix, put $n = 2$ and

$$P(\lambda) = \lambda^3 - u_1\lambda^2 + u_2\lambda - u_3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

Then equation (3.9) has the form:

$$\mathcal{A}^{1/2} = \frac{1}{2\,\pi\,\mathrm{i}} \left[\oint_\gamma \frac{\lambda^{1/2}\, d\lambda}{P(\lambda)}\, \mathcal{A}^2 + \oint_\gamma \frac{\lambda^{1/2}(\lambda - u_1)\, d\lambda}{P(\lambda)}\, \mathcal{A} + \oint_\gamma \frac{\lambda^{1/2}(\lambda^2 - u_1\lambda + u_2)\, d\lambda}{P(\lambda)}\, \mathcal{I} \right] ,$$

and by using Cauchy's residue theorem we find:

$$
\begin{aligned}
\mathcal{A}^{1/2} = {} & \frac{\lambda_1^{1/2}(\lambda_2-\lambda_3)-\lambda_2^{1/2}(\lambda_1-\lambda_3)+\lambda_3^{1/2}(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\,\mathcal{A}^2 + \\
& + \frac{\lambda_1^{1/2}(\lambda_2-\lambda_3)(\lambda_1-u_1)-\lambda_2^{1/2}(\lambda_1-\lambda_3)(\lambda_2-u_1)+\lambda_3^{1/2}(\lambda_1-\lambda_2)(\lambda_3-u_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\,\mathcal{A} + \\
& + \Bigg[\frac{\lambda_1^{1/2}(\lambda_2-\lambda_3)(\lambda_1^2-u_1\lambda_1+u_2)-\lambda_2^{1/2}(\lambda_1-\lambda_3)(\lambda_2^2-u_1\lambda_2+u_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} + \\
& + \frac{\lambda_3^{1/2}(\lambda_1-\lambda_2)(\lambda_3^2-u_1\lambda_3+u_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\Bigg]\,\mathcal{I}\,.
\end{aligned}
\tag{4.1}
$$

4.1.1. Numerical example 1.

Consider the matrix

$$
\mathcal{A} = \begin{Bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{Bmatrix}, \qquad \text{so that} \qquad \mathcal{A}^2 = \begin{Bmatrix} 2 & 2 & 1 \\ -3 & -2 & 0 \\ 2 & 1 & 0 \end{Bmatrix}. \tag{4.2}
$$

The invariants are:

$$
u_1 = 2\,, \quad u_2 = 2, \quad u_3 = 1\,. \tag{4.3}
$$

The characteristic equation is:

$$
\lambda^3 - 2\lambda^2 + 2\lambda - 1 = 0\,, \tag{4.4}
$$

and the roots are:

$$
\lambda_1 = \frac{1+\mathrm{i}\sqrt{3}}{2}\,, \quad \lambda_2 = \frac{1-\mathrm{i}\sqrt{3}}{2}\,, \quad \lambda_3 = 1\,. \tag{4.5}
$$

According to equation (4.1), choosing the positive sign for the square roots appearing below, we find:

$$
\lambda_1-\lambda_2 = \mathrm{i}\sqrt{3}\,, \qquad \lambda_1-\lambda_3 = \frac{1}{2}(-1+\mathrm{i}\sqrt{3})\,, \qquad \lambda_2-\lambda_3 = \frac{1}{2}(-1-\mathrm{i}\sqrt{3})\,.
$$

$$
(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3) = \mathrm{i}\sqrt{3}\,,
$$

$$
\lambda_1^{1/2} = \frac{1}{2}(\mathrm{i}+\sqrt{3})\,, \qquad \lambda_2^{1/2} = \frac{1}{2}(-\mathrm{i}+\sqrt{3})\,, \qquad \lambda_3^{1/2} = 1\,, \tag{4.6}
$$

$$
\lambda_1^{1/2}(\lambda_2-\lambda_3) = -\mathrm{i}\,, \qquad \lambda_2^{1/2}(\lambda_1-\lambda_3) = \mathrm{i}\,, \qquad \lambda_3^{1/2}(\lambda_1-\lambda_2) = \mathrm{i}\sqrt{3}\,,
$$

so that the coefficient of $\mathcal{A}^2$ is: $\frac{-2\,\mathrm{i}+\mathrm{i}\sqrt{3}}{\mathrm{i}\sqrt{3}} = \frac{\sqrt{3}-2}{\sqrt{3}}$.
Moreover, recalling that $u_1 = u_2 = 2$, by elementary computations, we find the other coefficients in equation (4.1).
The coefficient of $\mathcal{A}$ is: $\frac{3-\sqrt{3}}{\sqrt{3}}$, and the coefficient of $\mathcal{I}$ is: $\frac{\sqrt{3}-1}{\sqrt{3}}$.
Then equation (4.1) has the form:

$$\mathcal{A}^{1/2} = \frac{\sqrt{3}-2}{\sqrt{3}}\,\mathcal{A}^2 + \frac{3-\sqrt{3}}{\sqrt{3}}\,\mathcal{A} + \frac{\sqrt{3}-1}{\sqrt{3}}\,\mathcal{I}\,,$$

that is:

$$\mathcal{A}^{1/2} = \frac{1}{\sqrt{3}} \begin{Bmatrix} \sqrt{3}+1 & \sqrt{3}-1 & \sqrt{3}-2 \\ -\sqrt{3} & 3-\sqrt{3} & 3-\sqrt{3} \\ \sqrt{3}-1 & \sqrt{3}-2 & \sqrt{3}-1 \end{Bmatrix}. \tag{4.7}$$

It is easily seen that by equation (4.7) it follows:

$$\left[\mathcal{A}^{1/2}\right]^2 = \mathcal{A}\,.$$

By choosing different signs for the square roots considered in equation (4.6), we find other possible determinations for the square root of $\mathcal{A}$:

$$\mathcal{A}^{1/2} = \frac{1}{\sqrt{3}} \begin{Bmatrix} -\sqrt{3}+1 & -\sqrt{3}-1 & -\sqrt{3}-2 \\ \sqrt{3} & 3+\sqrt{3} & 3+\sqrt{3} \\ -\sqrt{3}-1 & -\sqrt{3}-2 & -\sqrt{3}-1 \end{Bmatrix}, \tag{4.8}$$

and

$$\mathcal{A}^{1/2} = \frac{1}{\sqrt{3}} \begin{Bmatrix} -\sqrt{3}-1 & -\sqrt{3}+1 & -\sqrt{3}+2 \\ \sqrt{3} & -3+\sqrt{3} & -3+\sqrt{3} \\ -\sqrt{3}+1 & -\sqrt{3}+2 & -\sqrt{3}+1 \end{Bmatrix}, \tag{4.9}$$

and

$$\mathcal{A}^{1/2} = \frac{1}{\sqrt{3}} \begin{Bmatrix} \sqrt{3}-1 & \sqrt{3}+1 & \sqrt{3}+2 \\ -\sqrt{3} & -3-\sqrt{3} & -3-\sqrt{3} \\ \sqrt{3}+1 & \sqrt{3}+2 & \sqrt{3}+1 \end{Bmatrix}, \tag{4.10}$$

4.2. *A cubic root for a 2×2 matrix*

4.2.1. Numerical example 2.

Consider the matrix

$$\mathcal{A} = \begin{Bmatrix} 9 & 1 \\ -8 & 0 \end{Bmatrix} \tag{4.11}$$

The invariants are:

$$u_1 = 9\,, \quad u_2 = 8\,. \tag{4.12}$$

The characteristic equation is:

$$\lambda^2 - 9\lambda + 8 = 0\,, \tag{4.13}$$

and the roots are:

$$\lambda_1 = 8\,, \quad \lambda_2 = 1\,. \tag{4.14}$$

We will consider only the real cubic roots for the eigenvalues, so that we find only the real solution for $\mathcal{A}^{1/3}$. Of course the possible cubic roots are $3^2 = 9$.
Equation (3.9) has the form:

$$\mathcal{A}^{1/3} = \frac{\lambda_1^{1/3} - \lambda_2^{1/3}}{\lambda_1 - \lambda_2}\,\mathcal{A} + \frac{\lambda_1^{4/3} - \lambda_2^{4/3} - u_1(\lambda_1^{1/3} - \lambda_2^{1/3})}{\lambda_1 - \lambda_2}\,\mathcal{I}\,, \tag{4.15}$$

that is:

$$\mathcal{A}^{1/3} = \frac{1}{7}\,\mathcal{A} + \frac{6}{7}\,\mathcal{I}\,, \tag{4.16}$$

so that

$$\mathcal{A}^{1/3} = \frac{1}{7}\left\{\begin{array}{cc} 15 & 1 \\ -8 & 6 \end{array}\right\}\,. \tag{4.17}$$

It is easily seen that

$$\left[\mathcal{A}^{1/3}\right]^3 = \frac{1}{343}\left\{\begin{array}{cc} 3087 & 343 \\ -2744 & 0 \end{array}\right\} = \mathcal{A}\,.$$

4.3. *The special case of Identity matrices*

4.3.1. Numerical example 3.

We consider here, for instance, the identity 3×3 matrix

$$\mathcal{I}_3 = \mathcal{I}_3^2 = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\}\,. \tag{4.18}$$

The invariants are:

$$u_1 = 3\,, \quad u_2 = 3, \quad u_3 = 1\,. \tag{4.19}$$

The characteristic equation is:

$$(\lambda - 1)^3 = 0\,, \tag{4.20}$$

and the eigenvalues are the cubic roots of unity:

$$\varepsilon_0 := \lambda_1 = 1\,, \quad \varepsilon_1 := \lambda_2 = -\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\,, \quad \varepsilon_2 := \lambda_3 = -\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}\,. \tag{4.21}$$

by using equations (3.6) and (3.10) because of the multiple root, we find the cubic roots of $\mathcal{I}_3$ in the form:

$$\mathcal{I}_3^{1/3} = \left\{ \begin{matrix} \varepsilon_i & 0 & 0 \\ 0 & \varepsilon_j & 0 \\ 0 & 0 & \varepsilon_k \end{matrix} \right\}\,, \tag{4.22}$$

where (i, j, k) are the arrangements with repetitions of numbers $(0, 1, 2)$, so that we have in total $3^3 = 27$ cubic roots, including $\mathcal{I}_3$ itself, derived by this method.

Remark 4.1 The same procedure can be extended in a direct way to the n-th roots of $\mathcal{I}_r$, therefore finding n^r roots, including $\mathcal{I}_r$ itself.
As we have noticed before, the particular cases of k-matrices [29] and of other special matrices cannot be found by using the "canonical" method proposed in Sect. 3.

Remark 4.2 Several other checks has been made on the equation (3.9), and in particular some case of singular matrices - for which the roots exist - was checked. The results are found correct.

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5. Conclusion

A general method for computing the n-th roots of complex matrices has been shown. The method is based on properties of the $F_{k,n}$ functions, which are the basic solutions of linear recurrence relations and on Cauchy's residue theorem. In the author opinion, the FKN seems to be naturally connected with the problem of computing matrix roots. The efficiency of the procedure is not a surprise, since the used equation (3.6) is nothing but a particular case of the Dunford-Taylor (also called of Riesz-Fantappiè) formula.

Compliance with ethical standards

Conflict of interest. The author declares that he has not received funds from any institution and that he has no conflicts of interest.

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