# Contents

# 1 Introduction

The exponential function $e^{ax}$ is an eigenfunction of the derivative operator, since

$$De^{ax} = ae^{ax} \tag{1.1}$$

where $D := d/dx$, and $a$ denotes a real or complex arbitrary constant.

Another interesting differential operator exists in literature, namely the *Laguerre derivative*, denoted in the following by $\hat{D}_L$, and defined by

$$\hat{D}_L := DxD = \frac{d}{dx}x\frac{d}{dx}. \tag{1.2}$$

In preceding articles, we have shown the role of the Laguerre derivative in the framework of the so called *monomiality principle* [1], [2] and its application to the multidimensional Hermite (Hermite-Kampé de Fériet or Gould-Hopper polynomials) [3], [4], [5] or Laguerre polynomials [1], [6].

It is easily seen, by induction, that the Laguerre derivative verifies:

$$(DxD)^n = D^n x^n D^n \tag{1.3}$$

Furthermore, introducing the Tricomi function $C_0(x)$, of order zero, and the relevant Bessel function:

$$C_0(x) := \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{(k!)^2} = J_0(2\sqrt{x}), \tag{1.4}$$

and putting:

$$e_1(x) \equiv C_0(-x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}, \tag{1.5}$$

we obtain the result:

**Theorem 1.1** *The function $e_1(ax)$ is an eigenfunction of the Laguerre derivative operator, i.e.*

$$\hat{D}_L\, e_1(ax) = ae_1(ax) \tag{1.6}$$

**Proof.** Note that

$$\hat{D}_L = D + xD^2, \tag{1.7}$$

and consequently

$$\hat{D}_L\, e_1(ax) = \left(D + xD^2\right) \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^2} = \tag{1.8}$$

$$= \sum_{k=1}^{\infty} (k + k(k-1))\; a^k \frac{x^{k-1}}{(k!)^2} = \sum_{k=1}^{\infty} k^2\; a^k \frac{x^{k-1}}{(k!)^2} =$$

$$= a \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^2} = ae_1(ax).$$

□

Note that the preceding conclusion depends on the coefficients of the combination expressing the Laguerre derivative $\hat{D}_L$ in terms of $D$, and $xD^2$, so that it turns out the identity: $(k + k(k-1)) = k^2$.

In the following we will show that the above technique can be iterated, producing Laguerre classes of exponential-type functions, called *L-exponentials*, and the relevant *L-circular*, and *L-hyperbolic* functions.

Further extensions are given in the concluding section, and applications to the solution of generalized evolution problems is touched on.

# 2 Generalizations of the Laguerre derivative and $L$-exponential functions

In this section, we generalize the Laguerre derivative, and define the relevant $L$-exponential functions.

We start by considering the operator:

$$\hat{D}_{2L} := DxDxD = D\left(xD + x^2D^2\right) = D + 3xD^2 + x^2D^3, \tag{2.1}$$

and the function:

$$e_2(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^3}. \tag{2.2}$$

The following theorem holds true:

**Theorem 2.1** *The function* $e_2(ax)$ *is an eigenfunction of the operator* $\hat{D}_{2L}$ *i.e.*

$$\hat{D}_{2L}\, e_2(ax) = ae_2(ax) \tag{2.3}$$

**Proof.** Note that

$$\hat{D}_{2L}\, e_2(ax) = \left(D + 3xD^2 + x^2D^3\right) \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^3} =$$

$$= \sum_{k=1}^{\infty} (k + 3k(k-1) + k(k-1)(k-2))\; a^k \frac{x^{k-1}}{(k!)^3} = \sum_{k=1}^{\infty} k^3\, a^k \frac{x^{k-1}}{(k!)^3} =$$

$$= a\sum_{k=0}^{\infty} a^k \, \frac{x^k}{(k!)^3} = ae_2(ax). \qquad (2.4)$$

□

Even in this case, the conclusion depends on the identity:

$$k + 3k(k-1) + k(k-1)(k-2) = k^3,$$

so that, it can be recognized that the coefficients of the combination expressing the $2L$-derivative $\hat{D}_{2L}$ in terms of $D$, $xD^2$, and $x^2D^3$, are the *Stirling numbers of the second kind*, $S(3,1), S(3,2), S(3,3)$, (see: [7] and [8], p. 835, for an extended table).

We can consequently extend the above results as follows.

Considering the operator:

$$\begin{aligned}\hat{D}_{(n-1)L} &:= Dx\cdots DxDxD = D\left(xD + x^2D^2 + \ldots + x^{n-1}D^{n-1}\right) = \\ &= S(n,1)D + S(n,2)xD^2 + \ldots + S(n,n)x^{n-1}D^n, \end{aligned} \qquad (2.5)$$

and the function:

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}. \qquad (2.6)$$

we can state the following theorem:

**Theorem 2.2** *The function* $e_n(ax)$ *is an eigenfunction of the operator* $\hat{D}_{nL}$ *i.e.*

$$\hat{D}_{nL}\, e_n(ax) = ae_n(ax) \qquad (2.7)$$

**Proof.** Proceeding by induction, i.e. assuming eq. (2.5) to be true, and recalling the above remarks, it is sufficient to prove that the coefficients of the combination expressing the $nL$-derivative $\hat{D}_{nL}$ in terms of $D$, $xD^2$, …, and $x^nD^{n+1}$, verify the same induction property as the Stirling numbers of the second kind, namely [7]:

$$S(n+1,h) = S(n,h-1) + hS(n,h). \qquad (2.8)$$

This is clearly true, since, considering in eq.

$$\hat{D}_{nL} := D\left(S(n,1)xD + S(n,2)x^2D^2 + \ldots + S(n,n)x^nD^n\right), \qquad (2.9)$$

the general terms, i.e.

$$D\left(S(n,h-1)x^{h-1}D^{h-1} + S(n,h)x^hD^h\right), \qquad (2.10)$$

we find:

$$(h-1)S(n,h-1)x^{h-2}D^{h-1}+\ S(n,h-1)x^{h-1}D^h +$$
$$+\ hS(n,h)x^{h-1}D^h + S(n,h)x^hD^{h+1},$$

so that, the coefficient of $x^{h-1}D^h$ is given by $S(n,h-1)+hS(n,h)$ and must coincide with $S(n+1,h)$, and then the recursion (2.8) holds true.
□

**Remark 2.1** *The above results show that, for every positive integer $n$, we can define a Laguerre-exponential function, satisfying an eigenfunction property, which is an analog of the elementary property (1.1) of the exponential. This function, denoted by $e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}$, reduces to the exponential function when $n = 0$, so that we can put by definition:*

$$e_0(x) := e^x, \qquad \hat{D}_{0L} := D.$$

*Obviously, $\hat{D}_{1L} := \hat{D}_L$.*

*For this reason we will refer to such functions as* $L$-exponential functions, *or shortly* $L$-exponentials.

## 2.1 The isomorphism $\mathcal{T}_x$

In recent articles, [9], [10], [11], [12], [13], we have considered a differential isomorphism, denoted by the symbol $\mathcal{T} := \mathcal{T}_x$, acting onto the space $\mathcal{A} := \mathcal{A}_x$ of analytic functions of the $x$ variable by means of the correspondence:

$$D := \frac{d}{dx} \quad \rightarrow \quad \hat{D}_L := DxD; \qquad\qquad x\cdot \quad \rightarrow \quad \hat{D}_x^{-1}, \tag{2.11}$$

where

$$\hat{D}_x^{-1} f(x) := \int_0^x f(\xi)d\xi, \tag{2.12}$$

$$\hat{D}_x^{-n} f(x) := \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)d\xi, \tag{2.13}$$

so that

$$\mathcal{T}_x(x^n) = \hat{D}_x^{-n}(1) := \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}d\xi = \frac{x^n}{n!}. \tag{2.14}$$

Note that:

$$\mathcal{T}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x) \tag{2.15}$$

$$\mathcal{T}_x^2(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{(k!)^2} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^3} = e_2(x), \tag{2.16}$$

and so on.

A straightforward computation gives:

$$\hat{D}_L \frac{x^n}{n!} = DxD \frac{x^n}{n!} = n \frac{x^{n-1}}{(n-1)!}, \tag{2.17}$$

i.e. the Laguerrian derivative is a lowering shift operator for the polynomial family $p_n(x) := \frac{x^n}{n!}$.

# 3 The monomiality principle and some relevant examples

The study of the properties of ordinary and generalized polynomials is simplified by the use of the so called *monomiality principle*, according to which a given polynomial $p_n(x)$ ($n \in \mathbf{N}$ and $x \in \mathbf{C}$) is defined a "quasi monomial" if two operators $\hat{P}$ and $\hat{M}$, called from now on "derivative" and "multiplicative" operator respectively, can be defined in such a way that

$$\hat{P}(p_n(x)) = np_{n-1}(x), \tag{3.1}$$

$$\hat{M}(p_n(x)) = p_{n+1}(x). \tag{3.2}$$

The $\hat{P}$ and $\hat{M}$ operators are shown to satisfy the commutation property

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}, \tag{3.3}$$

and thus display a Weyl group structure.

The properties of $p_n(x)$ can be deduced from those of the $\hat{P}$ and $\hat{M}$ operators. If $\hat{P}$ and $\hat{M}$ possess a differential realization, then the polynomial $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}(p_n(x)) = np_n(x). \tag{3.4}$$

If $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as

$$p_n(x) = \hat{M}^n(1). \tag{3.5}$$

The identity (3.5) implies that the exponential generating function of $p_n(x)$ can be cast in the form

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x). \tag{3.6}$$

## 3.1 The Hermite-Kampé de Fériet polynomials

Consider the Hermite-Kampé de Fériet, or *heat* polynomials:

$$\begin{aligned} H_n(x,y) &:= n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \\ H_n(x,0) &= x^n \end{aligned} \tag{3.7}$$

which are related to the Hermite polynomials by

$$H_n(x,-\tfrac{1}{2}) = He_n(x) := n! \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r x^{n-2r}}{(n-2r)!r!2^r}$$

$$H_n(2x,-1) = H_n(x) := n! \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r (2x)^{n-2r}}{(n-2r)!r!}.$$

The $H_n(x,y)$ polynomials are quasi-monomial with respect to the operators:

$$\begin{aligned} \hat{P} &:= \frac{\partial}{\partial x} \\ \hat{M} &:= x + 2y\frac{\partial}{\partial x}. \end{aligned} \tag{3.8}$$

## 3.2 The Gould-Hopper polynomials

An extension of the H-KdF polynomials is given by assuming:

$$H_n^{(j)}(x,y) := n! \sum_{r=0}^{[\frac{n}{j}]} \frac{y^r x^{n-jr}}{r!(n-jr)!}$$

The $H_n^{(j)}(x,y)$ polynomials are quasi-monomial with respect to the operators:

$$\begin{aligned} \hat{P} &:= \frac{\partial}{\partial x} \\ \hat{M} &:= x + jy\frac{\partial^{j-1}}{\partial x^{j-1}}. \end{aligned}$$

Note that, for $j=1$, we simply have:

$$H_n^{(1)}(x,y) := n! \sum_{r=0}^{n} \frac{y^r x^{n-r}}{r!(n-r)!} = (x+y)^n.$$

### 3.3 The two variable Laguerre polynomials

Consider the *two variable Laguerre polynomials*, defined by :

$$\begin{aligned} L_n(x,y) &:= n!\sum_{r=0}^{n}\frac{y^{n-r}x^r}{(n-r)!(r!)^2} \\ L_n(x,0) &= \tfrac{x^n}{n!}, \end{aligned} \tag{3.9}$$

which are related to the ordinary Laguerre $L_n(x)$ by:

$$\begin{aligned} L_n(x,1) &= L_n(-x) \\ L_n(x,y) &= y^n L_n\left(-\tfrac{x}{y}\right). \end{aligned} \tag{3.10}$$

The $L_n(x,y)$ polynomials are quasi-monomial with respect to the operators:

$$\begin{aligned} \hat{P}_* &:= D_x x D_x \\ \hat{M}_* &:= y + \hat{D}_x^{-1} \end{aligned} \tag{3.11}$$

**Remark 3.1** The monomiality property of the polynomials $L_n(x,y)$ can be derived directly observing that, substituting:

$$\begin{array}{ccc} \hat{P} := D_x & \text{with} & \hat{P}_* := D_x x D_x, \\ \hat{M} := x\cdot & \text{with} & \hat{M}_* := \hat{D}_x^{-1} \\ x^n & \text{with} & \dfrac{x^n}{n!}, \end{array}$$

then

$$H_n^{(1)}(x,y) := (y+x)^n = e^{yD_x}x^n$$

becomes

$$L_n(x,y) := (y+\hat{D}_x^{-1})^n(1) = e^{yD_x x D_x}\left(\frac{x^n}{n!}\right).$$

Consequently we deduce that the Laguerre polynomials $L_n(x,y)$ correspond to the Hermite-Kampé de Fériet polynomials $H_n^{(1)}(x,y) := (x+y)^n$ with respect to the isomorphism $\mathcal{T}_x$.

The polynomials $\mathcal{L}_n(x,y)$ was applied in several problems in Physics, connected with electromagnetic waves, diffusion problems, and so on.

# 4 Iterations of the isomorphism $\mathcal{T}_x$

It is worth to note that the isomorphism $\mathcal{T} := \mathcal{T}_x$, defined in Section 2, can be iterated producing a set of generalized Laguerre derivatives as follows.

By using equations (2.11), we find:

$$\mathcal{T}_x\hat{D}_L = \mathcal{T}_x(DxD) = DxD\hat{D}_x^{-1}DxD = DxDxD =: \hat{D}_{2L}, \tag{4.1}$$

$$\mathcal{T}_x \hat{D}_{2L} = \mathcal{T}_x(DxDxD) = DxDxDxD =: \hat{D}_{3L}, \tag{4.2}$$

and, in general, by induction:

$$\mathcal{T}_x^{s-1} \hat{D}_L = \mathcal{T}_x^{s-1}(DxD) = DxD\hat{D}_x^{-1}DxD = DxDxD\cdots xD =: \hat{D}_{sL},$$

where the last operator contains $s+1$ ordinary derivatives, always denoted by the symbol $D$. It is convenient, in the following, to introduce a suitable notation regarding the iterations of the isomorphism $\mathcal{T}_x$, showing their action on powers, and consequently on all functions belonging to $\mathcal{A} := \mathcal{A}_x$. In fact, according to the above definition we can write:

$$\mathcal{T}_x = \hat{D}_x^{-1} = \hat{D}_x^{-1}(1) \tag{4.3}$$

$$\mathcal{T}_x^2 = \mathcal{T}_x \hat{D}_x^{-1}(1) = \hat{D}_{\mathcal{T}_x}^{-1}(1) \qquad \Rightarrow \qquad \hat{D}_{\mathcal{T}_x}^{-n}(1) = \frac{x^n}{(n!)^2}, \tag{4.4}$$

and, by induction:

$$\mathcal{T}_x^s = \mathcal{T}_x^{s-1} \hat{D}_x^{-1}(1) = \hat{D}_{\mathcal{T}_x^{s-1}}^{-1}(1) \qquad \Rightarrow \qquad \hat{D}_{\mathcal{T}_x^{s-1}}^{-n}(1) = \frac{x^n}{(n!)^s}. \tag{4.5}$$

It is easily seen that, $\forall k \in \mathbf{N}$, $\forall s \in \mathbf{N}$,

$$\hat{D}_{\mathcal{T}_x}^{-1}(x^k) = \frac{k!\ x^{k+1}}{[(k+1)!]^2}, \quad \ldots, \quad \hat{D}_{\mathcal{T}_x^s}^{-1}(x^k) = \frac{k!\ x^{k+1}}{[(k+1)!]^{s+1}}, \tag{4.6}$$

and, $\forall h \in \mathbf{N}$,

$$\hat{D}_{\mathcal{T}_x}^{-h}(x^k) = \frac{k!\ x^{k+h}}{[(k+h)!]^2}, \quad \ldots, \quad \hat{D}_{\mathcal{T}_x^s}^{-h}(x^k) = \frac{k!\ x^{k+h}}{[(k+h)!]^{s+1}}.$$

# 5 Higher order Laguerre polynomials

According to the above mentioned properties, by applying the isomorphism $\mathcal{T}_x^s$ with respect to the $x$ variable, the following higher order Laguerre polynomials corresponding to the Gould-Hopper $H^{(j)}(x,y)$ ones appear:

$$L_m^{(j;s)}(x,y) := L_m^{(j;s;0)}(x,y) := \mathcal{T}_x^s \left( H_m^{(j)}(x,y) \right). \tag{5.1}$$

The relevant properties are given by the following theorem

**Theorem 5.1** *The polynomials* $L_m^{(j;s)}(x,y)$ *are explicitly given by*

$$L_m^{(j;s)}(x,y) = H_m^{(j)}(\mathcal{T}_x^s, y) = m! \sum_{k=0}^{\left[\frac{m}{j}\right]} \frac{y^k\ x^{m-jk}}{k!\ [(m-jk)!]^{s+1}} \tag{5.2}$$

*Furthermore, they are defined by the operational identity:*

$$\begin{aligned} L_m^{(j;s)}(x,y) = \exp\{y\left(\hat{D}_{sL}\right)_x^j\}x^m = \exp\{yD_x^j x^j D_x^j \cdots x^j D_x^j\}x^m = \\ = \left(\mathcal{T}_x^s + jy\left(\hat{D}_{sL}\right)_x^{j-1}\right)^m (1) \end{aligned} \tag{5.3}$$

*so that they are quasi-monomial with respect to the operators:*

$$\begin{aligned} \hat{P} := \left(\hat{D}_{sL}\right)_x \\ \hat{M} := \left(\mathcal{T}_x^s + jy\left(\hat{D}_{sL}\right)_x^{j-1}\right) \end{aligned} \tag{5.4}$$

*and satisfy the differential equation:*

$$\frac{\partial}{\partial y} L_m^{(j;s)}(x,y) = \left(\hat{D}_{sL}\right)_x^j L_m^{(j;s)}(x,y). \tag{5.5}$$

*The generating function is given by*

$$\sum_{m=0}^{\infty} L_m^{(j;s)}(x,y)\frac{t^n}{n!} = \mathcal{T}_x^s \exp\{xt + yt^j\} = e^{yt^j}\sum_{k=0}^{\infty}\frac{x^k t^k}{[k!]^{s+1}} = e^{yt^j} e_s(xt). \tag{5.6}$$

The proof follows by a straightforward application of the iterated isomorphism $\mathcal{T}_x^s$ starting from the corresponding properties of the Gould-Hopper polynomials $H_m^{(j)}(x,y)$ and using equations (2.5)-(2.15).

In a similar way, by applying the isomorphisms $\mathcal{T}_x^s$ and $\mathcal{T}_y^\sigma$ to the same polynomials we can define a more general class of higher order Laguerre polynomials:

$$L_m^{(j;s;\sigma)}(x,y) = \mathcal{T}_x^s \mathcal{T}_y^\sigma \left(H_m^{(j)}(x,y)\right). \tag{5.7}$$

which are explicitly expressed by

$$L_m^{(j;s;\sigma)}(x,y) = H_m^{(j)}(\mathcal{T}_x^s, \mathcal{T}_y^\sigma) = m! \sum_{k=0}^{\left[\frac{m}{j}\right]} \frac{y^k \ x^{m-jk}}{[k!]^{\sigma+1} \ [(m-jk)!]^{s+1}} \tag{5.8}$$

and are given by the generating function:

$$\sum_{m=0}^{\infty} L_m^{(j;s;\sigma)}(x,y)\frac{t^m}{m!} = \mathcal{T}_x^s \mathcal{T}_y^\sigma \exp\{xt + yt^j\} = e_s(xt)\ e_\sigma(yt^j). \tag{5.9}$$

Further properties are obtained by using the same procedure as above.

# 6 Laguerre-type Bessel functions

Noting that the isospectral transformation $\mathcal{T} := \mathcal{T}_x$, defined by equation (1.1) implies, according equation (1.2):

$$x^n \quad \rightarrow \quad \hat{D}_x^{-n}(1) = \frac{x^n}{n!} =: p_n(x). \tag{6.1}$$

Then the polynomial family $\{p_n(x)\}$ is quasi-monomial under the action of the operators:

$$\hat{P} := \hat{D}_x^{-1}, \qquad \hat{M} := \hat{D}_L. \tag{6.2}$$

In fact:

$$\hat{P}p_n(x) = \hat{D}_x^{-1}\frac{x^n}{n!} = \frac{x^{n+1}}{(n+1)!} = p_{n+1}(x), \tag{6.3}$$

$$\hat{M}p_n(x) = \hat{D}_L\frac{x^n}{n!} = DxD\frac{x^n}{n!} = n\frac{x^{n-1}}{(n-1)!} = np_{n-1}(x). \tag{6.4}$$

Furthermore

$$\left[\hat{P}, \hat{M}\right] = \left[\hat{D}_x^{-1}, \hat{D}_L\right] = 1.$$

Then the main properties of the polynomial family $\{p_n(x)\}$, (since $p_0 = 1$), can be deduced by using the standard monomiality technique:

- Generating function.

$$\exp\left\{t\hat{D}_x^{-1}\right\}(1) = \sum_{n=0}^{\infty}\frac{t^n}{n!}p_n(x) = \sum_{n=0}^{\infty}\frac{t^n\ x^n}{(n!)^2} = e_1(xt). \tag{6.5}$$

- Differential equation.

$$\hat{D}_x^{-1}\hat{D}_L\frac{x^n}{n!} = n\frac{x^n}{n!}$$

- Operational definition.

$$p_n(x) = \hat{D}_x^{-n}(1) \quad \Leftrightarrow \quad \hat{D}_x^{-n}(1) := \frac{x^n}{n!}.$$

Consider the classical first kind Bessel functions, defined by

$$J_n(x) := \sum_{n=0}^{\infty}\frac{(-1)^h x^{n+2h}(x)}{2^{n+2h}\ h!(n+h)!}. \tag{6.6}$$

According to the monomiality principle, it is possible to construct a class of Bessel-type functions, by considering the $p$-based Bessel functions, defined as follows:

$$ {}_pJ_n(x) := \sum_{n=0}^{\infty} \frac{(-1)^h p_{n+2h}(x)}{2^{n+2h}\ h!(n+h)!}, $$

where $p$ is a symbol denoting the considered polynomial family $p \equiv \{p(x)\} = \left\{\frac{x^n}{n!}\right\}$, so that:

$$ {}_pJ_n(x) := \sum_{n=0}^{\infty} \frac{(-1)^h x^{n+2h}}{2^{n+2h}\ h!(n+h)!(n+2h)!}. \tag{6.7} $$

The Bessel-type functions defined by equation (6.7) are essentially determined by applying the above mentioned isomorphism $\mathcal{T}$ to both sides of equation (6.6), so that we will refer to them as *Laguerre-type Bessel functions* (shortly *L-Bessel functions*).

Then we can prove the following theorems.

**Theorem 6.1** *The generating function of the Laguerre-type Bessel functions* ${}_pJ_n(x)$ *is given by:*

$$ e_1\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} {}_pJ_n(x)\ t^n, \tag{6.8} $$

*which is obtained by applying the isomorphism* $\mathcal{T}$ *to both sides of the generating function of the ordinary first kind Bessel functions:*

$$ \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\ t^n. $$

**Proof.** In fact, using definition (1.4) and the Binomial theorem, we have:

$$ e_1\left[x\left(\tfrac{t}{2}-\tfrac{1}{2t}\right)\right] = \sum_{k=0}^{+\infty} \frac{x^k}{(k!)^2} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \left(\frac{t}{2}\right)^{k-h} \left(\frac{1}{2t}\right)^h = $$

$$ = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{h=0}^{k} \frac{(-1)^h}{h!\ (k-h)!} \frac{t^{k-2h}}{2^k}. $$

Putting $n = k-2h$, and noting that $n$ runs from $-\infty$ to $+\infty$, the above formula can be written as:

$$ e_1\left[x\left(\frac{t}{2}-\frac{1}{2t}\right)\right] = \sum_{n=-\infty}^{+\infty} \left(\sum_{h=0}^{+\infty} \frac{(-1)^h t^{n+2h}}{2^{n+2h}\ h!\ (n+h)!\ (n+2h)!}\right) t^n. $$

Then the proof follows by comparing the last formula with equation (6.7).

The same result can be shown starting from $\exp\left[\frac{\hat{D}_x^{-1}}{2}\left(t-\frac{1}{t}\right)\right]$, and using the property (6.9) of the inverse derivative operator.

□

**Theorem 6.2** *The Laguerre-type Bessel functions* ${}_pJ_n(x)$ *satisfy the recurrence relations:*

$$\begin{cases} \hat{D}_x^{-1}\left[{}_pJ_{n-1}(x) + {}_pJ_{n+1}(x)\right] = 2n\ {}_pJ_n(x) \\ {}_pJ_{n-1}(x) - {}_pJ_{n+1}(x) = 2\hat{D}_L\ {}_pJ_n(x). \end{cases} \tag{6.9}$$

**Proof.** Since

$$\begin{aligned} e_1\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] &= \exp\left[\frac{\hat{D}_x^{-1}}{2}\left(t-\frac{1}{t}\right)\right] = \\ &= \sum_{n=-\infty}^{+\infty} {}_pJ_n(x)\ t^n = \sum_{n=-\infty}^{+\infty} J_n(\hat{D}_x^{-1})\ t^n, \end{aligned} \tag{6.10}$$

the first recurrence can be proved by differentiating the first and third term of equation (6.10) with respect to $t$. The second recurrence relation follows by applying the Laguerrian derivative $\hat{D}_L := D_x x D_x$ to the second and fourth term of equation (6.10). This last computation is quite tedious, and can be simplified by using the result of Lemma 1.1, which allows us to obtain the same result by differentiating formally to the second and fourth term of equation (6.10) with respect to $\hat{D}_x^{-1}$.

□

Then, we can derive the following result:

**Theorem 6.3** *The shift operators* $\hat{E}_+$ *and* $\hat{E}_-$ *of the Laguerre-type Bessel functions* ${}_pJ_n(x)$ *are given by the equations:*

$$\begin{cases} \hat{D}_x^{-1}\hat{E}_+ := n\ \hat{I} - \hat{D}_x^{-1}\hat{D}_L \\ \hat{D}_x^{-1}\hat{E}_- := n\ \hat{I} + \hat{D}_x^{-1}\hat{D}_L, \end{cases} \tag{6.11}$$

*where* $\hat{I}$ *denotes the identity operator.*

**Theorem 6.4** *The differential equation satisfied by the Laguerre-type Bessel functions* ${}_pJ_n(x)$ *is given by:*

$$\left(\hat{D}_L^2 + \hat{D}_x\hat{D}_L - n^2\hat{D}_x^2 + \hat{I}\right)\ {}_pJ_n(x) = 0, \tag{6.12}$$

*and can be derived by applying the isomorphism* $\mathcal{T}$ *to both sides of the differential equation of the ordinary first kind Bessel functions:*

$$\left[x^2\hat{D}_x^2 + x\hat{D}_x + (x^2-n^2)\hat{I}\right] J_n(x) = 0,$$

and using straightforward modifications.

**Proof.** In fact, using the isomorphism (1.1), the differential equation of the ordinary first kind Bessel functions becomes:

$$\left[\hat{D}_x^{-2}\hat{D}_L^2+\hat{D}_x^{-1}\hat{D}_L+(\hat{D}_x^{-2}-n^2)\hat{I}\right]J_n(x)=0,$$

and the result follows differentiating twice both sides of the last equation.

The same result can be proved by using the shift operators (6.13) and the usual technique connected with the monomiality principle, starting from the equation:

$$\hat{E}_+\left[\hat{E}_-\left({}_pJ_n(x)\right)\right]=\hat{E}_+\left({}_pJ_{n-1}(x)\right)={}_pJ_n(x).$$

□

# 7 2D Appell Polynomials

For any $j\geq 2$, the 2D Appell polynomials $R_n^{(j)}(x,y)$ are defined by means of the generating function:

$$G_A^{(j)}(x,y;t):=A(t)\;e^{xt+yt^j}=\sum_{n=0}^{\infty}R_n^{(j)}(x,y)\frac{t^n}{n!} \tag{7.1}$$

Even in this general case, the polynomial $R_n^{(j)}(x,y)$, is *isobaric of weight* $n$, so that it does not contain the variable $y$, for every $n=0,1,...,j-1$.

- **Explicit forms of the polynomials $R_n^{(j)}$ in terms of the Gould-Hopper polynomials $H_n^{(j)}$ and vice-versa.**

  The following formulas hold:

$$\begin{aligned} R_n^{(j)}(x,y) &= \sum_{h=0}^{n}\binom{n}{h}\mathcal{R}_{n-h}H_h^{(j)}(x,y)= \\ &= n!\sum_{h=0}^{n}\frac{\mathcal{R}_{n-h}}{(n-h)!}\sum_{r=0}^{[\frac{h}{j}]}\frac{x^{h-jr}y^r}{(h-jr)!r!}, \end{aligned} \tag{7.2}$$

where the $\mathcal{R}_k$ are the "Appell numbers" appearing into the definition:

$$A(t)=\sum_{k=0}^{\infty}\frac{\mathcal{R}_k}{k!}t^k,\quad (A(0)\neq 0),$$

$$H_n^{(j)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}Q_{n-k}R_k^{(j)}(x,y),$$

where the $Q_k$ are the coefficients of the Taylor expansion in a neighborhood of the origin of the reciprocal function $1/A(t)$.

- **Recurrence relation.**

It is useful to introduce the coefficients of the Taylor expansion:

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{7.3}$$

The following linear homogeneous recurrence relation for the generalized Appell polynomials $R_n^{(j)}(x,y)$ holds:

$$\begin{aligned} R_0^{(j)}(x,y) &= 1, \\ R_n^{(j)}(x,y) &= (x+\alpha_0)R_{n-1}^{(j)}(x,y) + \binom{n-1}{j-1} jy\ R_{n-j}^{(j)}(x,y) \\ &+ \sum_{k=0}^{n-2} \binom{n-1}{k} \alpha_{n-k-1} R_k^{(j)}(x,y). \end{aligned} \tag{7.4}$$

- **Shift operators.**

$$L_n^- := \frac{1}{n} D_x, \tag{7.5}$$

$$L_n^+ := (x+\alpha_0) + \frac{j}{(j-1)!}\ y\ D_x^{j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!}\ D_x^{n-k},$$

- **Differential equation.**

$$\begin{aligned} &\left[ \frac{\alpha_{n-1}}{(n-1)!} D_x^n + \ldots + \frac{\alpha_j}{j!} D_x^{j+1} + \left( \frac{\alpha_{j-1} + jy}{(j-1)!} \right) D_x^j \right. \\ &\left. + \frac{\alpha_{j-2}}{(j-2)!} D_x^{j-1} + \ldots + (x+\alpha_0)\, D_x - n \right] R_n^{(j)}(x,y) = 0, \end{aligned} \tag{7.6}$$

# 8 Higher order Appell polynomials

According to the above mentioned properties, we can define a more general class of higher order Appell polynomials. Namely, the following result holds:

**Theorem 8.1** *The polynomials* $R_n^{(j;s;\sigma)}(x,y)$*, defined by the generating function*

$$A(t)\ e_s(xt) e_\sigma(yt^j) = \sum_{n=0}^{\infty} R_n^{(j;s;\sigma)}(x,y) \frac{t^n}{n!}, \tag{8.1}$$

*where:*

$$R_n^{(j;s;\sigma)}(x,y) := R_n^{(j)}\left(\mathcal{T}_x^s, \mathcal{T}_y^\sigma\right), \tag{8.2}$$

*are explicitly expressed by*

$$R_n^{(j;s;\sigma)}(x,y) = \sum_{h=0}^{n} \binom{n}{h} R_{n-h} L_h^{(j;s;\sigma)}(x,y) \tag{8.3}$$

*where the coefficients* $R_k$ *are the Appell numbers associated with the function* $A(t)$ (see equation (2.2)), *and the higher order Laguerre polynomials* $L_h^{(j;s;\sigma)}(x,y)$, *defined by equation* (1.14), *come into play:*

$$L_h^{(j;s;\sigma)}(x,y) = \mathcal{T}_x^s \mathcal{T}_y^\sigma H_h^{(j)}(x,y). \tag{8.4}$$

**Proof.** Applying the isomorphisms $\mathcal{T}_x^s$ and $\mathcal{T}_y^\sigma$ to both sides of the generating function of the polynomials $R_n^{(j)}(x,y)$, yields

$$\mathcal{T}_x^s \mathcal{T}_y^\sigma \; A(t)\; e^{xt+yt^j} = \sum_{n=0}^{\infty} \mathcal{T}_x^s \mathcal{T}_y^\sigma R_n^{(j)}(x,y) \frac{t^n}{n!},$$

and therefore:

$$A(t)\; e_s(xt) e_\sigma(yt^j) = \sum_{n=0}^{\infty} R_n^{(j)}\left(\mathcal{T}_x^s, \mathcal{T}_y^\sigma\right) \frac{t^n}{n!},$$

so that equations (8.1)-(8.2) hold. Equation (8.4) is a consequence of (2.2) and (8.3).
□

Further properties are obtained by using the same procedure as above, and can be summarized as follows.

**Theorem 8.2** *The polynomials* $R_n^{(j;s;\sigma)}(x,y)$, *verify the recurrence relation*

$$\begin{aligned} R_0^{(j;s;\sigma)}(x,y) &= 1, \\ R_n^{(j;s;\sigma)}(x,y) &= \left(\hat{D}_{\mathcal{T}_x^{s-1}}^{-1} + \alpha_0\right) R_{n-1}^{(j;s;\sigma)}(x,y) + \binom{n-1}{j-1} jy\; R_{n-j}^{(j;s;\sigma)}(x,y) \\ &+ \sum_{k=0}^{n-2} \binom{n-1}{k} \alpha_{n-k-1} R_k^{(j;s;\sigma)}(x,y), \end{aligned} \tag{8.5}$$

*where the operator* $\hat{D}_{\mathcal{T}_x^{s-1}}^{-1}$, *acting on* $R_{n-1}^{(j;s;\sigma)}(x,y)$, *is defined by equations* (1.9)-(1.10), *and the* $\alpha_k$ *are the coefficients of the expansion* (2.3).

**Theorem 8.3** *Shift operators for the polynomials* $R_n^{(j;s;\sigma)}(x,y)$, *are given by:*

$$L_n^- := \frac{1}{n}\hat{D}_{sL}, \tag{8.6}$$

$$L_n^+ := \left(\hat{D}^{-1}_{\mathcal{T}_x^{s-1}} + \alpha_0\right) + \frac{j}{(j-1)!}\; y\; \hat{D}_{sL}^{\,j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!}\; \hat{D}_{sL}^{\,n-k}.$$

**Theorem 8.4** *A differential equation satisfied by the polynomials* $R_n^{(j;s;\sigma)}(x,y)$, *is given by:*

$$\left[\frac{\alpha_{n-1}}{(n-1)!}\hat{D}_{sL}^{\,n} + \ldots + \frac{\alpha_j}{j!}\hat{D}_{sL}^{\,j+1} + \left(\frac{\alpha_{j-1}+jy}{(j-1)!}\right)\hat{D}_{sL}^{\,j} \right. \tag{8.7}$$
$$\left. + \frac{\alpha_{j-2}}{(j-2)!}\hat{D}_{sL}^{\,j-1} + \ldots + \left(\hat{D}^{-1}_{\mathcal{T}_x^{s-1}} + \alpha_0\right)\hat{D}_{sL} - n\right] R_n^{(j;s;\sigma)}(x,y) = 0.$$

# 9 Applications to a class of Laguerre-type evolution problems

We show in this section a simple application to the solution of evolution equations [14].

**Theorem 9.1** *For any fixed integral* $s \geq 1$, *consider the problem:*

$$\begin{cases} \hat{D}_{sL}\; S(x,t) = \dfrac{\partial S}{\partial t}, & \text{in the half} - \text{plane} \quad x > 0, \\ S(0,t) = g(t), \end{cases} \tag{9.1}$$

*with analytic boundary condition* $g(t)$.

*The operational solution of equation* (9.1) *is given by:*

$$S(x,t) = e_s\left(x\frac{\partial}{\partial t}\right) g(t) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{s+1}} \frac{d^k}{dt^k} g(t) \tag{9.2}$$

Representing $g(t) = \sum_{k=0}^{\infty} a_k t^k$, from equation (9.2) we find, in particular:

$$S(x,0) = \sum_{k=0}^{\infty} a_k \frac{x^k}{(k!)^s}.$$

Note that the operational solution becomes an effective solution whenever the series in equation (9.2) is convergent. The validity of this condition depends on the growth of the coefficients $a_k$ of the boundary data $g(t)$ (see [15]), but it is usually satisfied in physical problems.

More generally the following results holds:

**Theorem 9.2** *Let* $\hat{\Omega}_x$ *be a differential operator with respect to the* $x$ *variable, and denote by* $\psi(x)$ *an eigenfunction of* $\hat{\Omega}_x$ *such that:*

$$\hat{\Omega}_x\ \psi(ax) = a\ \psi(ax), \qquad \psi(0) = 1, \tag{9.3}$$

*then the evolution problem:*

$$\begin{cases} \hat{\Omega}_x\ S(x,t) = \dfrac{\partial S}{\partial t}, & \text{in the half} - \text{plane} \quad x > 0, \\ S(0,t) = g(t), \end{cases} \tag{9.4}$$

*with analytic boundary condition* $g(t)$*, admits the operational solution:*

$$S(x,t) = \psi\left(x\frac{\partial}{\partial t}\right) g(t) \tag{9.5}$$

**Proof**. The eigenfunction property of $\psi$, implies:

$$\hat{\Omega}_x\ S(x,t) = \hat{\Omega}_x\ \psi\left(x\frac{\partial}{\partial t}\right) g(t) = \frac{\partial}{\partial t}\ \psi\left(x\frac{\partial}{\partial t}\right) g(t) = \frac{\partial}{\partial t}\ S(x,t),$$

since $\frac{\partial}{\partial t}$ commutes with $\psi\left(x\frac{\partial}{\partial t}\right)$.

Furthermore, as a consequence of the hypothesis $\psi(0) = 1$, the boundary condition, for $x = 0$, is trivially satisfied.
□

# 10 Further results

In this Section we apply the preceding results to the solution of further Laguerre-type evolution problems. Some other extensions can be found in [16].

We start from the operational definitions of generalized and higher order Laguerre polynomials:

$$e^{t\hat{D}_L}\frac{x^m}{m!} = L_m^{(1;1)}(x,t) = m!\sum_{n=0}^{m}\frac{t^n\ x^{m-n}}{n!\ [(m-n)!]^2}, \tag{10.1}$$

and, for every integer $j \geq 1$:

$$e^{t\hat{D}_L^j}\frac{x^m}{m!} = L_m^{(j;1)}(x,t) = m!\sum_{n=0}^{\left[\frac{m}{j}\right]}\frac{t^n\ x^{m-jn}}{n!\ [(m-jn)!]^2}. \tag{10.2}$$

More generally, for every integers $j \geq 1$ and $s \geq 1$:

$$e^{t\hat{D}_{sL}^j}\frac{x^m}{(m!)^s} = L_m^{(j;s)}(x,t) = m!\sum_{n=0}^{\left[\frac{m}{j}\right]}\frac{t^n\ x^{m-jn}}{n!\ [(m-jn)!]^{s+1}}. \tag{10.3}$$

We note that, from the above equations, if we consider an analytic function admitting the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m \frac{x^m}{(m!)^s}, \tag{10.4}$$

then we can write

$$e^{t\hat{D}^j_{sL}} f(x) = \sum_{m=0}^{\infty} a_m \; L_m^{(j;s)}(x,t). \tag{10.5}$$

Sufficient conditions for the convergence of last series expansions can be found in [16], showing connections with the corresponding ones considered in [15].

Therefore, we can solve the following Laguerre-type evolution problems.

**Theorem 10.1** *The problem:*

$$\begin{cases} \dfrac{\partial S}{\partial t} = \dfrac{\partial^j}{\partial x^j} x^j \dfrac{\partial^j}{\partial x^j} S = \hat{D}^j_L S, \quad \text{in the half} - \text{plane} \quad t > 0, \\ S(x,0) = f(x), \end{cases} \tag{10.6}$$

*with analytic initial condition* $f(x) = \sum_{m=0}^{\infty} a_m \frac{x^m}{m!}$ *admits the operational solution:*

$$S(x,t) = e^{t\hat{D}^j_L} f(x). \tag{10.7}$$

Consequently, the solution (10.7) can be found in terms of the series expansion:

$$S(x,t) = \sum_{m=0}^{\infty} a_m \; L_m^{(j;1)}(x,t). \tag{10.8}$$

More generally

**Theorem 10.2** *The problem:*

$$\begin{cases} \dfrac{\partial S}{\partial t} = \hat{D}^j_{sL} S, \quad \text{in the half} - \text{plane} \quad t > 0, \\ S(x,0) = f(x), \end{cases} \tag{10.9}$$

*with analytic initial condition* $f(x) = \sum_{m=0}^{\infty} a_m \frac{x^m}{(m!)^s}$ *admits the operational solution:*

$$S(x,t) = e^{t\hat{D}^j_{sL}} f(x). \tag{10.10}$$

Consequently, the solution (10.10) can be found in terms of the series expansion:

$$S(x,t) = \sum_{m=0}^{\infty} a_m \; L_m^{(j;s)}(x,t). \tag{10.11}$$

Following the same methods developed in [15], i.e. introducing suitable hyperbolic or circular operators of the generalized Laguerre derivative, we have found in [16] explicit solutions of Laguerre-hyperbolic or Laguerre-elliptic type problems too.

# 11 Eigenfunctions of generalized Laguerre-type derivatives

We consider in this section operators of the kind

$$D^{h+1} x^j D^{j-h}, \qquad h,j \in \mathbf{N}_0 \equiv \mathbf{N} \cup \{0\}, \quad j > h. \tag{11.1}$$

Note that, as particular cases, when $h = 0$ we find the operator $Dx^jD^j$, and when $j = h + 1 = \ell$ the operator $D^\ell x^\ell D$.

We prove the following result

**Theorem 11.1** *The operator* (11.1) *admits the eigenfunction*

$$e_{(h+1,j,j-h)}(x) :=$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2(k-1)!(k-2)!\cdots(k+h-j+1)!(k+h)!(k+h-1)!\cdots(k+1)!}.$$

Note that in the denominator of the right hand side of the last equation, $(k!)^2$ is multiplied by a total of $(j-h-1)+h = j-1$ factorials.

**Proof.** It is a straightforward consequence of equation:

$$\begin{aligned} &D^{h+1}x^jD^{j-h}x^k = \\ &= k^2(k-1)(k-2)\cdots(k+h-j+1)(k+h)(k+h-1)\cdots(k+1)x^{k-1}. \end{aligned} \tag{11.2}$$

□

**Remark 11.1** *In the series expansion of the eigenfunction of Theorem 11.1, and in the following as well, we assume the usual convention* $\Gamma(s) = 0$, for $s \leq 0$ *for the running index.*

Consider now the positive integers $j_1 \geq j_2 \ldots \geq j_r$, and denote by $j_{s_1}, j_{s_2}, \ldots, j_{s_r}$ a rearrangement of the same integers according the permutation: $s_1, s_2, \ldots, s_r$.

**Theorem 11.2** *The operator* $Dx^{j_{s_1}}D^{j_{s_1}}\cdots x^{j_{s_r}}D^{j_{s_r}}$ *admits the eigenfunction*

$$e_{(1,j_{s_1},j_{s_1},\ldots,j_{s_r},j_{s_r})}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{r+1}\ [F_1]^r \times [F_2]^{r-1} \times \cdots \times F_r},$$

*where* $F_1, F_2, \ldots, F_r$ *are given by*

$$\begin{aligned} &F_1 := (k-1)!(k-2)!\cdots(k-j_1+1)!,\\ &F_2 := (k-j_1)!(k-j_1-1)!\cdots(k-j_2+1)!,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &F_r := (k-j_{r-1})!(k-j_{r-1}-1)!\cdots(k-j_r+1)! \end{aligned}$$

**Proof.** The result is a consequence of equation:

$$\begin{aligned} &Dx^{j_{s_1}}D^{j_{s_1}}\cdots x^{j_{s_r}}D^{j_{s_r}}x^k =\\ &= k^r\ (k-1)(k-2)\cdots(k-j_1+1)\ (k-1)(k-2)\cdots(k-j_2+1)\cdots \\ &\quad\cdots(k-1)(k-2)\cdots(k-j_r+1)x^{k-1}. \end{aligned} \tag{11.3}$$

□

In a similar way, considering the positive integers $\ell_1 \le \ell_2 \ldots \le \ell_r$, and denoting by $\ell_{s_1}, \ell_{s_2}, \ldots, \ell_{s_r}$ a rearrangement of the same integers according the permutation: $s_1, s_2, \ldots, s_r$, we can prove the following result

**Theorem 11.3** *The operator* $D^{\ell_{s_1}}x^{\ell_{s_1}}\cdots D^{\ell_{s_r}}x^{\ell_{s_r}}D$ *admits the eigenfunction*

$$e_{(\ell_{s_1},\ell_{s_1},\ldots,\ell_{s_r},\ell_{s_r},1)}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{r+1}\ [G_1]^r \times [G_2]^{r-1} \times \cdots \times G_r},$$

*where* $G_1, G_2, \ldots, G_r$ *are given by*

$$\begin{aligned} &G_1 := (k+1)!(k+2)!\cdots(k+\ell_1-1)!,\\ &G_2 := (k+\ell_1)!(k+\ell_1+1)!\cdots(k+\ell_2-1)!,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &G_r := (k+\ell_{r-1})!(k+\ell_{r-1}+1)!\cdots(k+\ell_r-1)! \end{aligned}$$

**Remark 11.2** *The above eigenfunctions belong to the class of the so called multi-index Bessel functions we considered in [17], but we use at present a more suitable notation for indices.*

# 12 Applications to generalized Laguerre-type evolution problems

In this Section we apply the preceding result to the solution of generalized Laguerre-type evolution problems.

The possibility of obtaining such applications is based on the following theorem generalizing a known result recalled in [9]:

**Theorem 12.1** *Let* $\hat{\Omega}_x$ *be a differential operator with respect to the* $x$ *variable, and denote by* $\psi(x)$ *an eigenfunction of* $\hat{\Omega}_x$ *such that:*

$$\hat{\Omega}_x\ \psi(ax) = a\ \psi(ax), \qquad K \lim_{x\to 0^+} x^{-H}\psi(x) = 1, \tag{12.1}$$

*where* $H$ *and* $K$ *denote positive constants, then the evolution problem:*

$$\begin{cases} \hat{\Omega}_x\ S(x,t) = \dfrac{\partial S}{\partial t}, & \text{in the half} - \text{plane} \quad x > 0, \\ K \displaystyle\lim_{x\to 0^+} x^{-H} S(x,t) = g(t), \end{cases} \tag{12.2}$$

*where* $g(t)$ *denotes an analytic function, admits the operational solution:*

$$S(x,t) = \psi\left(x\frac{\partial}{\partial t}\right) g(t) \tag{12.3}$$

**Proof**. The eigenfunction property of $\psi$, implies:

$$\hat{\Omega}_x\ S(x,t) = \hat{\Omega}_x\ \psi\left(x\frac{\partial}{\partial t}\right) g(t) = \frac{\partial}{\partial t}\ \psi\left(x\frac{\partial}{\partial t}\right) g(t) = \frac{\partial}{\partial t}\ S(x,t),$$

since $\frac{\partial}{\partial t}$ commutes with $\psi\left(x\frac{\partial}{\partial t}\right)$.

Furthermore, as a consequence of the hypothesis $(12.1)_2$ the boundary condition, for $x \to 0$, is trivially satisfied.
□

Accordingly, we can state the following results:

**Theorem 12.2** *The evolution problem*

$$\begin{cases} D_x^{h+1} x^j D_x^{j-d} S(x,t) = D_t S(x,t), & \text{in the half} - \text{plane} \quad x > 0, \\ \\ K \displaystyle\lim_{x\to 0^+} x^{h-j+1} S(x,t) = g(t), \end{cases}$$

*where* $K = [(j-h-1)!]^2 (j-h-2)!(j-h-3)! \cdots 0!(j-1)!(j-2)! \cdots (j-h)!$, *and* $g(t)$ *denotes an analytic function, admits the operational solution:*

$$\begin{aligned} S(x,t) &= e_{(h+1,j,j-h)}\left(xD_t\right) g(t) = \\ &= \sum_{k=j-h-1}^{\infty} \frac{x^k\ D_t^k g(t)}{(k!)^2 (k-1)! \cdots (k+h-j+1)!(k+h)! \cdots (k+1)!} \end{aligned}$$

Further results can be found in [18].

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