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ON SOME NONCLASSICAL PROBLEMS FOR DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS TO THE THEORY OF CUSPED PRISMATIC SHELLS

Preface

The present Lecture Notes contains extended material mainly based on the lectures presented at the Workshop on Mathematical Methods for Elastic Cusped Plates and Bars (Tbilisi, September 27–28, 2001).

The work consists of the list of notation, introduction, three chapters and references.

The Introduction contains a survey of results related to the subject and a brief presentation of results of the present work.

In Chapter 1 some auxiliary materials are given which are used in Chapters 2 and 3.

Chapter 2 deals with the problems of cylindrical bending and bending vibration of a cusped plate. Bending problems of cusped plates fall outside of the limits of classical bending theory. The aim of this chapter is to study the problem of well-possedness of boundary value problems and initial boundary value problems in case of cylindrical bending of shells with two cusped edges and in some cases to solve these problems in explicit forms.

Chapter 3 is dedicated to the interface problem of the interaction of a plate with two cusped edges and a flow of an incompressible fluid.

Acknowledgments. The author is very grateful to Prof. G. Jaiani, Prof. S. Kharibegashvili, and Prof. D. Natroshvili for their useful discussions.

Author

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List of Notations

$\mathbb{N} := \{1, 2, \cdots\}$,

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$\mathbb{R}^n$ $n-$dimensional Euclidean space ($n \in \mathbb{N}$)

$\Omega := \{(x_1, x_2, x_3) : \ -\infty < x_1 < \infty, \ \ 0 < x_2 < l, \ \ x_3 = 0\}$ - the projection of a plate on the plane $x_3 = 0$

$I := \{[0, l] \times \{0\}\}$

$\Omega^f := \{x_1, x_2, x_3 : x_1 = 0, \ x_2 := (x_2, x_3) \in \mathbb{R}^2 \backslash I\}$ - space which occupies the fluid

$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x)$ - thickness of a plate at point x

ω - oscillation frequency

$D(x_2)$ - flexural rigidity

ρ - density of a plate

$w(x_2, t)$ - deflection of a plate

$q(x_2, t)$ - lateral load

$M_2(x_2, t)$ - bending moment

$Q_2(x_2, t)$ - intersecting force

E - Young's modulus

σ - Poisson's ratio

$F := (F_2, F_3)$ - plane volume forces

δ_{ij} - Kroneker Delta

ρ^f - density of a fluid

$u := (u_1, u_2, u_3)$ - displacement vector of a fluid

$v := (v_1, v_2, v_3)$ - velocity vector of a fluid

p - pressure of a fluid

$p(x_2, \overset{(+)}{h}(x_2), t) \ (p(x_2, \overset{(-)}{h}(x_2), t))$ - the value of the pressure on the upper (lower) surface of the plate

$v_{3\infty}(t), \ p_\infty(t)$ - values of the velocity vector component and pressure at infinity

$\sigma^f_{jk} = -p\delta_{jk} + \mu \left(\dfrac{\partial v_j}{\partial x_k} + \dfrac{\partial v_k}{\partial x_j} \right)$ - stress tensor of a fluid

$\nu, \ \mu$ - coefficients of viscosity

$\Delta := \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$

$w_{,t} := \dfrac{\partial w}{\partial t}$

$w_{,i} := \dfrac{\partial w}{\partial x_i}$, $i = 1, 2, 3$

$C^n(]0, l[)$ $(C^n([0, l]))$ - n-times continuously differentiable functions in $]0, l[$ (on $[0, l]$)

$C^n(\Omega^f)$ - n-times continuously differentiable functions in Ω^f with respect to x_2 and x_3

$C(t > 0)$ - continuous functions with respect to t for $t > 0$

$H([0, l])$ - class of Hölder continuous functions

$L_2([0, l])$ - class of square integrable functions on $[0, l]$

Introduction

In 1955 I.Vekua [95]-[97] raised the problem of investigation of cusped plates, i.e.
such ones whose thickness on the part of the plate boundary or on the whole one
vanishes. The problem mathematically leads to the question of setting and solving of
boundary value problems for even order equations and systems of elliptic type with
the order degeneration in the statical case and of initial boundary value problems
for even order equations and systems of hyperbolic type with the order degenera-
tion in the dynamical case (for corresponding investigations see the survey [35] and
also I. Vekua's comments in [97, p.86]). There exists a wide literature devoted to
the theory of degenerate and mixed type equations (see, e.g., [5], [30]), which was
developed intensively in the period from early 50-ies till early 70-ies but it could not
cover the above equations and systems because of distinct peculiarities of the latter
caused by the geometry of the mechanical problem.

The first work concerning classical bending of cusped elastic plates was done by
E. Makhover [67], [68] and S. Mikhlin [71].

In 1957 E. Makhover [67], [68], by using the results of S. Mikhlin [71], had
considered such a cusped plate with the stiffness $D(x_1, x_2)$ satisfying

$$D_1 x_2^{\kappa_1} \leq D(x_1, x_2) \leq D_2 x_2^{\kappa_1}, \quad D_1, D_2, \ \kappa_1 = const > 0, \tag{1}$$

within the framework of classical bending theory. She particularly studied in which
cases the deflection ($\kappa_1 < 2$) or its normal derivative ($\kappa_1 < 1$) on the cusped edge
of the plate can be given. In 1971, A. Khvoles [62] represented the forth order Airy
stress function operator as the product of two second order operators in the case
when the plate thickness $2h$ is given by

$$2h = h_0 x_2^{\kappa_2}, \ h_0, \kappa_2 = const > 0, \ x_2 \geq 0, \tag{2}$$

and investigated the general representation of corresponding solutions. Since 1972
the work of G. Jaiani in [36]–[51] is also devoted to these problems. By using more
natural spaces than E. Makhover, G. Jaiani in [48] has analyzed in which cases
the cusped edge can be freed ($\kappa_1 > 0$) or freely supported ($\kappa_1 < 2$). Moreover,
he established well–posedness and the correct formulation of all admissible princi-
pal boundary value problems (BVPs). In [41], [42], [47] he also investigated the
tension–compression problem of cusped plates, based on I. Vekua's model of shallow
prismatic shells. G. Jaiani's results can be summarized as follows.

Let n be the inward normal of the plate boundary. In the case of the tension-compression problem on the cusped edge, where

$$0 \leq \frac{\partial h}{\partial n} < +\infty \text{(in the case (2) this means } \kappa_2 \geq 1),$$

which will be called a sharp cusped edge, one can not prescribe the displacement vector; while on the cusped edge, where

$$\frac{\partial h}{\partial n} = +\infty \text{(in the case (2) this means } \kappa_2 < 1),$$

called a blunt cusped edge, the displacement vector can be prescribed. In the case of the classical bending problem with a cusped edge, where

$$\frac{\partial h}{\partial n} = O(d^{\kappa-1}) \text{as d} \to 0, \ \kappa = \text{const} > 0 \tag{3}$$

and where d is the distance between an interior reference point of the plate projection and the cusped edge, the edge can not be fixed if $\kappa \geq \frac{1}{3}$, but it can be fixed if $0 < \kappa < \frac{1}{3}$; it can not be freely supported if $\kappa \geq \frac{2}{3}$, and it can be freely supported if $0 < \kappa < \frac{2}{3}$; it can be free or arbitrarily loaded by a shear force and a bending moment if $\kappa > 0$. Note that in the case (2), the condition (3) implies that $d_2 = x_2$ and $\kappa = \kappa_2 = \frac{\kappa_1}{3}$.

For the specific cases of cusped cylindrical and conical shell bending, the above results remain valid as it has been shown by G. Tsiskarishvili and N. Khomasuridse [89]-[92]. These results also remain valid in the case of classical bending of orthotropic cusped plates (see [51]). However, for general cusped shells and also for general anisotropic cusped plates, the corresponding analysis is done.

The problems involving cusped plates lead to correct mathematical formulations of BVPs for even order elliptic equations and systems whose orders degenerate at the boundary (see [47], [52]-[53]).

Applying the functional–analytic method developed by G. Fichera in [28], [29] (see also [21], [22]), in [47] the particular case of Vekua's system for general cusped plates has been investigated.

The classical bending of plates with the stiffness (1) in energetic and in weighted Sobolev spaces has been studied by G. Jaiani in [48], [50]. In the energetic space some restrictions on the lateral load has been relaxed by G. Devdariani in [20]. G. Tsiskarishvili [90] characterized completely the classical axial symmetric bending of specific circular cusped plates without or with a hole.

In the case (2), the basic BVPs have been explicitly solved in [43] and [53] with the help of singular solutions depending only on the polar angle.

If we consider the cylindrical bending of a plate, in particular of a cusped one, with rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq \ell$, then we actually get the corresponding results also for cusped beams (see [49], [43], [93], [73]-[77], [12], [13], [54], [55]).

In 1999-2001 two contact problems were considered by N. Shavlakadze [86], [87], namely, the contact problem for an unbounded elastic medium composed of two half-planes $x_1 > 0$ and $x_1 < 0$ having different elastic constants and strengthened on the semi-axis $x_2 > 0$ by an inclusion of variable thickness (cusped beam) with constant Young's modulus and Poisson's ratio. It was assumed that the plate is subjected to plane deformation, the flexural rigidity D had the form

$$D = D_0 x_2^{\varkappa}, \;\; D_0, \;\; \varkappa = \text{const} > 0,$$

and the cusped end $x_2 = 0$ of the beam was free.

At the same time (in the fifties of the twentieth century), I.Vekua [95] introduced a new mathematical model for elastic prismatic shells (i.e., of plates of variable thickness) which was based on expansions of the three–dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier-Legendre series with respect to the variable plate thickness. By taking only the first $N + 1$ terms of the expansions, he introduced the so–called N–th approximation. Each of these approximations for $N = 0, 1, \dots$ can be considered as an independent mathematical model of plates. In particular, the approximation for $N = 1$ corresponds to the classical Kirchhoff plate model. In the sixties, I. Vekua [96] developed the analogous mathematical model for thin shallow shells. All his results concerning plates and shells are collected in his monograph [97]. Works of I. Babuška, D. Gordeziani, V. Guliaev, I. Khoma, A. Khvoles, T. Meunargia, C. Schwab, T. Vashakmadze, V. Zhgenti, and others (see [2], [31], [33], [61], [62], [69], [84], [85], [94], [100] and the references therein) are devoted to further analysis of I.Vekua's models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (to non-shallow shells, to the anisotropic case, etc.).

In [56] variational hierarchical two–dimensional models for cusped elastic plates are constructed. With the help of variational methods, existence and uniqueness theorems for the corresponding two–dimensional boundary value problems are proved in appropriate weighted functional spaces. By means of the solutions of these two–dimensional boundary value problems, a sequence of approximate solutions in the corresponding three-dimensional region is constructed. This sequence converges in the Sobolev space H^1 to the solution of the original three-dimensional boundary value problem. The systems of differential equations corresponding to the two-dimensional variational hierarchical models are explicitly given for a general orthogonal system and for Legendre polynomials, in particular.

Recently N.Chinchaladze, R. Gilbert, G. Jaiani, S. Kharibegashvili and D. Natroshvili have studied the well posedness of boundary value problems for elastic cusped prismatic shells in the Nth approximation of I. Vekua's hierarchical models under (all reasonable) boundary conditions at the cusped edge and given displacements at the non-cusped edge and stresses at the upper and lower faces of the shell [19].

For the last decades the direct and inverse problems connected with the interaction between difference vector fields have received much attention in the mathe-

matical and engineering scientific literature and have been intensively investigated. They arise in many physical and mechanical models describing the interaction of two different media where the whole process is characterized by a vector-function of dimension k in one medium and by a vector-function of dimension n in the other (for example, fluid-structure interaction where a streamlined body is an elastic obstacle, scattering of acoustic and electromagnetic waves by an elastic obstacle, interaction between an elastic body and seismic waves, etc.).

A lot of authors have considered and studied in detail the direct problems of interaction between an elastic isotropic body occupying a bounded region $\overline{\Omega}$ with a three-dimensional elastic vector field to be defined, and some isotropic medium (say fluid) occupying the unbounded exterior region, the compliment of Ω with respect to the whole space, where a scalar field is to be defined. The time-harmonic dependent unknown vector and scalar fields are coupled by some kinematic and dynamic conditions on the boundary $\partial\Omega$, which lead to various type of non-classical interface problems of steady oscillations for a piecewise homogeneous isotropic medium. An exhaustive information in this direction concerning theoretical and numerical results can be found in [4], [6], [7], [24], [25], [59], [60], [32], [34] [26], [27], [78], [84].

Some particular cases where the elastic body under consideration is anisotropic have been treated in [57], [58], [79].

Various authors dedicated their works to the solid-fluid (see e.g. [79], [83], [98]-[99], [80]-[82], [9]-[11]), [14]-[18] contact problems. The present work is devoted to the interaction problems when profile of an elastic part is cusped on some part boundary.

Bending problems of cusped plates fall outside of the limits of classical bending theory. The aim of the dissertation is to study the problem of well-possedness of boundary value problems and initial boundary value problems in case of cylindrical bending of shells with two cusped edges and in some cases to solve these problems in explicit forms.

The work consists of the list of notations, introduction, three chapters and bibliography.

The Introduction contains a survey of results related to the subject and a brief presentation of results of the present work.

In Chapter 1 some auxiliary materials are given used in Chapters 2 and 3.

Chapter 2 deals with the problems of cylindrical bending and bending vibration of a plate.

Let us consider the plate whose projection on $x_3 = 0$ occupies the domain Ω

$$\Omega = \{(x_1, x_2, x_3) : \ -\infty < x_1 < \infty, \ \ 0 < x_2 < l, \ \ x_3 = 0\},$$

and where the thickness of the plate are given by the equation

$$2h(x_2) = h_0 x_2^{\alpha/3}(l - x_2)^{\beta/3}, \quad h_0, \, l, \, \alpha, \, \beta = \text{const}, \ \ h_0, \, l > 0, \quad \alpha, \, \beta \geq 0.$$

When $\alpha^2 + \beta^2 > 0$ a plate is called a cusped plate. A profile of the plate under consideration has one of the forms shown in Figures 4-12.

The equation of cylindrical bending of the plate has the form (see, e.g., [88])

$$(D(x_2)w,_{22}(x_2)),_{22} = q(x_2), \quad 0 < x_2 < l, \qquad (4)$$

where $w(x_2)$ is a deflection of the plate, $q(x_2)$ is a load, $D(x_2)$ is a flexural rigidity of the plate, and by $w,_i$ we denote $w,_i := \dfrac{\partial w}{\partial x_i}$.

In general,

$$D(x_2) := \frac{2Eh^3(x_2)}{3(1-\sigma^2)},$$

where E is a Young's modulus, σ is a Poisson's ratio. Let $E =$const, $\sigma =$const, and

$$D(x_2) = D_0 x_2^\alpha (l - x_2)^\beta, \quad D_0 = \text{const} > 0.$$

In the case of cylindrical bending of an isotropic plate, the bending moment $M_2(x_2)$ and the intersection force $Q_2(x_2)$ are given by the formulae (see [88])

$$M_2(x_2) := -D(x_2)w,_{22}(x_2), \quad Q_2(x_2) := M_{2,2}(x_2). \qquad (5)$$

Section 2.1 is devoted to the investigation of properties of equation (4) and formulation of all admissible classical bending boundary value problems (BVPs).

If $q(x_2) \in C([0,l])$ then

$$M_2(x_2), \quad Q_2(x_2) \in C([0,l]),$$

the behaviour of the $w,_2(x_2)$ and $w(x_2)$ when $x_2 \to 0_+$ and $x_2 \to l_-$ depends on α and β. As a result of the corresponding analysis we obtain that, e.g., at the point $x_2 = 0$ the following classical bending boundary conditions are admissible

$$\begin{aligned}
&1.\ w(0) = w'(0) = 0 \ \ \text{iff(if and only if)} \ \ \alpha < 1; && (6)\\
&2.\ w'(0) = Q_2(0) = 0 \ \ \text{iff} \ \ \alpha < 1; && (7)\\
&3.\ w(0) = M_2(0) = 0 \ \ \text{iff} \ \ \alpha < 3; && (8)\\
&4.\ M_2(0) = Q_2(0) = 0 \ \ \text{for any} \ \ \alpha. && (9)
\end{aligned}$$

Similar conditions we have at the point $x_2 = l$, under the same restrictions on β. All BVPs are solved in the explicit integral forms. Using these integral representations and the difference equation corresponding to (4) by means of MATLAB we get numerical results for the deflection, the bending moment and the intersecting force for different materials (see Figures 13-16).

In Section 2.2 a dynamical problem is investigated for the above cusped plate. The corresponding equation has the following form

$$(D(x_2)w,_{22}(x_2,t)),_{22} = q(x_2,t) - 2\rho h(x_2)\frac{\partial^2 w(x_2,t)}{\partial t^2}, \quad 0 < x_2 < l, \qquad (10)$$

where ρ is a density of the plate.

We solve equation (10) under the following initial conditions (IC)

$$w(x_2, 0) = \varphi_1(x_2), \quad w_{,t}(x_2, 0) = \varphi_2(x_2), \quad x_2 \in [0, l], \tag{11}$$

where $\varphi_1(x_2)$, $\varphi_1(x_2) \in C([0, l])$ are given functions.

In this case the bending moment and the intersecting force are given by the expressions

$$M_2(x_2, t) := -D(x_2)w_{,22}(x_2, t), \tag{12}$$
$$Q_2(x_2, t) := M_{2,2}(x_2, t). \tag{13}$$

Since of (10) is not degenerate equation with respect to $t = 0$, taking into account (6)-(9), the following initial boundary value problems (IBVPs) are admissible

Problem 11 *Let $0 \leq \alpha < 3$, $0 \leq \beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$w(\cdot, t) \in C^4(]0, l[) \cap C([0, l]) \cap C^1(]0, l[)$, $\quad M_2(\cdot, t) \in C([0, l])$, $\quad Q_2(\cdot, t) \in C([0, l])$,
$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0)$,
$\quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0)$,

equation (10), the BCs

$$w(0, t) = M_2(0, t) = w_{,2}(l, t) = Q_2(l, t) = 0, \quad t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l[) \cap C([0, l]),$$

$$\begin{aligned}\varphi_i(0) &= -D(x_2)\varphi_i''(x_2)|_{x_2=0_+} = \varphi_i'(l) \\ &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=l_-} = 0, i = 1, 2.\end{aligned}$$

Problem 12 *Let $0 \leq \alpha$, $\beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l])$,
$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0)$, $\quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0)$,

equation (10), the boundary conditions (BCs)

$$w(0, t) = w_{,2}(0, t) = w(l, t) = w_{,2}(l, t) = 0, \quad t > 0,$$

and ICs (11), where
$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = \varphi_i'(l) = 0, \quad i = 1, 2.$$

Problem 13 *Let $0 \leq \alpha,\ \beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]),\quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\quad w(x_2, t) \in C(0 \leq x_2 \leq l,\ t \geq 0),$$

equation (10), the BCs

$$w(0, t) = w_{,2}(0, t) = w_{,2}(l, t) = Q_2(l, t) = 0,\quad t > 0,$$

and ICs (11), where
$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$\varphi_i(0) = \varphi_i'(0) = \varphi_i'(l) = (-D(x_2)\varphi_i''(x_2))' \,|_{x_2 = l_-} = 0,\ i = 1, 2.$$

Problem 14 *Let $0 \leq \alpha, < 1,\ 0 \leq \beta < 3$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]) \cap C([0, l]),\quad M_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\quad w(x_2, t) \in C(0 \leq x_2 \leq l,\ t \geq 0),$$

equation (10), the BCs

$$w(0, t) = w_{,2}(0, t) = w(l, t) = M_2(l, t) = 0,\quad t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]) \cap C([0, l]),$$

$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = (-D(x_2)\varphi_i''(x_2)) \,|_{x_2 = l_-} = 0,\ i = 1, 2.$$

Problem 15 *Let $0 \leq \alpha < 1,\ \beta \geq 0$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]),\quad M_2(\cdot, t) \in C([0, l]),\quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\quad w(x_2, t) \in C(0 \leq x_2 < l,\ t \geq 0),$$

equation (10), the BCs

$$w(0, t) = w_{,2}(0, t) = M_2(l, t) = Q_2(l, t) = 0,\quad t > 0,$$

and ICs (11), where
$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$

$$\begin{aligned}
\varphi_i(0) &= \varphi_i'(0) = (-D(x_2)\varphi_i''(x_2)) \,|_{x_2 = l_-} \\
&= (-D(x_2)\varphi_i''(x_2))' \,|_{x_2 = l_-} = 0,\ i = 1, 2.
\end{aligned}$$

Problem 16 *Let $0 \leq \alpha, \ \beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]), \quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (10), the BCs

$$w_{,2}(0, t) = Q_2(0, t) = w(l, t) = w_{,2}(l, t) = 0, \quad t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$\varphi_i'(0) = (-D(x_2)\varphi_i''(x_2))' \big|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0, \ i = 1, 2.$$

Problem 17 *Let $0 \leq \alpha < 1, \ 0 \leq \beta < 3$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l[) \cap C([0, l]), \quad M_2(\cdot, t) \in C([0, l]), \quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (10), the BCs

$$w_{,2}(0, t) = Q_2(0, t) = w(l, t) = M_2(l, t) = 0, \quad t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l[) \cap C([0, l]),$$

$$\begin{aligned}
\varphi_i'(0) &= (-D(x_2)\varphi_i''(x_2))' \big|_{x_2=0_+} = \varphi_i(l) \\
&= (-D(x_2)\varphi_i''(x_2)) \big|_{x_2=l_-} = 0, \ i = 1, 2.
\end{aligned}$$

Problem 18 *Let $0 \leq \alpha < 3, \ 0 \leq \beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1(]0, l]) \cap C([0, l]), \quad M_2(\cdot, t) \in C([0, l]), \quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (10), the BCs

$$w(0, t) = M_2(0, t) = w(l, t) = w_{,2}(l, t) = 0, \quad t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l]) \cap C([0, l]),$$
$$\varphi_i(0) = (-D(x_2)\varphi_i''(x_2)) \big|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0, \ i = 1, 2.$$

Problem 19 *Let $0 \leq \alpha,\ \beta < 3$. Find a function $w(x_2, t)$, which the satisfies follow-ing smoothness conditions*

$$
w(\cdot, t) \in C^4(]0, l[) \cap C([0, l]),\quad M_2(\cdot, t) \in C([0, l]),
$$
$$
w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\quad w(x_2, t) \in C(0 \leq x_2 \leq l,\ t \geq 0),
$$

equation (10), the BCs

$$
w(0, t) = M_2(0, t) = w(l, t) = M_2(l, t) = 0,\quad t > 0,
$$

and ICs (11), where

$$
\varphi_i(x_2) \in C^4(]0, l[) \cap C([0, l]),
$$

$$
\begin{aligned}
\varphi_i(0) &= \left. (-D(x_2)\varphi_i''(x_2)) \right|_{x_2=0_+} = \varphi_i(l) \\
&= \left. (-D(x_2)\varphi_i''(x_2)) \right|_{x_2=l_-} = 0,\ i = 1, 2.
\end{aligned}
$$

Problem 20 *Let $\alpha \geq 0,\ 0 < \beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$
w(\cdot, t) \in C^4(]0, l[) \cap C^1(]0, l[),\quad M_2(\cdot, t) \in C([0, l]),\quad Q_2(\cdot, t) \in C([0, l]),
$$
$$
w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\quad w(x_2, t) \in C(0 < x_2 \leq l,\ t \geq 0),
$$

equation (10), the BCs

$$
M_2(0, t) = Q_2(0, t) = w(l, t) = w_{,2}(l, t) = 0,\quad t > 0,
$$

and ICs (11), where

$$
\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l[),
$$

$$
\begin{aligned}
(-D(x_2)\varphi_i''(x_2)) &= \left. (-D(x_2)\varphi_i''(x_2))' \right|_{x_2=0_+} \\
&= \varphi_i(l) = \varphi_i'(l) = 0,\ i = 1, 2.
\end{aligned}
$$

Let $q \equiv 0$. Using the Fourier method, we look for $w(x_2, t)$ in the following form

$$
w(x_2, t) = X(x_2)T(t),
$$

where $T(t)$ and $X(x_2)$ are satisfying the following equations

$$
T''(t) + \lambda T(t) = 0,
$$

and

$$
X(x_2) = \lambda \int_0^l g(\xi)K(x_2, \xi)X(\xi)d\xi,\quad g(x_2) := 2\rho h(x_2), \tag{14}
$$

where $K(x_2, \xi) \in C([0, l] \times [0, l])$ is constructed explicitly and it depends on the coefficients of equation (10) and the type of boundary conditions in Problems 11-20.

We denote by λ_n and X_n the corresponding eigenvalues and eigenfunctions of (14).

The following propositions hold.

Proposition 2.2 $K(x_2, \xi)$ *is symmetric with respect to* x_2 *and* ξ.

Proposition 2.3 *Number of eigenvalues* λ_n *of (14) is not finite.*

Proposition 2.4 *All* λ_n *are positive.*

The solution of equation (10) under the initial conditions (11) and one of the boundary conditions (see Problems 11-20) can be written as follows [15]

$$w(x_2, t) = \sum_{n=1}^{\infty} X_n(x_2) \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right), \tag{15}$$

where

$$b_1^n = \frac{1}{\sqrt{\lambda_n}} \int_0^l g(x_2) X_n(x_2) \varphi_2(x_2) dx_2, \quad b_2^n = \int_0^l g(x_2) X_n(x_2) \varphi_1(x_2) dx_2. \tag{16}$$

Let us consider one of the IBVP. For the sake of simplicity we consider Problem 11.

Further, if we suppose that $\psi_i(x_2) := \frac{(D\varphi_i'')''}{\sqrt{g(x_2)}} \in C([0, l])$ $(i = 1, 2)$, we can prove the following theorems [15]

Theorem 2.5 *The series (15) converges absolutely and uniformly on* $[0, l]$. *Moreover, the series*

$$w_{,t}(x_2, t) = \sum_{n=1}^{\infty} X_n(x_2) \sqrt{\lambda_n} \left(b_1^n \cos(\sqrt{\lambda_n} t) - b_2^n \sin(\sqrt{\lambda_n} t) \right)$$

and

$$w_{,tt}(x_2, t) = -\sum_{n=1}^{\infty} X_n(x_2) \lambda_n \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right)$$

converge absolutely and uniformly on any $[a, b] \subset]0, l[$ *if the functions*

$$\Psi_i(x_2) := \frac{\psi_i(x_2)}{\sqrt{g(x_2)}} \text{for} i = 1, 2 \text{satisfyBCsgiveninProblem11} \tag{17}$$

and the functions

$$\chi_i(x_2) \sqrt{g(x_2)} := (D(x_2) \Psi_i''(x_2))'', \quad i = 1, 2, \quad \text{are integrable on }]0, l[\tag{18}$$

(For this, e.g., it is sufficient that $\frac{d^j}{dx_2^j} \varphi_i(x_2) = O(x_2^{\gamma_{ij}})$, $\gamma_{ij} = \text{const} > 7 - j - \frac{5\alpha}{3}$, $x_2 \to 0_+$, $\frac{d^j}{dx_2^j} \varphi_i(x_2) = O((l - x_2)^{\delta_{ij}})$, $\delta_{ij} = \text{const} > 7 - j - \frac{5\beta}{3}$, $x_2 \to l_-$, $i = 1, 2$; $j = \overline{2, 8}$).*

Theorem 2.6 *The series*

$$\frac{\partial^i}{\partial x_2^i} w(x_2, t) = \sum_{n=1}^{\infty} \frac{d^i}{dx_2^i} X_n(x_2) \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right), \quad i = 1, 2, 3, 4,$$

are convergent absolutely and uniformly on any $[a, b] \subset]0, l[$, *while the series*

$$\frac{\partial^{i-1}}{\partial x_2^{i-1}} (D(x_2) w_{,x_2 x_2} (x_2, t)) = \sum_{n=1}^{\infty} \frac{d^{i-1}}{dx_2^{i-1}} (D(x_2) X_n''(x_2)) \left(b_1^n \sin(\sqrt{\lambda_n} t) + \right.$$

$$\left. + b_2^n \cos(\sqrt{\lambda_n} t) \right), \quad i = 1, 2$$

are convergent absolutely and uniformly on $[0, l]$.

Thus, (15) is the solution of the Problem 11 for $q(x_2, t) \equiv 0$.

Let us consider the case when $q(x_2, t) \not\equiv 0$, $\varphi_i = 0$, and let $\frac{q}{\sqrt{g}}(\cdot, t) \in L_2(0, l)$. Then $q(x_2, t)$ can be represented as a convergent series in $L_2(0, l)$:

$$q(x_2, t) = \sum_{n=1}^{\infty} g(x_2) X_n(x_2) q_n(t), \quad q_n(t) := \int_0^l q(x_2, t) X_n(x_2) dx_2.$$

Further, we look for the solution in the form $w(x_2, t) = \sum_{n=1}^{\infty} w_n(x_2, t)$, where $w_n(x_2, t)$ is a solution of the equation (10) under the homogeneous initial conditions and under the boundary conditions given in Problem 11 with $q(x_2, t)$ replaced by $g(x_2) X_n(x_2) q_n(t)$. Now, using the method of separation of variables we can write

$$w_n(x_2, t) = X_n(x_2) T_{1n}(t),$$

where

$$T_{1n}''(t) + \lambda_n T_{1n}(t) = q_n(t).$$

Therefore, $w(x_2, t)$ can be expressed as follows

$$w(x_2, t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} X_n \int_0^t \sin(\sqrt{\lambda_n}(t - \tau)) q_n(\tau) d\tau. \tag{19}$$

Similarly to Theorems 2.5 and 2.6, if the following conditions are fulfilled

$$\tau(x_2, t) := \frac{1}{\sqrt{g(x_2)}} \left(D(x_2) \left(\frac{q(x_2, t)}{g(x_2)} \right)_{,x_2 x_2} \right)_{,x_2 x_2} \in C[0, l],$$

$$\tag{20}$$

and $\dfrac{\tau(x_2, t)}{\sqrt{g(x_2)}}$ satisfies the BCs given in Problem 11

(For this, e.g., it is sufficient that $\frac{\partial^j}{\partial x_2^j}q(x_2,t) = O(x_2^{\gamma_j})$ $x_2 \to 0_+$, $\gamma_j > 7 - j - \frac{2\alpha}{3}$, $\frac{\partial^j}{\partial x_2^j}q(x_2,t) = O((l-x_2)^{\delta_j})$ $x_2 \to l_-$, $\gamma_j > 7 - j - \frac{2\beta}{3}$, $j = \overline{0,8}$) we get the absolute and uniform convergence of the series (19) and

$$\frac{\partial^i}{\partial x_2^i}(D(x_2)w_{,x_2 x_2}(x_2,t)) = \sum_{n=1}^{\infty}\frac{d^i}{dx_2^i}(D(x_2)X_n'')T_{1n}(t), \quad i = 0,1,$$

on $[0,l]$, and absolute and uniform convergence of

$$\frac{\partial^i}{\partial x_2^i}w_{x_2}(x_2,t) = \sum_{n=1}^{\infty}\frac{d^i}{dx_2^i}X_n(x_2)T_{1n}(t), \quad i = 1,...,4,$$

$$\frac{\partial^i}{\partial t^i}w(x_2,t) = \sum_{n=1}^{\infty}X_n(x_2)\frac{d^i}{dt^i}T_{1n}(t), \quad i = 1,2,$$

on any $[a,b] \subset]0,l[$.

Now, let $q(x_2,t) \not\equiv 0$, $\varphi_i(x_2) \not\equiv 0$. If conditions (20), (17), and (18) are satisfied then the solution of Problem 11 can be expressed as follows

$$w(x_2,t) = \sum_{n=1}^{\infty}w_n(x_2,t),$$

where

$$w_n(x_2,t) = X_n(x_2)(T_{1n}(t) + T_n(t)),$$

$w^1(x_2,t) := X_n T_{1n}(t)$ is given by the formula (19) and $w^2(x_2,t) := X_n T_n(t)$ is given by the formula (15).

Remark 1 *Similarly are solved IBVPs corresponding to the Problems 12-20.*

We can avoid the restrictions (20) if we consider harmonic vibration. In this case

$$w(x_2,t) = e^{i\omega t}w_0(x_2), \quad q(x_2,t) = e^{i\omega t}q_0(x_2),$$

where $\omega = const$ is an oscillation frequency, $q_0(x_2) \in C([0,l])$ is a given function. E.g., in the case of Problem 11, for $w_0(x_2)$ we get the following problem

$$\begin{aligned}(D(x_2)w_0''(x_2))'' &= q_0(x_2) + 2\omega^2 \rho h(x_2)w_0(x_2), & (21)\\ w_0(0) = M_2(0) &= w'(l) = Q_2(l) = 0, \quad 0 \le \alpha < 2, \ 0 \le \beta < 1,\\ w_0(x_2) &\in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l[).\end{aligned}$$

This problem is equivalent to the integral equation

$$w_0(x_2) - \omega^2 \int_0^l K(x_2,\xi)\,g(\xi)\,w_0(\xi)d\xi = F(x_2), \tag{22}$$

where

$$F(x_2) := \int_0^l K(x_2, \xi)\, q_0(\xi) d\xi,$$

$K(x_2, \xi)$ has the same form as in integral equation (14).

If $\omega^2 \neq \lambda_n$, the unique solution of (22) can be written as follows (see, e.g., [66])

$$
\begin{aligned}
w_1(x_2) &= F(x_2)\sqrt{g(x_2)} \\
&+ \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi)\, \sqrt{g(x_2)}\, Y_n(\xi) d\xi \right] Y_n(x_2),
\end{aligned}
\tag{23}
$$

It is shown that the series in the right hand side of (23) is absolutely and uniformly convergent on $[0, l]$, because of $q_0 \in C([0, l])$.

Using the difference equation corresponding to (21), by means of MATLAB we get numerical and graphical results for harmonic vibration problems.

Chapter 3 is dedicated to the interface problem of the interaction of a plate with two cusped edges and a flow of a fluid.

We assume that the flow is independent of x_1, parallel to the plane $0x_2x_3$, i.e. $v_1 \equiv 0$, and generates a bending of the plate. Let at infinity, for pressure we have

$$p(x_2, x_3, t) \to p_\infty(t), \quad \text{when } |x| \to \infty, \tag{24}$$

and let for the velocity components conditions at infinity be either

$$v_2(x_2, x_3, t) = O(1), \quad v_3(x_2, x_3, t) \to v_{3\infty}(t), \tag{25}$$

or

$$v_j(x_2, x_3, t) = O(1), \quad j = 2, 3, \tag{26}$$

where $v := (v_2, v_3)$ is a velocity vector of the fluid, $p(x_2, x_3, t)$ is a pressure, and $v_{3\infty}(t)$, $p_\infty(t)$ are given functions.

Let us introduce the following notations

$$
\begin{aligned}
I &:= \{[0, l] \times 0\}, \\
\Omega^f &:= \left\{ x_1, x_2, x_3 : x_1 = 0,\ x := (x_2, x_3) \in \mathbb{R}^2 \backslash I \right\}.
\end{aligned}
$$

If the middle plane of the plate lies in the plane $0x_1x_2$ and the flow of moving fluid involves bending of the plate then transmission conditions could have the form:

$$\sigma_{N3}^f \left(x_1, x_2, \overset{(+)}{h}(x_1, x_2), t \right) - \sigma_{N3}^f \left(x_1, x_2, \overset{(-)}{h}(x_1, x_2), t \right) = q(x_1, x_2, t), \tag{27}$$

$$v_3 \left(x_1 - \overset{(+)}{h}(x_1, x_2) w_{,1}(x_1, x_2, t), x_2 - \overset{(+)}{h}(x_1, x_2) w_{,2}(x_1, x_2, t), \overset{(+)}{h}(x_1, x_2) \right.$$

$$+w(x_1, x_2, t), t\bigg) = v_3 \left(x_1 - \overset{(-)}{h}(x_1, x_2)w,_1(x_1, x_2, t), x_2 \right.$$
$$\left. - \overset{(-)}{h}(x_1, x_2)w,_2(x_1, x_2, t), \overset{(-)}{h}(x_1, x_2) + w(x_1, x_2, t), t \right) = \frac{\partial w(x_1, x_2, t)}{\partial t}, \tag{28}$$

(the first of the last pair of equalities is valid since deflection of plate w is independent of x_3).

After corresponding analysis we arrive at the conclusion that, for the normal component of the velocity vector and the pressure in the case of an ideal fluid, we have the following transmission conditions (compare with [65], [99], [83])

$$v_3(x_2, 0, t) = \frac{\partial w(x_2, t)}{\partial t}, \quad x_2 \in]0, l[, \quad t \geq 0. \tag{29}$$

$$- p(x_2, \overset{(-)}{h}(x_2), t) \cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)), x_3)$$
$$- p(x_2, \overset{(+)}{h}(x_2), t) \cos(\overrightarrow{n}(x_2, \overset{(+)}{h}(x_2)), x_3) = q(x_2, t), \quad x_2 \in]0, l[. \tag{30}$$

In the case of a viscous fluid we add to (29) the transmission condition for the tangential component of the velocity vector

$$v_2(x_2, 0, t) = 0, \quad x_2 \in]0, l[, \quad t \geq 0. \tag{31}$$

In Section 3.1 the solution of the interaction problem in the case of an ideal fluid is given [2].

For the potential motion of the flow there exists a complex function $\Phi = -\psi + i\varphi$ such that

$$\frac{\partial \varphi(x_2, x_3, t)}{\partial x_2} = \frac{\partial \psi(x_2, x_3, t)}{\partial x_3} = v_2(x_2, x_3, t),$$
$$\frac{\partial \varphi(x_2, x_3, t)}{\partial x_3} = -\frac{\partial \psi(x_2, x_3, t)}{\partial x_2} = v_3(x_2, x_3, t). \tag{32}$$

The pressure is given by the formula

$$p(x_2, x_3, t) = \rho^f \left[\frac{v_\infty^2}{2} + \frac{p_\infty}{\rho^f} + \frac{\partial \varphi_\infty}{\partial t} - \frac{\partial \varphi}{\partial t} - \frac{1}{2}(v_2^2 + v_3^2) \right]. \tag{33}$$

We calculate $w(x_2, t)$ from the equation (10).

Problem 21 *Find a function $w(\cdot, t) \in C^4(]0, l[)$ (and additional smoothness conditions indicated in Problems 11-20), also the functions $v_2(x_2, x_3, t) \in C^2(\Omega^f) \cup C^1(t > 0)$, $v_3(x_2, x_3, t) \in C^2(\Omega^f) \cup C^1(t > 0)$ and $p(x_2, x_3, t) \in C(\Omega^f) \cup C(t > 0)$ which satisfy the system of equations (10), (32), (33), transmission conditions (29), (30), conditions at infinity (24), (25) and one of the BCs given in Problems 11-20.*

For $\Phi_{,2}(x_2, x_3, t) = v_3 + iv_2$, in view of (25) and (29), we get the following expression [72]

$$\Phi_{,2} = -\frac{1}{\pi i \sqrt{(x_2 + ix_3)(x_2 + ix_3 - l)}} \int\limits_0^l \frac{\sqrt{(\xi_2 + ix_3)(\xi_2 + ix_3 - l)}}{(\xi_2 - x_2) - ix_3} w_{,t}(\xi_2, t) d\xi_2$$

$$+ v_{3\infty} \frac{x_2 + ix_3 - l/2}{\sqrt{(x_2 + ix_3)(x_2 + ix_3 - l)}}. \tag{34}$$

Let

$$w(x_2, t) = e^{i\omega t} w_0(x_2), \quad q(x_2, t) = e^{i\omega t} q_0(x_2),$$
$$p(x_2, x_3, t) = e^{i\omega t} p_0(x_2, x_3),$$

$$u_2(x_2, x_3, t) = e^{i\omega t} u_2^0(x_2, x_3), \quad u_3(x_2, x_3, t) = e^{i\omega t} u_3^0(x_2, x_3),$$
$$\varphi(x_2, x_3, t) = i e^{i\omega t} \varphi_0(x_2, x_3), \quad \psi(x_2, x_3, t) = i e^{i\omega t} \psi_0(x_2, x_3),$$

$$v_2(x_2, x_3, t) = i e^{i\omega t} v_2^0(x_2, x_3), \quad v_3(x_2, x_3, t) = i e^{i\omega t} v_3^0(x_2, x_3),$$

$$p_\infty(t) = e^{i\omega t} p_\infty^0, \quad v_{3\infty}(t) = i e^{i\omega t} v_{3\infty}^0, \quad p_\infty^0, \ v_{3\infty}^0 = \text{const},$$

where $\omega = \text{const} > 0$ is an oscillation frequency, $v_2 = u_{2,t} \ (v_3 = u_{3,t})$.

After separating real and imaginary parts of (34), we obtain the expressions for v_2 and v_3. By means of the latter, in view of (32), we can calculate φ and then substitute it into (33). Then substituting the obtained expression for $p(x_2, x_3, t)$ into (30), we get the expression for $q(x_2)$. Therefore, all the mechanical quantities in the fluid part and the lateral load are calculated by means of deflection. In the case of harmonic vibration for deflection we get the second order Fredholm type linear integral equation [2]

$$w_0(x_2) - \omega^2 \int\limits_0^l K_1(x_2, \xi) w_0(\xi) d\xi = f_1(x_2), \tag{35}$$

where $K_1(x_2, \xi_2) \in C([0, l] \times [0, l])$ and $f_1(x_2) \in C([0, l])$ are defined explicitly. They depend on the coefficients of the equation (21), on the type of the boundary conditions in Problems 11-20, and the conditions at infinity (24)-(25).

The following proposition is valid

Proposition 3.2 *Problem of the harmonic vibration corresponding to the Problem 17 has a unique solution when*

$$\omega^2 < \frac{1}{Ml},$$

where

$$M := \max_{x_2, \xi \in [0, l]} \{|K_1(x_2, \xi)|\}.$$

Remark 2 *If the plate thickness is sufficiently small, we can assume that:*
1. the fluid occupies $\mathbb{R}^2 \backslash I$;
2. the plate occupies I (its geometry depending on the thickness is taken into account in the coefficient of the bending equation);
3. $\overset{(\pm)}{h}$ can be neglected. Since the normals of I are $(0,0,1)$ and $(0,0,-1)$, (30) can be rewritten as follows

$$-p(x_2, 0_+, t) + p(x_2, 0_-, t) = q(x_2, t), \quad x_2 \in]0, l[.$$

In Section 3.2 an interaction problem for the case of an incompressible viscous fluid is solved [3]. We consider the case when the motion of the fluid is sufficiently slow, i.e., v_j and $v_{j,k}$ $(j, k = 2, 3)$ are so small that linearized Navier-Stokes equations can be applied

$$\frac{\partial v_2}{\partial t} = -\frac{1}{\rho^f} \frac{\partial p}{\partial x_2} + \nu \Delta v_2,$$

$$\frac{\partial v_3}{\partial t} = -\frac{1}{\rho^f} \frac{\partial p}{\partial x_3} + \nu \Delta v_3,$$

(36)

where $\nu = \mu/\rho^f$ is a coefficient of viscosity, $\Delta = \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$.

We add to the (36) the following equation

$$\operatorname{div} v(x_2, x_3, t) = 0, \quad (x_2, x_3) \in \Omega^f, \quad t \geq 0.$$

Let us consider the problem of harmonic vibration.

Problem 22 *Find a function $w_0(x_2)$ on I, which satisfies the equation (21), one of the boundary conditions given in the Problems 11-20, and also find functions $u_i^0(x_2, x_3)$, $p_0(x_2, x_3)$, $q_0(x_2)$ on Ω^f, which satisfy the following system of equations*

$$\Delta p_0(x_2, x_3) = 0,$$
$$-\omega^2 u_j^0 = -\frac{1}{\rho^f} \frac{\partial p_0}{\partial x_j} + \nu i \omega \Delta u_j^0, \quad j = 2, 3,$$

smoothness conditions

$$u_i^0 \in C^2(\Omega^f) \cap C(\mathbb{R}^2) \cap C(t > 0), \quad i = 2, 3;$$
$$p_0 \in C^2(\Omega^f);$$
$$q_{0,2}(\cdot, t) \in C^\gamma([0, l]), \quad 0 < \gamma \leq 1,$$

following conditions at infinity

$$p_0|_{|x| \to \infty} = O(1), \quad u_j^0|_{|x| \to \infty} = O(1), \quad j = 2, 3,$$

and transmissions conditions as follows

$$-p_0(x_2, 0_+) + p_0(x_2, 0_-) = q_0(x_2), \quad x_2 \in]0, l[,$$
$$u_3^0(x_2, 0) = w_0(x_2), \quad u_2^0 = 0, \quad x_2 \in]0, l[,$$

where $q(x_2, t) = e^{i\omega t} q_0(x_2)$.

In this case all the mechanical quantities in question are calculated by means of the lateral load q_0, for which we obtain a second order supersingular integral equation

$$\int\limits_0^l \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2 + 2\pi\omega^2\rho^f \int\limits_0^l K_2(x_2, \xi_2) q_0(\xi_2) d\xi_2 = f_2(x_2),$$

where the supersingular integral is defined in H'adamard's finite part sense, $K_2(x_2, \xi_2) \in C([0, l] \times [0, l])$ and $f_2(x_2) \in C([0, l])$ are quite definite functions. If $q_{0,2}(x_2) \in C^\gamma([0, l])$ $0 < \gamma \leq 1$, this equation is solved by a method developed by Boikov, I.V., Dobrynin, N.F., Domnin, L. in [9] (see also Chapter 1, section 1.3).

Chapter 1

Preliminary Materials

1.1. Cusped Plates

Let $0x_1x_2x_3$ be the Cartesian coordinate system, and Ω be a domain in the plane $0x_1x_2$ with a piecewise smooth boundary.

The body bounded upper by the surface

$$x_3 = \overset{(+)}{h}(x_1, x_2) \geq 0, \quad (x_1, x_2) \in \Omega,$$

lower by the surface

$$x_3 = \overset{(-)}{h}(x_1, x_2) \geq 0, \quad (x_1, x_2) \in \Omega,$$

and from the side by a cylindrical surface parallel to the x_3-axis, will be called *a cusped plate*.
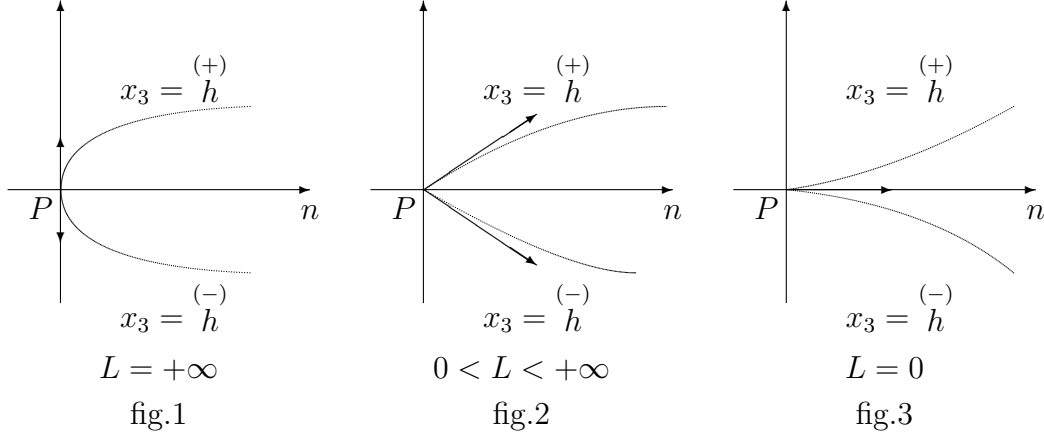
$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2), \quad (x_1, x_2) \in \Omega,$$

is the *thickness of the plate*.

The points $P \in \partial\Omega$, at which plate thickness $2h(x_1, x_2) = 0$, will be called *plate cusps*. If $h \in C^1(\Omega)$, obviously,

$$0 \leq L := \lim_{Q \to P} \frac{\partial 2h(Q)}{\partial n} \leq +\infty, \quad Q \in \Omega, \ P \in \partial\Omega,$$

provided that the finite or infinite limit L exists; if P is an angular point of the boundary $\partial\Omega$ under the inward to $\partial\Omega$ normal n we mean bisector of angle between unilateral tangents to $\partial\Omega$ at P. Ω will be called *a projection of the plate*. $\partial\Omega$ will be called *a plate boundary*. On the figures 1-3 are represented the possible normal sections (profiles) of a symmetric plate ($\overset{(+)}{h}(x_1, x_2) = -\overset{(-)}{h}(x_1, x_2)$) at the point P in its neighborhood.

$x_3 = \overset{(+)}{h}$ $x_3 = \overset{(+)}{h}$ $x_3 = \overset{(+)}{h}$

$x_3 = \overset{(-)}{h}$ $x_3 = \overset{(-)}{h}$ $x_3 = \overset{(-)}{h}$

$L = +\infty$ $0 < L < +\infty$ $L = 0$

fig.1   fig.2   fig.3

Let us now consider an isotropic cusped plate.

The equation of the classical bending theory for an isotropic plate has the following form (see [88], p.364)

$$
\begin{aligned}
(Dw,_{11}),_{11} \;+\;& (Dw,_{22}),_{22} +\nu(Dw,_{22}),_{11} +\nu(Dw,_{11}),_{22} \\
+\;& 2(1-\nu)(Dw,_{12}),_{12} = q(x_1, x_2),
\end{aligned} \tag{1.1}
$$

where $D \in C^2(\Omega)$, and

$$
D := \frac{2Eh^3}{3(1-\nu^2)},
$$

where E is Young's modulus and ν is Poisson's ratio.

We recall (see [88]) that

$$
\begin{aligned}
M_\alpha &= -(D_{\alpha,\alpha\underline{\alpha}} +D_3w,_{\beta\underline{\beta}}), \;\; \alpha \neq \beta, \;\; \alpha, \; \beta = 1,2, \\
M_{12} &= -M_{21} = 2D_4w,_{12}, \\
Q_\alpha &= M_{\alpha,\underline{\alpha}} + M_{12,\beta}, \;\; \alpha \neq \beta, \;\; \alpha, \; \beta = 1,2, \\
Q_\alpha^* &= Q_\alpha + M_{21,\beta}, \;\; \alpha \neq \beta, \;\; \alpha, \; \beta = 1,2,
\end{aligned} \tag{1.2}
$$

where M_α are bending moments, $M_{\alpha\beta}, \alpha \neq \beta$, are twisting moments, Q_α are shearing forces and Q_α^* are generalized shearing forces (bars under repeated indices mean that we do not sum with respect to these indices).

At the points of the boundary, where the thickness vanishes, all quantities will be defined as limits from inside of Ω.

1.2.   Hilbert-Schmidt Theorems

Recall the following three Hilbert-Schmidt theorems (see [1], [66], [70])

Theorem 1.1 *If $u(x_2)$ has the form*

$$
u(x_2) = \lambda \int_0^l R(x_2, \xi)f(\xi)d\xi,
$$

with $f(x_2)$ piece-wise continuous on $[0, l]$, and a symmetric kernel $R(x_2, \xi) \in C([0, l] \times [0, l])$, then

$$u(x_2) = \sum_{n=1}^{\infty} (u, Y_n) Y_n(x_2), \tag{1.3}$$

where

$$(u, Y_n) := \int_0^l u(x_2) Y_n(x_2) dx_2,$$

Y_n is an eigenfunction of $R(x_2, \xi)$, and the series on the right-hand side of (1.3) is convergent absolutely and uniformly on $[0, l]$.

Theorem 1.2 *If the number of eigenvalues λ_n of the symmetric kernel is finite then*

$$R(x_2, \xi) = \sum_{n=1}^{N} \frac{Y_n(x_2) Y_n(\xi)}{\lambda_n}.$$

Theorem 1.3 *If $f(x_2) \in C([0, l])$, then*

$$\int_0^l R(x_2, \xi) f(\xi) d\xi = \sum_{n=1}^{\infty} \frac{(f, Y_n)}{\lambda_n} Y_n,$$

and the series is convergence absolutely and uniformly, here $R(x_2, \xi)$ is a symmetric kernel with respect to x_2, ξ, Y_n are eigenfunctions of R corresponding to eigenvalues λ_n.

Definition 1.4 *Kernel $R(x_2, \xi) \in C([0, l] \times [0, l])$ is called positive (negative) definite if for any $f(x_2)$ piecewise continuous function the following integral form*

$$J =: \int_0^l \int_0^l R(x_2, \xi) f(x_2) f(\xi) d\xi dx_2$$

is positive (negative).

Theorem 1.5 *$R(x_2, \xi)$ is a positive definite if and only if all eigenvalues λ_n of $R(x_2, \xi)$ are positive.*

Theorem 1.6 *If $R(x_2, \xi) \in C([0, l] \times [0, l])$ is positive definite kernel, then it can be represented as follows*

$$R(x_2, \xi) = \sum_{n=1}^{\infty} \frac{Y_n(x_2) Y_n(\xi)}{\lambda_n}, \tag{1.4}$$

where Y_n are eigenfunctions of $R(x_2, \xi)$, and λ_n are eigenvalues of $R(x_2, \xi)$. The series in the right-hand side of (1.4) is convergent uniformly on $[0, l]$.

Let us consider the following integral equation

$$\varphi(x_2) - \lambda \int_0^l R(x_2, \xi)\varphi(\xi)d\xi = f(x_2), \tag{1.5}$$

where $R(x_2, \xi) \in C([0, l] \times [0, l])$ and $f(x_2) \in C([0, l])$.

The solution of (1.5) has the following form

$$\varphi(x_2) = f(x_2) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} Y_n(x_2),$$

where

$$f_n = \int_0^l f(\xi)Y_n(\xi)d\xi.$$

Proposition 1.7 *If $R(x_2, \xi)$ is a positive definite kernel then*

$$\Gamma(x_2, \xi; \lambda) = \sum_{n=1}^{\infty} \frac{Y_n(x_2)Y_n(\xi)}{\lambda_n - \lambda},$$

where $\Gamma(x_2, \xi; \lambda)$ is a resolvent of the equation (1.5)

1.3.   Singular and Supersingular Integral Equations

Definition 1.8 *We say that a function $\varphi(x)$ satisfies the condition $H(\mu)$ (Hölder continuous function) on $[0, l]$, if for arbitrary points $x_1, x_2 \in [0, l]$ we have*

$$|\varphi(x_1) - \varphi(x_2)| \leq A|x_1 - x_2|^{\mu},$$

where $A, \mu = \text{const} > 0$, $0 < \mu \leq 1$. A is a coefficient, μ is an exponent of the condition $H(\mu)$.

Let us denote
$$S := \mathbb{R}^2 \backslash L,$$

where $L = \cup L_j := a_j b_j$, $a_j b_j \in 0x_2$, $j = \overline{1, p}$, $a_1 < b_1 < a_2 < b_2 < \dots$.
Problem (Dirichlet Problem). (see [72]) *Find such a harmonic function $u(x_2, x_3)$, which is a bounded function everywhere on S and satisfying the following conditions*

$$u^+ = f^+(x_2), \quad u^- = f^-(x_2), \quad \text{on } L,$$

where $f^+(x_2)$ and $f^-(x_2)$ are given real functions and $f^+(x_2)$, $f^-(x_2) \in H$.
Solution. We assume that derivations of $f^+(x_2)$ and $f^-(x_2)$ exist and we denote by $\Phi(z)$ the analytic function whose real part is $u(x_2, x_3)$,

$$\Phi(z) := u(x_2, x_3) + iv(x_2, x_3),$$
$$\Phi'(z) := \frac{\partial u(x_2, x_3)}{\partial x_2} + i\frac{\partial v(x_2, x_3)}{\partial x_2}.$$

The solution of the Problem is given by the following formulae

$$\Phi' = \frac{1}{\pi i \sqrt{R(z)}} \int_L \frac{\sqrt{R(\xi)}f'(\xi)d\xi}{\xi - \zeta} + \frac{1}{\pi i} \int_L \frac{g'(\xi)d\xi}{\xi - \zeta} - \frac{c_1 z^{p-1} + \ldots + c_p}{\sqrt{R(z)}}, \qquad (1.6)$$

where

$$R(z) = \prod_{j=1}^{q}(z - c_j), \quad c_j = \{a_j b_j\}, \quad q = 2p,$$
$$2f(x_2) := f^+(x_2) + f^-(x_2), \quad 2g(x_2) := f^+(x_2) - f^-(x_2).$$

Thus, from (1.6) we get

$$\Phi(z) = \int_0^z \Phi'(z)dz + c. \qquad (1.7)$$

$c_1, \ldots, c_p, c$ are real constants. We define c_1 from the conditions at infinity, and $c_2, \ldots, c_p, c$ from the following conditions (see [72])

$$\mathrm{Re} \int_0^{a_k} \Phi'(z)dz + c_k = f(a_k).$$

the last system is uniquely solvable.

Let $\varphi'(x) \in H([0, l])$ and let us consider the following integral

$$I(x_0) = \int_0^l \frac{\varphi(x)dx}{(x - x_0)^n}, \quad n = \mathrm{const} \geq 2,$$

which we define as H'adamard integral as follows (see [8], [3])

$$I(x_0) = \lim_{\varepsilon \to 0} \left(\int_{L_*} \frac{\varphi(x)dx}{(x - x_0)^n} + \frac{2\varphi(x_0)}{\varepsilon} \right),$$

where $L_* := [0, l] \backslash Q(\varepsilon, x_0)$, $Q(\varepsilon, x_0) := (x_0 - \varepsilon, x_0 + \varepsilon)$.

Let us consider the following supersingular integral equation

$$\int\limits_{-1}^{1} \frac{X(\tau)}{(\tau-x)^2} d\tau + \int\limits_{-1}^{1} K(x,\tau)X(\tau)d\tau = f(x), \tag{1.8}$$

where $K(x,\tau) \in C([-1,1] \times [-1,1])$, $f(x) \in C([-1,1])$.

The approximate solution of (1.8) is given in book [8] for $X'(x) := (dX(x)/dx) \in H([-1,1])$. Below we briefly state this result.

Let us divide interval $[-1,1]$ into N parts as follows

$$y'_k := -1 + \frac{2k}{N}, \quad k = \overline{0,N}, \quad y_k := -1 + \frac{2k+1}{2N}, \quad k = \overline{0,N-1},$$

$$X_N := (X(y_0), ..., X(y_{N_1})),$$

we will call X_N an approximate solution of (1.8). For X_N we get the following system of linear equations

$$-2NX(y_i) - \sum_{\substack{j=0 \\ j\neq i}}^{N-1} X(y_j) \left[\frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right]$$

$$+ \frac{2}{N} \sum_{j=0}^{N-1} K(y_i, y_j) X(y_j) = f(y_i), \quad i = \overline{0, N-1}. \tag{1.9}$$

The system (1.9) is uniquely solvable [8].

Let us denote by X^* the solution of (1.8), by X_N^* the solution of (1.9) and let $\hat{X}_N^*$ be a projection of X_0^* on y_k. In order to get the error estimate of the approximate solution of the equation (1.8), we consider

$$\left| -2N \left(X_N^*(y_i) - \hat{X}_N^*(y_i) \right) - \sum_{\substack{j=0 \\ j\neq i}}^{N-1} \left\{ X_N^*(y_j) - \hat{X}_N^*(y_j) \right\} \left\{ \frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right\} \right|$$

$$= \left| \int\limits_{0}^{l} \frac{X^*(\xi)}{(\xi - y_i)^2} d\xi - 2N\hat{X}_N^*(y_i) + \sum_{\substack{j=0 \\ j\neq i}}^{N-1} \hat{X}_N^*(y_j) \left\{ \frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right\} \right|$$

$$\leq \left| \int\limits_{y'_i}^{y'_{i+1}} \frac{X^*(\xi) - X^*(y_i)}{(\xi - y_i)^2} d\xi \right| + \sum_{\substack{j=0 \\ j\neq i}}^{N-1} \left| \int\limits_{y'_i}^{y'_{i+1}} \frac{X^*(\xi) - X^*(y_j)}{(\xi - y_j)^2} d\xi \right| =: I_1 + I_2.$$

Therefore, since $X'(x) \in H([0,l])$, we have that there exist $A = \text{const} > 0$, and $\alpha_1 = \text{const }0 < \alpha_1 < 1$ such that

$$|X'(y_1) - X'(y_2)| \leq A|y_1 - y_2|^{\alpha_1}.$$

Using the following expression

$$\int\limits_{y_i'}^{y_{i+1}'} \frac{d\xi}{\xi - y_i} = \ln|\xi - y_i|\big|_{y_i'}^{y_{i+1}'} = \ln\frac{l/(2N)}{l/(2N)} = 0,$$

we obtain

$$
\begin{aligned}
I_1 &= \left| \int\limits_{y_i'}^{y_{i+1}'} \frac{X^*(\xi) - X^*(y_i) - (\xi - y_i)\left\{ \frac{dX^*(\xi)}{d\xi}\big|_{\xi=y_i} \right\}}{(\xi - y_i)^2} d\xi \right| \\
&= \left| \int\limits_{y_i'}^{y_{i+1}'} \frac{\frac{dX^*(\xi)}{d\xi} - \frac{dX^*(\xi)}{d\xi}\big|_{\xi=y_i}}{\xi - y_i} d\xi \right| \leq A \left(\frac{N}{2} \right)^{-\alpha_1},
\end{aligned}
\tag{1.10}
$$

Analogously, we get

$$I_2 \leq A(N-1) \left(\frac{N}{2} \right)^{-\alpha_1}. \tag{1.11}$$

From (1.10) and (1.11) we obtain that the error of this method might be too large. For getting the most better results instead of the system (1.9) we consider the following system

$$
\begin{aligned}
a_{ii} X(y_i) &- \sum_{j=0}^{N-1}{}' X(y_j) \left[\frac{1}{y_{j+i}' - y_i} - \frac{1}{y_j' - y_i} \right] \\
&+ \frac{2}{N} \sum_{j=0}^{N-1} K(y_i, y_j) X(y_j) = f(y_i), \quad i = \overline{0, N-1}.
\end{aligned}
\tag{1.12}
$$

where

$$a_{ii} := -2N \int\limits_{\Delta_{ii}} \frac{d\xi_2}{(\xi_2 - y_i)^2}, \quad \Delta_{ii} := [-1, 1] \cap \left[y_i' - \frac{n}{N}, y_{i+1}' + \frac{n}{N} \right],$$

$$n := \sqrt{N} \quad \sum{}' := \sum_{\substack{j=0 \\ j \neq i-1, \, i, \, i+1}}^{N-1} .$$

In this case, after repeating the above calculations, the error of the approximate solution of (1.8) will be

$$|X^* - X_N^*| \leq An^{-\alpha_1},$$

where X^* and X_N^* are the solutions of equations (1.8) and (1.12), respectively.

Chapter 2

Bending of a Cusped Plate

2.1.   Cylindrical Bending of a Cusped Plate

In this chapter we will consider a plate, whose projection on $x_3 = 0$ occupies the domain Ω

$$\Omega = \{(x_1, x_2, x_3): \ -\infty < x_1 < \infty, \ 0 < x_2 < l, \ x_3 = 0\}.$$

The equation of the cylindrical bending of plates has the following form [see, Chapter 1, equation (1.1)]

$$(D(x_2)w,_{22}(x_2)),_{22} = q(x_2), \quad 0 < x_2 < l. \tag{2.1}$$

In general $D(x_2)$ is given by the equation

$$D(x_2) := \frac{2Eh^3(x_2)}{3(1-\nu^2)}, \tag{2.2}$$

We consider the case when $E =$const, $\nu =$const, and

$$D(x_2) = D_0 x_2^\alpha (l-x_2)^\beta, \quad D_0, \ \alpha, \ \beta = \text{const}, \ \ D_0 > 0, \ \ \alpha, \ \beta \geq 0. \tag{2.3}$$

Then

$$2h(x_2) = h_0 x_2^{\alpha/3}(l-x_2)^{\beta/3}, \quad h_0 = \text{const} > 0.$$

If $\alpha^2 + \beta^2 > 0$, equation (2.1) becomes degenerate one. Such plates are called *cusped plates*.

The profile of the plate under consideration has one of the forms shown in Figures 4-12.

In case under consideration [see Chapter 1, formulas (1.2)]

$$
\begin{aligned}
M_2(x_2) &:= -D(x_2)w,_{22}(x_2), & (2.4)\\
Q_2(x_2) &:= M_{2,2}(x_2), & (2.5)
\end{aligned}
$$

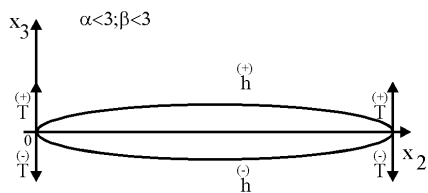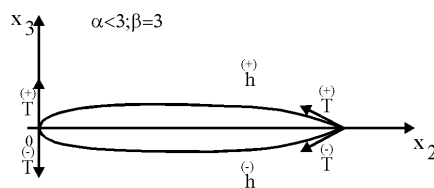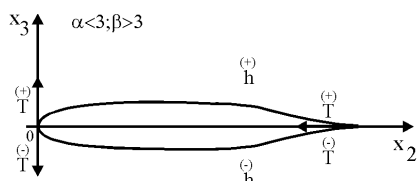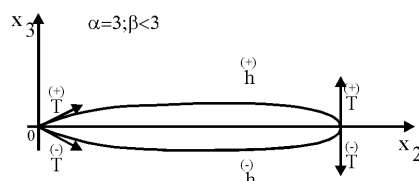Fig.4

Fig.5

Fig.6

Fig.7
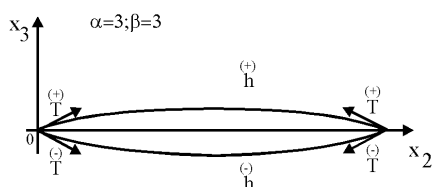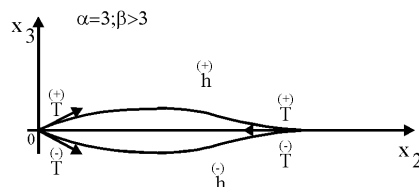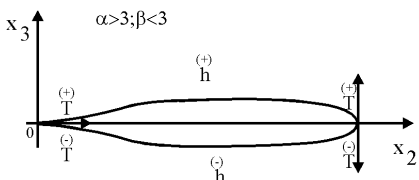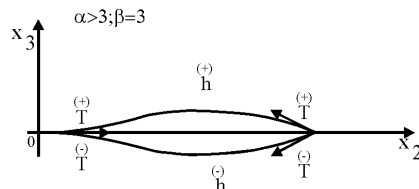
Fig.8

Fig.9

Fig.10

Fig.11
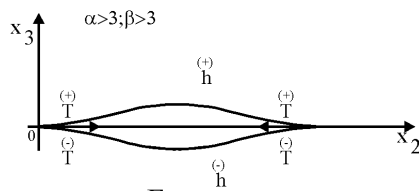
Fig.12

where $M_2(x_2)$ is a bending moment, $Q_2(x_2)$ is a shearing force.

Obviously, if we assume that $q(x_2) \in C([0, l])$, $w(x_2) \in C^4(]0, l[)$, for $Q_{2,2}(x_2)$, $M_{2,2}(x_2)$, $w_{,2}(x_2)$, and $w(x_2)$ we have

$$Q_2(x_2) \;:=\; -\int_{x_2^0}^{x_2} q(\xi)d\xi - c_1, \tag{2.6}$$

$$M_2(x_2) \;:=\; -\int_{x_2^0}^{x_2} (x_2 - \xi)q(\xi)d\xi - c_1 x_2 - c_2, \tag{2.7}$$

$$w_{,2}(x_2) \;:=\; \int_{x_2^0}^{x_2} \left\{ \left[-\int_{x_2^0}^{\xi} \eta q(\eta)d\eta + c_2 \right] + \xi \left[\int_{x_2^0}^{\xi} q(\eta)d\eta + c_1 \right] \right\} D^{-1}(\xi)d\xi \tag{2.8}$$
$$+\;\; c_3,$$

$$w(x_2) \;:=\; \int_{x_2^0}^{x_2} (x_2 - \xi) \left\{ \left[-\int_{x_2^0}^{\xi} \eta q(\eta)d\eta + c_2 \right] + \xi \left[\int_{x_2^0}^{\xi} q(\eta)d\eta + c_1 \right] \right\} D^{-1}(\xi)d\xi$$
$$+\;\; c_3 x_2 + c_4, \quad x_2^0 \in]0, l[. \tag{2.9}$$

At points 0, l all above quantities are defined as the corresponding limits when $x_2 \to 0_+$ and $x_2 \to l_-$.

Obviously,

$$Q_2(x_2), \;\; M_2(x_2) \in C([0, l]),$$
$$w(x_2), \;\; w_{,2}(x_2) \in C(]0, l[),$$

the behavior of the $w_{,2}(x_2)$ and $w(x_2)$ when $x_2 \to 0_+$ and $x_2 \to l_-$ depends, in view of (2.8), (2.9), on α, β.

As a result of the corresponding analysis we arrive at the admissible classical bending BVPs:

Problem 1 *Let $\alpha < 1$, $\beta < 1$. Find $w \in C^4(]0, l[) \cap C^1([0, l])$ satisfying (2.1) and the following boundary conditions (BCs):*

$$w(0) = g_{11}, \;\; w_{,2}(0) = g_{21}, \;\; w(l) = g_{12}, \;\; w_{,2}(l) = g_{22}; \tag{2.10}$$

Problem 2 *Let $\alpha < 1$, $\beta < 1$. Find $w \in C^4(]0, l[) \cap C^1([0, l])$ satisfying (2.1) and BCs:*

$$w(0) = g_{11}, \;\; w_{,2}(0) = g_{21} \;\; w_{,2}(l) = g_{22} \;\; Q_2(l) = h_{22};$$

Problem 3 *Let $0 \le \alpha < 1$, $\quad 0 \le \beta < 2$. Find $w \in C^4(]0, l[) \cap C^1([0, l[) \cap C([0, l])$ satisfying (2.1) and BCs:*

$$w(0) = g_{11}, \;\; w_{,2}(0) = g_{21}, \;\; w(l) = g_{12}, \;\; M_2(l) = h_{12};$$

Problem 4 *Let $0 \leq \alpha < 1$, $\beta \geq 0$. Find $w \in C^4(]0, l[) \cap C^1([0, l[)$ satisfying (2.1) and the following BCs:*

$$w(0) = g_{11}, \quad w_{,2}(0) = g_{21} \quad M_2(l) = h_{12}, \quad Q_2(l) = h_{22};$$

Problem 5 *Let $0 \leq \alpha$, $\beta < 1$. Find $w \in C^4(]0, l[) \cap C^1([0, l])$ satisfying (2.1) and the following BCs:*

$$w_{,2}(0) = g_{21} \quad Q_2(0) = h_{21}, \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22};$$

Problem 6 *Let $0 \leq \alpha < 1$, $0 \leq \beta < 2$. Find $w \in C^4(]0, l[) \cap C^1([0, l[) \cap C([0, l])$ satisfying (2.1) and the following BCs:*

$$w_{,2}(0) = g_{21}, \quad Q_2(0) = h_{21}, \quad w(l) = g_{12}, \quad M_2(l) = h_{12};$$

Problem 7 *Let $0 \leq \alpha < 2$, $\quad 0 \leq \beta < 1$. Find $w \in C^4(]0, l[) \cap C^1(]0, l]) \cap C([0, l])$ satisfying (2.1) and the following BCs:*

$$w(0) = g_{11}, \quad M_2(0) = h_{11}, \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22};$$

Problem 8 *Let $0 \leq \alpha < 2$, $0 \leq \beta < 1$. Find $w \in C^4(]0, l[) \cap C([0, l]) \cap C^1(]0, l])$ satisfying (2.1) and the following BCs:*

$$w(0) = g_{11}, \quad M_2(0) = h_{11}, \quad w_{,2}(l) = g_{22}, \quad Q_2(l) = h_{22}; \tag{2.11}$$

Problem 9 *Let $0 \leq \alpha$, $\beta < 2$. Find $w \in C^4(]0, l[) \cap C([0, l])$ satisfying (2.1) and the following BCs:*

$$w(0) = g_{11}, \quad M_2(0) = h_{11} \quad w(l) = g_{12}, \quad M_2(l) = h_{12};$$

Problem 10 *Let $\alpha \geq 0$, $\quad 0 \leq \beta < 1$. Find $w \in C^4(]0, l[) \cap C^1(]0, l])$ satisfying (2.1) and the following BCs:*

$$M_2(0) = h_{11}, \quad Q_2(0) = h_{22} \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22}.$$

In all these problems g_{ij}, h_{ij} $(i, j = 1, 2)$ are given constants.

All above problems are solved explicitly. Let us solve typical ones. For the sake of simplicity we consider homogeneous BCs.

Solution of Problem 1:

By virtue of (2.8) and homogeneous boundary conditions for $w_{,2}$ we have

$$c_3 = \int_0^{x_2^0} \left[\int_{x_2^0}^{\xi} (\eta) q(\eta) d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi) d\xi, \tag{2.12}$$

$$c_3 = \int_l^{x_2^0} \left[\int_{x_2^0}^{\xi} (\xi - \eta) q(\eta) d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi) d\xi. \tag{2.13}$$

Taking into account of (2.9) and homogeneous conditions (2.10) for w, we obtain

$$c_4 = -\int_0^{x_2^0} \xi \left[\int_{x_2^0}^{\xi} (\xi - \eta)q(\eta)d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi)d\xi, \qquad (2.14)$$

$$c_4 = -\int_l^{x_2^0} \xi \left[\int_{x_2^0}^{\xi} (\xi - \eta)q(\eta)d\eta + c_2 + \xi c_1 \right] D^{-1}(\xi)d\xi. \qquad (2.15)$$

Obviously, from (2.12)-(2.15), for c_1 and c_2 we have the following system

$$c_1 \int_0^l \xi D^{-1}(\xi)d\xi \;+\; c_2 \int_0^l D^{-1}(\xi)d\xi = -\int_{x_2^0}^l q(\eta)d\eta \int_\eta^l (\xi - \eta)D^{-1}(\xi)d\xi$$

$$+ \int_0^{x_2^0} q(\eta)d\eta \int_0^\eta (\xi - \eta)D^{-1}(\xi)d\xi =: d_1, \qquad (2.16)$$

$$c_1 \int_0^l \xi^2 D^{-1}(\xi)d\xi \;+\; c_2 \int_0^l \xi D^{-1}(\xi)d\xi = -\int_{x_2^0}^l q(\eta)d\eta \int_\eta^l \xi(\xi - \eta)D^{-1}(\xi)d\xi$$

$$+ \int_0^{x_2^0} q(\eta)d\eta \int_0^\eta \xi(\xi - \eta)D^{-1}(\xi)d\xi =: d_2. \qquad (2.17)$$

The determinant of this system is equal to

$$\Delta_1 = \left\{ \int_0^l \xi D^{-1}(\xi)d\xi \right\}^2 - \int_0^l D^{-1}(\xi)d\xi \int_0^l \xi^2 D^{-1}(\xi)d\xi < 0. \qquad (2.18)$$

The last assertion follows from the Hölder inequality which is strict since $\xi D^{-\frac{1}{2}}(\xi)$ and $D^{-\frac{1}{2}}(\xi)$ are positive on $]0, l[$, and $\xi^2 D^{-1}(\xi)$ and $D^{-1}(\xi)$ differ from each other by a nonconstant factor ξ^2.

Further,

$$c_1 = \frac{d_1 \int_0^l \frac{\xi}{\xi^\alpha (l-\xi)^\beta} d\xi - d_2 \int_0^l \frac{1}{\xi^\alpha (l-\xi)^\beta} d\xi}{\Delta_1},$$

$$c_2 = \frac{d_2 \int_0^l \frac{\xi}{\xi^\alpha (l-\xi)^\beta} d\xi - d_1 \int_0^l \frac{\xi^2}{\xi^\alpha (l-\xi)^\beta} d\xi}{\Delta_1}.$$

After substitution the c_1 and c_2 into (2.12) and (2.14) we get the expressions for c_3 and c_4. It is obvious, that the last integral of the expression c_1 exists if and only if $\alpha < 1$, $\beta < 1$.

Solution of Problem 8: From (2.6), (2.7) and BC we get

$$c_1 = - \int_{x_2^0}^{l} q(\xi)d\xi, \quad c_2 = - \int_{0}^{x_2^0} \xi q(\xi)d\xi, \tag{2.19}$$

hence

$$Q_2(x_2) = \int_{x_2}^{l} q(\xi)d\xi, \quad M_2(x_2) = \int_{0}^{x_2} \xi q(\xi)d\xi + x_2 \int_{x_2}^{l} q(\xi)d\xi,$$

Substituting (2.19) in (2.8) and (2.9) and taking into account BCs, after using Dirichlet formula we have

$$c_3 = \int_{x_2^0}^{l} \left[\int_{0}^{\xi} \eta q(\eta)d\eta + \xi \int_{\xi}^{l} q(\eta)d\eta \right] D^{-1}(\xi)d\xi + \int_{x_2^0}^{l} \xi q(\xi) \int_{x_2^0}^{l} \eta D^{-1}(\eta)d\eta d\xi$$

$$- \int_{x_2^0}^{l} q(\xi) \int_{\xi}^{l} (\eta - \xi)D^{-1}(\eta)d\eta d\xi + \int_{0}^{x_2^0} q(\xi) \int_{x_2^0}^{l} D^{-1}(\eta)d\eta d\xi,$$

$$c_4 = \int_{0}^{x_2^0} q(\xi) \int_{0}^{\xi} \eta(\eta - \xi)D^{-1}(\eta)d\eta d\xi + \int_{x_2^0}^{l} q(\xi) \int_{0}^{x_2^0} \eta^2 D^{-1}(\eta)d\eta d\xi$$

$$+ \int_{0}^{x_2^0} \xi q(\xi) \int_{0}^{x_2^0} \eta D^{-1}(\eta)d\eta d\xi,$$

and

$$w_{,2}(x_2) = \int_{x_2}^{l} \frac{\int_{0}^{\xi} \eta q(\eta)d\eta + \xi \int_{\xi}^{l} q(\eta)d\eta}{D_0 \xi^{\alpha}(l - \xi)^{\beta}} d\xi,$$

$$w(x_2) = \tag{2.20}$$

$$\int_{x_2}^{l} \frac{q(\xi)}{D_0} \left[-x_2 \int_{\xi}^{x_2} \frac{d\eta}{\eta^{\alpha-1}(l - \eta)^{\beta}} + \int_{0}^{x_2} \frac{d\eta}{\eta^{\alpha-2}(l - \eta)^{\beta}} + x_2 \xi \int_{\xi}^{l} \frac{d\eta}{\eta^{\alpha}(l - x)^{\beta}} \right] d\xi$$

$$+ \int_{l}^{x_2} \frac{q(\xi)}{D_0} \left[\xi \int_{\xi}^{x_2} \frac{d\eta}{\eta^{\alpha-1}(l - \eta)^{\beta}} + \int_{0}^{\xi} \frac{d\eta}{\eta^{\alpha-2}(l - \eta)^{\beta}} + x_2 \xi \int_{\xi}^{l} \frac{d\eta}{\eta^{\alpha}(l - x)^{\beta}} \right] d\xi.$$

It is easy to see that $w(x_2)$ and $w_{,2}(x_2)$ belong to $C([0, l])$, since

$$\lim_{\xi \to 0+} \frac{\int_0^\xi \eta q(\eta) d\eta}{\xi^\alpha (l - \xi)^\beta} = \lim_{\xi \to 0+} \frac{\xi q(\xi)}{\alpha \xi^{\alpha-1}(l - \xi)^\beta - \beta \xi^\alpha (l - \xi)^{\beta-1}}$$

$$= \lim_{\xi \to 0+} \frac{q(\xi)}{\xi^{\alpha-2}\left[\alpha(l - \xi)^\beta - \beta \xi(l - \xi)^{\beta-1}\right]}.$$

Solution of Problem 9:

$$Q_2(x_2) = \int_{x_2}^l q(\xi) d\xi - \frac{1}{l} \int_0^l \xi q(\xi) d\xi,$$

$$M_2(x_2) = x_2 \int_{x_2}^l q(\xi) d\xi + \int_0^{x_2} \xi q(\xi) d\xi - \frac{x_2}{l} \int_0^l \xi q(\xi) d\xi,$$

$$w_{,2}(x_2) = -\int_{x_2}^l \frac{R_1(\xi)}{\xi^\alpha (l - \xi)^\beta} d\xi + \frac{1}{l} \int_0^l \frac{R_1(\xi)}{\xi^{\alpha-1}(l - \xi)^\beta} d\xi,$$

$$w(x_2) = -x_2 \int_{x_2}^l \frac{R_1(\xi)}{\xi^\alpha (l - \xi)^\beta} d\xi - \int_0^{x_2} \frac{R_1(\xi)}{\xi^{\alpha-1}(l - \xi)^\beta} d\xi$$

$$+ \frac{x_2}{l} \int_0^l \frac{R_1(\xi)}{\xi^{\alpha-1}(l - \xi)^\beta} d\xi,$$

where

$$R_1(\xi) := -\frac{1}{l}(l - \xi) \int_0^\xi \eta q(\eta) d\eta - \frac{\xi}{l} \int_\xi^l (l - \eta) q(\eta) d\eta.$$

Solution of Problem 10:

$$Q_2(x_2) = -\int_0^{x_2} q(\xi) d\xi, \quad M_2(x_2) = -\int_0^{x_2} (x_2 - \xi) q(\xi) d\xi,$$

$$w_{,2}(x_2) = \int_{x_2}^l \frac{\int_0^\xi (\xi - \eta) q(\eta) d\eta}{\xi^\alpha (l - \xi)^\beta} d\xi,$$

$$w(x_2) = -\int_{x_2}^l (x_2 - \xi) \frac{\int_0^\xi (\xi - \eta) q(\eta) d\eta}{\xi^\alpha (l - \xi)^\beta} d\xi,$$

$w_{,2}$ and w are bounded as $x_2 \to 0_+$ if

$$\exists q^{([\alpha]-2)}, \quad \text{such that} \quad \lim_{\xi \to 0_+} q^{([\alpha]-2)}(\xi) \neq \infty, \quad \lim_{\xi \to 0_+} q^{(i-3)} = 0, \quad i = \overline{3, [\alpha]}, \quad \alpha \geq 3,$$

is fullfiled since

$$\lim_{\xi \to 0+} \frac{\int_0^\xi \eta q(\eta) d\eta}{\xi^\alpha (l - \xi)^\beta} = \lim_{\xi \to 0+} \frac{q(\xi)}{\xi^{\alpha-2} \left[\alpha(l - \xi)^\beta - \beta\xi(l - \xi)^{\beta-1} \right]}.$$

For homogeneous BCs solutions of all the above problems can be represented as follows [14]

$$w(x_2) = \int_0^l K(x_2, \xi) q(\xi) d\xi, \tag{2.21}$$

where

$$K(x_2, \xi) = \begin{cases} K_3(\xi, x_2), & 0 \leq \xi \leq x_2, \\ K_3(x_2, \xi), & x_2 \leq \xi \leq l. \end{cases} \tag{2.22}$$

$K_3(x_2, \xi)$ has different forms for different problems, e.g.,

Problem 1.

$$K_3(x_2, \xi) = \int_0^{x_2} (\eta - x_2)(\eta - \xi) D^{-1}(\eta) d\eta$$

$$+ \left\{ \int_0^\xi (\xi - \eta) D^{-1} d\eta \int_0^{x_2} (x_2 - \eta)\eta D^{-1}(\eta) d\eta \right.$$

$$+ \int_0^\xi \eta(\xi - \eta) D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta) D^{-1}(\eta) d\eta \right\} \frac{\int_0^l \eta D^{-1}(\eta) d\eta}{\Delta_1}$$

$$- \int_0^\xi (\xi - \eta)\eta D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta)\eta D^{-1}(\eta) d\eta \frac{\int_0^l D^{-1}(\eta) d\eta}{\Delta_1}$$

$$+ \int_0^\xi (\xi - \eta) D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta) D^{-1}(\eta) d\eta \frac{\int_0^l \eta^2 D^{-1}(\eta) d\eta}{\Delta_1}, \tag{2.23}$$

where Δ_1 is given by (2.18).

Problem 2.

$$K_3(x_2, \xi) = \int_0^{x_2} (x_2 - \eta)(\xi - \eta)D^{-1}d\eta$$

$$- \frac{1}{\int_0^l (l - \eta)D^{-1}(\eta)d\eta} \int_0^\xi (\xi - \eta)D^{-1}(\eta)d\eta$$

$$\times \int_0^{x_2} (x_2 - \eta)D^{-1}(\eta)d\eta. \tag{2.24}$$

Problem 3.

$$K_3(x_2, \xi) = \int_0^{x_2} (x_2 - \eta)(\xi - \eta)D^{-1}(\eta)d\eta$$

$$- \frac{1}{\int_0^l (l - \eta)^2 D^{-1}(\eta)d\eta} \int_0^{x_2} (x_2 - \eta)(l - \eta)D^{-1}(\eta)d\eta$$

$$\times \int_0^\xi (\xi - \eta)(l - \eta)D^{-1}(\eta)d\eta. \tag{2.25}$$

Problem 4.

$$K_3(x_2, \xi) = \int_0^{x_2} (\xi - \eta)(x_2 - \eta)D^{-1}(\eta)d\eta. \tag{2.26}$$

Problem 5.

$$K_3(x_2, \xi) = \int_{x_2}^l (x_2 - \eta)(\xi - \eta)D^{-1}(\eta)d\eta$$

$$- \frac{1}{\int_0^l \eta D^{-1}(\eta)d\eta} \int_{x_2}^l (x_2 - \eta)D^{-1}(\eta)d\eta$$

$$\times \int_\xi^l (\xi - \eta)D^{-1}(\eta)d\eta. \tag{2.27}$$

Problem 6.

$$K_3(x_2, \xi) = -(l - x_2) \int_\xi^{x_2} \eta D^{-1}(\eta) d\eta + \int_l^{x_2} \eta^2 D^{-1}(\eta) d\eta$$

$$+ (l - x_2)\xi \int_\xi^0 D^{-1}(\eta) d\eta. \tag{2.28}$$

Problem 7.

$$K_3(x_2, \xi) = \int_{x_2}^l (x_2 - \eta)(\xi - \eta) D^{-1}(\eta) d\eta$$

$$- \frac{1}{\int_0^l \eta^2 D^{-1}(\eta) d\eta} \int_{x_2}^l (x_2 - \eta)\eta D^{-1}(\eta) d\eta$$

$$\times \int_\xi^l (\xi - \eta)\eta D^{-1}(\eta) d\eta. \tag{2.29}$$

Problem 8.

$$K_3(x_2, \xi) := -x_2 \int_\xi^{x_2} \eta D^{-1}(\eta) d\eta + \int_0^{x_2} \eta^2 D^{-1}(\eta) d\eta$$

$$+ x_2\xi \int_\xi^l D^{-1}(\eta) d\eta. \tag{2.30}$$

Problem 9.

$$K_3(x_2, \xi) = \frac{x_2\xi}{l^2} \int_\xi^l (l - \eta) D^{-1}(\eta) d\eta + \frac{x_2(l - \xi)}{l^2} \int_\xi^{x_2} (l - \eta)\eta D^{-1}(\eta) d\eta$$

$$+ \frac{(l - x_2)(l - \xi)}{l^2} \int_0^{x_2} \eta^2 D^{-1}(\eta) d\eta. \tag{2.31}$$

Problem 10.

$$K_3(x_2, \xi) = - \int_\xi^l (x_2 - \eta)(\eta - \xi) D^{-1}(\eta) d\eta. \tag{2.32}$$

Obviously, taking into account (2.23)-(2.32), we have (see (2.22))

$$K(x_2, \xi) \in \begin{cases} C([0, l] \times [0, l]), & \text{in case of Problems } 1 - 3, \ 5 - 9; \\ C(]0, l] \times]0, l]), & \text{in case of Problems } 10; \\ C([0, l[\times [0, l[), & \text{in case of Problems } 4, \end{cases} \qquad (2.33)$$

and

$$K'_{,2}(x_2, \xi) \in \begin{cases} C([0, l] \times [0, l]), & \text{in case of Problems } 1, \ 2, \ 5; \\ C(]0, l] \times]0, l]), & \text{in case of Problems } 7, \ 8, \ 10; \\ C([0, l[\times [0, l[), & \text{in case of Problems } 3, \ 4, \ 6; \end{cases} \qquad (2.34)$$

Remark 2.1 *Problems 1-10 are not correct for the different from the indicated in Problems 1-10 values of α and β. It is evident from the fact that in the above cases, in general, the limits of w and $w_{,2}$ as $x_2 \to 0_+$, l_- do not exist. The last assertions easily follow from the general representations (2.9) and (2.8) of w and $w_{,2}$ with (2.3).*

Using integral representations and the difference equation corresponding to (2.1) by means of MATLAB we get numerical results and corresponding graphical results for deflection, bending moment and intersecting force for different materials. These numerical results coincide up to 10^{-3}. In Figures 13-17 is shown graphical results taking into difference equation corresponding to (2.1), In Figure 18 is shown the graphical results for deflection using integral representation corresponding to Figures 13-17.

Problem, 1 Aluminium
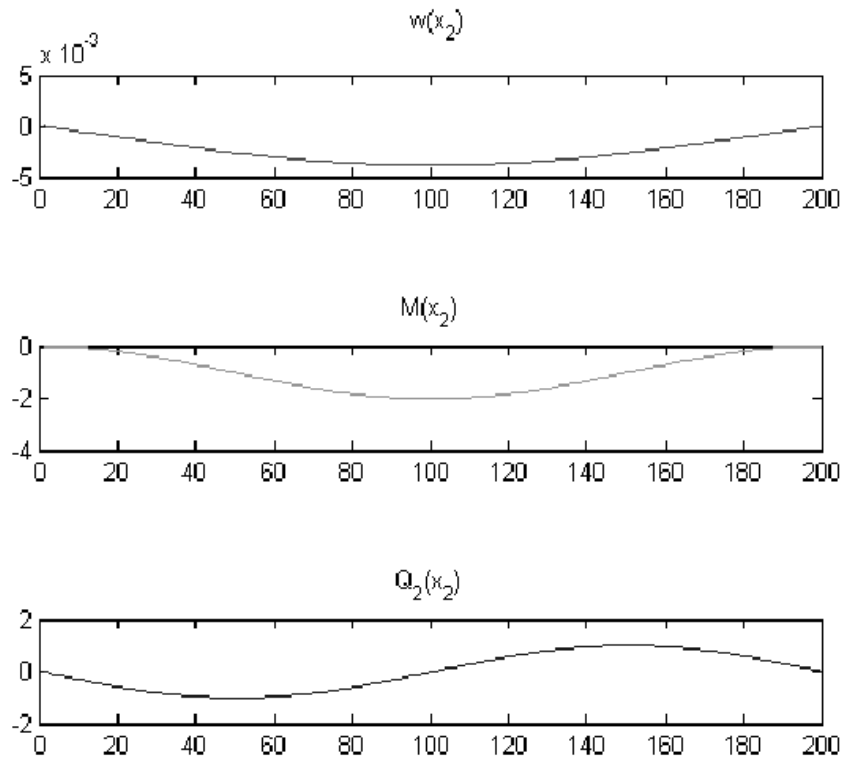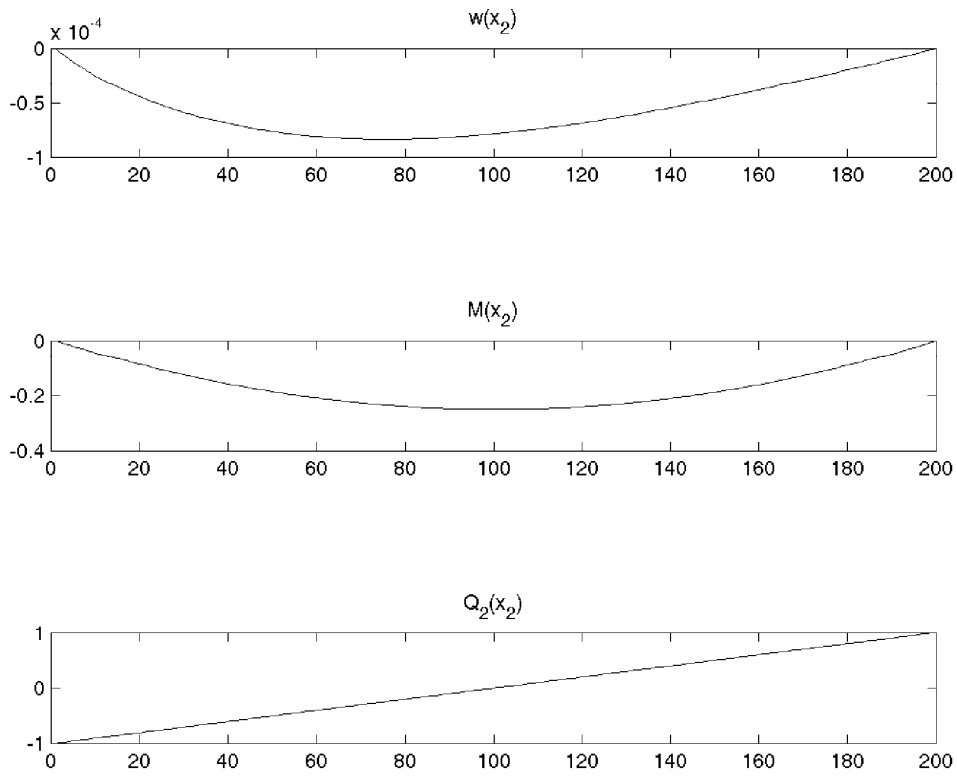$\alpha=0.3, \quad \beta=0.3$
$l=2\pi, \quad q=\cos(x_2)$
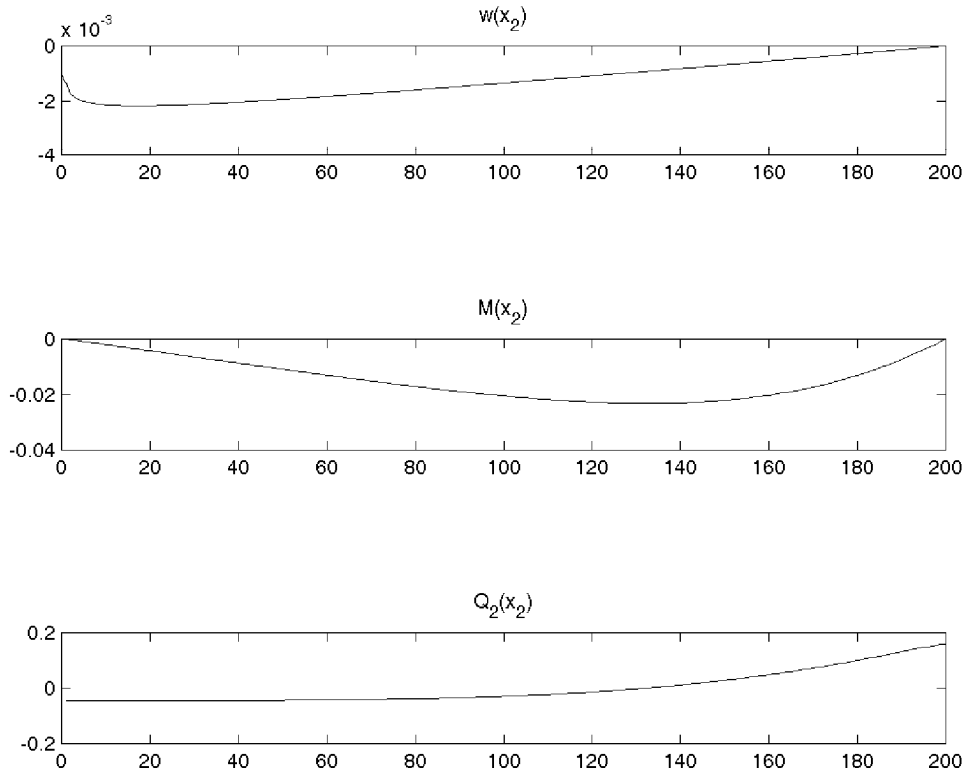

Fig. 13

Problem 8, Iron
α=1.5, β=0.1;
l=1; q=const

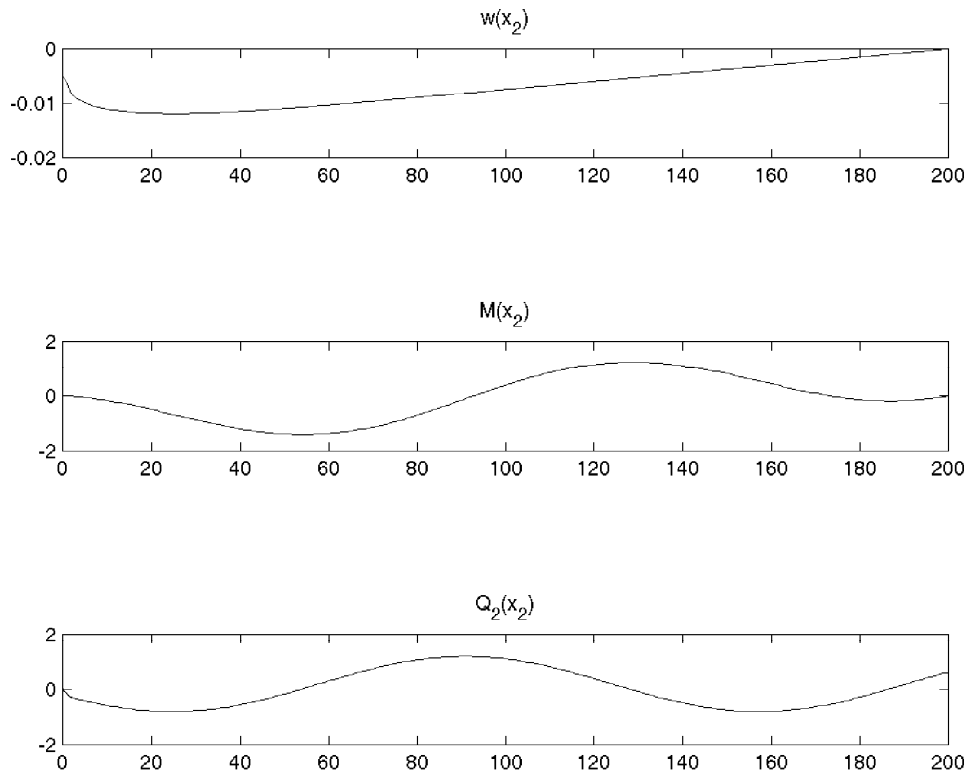
Fig. 14

Problem 10, Aluminium
$\alpha=3$, $\beta=0.1$;
$l=1$; $q=h(x_2)$

Fig. 15

Problem 10, Aluminium
$\alpha=3$, $\beta=0.1$;
$l=3\pi$; $q=\sin(x_2)$

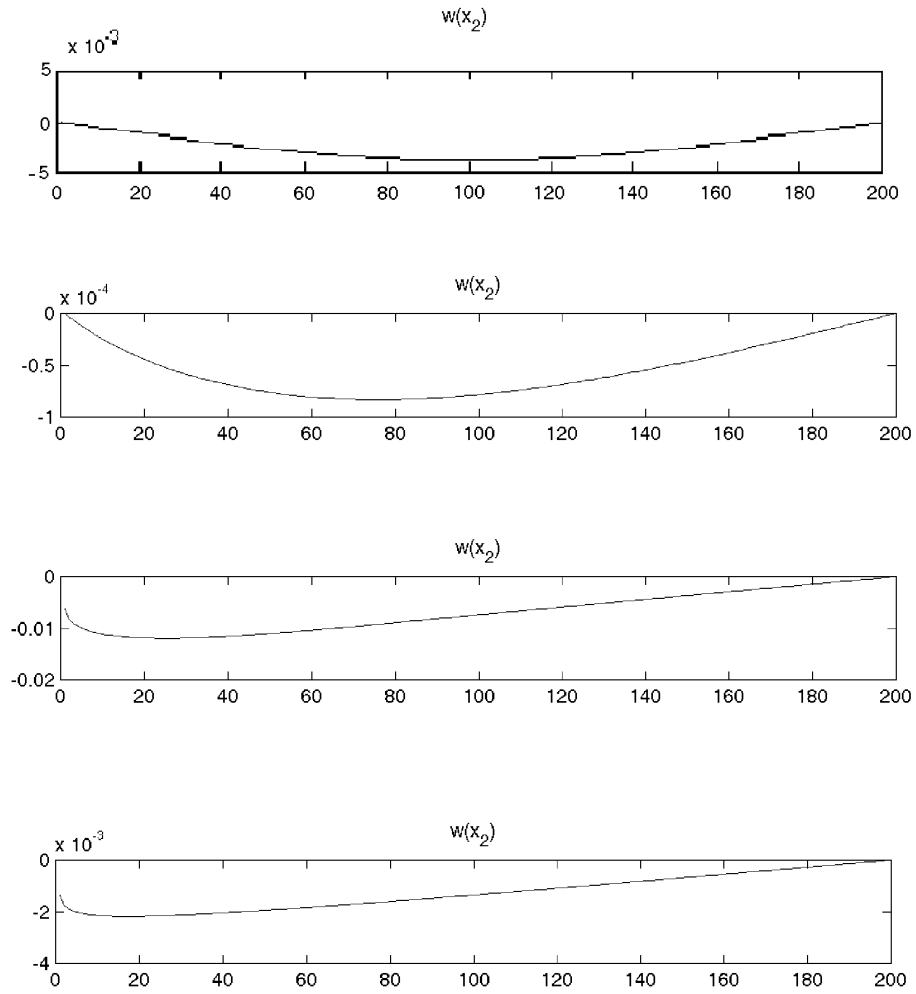$w(x_2)$

$M(x_2)$

$Q_2(x_2)$

Fig. 16

.

Fig. 17

.

2.2.   Vibration of the Plate with Two Cusped Edges

The equation of bending vibration has the following form

$$(D(x_2)w,_{22}(x_2,t)),_{22} = q(x_2,t) - 2\rho h(x_2)\frac{\partial^2 w(x_2,t)}{\partial t^2}, \quad 0 < x_2 < l, \qquad (2.35)$$

where ρ is a density of the shell.

In this case we have to add to the BCs of Problems 1-10 the initial conditions

$$w(x_2,0) = \varphi_1(x_2), \quad w,_t(x_2,0) = \varphi_2(x_2), \qquad (2.36)$$

where $\varphi_i(x_2) \in C^4(]0,l[)$, $i = 1,2$, are given functions..

Let us consider the following initial boundary value problem (IBVP):

Problem 11 *Let $0 \le \alpha < 2$, $0 \le \beta < 1$. Find*

$$w(\cdot,t) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l[), \quad M_2(\cdot,t) \in C([0,l]), \quad Q_2(\cdot,t) \in C([0,l]),$$
$$w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \le x_2 \le l, \ t \ge 0)$$

$$(2.37)$$

satisfying equation (2.35), the BCs

$$w(0,t) = M_2(0,t) = w,_2(l,t) = Q_2(l,t) = 0, \qquad (2.38)$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l[), \quad i = 1,2. \qquad (2.39)$$

$$\varphi_i(0) = -D(x_2)\varphi_i''(x_2)|_{x_2=0_+} = \varphi_i'(l) = \qquad (2.40)$$
$$= (-D(x_2)\varphi_i''(x_2))'|_{x_2=l_-} = 0, i = 1,2.$$

Solution. In this section all quantities, in particular, in (2.4), (2.5) depend on x_2 and t.

Using the Fourier method, we will look for $w(x_2,t)$ in the following form

$$w(x_2,t) = X(x_2)T(t). \qquad (2.41)$$

Let firstly $q(x_2,t) \equiv 0$. Then from (2.35) we get

$$\frac{(D(x_2)X''(x_2))''}{g(x_2)X(x_2)} = -\frac{T''(t)}{T(t)} = \lambda = \text{const.}$$

Hence,

$$T''(t) + \lambda T(t) = 0, \qquad (2.42)$$

and

$$(D(x_2)X''(x_2))'' = \lambda g(x_2)X(x_2), \qquad (2.43)$$

where $g(x_2) := 2\rho h(x_2)$.

From (2.38) for $X(x_2)$ we obtain the following BCs

$$X(0) = -D(x_2)X''(x_2)|_{x_2=0} = X'(l) = (-D(x_2)X''(x_2))'|_{x_2=l} = 0. \qquad (2.44)$$

Now, in view of (2.37), we have to solve the following BVP:

Find

$$X(x_2) \in C^4(]0, l[) \cap C([0, l]) \cap C^1(]0, l[), \qquad (2.45)$$

which satisfies equation (2.43) and BCs (2.44).

If in (2.21) we replace $w(x_2)$ and $q(x_2)$ by $X(x_2)$ and $\lambda g(x_2)X(x_2)$, respectively, then, similarly to Section 2.1, for $X(x_2)$ we obtain

$$X(x_2) = \lambda \int_0^l g(\xi)K(x_2, \xi)X(\xi)d\xi, \qquad (2.46)$$

where

$$K(x_2, \xi) = \begin{cases} K_3(\xi, x_2), & 0 \le \xi \le x_2, \\ K_3(x_2, \xi), & x_2 \le \xi \le l. \end{cases}$$

$$K_3(x_2, \xi) := -x_2 \int_\xi^{x_2} \eta D^{-1}(\eta)d\eta + \int_0^{x_2} \eta^2 D^{-1}(\eta)d\eta + x_2\xi \int_\xi^l D^{-1}(\eta)d\eta. \qquad (2.47)$$

Proposition 2.2 $K(x_2, \xi)$ *is symmetric with respect to x_2 and ξ.*

Proof For z_1 and z_2, such that $0 < z_1, z_2 < l$ we get

$$K(z_1, z_2) = \begin{cases} K_3(z_2, z_1), & 0 \le z_2 \le z_1, \\ K_3(z_1, z_2), & z_1 \le z_2 \le l, \end{cases}$$

$$K(z_2, z_1) = \begin{cases} K_3(z_1, z_2), & z_1 \le z_2 \le l, \\ K_3(z_2, z_1), & 0 \le z_2 \le z_1, \end{cases}$$

i.e.,

$$K(z_1, z_2) = K(z_2, z_1), \quad \text{for any } z_1, z_2 \in [0, l].$$

□

(2.46) can be rewritten as follows

$$Y(x_2) = \lambda \int_0^l R(x_2, \xi)Y(\xi)d\xi, \qquad (2.48)$$

where

$$Y(x_2) = \sqrt{g(x_2)}X(x_2), \quad R(x_2, \xi) = \sqrt{g(x_2)}K(x_2, \xi)\sqrt{g(\xi)}. \qquad (2.49)$$

(2.48) is an integral equation with a symmetric kernel.

Proposition 2.3 *Number of eigenvalues λ_n of (2.48) is not finite.*

Proof Let it be finite, and $n = \overline{1, m}$. Then we can express $R(x_2, \xi)$ as follows (see Theorem 1.2)

$$R(x_2, \xi) = \sum_{n=1}^{m} \frac{Y_n(x_2)Y_n(\xi)}{\lambda_n},$$

where $Y_n(x_2) \in C^4(]0, l[)$, i.e.,

$$R(x_2, \xi) \in C^4 \left(]0, l[\times]0, l[\right). \tag{2.50}$$

On the other hand, by virtue of (2.47),

$$K_{x_2}'''(x_2, \xi)|_{\xi \to x_2-} - K_{x_2}'''(x_2, \xi)|_{\xi \to x_2+} = \frac{1}{D(x_2)},$$

then kernel

$$R(x_2, \xi) \notin C^4 \left(]0, l[\times]0, l[\right). \tag{2.51}$$

But, (2.50) and (2.51) contradict to each other, thus the number of λ_n is not finite.
□

Proposition 2.4 *All of λ_n are positive.*

Proof Obviously, if we denote by Y_n orthonormalized eigenfunctions (it can be assumed without loss of generality) of (2.48), then

$$X_n(x_2) = \frac{Y_n(x_2)}{\sqrt{g(x_2)}}$$

are eigenfunctions of (2.46) (i.e., of (2.43)). Let us multiply both sides of the following equation

$$(D(x_2)X_n''(x_2))'' = \lambda_n g(x_2)X_n(x_2), \tag{2.52}$$

by $X_n(x_2)$ and integrate it from 0 to l. Taking into account the first expression of (2.49), we obtain

$$
\begin{aligned}
\int_0^l X_n(x_2)(D(x_2)X_n''(x_2))'' dx_2 &= \lambda_n \int_0^l g(x_2)X_n(x_2)X_n(x_2)dx_2 \\
&= \lambda_n \int_0^l Y_n(x_2)Y_n(x_2)dx_2 = \lambda_n.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\lambda_n &= \int_0^l X_n(x_2)(D(x_2)X_n''(x_2))'' dx_2 = X_n(x_2)(D(x_2)X''(x_2))' \Big|_0^l \\
&\quad - \int_0^l X_n'(x_2)(D(x_2)X_n''(x_2))' dx_2 = \\
&\quad \text{(by virtue of the BCs (2.44))} \\
&= - \int_0^l X_n'(x_2)(D(x_2)X_n''(x_2))' dx_2 = X_n'(x_2)(D(x_2)X''(x_2)) \Big|_0^l \\
&\quad + \int_0^l D(x_2)(X_n'')^2(x_2) dx_2 = \int_0^l D(x_2)(X_n'')^2(x_2) dx_2 \geq 0.
\end{aligned}
$$

Hence, $\lambda_n > 0$ for any n, since in non trivial case $X_n \not\equiv 0$. $\square$

The solution of (2.42) can be written as follows

$$
T_n(t) = b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right), \quad b_i^n = \text{const}, \quad i = 1, 2.
$$

Now, we can find a solution of the Problem 11 in the form as follows

$$
w(x_2, t) = \sum_{n=1}^\infty \frac{Y_n(x_2)}{\sqrt{g(x_2)}} \left(b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right)\right) \tag{2.53}
$$

or, taking into account (2.49), in the following form

$$
w(x_2, t) = \sum_{n=1}^\infty X_n(x_2) \left(b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right)\right). \tag{2.54}
$$

In view of initial conditions (2.36), we formally have

$$
\sum_{n=1}^\infty Y_n(x_2)b_2^n = \varphi_1(x_2)\sqrt{g(x_2)}, \quad \sum_{n=1}^\infty \sqrt{\lambda_n}Y_n(x_2)b_1^n = \varphi_2(x_2)\sqrt{g(x_2)}. \tag{2.55}
$$

If $\psi_i(x_2) := \frac{(D\varphi_i'')''}{\sqrt{g(x_2)}} \in C[0, l]$, $(i = 1, 2)$, then after integration of the last expression, $\sqrt{g(x_2)}\varphi_i(x_2)$ can be expressed as follows

$$
\sqrt{g(x_2)}\varphi_i(x_2) = \int_0^l \sqrt{g(x_2)g(\xi)}K(x_2, \xi)\psi_i(\xi)d\xi,
$$

i.e.,

$$\sqrt{g(x_2)}\varphi_i(x_2) = \int\limits_0^l R(x_2,\xi)\psi_i(\xi)d\xi.$$

Hence, by virtue of Theorem 1.1, since $\psi_i(\xi) \in C([0,l])$ and symmetric $R(x_2,\xi) \in C([0,l] \times [0,l])$, we get absolutely and uniformly convergence of the series

$$\sqrt{g(x_2)}\varphi_i(x_2) = \sum_{n=1}^\infty \int\limits_0^l \sqrt{g(\xi)}\varphi_i(\xi)Y_n(\xi)d\xi \cdot Y_n(x_2),$$

i.e., of (2.55) on $[0,l]$, and

$$b_1^n = \frac{1}{\sqrt{\lambda_n}}\int\limits_0^l g(x_2)X_n(x_2)\varphi_2(x_2)dx_2, \ \ b_2^n = \int\limits_0^l g(x_2)X_n(x_2)\varphi_1(x_2)dx_2. \qquad (2.56)$$

Further, taking into account (2.45), $X(x_2) \in C([0,l])$. Then, by virtue of (2.49), we can rewrite (2.55) as follows

$$\varphi_1(x_2) = \sum_{n=1}^\infty X_n(x_2)b_2^n, \quad \varphi_2(x_2) = \sum_{n=1}^\infty \sqrt{\lambda_n}X_n(x_2)b_1^n. \qquad (2.57)$$

Evidently, last series will be absolutely and uniformly nt on $]0,l[$. Since there exists positive minimum of eigenvalues, from the convergence of the second series follows absolute and uniform convergence on $]0,l[$ of the series $\sum_{n=1}^N X_n(x_2)b_1^n$. Therefore, the series (2.54) is absolutely and uniformly convergent on $]0,l[$.

After formal differentiation of (2.54) with respect to t we get

$$w,_t (x_2,t) = \sum_{n=1}^\infty X_n(x_2)\sqrt{\lambda_n}\left(b_1^n\cos(\sqrt{\lambda_n}t) - b_2^n\sin(\sqrt{\lambda_n}t)\right), \qquad (2.58)$$

$$w,_{tt} (x_2,t) = -\sum_{n=1}^\infty X_n(x_2)\lambda_n\left(b_1^n\sin(\sqrt{\lambda_n}t) + b_2^n\cos(\sqrt{\lambda_n}t)\right). \qquad (2.59)$$

Theorem 2.5 *(2.57) and (2.54) converge absolutely and uniformly on $[0,l]$, and (2.58) - (2.59) converge absolutely and uniformly on any $[a,b] \in]0,l[$ if*

$$\Psi_i(x_2) := \frac{\psi_i(x_2)}{\sqrt{g(x_2)}} \ \text{ for } \ i = 1,2, \ \text{ are satisfying BCs } 2.40 \qquad (2.60)$$

$$\chi_i(x_2)\sqrt{g(x_2)} := (D(x_2)\Psi_i''(x_2))'', \ i = 1,2, \ \text{ are an integrable ones on }]0,l[\quad (2.61)$$

(for this, e.g. it is sufficient that $\dfrac{d^j}{dx_2^j}\varphi_i(x_2) = O(x_2^{\gamma_{ij}})$, $\gamma_{ij} = \text{const} > 7 - j - \dfrac{5\alpha}{3}$,

$x_2 \to 0_+$, $\dfrac{d^j}{dx_2^j}\varphi_i(x_2) = O((l - x_2)^{\delta_{ij}})$, $\delta_{ij} = \text{const} > 7 - j - \dfrac{5\beta}{3}$, $x_2 \to l_-$, $i = 1, 2$;

$j = \overline{2, 8}$).

Proof Substituting in (2.56) the function $g(x_2)X_n(x_2)$ found from (2.52), we get

$$b_1^n = \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l (D(x_2)X_n''(x_2))''\varphi_2(x_2)dx_2$$

(after integrating by parts 4-times, taking into account BCs, (2.40), (2.44), and (2.49))

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{(D(x_2)X_n''(x_2))'\varphi_2(x_2)|_0^l - \int_0^l (D(x_2)X_n''(x_2))'\varphi_2'(x_2)dx_2\right\}$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{-D(x_2)X_n''(x_2)\varphi_2'(x_2)|_0^l + \int_0^l D(x_2)X_n''(x_2)\varphi_2''(x_2)dx_2\right\}$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l X_n''(x_2)D(x_2)\varphi_2''(x_2)dx_2 = \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{X_n'(x_2)D(x_2)\varphi''(x_2)|_0^l\right.$$

$$-\int_0^l X_n'(x_2)(D(x_2)\varphi_2''(x_2))'dx_2\bigg\} = \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{-X_n(x_2)(D(x_2)\varphi_2''(x_2))'|_0^l\right.$$

$$+\int_0^l X_n(x_2)(D(x_2)\varphi_2''(x_2))''dx_2\bigg\} = \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l X_n(x_2)(D(x_2)\varphi_2''(x_2))''dx_2$$

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l Y_n(x_2)\psi_2(x_2)dx_2. \tag{2.62}$$

Analogously,

$$b_2^n = \frac{1}{\lambda_n}\int_0^l Y_n(x_2)\psi_1(x_2)dx_2. \tag{2.63}$$

In view of (2.61), $\Psi_i(x_2)$ can be expressed as follows

$$\Psi_i(x_2) = \int_0^l K(x_2, \xi)\sqrt{g(\xi)}\chi_i(\xi)d\xi, \quad i = 1, 2,$$

and by virtue of (2.60), (2.49) we obtain

$$\psi_i(x_2) = \int\limits_0^l R(x_2,\xi)\chi_i(\xi)d\xi, \quad i = 1,2.$$

According to the Theorem 1.1, the following series

$$\sum_{n=1}^\infty \beta_i^n Y_n(x_2),$$

where

$$\beta_i^n = \int\limits_0^l Y_n(x_2)\psi_i(x_2)dx_2, \quad i = 1,2, \tag{2.64}$$

is convergent absolutely and uniformly on $]0,l[$, i.e.,

$$\sum_{n=1}^\infty |\beta_i^n||Y_n(x_2)| < +\infty, \quad i = 1,2. \tag{2.65}$$

By view of $K(x_2,\xi)\sqrt{g(\xi)} \in C([0,l]\times[0,l])$, there exists such M that

$$M := \max_{0\le x_2,\xi\le l} \left| K(x_2,\xi)\sqrt{g(\xi)} \right| < +\infty.$$

Using (2.48), (2.63), (2.64) we have

$$
\begin{aligned}
|X_n(x_2)b_2^n| &= \left| \lambda_n \int\limits_0^l K(x_2,\xi)\sqrt{g(\xi)}Y_n(\xi)b_2^n d\xi \right| \\
&= \left| \int\limits_0^l K(x_2,\xi)\sqrt{g(\xi)}Y_n(\xi)\beta_2^n d\xi \right| \\
&\le \int\limits_0^l |K(x_2,\xi)\sqrt{g(\xi)}||Y_n(\xi)||\beta_2^n|d\xi := c_n^1
\end{aligned}
$$

On the other hand in virtue of (2.65) we obtain

$$
\begin{aligned}
\sum_{n=1}^\infty c_n^1 &= \sum_{n=1}^\infty c_n^1 \int\limits_0^l |K(x_2,\xi)\sqrt{g(\xi)}||Y_n(\xi)||\beta_2^n|d\xi \\
&\le M \int\limits_0^l \sum_{n=1}^\infty |Y_n(\xi)||\beta_2^n|d\xi \le MM_1 l < \infty.
\end{aligned}
$$

From the last two uniquality we get

$$|\varphi_1| \leq \sum_{n=1}^{\infty} |X_n(x_2)b_2^n| \leq \sum_{n=1}^{\infty} c_n^1 < +\infty.$$

Which means that φ_1 can be expressed as absolutely and uniformly convergent series. Analoguously, we can prove that φ_2 converges absolutely and uniformly on $[0, l]$.

Let, now consider (2.54) series. It is obviously that

$$|w(x_2, t)| \leq \sum_{n=1}^{\infty} |X_n(x_2)b_1^n| + \sum_{n=1}^{\infty} |X_n(x_2)b_2^n|,$$

and from the convergent of φ_1 and φ_2 we obtain that (2.54) converges absolutely and uniformly on $[0, l]$.

Further, from (2.58)

$$
\begin{aligned}
|w_{,t}(x_2, t)| &= \left| \sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n} \left(b_1^n \cos(\sqrt{\lambda_n}t) - b_2^n \sin(\sqrt{\lambda_n}t) \right) \right| \\
&\leq \left| \sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n}b_1^n \cos(\sqrt{\lambda_n}t) \right| \\
&\quad + \left| \sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n}b_2^n \sin(\sqrt{\lambda_n}t) \right| \\
&\leq \sum_{n=1}^{\infty} \left| X_n(x_2)\sqrt{\lambda_n}b_1^n \right| + \sum_{n=1}^{\infty} \left| X_n(x_2)\sqrt{\lambda_n}b_2^n \right|. \quad (2.66)
\end{aligned}
$$

According to Proposition 2.4, all of λ_n are positive. Therefore, we can find λ_0 such that $\lambda_0 \leq \min_{1 \leq i \leq \infty} \{\lambda_i\}$, and by virtue of (2.49), (2.62)-(2.65), we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} \left| X_n(x_2)\sqrt{\lambda_n}b_2^n \right| &= \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} \left| Y_n \sqrt{\lambda_n} \frac{1}{\lambda_n} \beta_1^n \right| \\
&\leq \frac{1}{\sqrt{\lambda_0}} \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} |Y_n||\beta_1^n| < \infty, \\
\sum_{n=1}^{\infty} \left| X_n(x_2)\sqrt{\lambda_n}b_1^n \right| &= \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} \left| Y_n \sqrt{\lambda_n} \frac{1}{\lambda_n \sqrt{\lambda_n}} \beta_2^n \right| \\
&\leq \frac{1}{\lambda_0} \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} |Y_n||\beta_2^n| < \infty, \quad x_2 \in]0, l[.
\end{aligned}
$$

Hence, the series in (2.66) are convergent. Thus, (2.58) is convergent absolutely and uniformly on $]0, l[$. Similarly, we get the absolute and uniform convergence of (2.59) on $]0, l[$. $\square$

Let us now differentiate (2.54) formally i-times with respect to x_2 and consider the following expressions

$$\frac{\partial^i}{\partial x_2^i} w(x_2, t) = \sum_{n=1}^{\infty} \frac{d^i}{dx_2^i} X_n(x_2) \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right),$$

$$i = 1, 2, 3, 4,$$

(2.67$_i$)

$$\frac{\partial^{i-1}}{\partial x_2^{i-1}} (D(x_2) w_{,x_2 x_2}(x_2, t)) = \sum_{n=1}^{\infty} \frac{d^{i-1}}{dx_2^{i-1}} (D(x_2) X_n''(x_2)) \left(b_1^n \sin(\sqrt{\lambda_n} t) \right.$$

$$\left. + b_2^n \cos(\sqrt{\lambda_n} t) \right), \; i = 1, 2.$$

(2.68$_i$)

Theorem 2.6 *The series* (2.67$_i$) *($i = 1, ..., 4$) are convergent absolutely and uniformly on any $[a, b] \in]0, l[$. The series (2.68$_i$) ($i = 1, 2$) are convergent absolutely and uniformly on $[0, l]$.*

Proof Obviously, in view of (2.44), after integration of (2.52), we get

$$X_n'(x_2) = \lambda_n \int_0^l R_1(x_2, \xi) X_n(\xi) d\xi,$$

(2.69)

where

$$R_1(x_2, \xi) = \begin{cases} \xi \int_{x_2}^l D^{-1}(\eta) d\eta, \; 0 \le \xi \le x_2, \\ -\int_\xi^{x_2} \eta D^{-1}(\eta) d\eta + \xi \int_\xi^l D^{-1}(\eta) d\eta, \; x_2 \le \xi \le l, \end{cases}$$

and

$$R_1(x_2, \xi) \in C([0, l] \times [0, l]),$$

(2.70)

because of $0 \le \alpha < 2$, $0 \le \beta < 1$.

Substituting (2.69) into (2.71$_1$) for $i = 1$, we obtain

$$\frac{\partial}{\partial x_2} w(x_2, t) = \sum_{n=1}^{\infty} \lambda_n \int_0^l R_1(x_2, \xi) X_n(\xi) d\xi \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right) =$$

$$= \int_0^l R_1(x_2, \xi) \left[\sum_{n=1}^{\infty} X_n(\xi) \lambda_n \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right) \right] d\xi,$$

(2.71)

since (2.59) is absolutely and uniformly convergent on $]0, l[$ and in view of (2.70) and $X_n(x_2) \in C([0, l])$ we conclude that the corresponding integral in (2.71) is absolutely convergent on $]0, l[$. Similarly, we can prove the convergence of the series (2.67_2), (2.67_3), (2.67_4), on $]0, l[$ and (2.68_i) $(i = 1, 2)$ on $[0, l]$. $\square$

Thus, (2.53) is the solution of the Problem 11 for $q(x_2, t) \equiv 0$.

Now, let us consider Problem 11 when $q(x_2, t) \not\equiv 0$, $\varphi_i = 0$, $i = 1, 2$, and let $\frac{q}{\sqrt{g}}(\cdot, t) \in L_2(0, l)$. Then $q(x_2, t)$ can be represented as a convergent series in $L_2(0, l)$:

$$\frac{q(x_2, t)}{\sqrt{g(x_2)}} = \sum_{n=1}^{\infty} \left(\frac{q(x_2, t)}{\sqrt{g(x_2)}}, Y_n \right) Y_n = \sum_{n=1}^{\infty} (q, X_n) X_n \sqrt{g},$$

hence,

$$q(x_2, t) = \sum_{n=1}^{\infty} g(x_2) X_n(x_2) q_n(t), \quad q_n(t) := \int_0^l q(x_2, t) X_n(x_2) dx_2.$$

Further, we look for the solution in the form

$$w(x_2, t) = \sum_{n=1}^{\infty} w_n(x_2, t),$$

where $w_n(x_2, t)$ is a solution of the Problem 11 with $q(x_2, t)$ replaced by $g(x_2) X_n(x_2) q_n(t)$. Using the method of separation of variables, we can write

$$w_n(x_2, t) = X_n(x_2) T_{1n}(t),$$

where

$$T_{1n}''(t) + \lambda_n T_{1n}(t) = q_n(t)$$

and $X_n(x_2)$ satisfies (2.46).

Therefore, $w(x_2, t)$ can be expressed as follows

$$w(x_2, t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} X_n \int_0^t \sin(\sqrt{\lambda_n}(t - \tau)) q_n(\tau) d\tau. \tag{2.72}$$

Now, similarly to the proofs of Theorems 2.5 and 2.6, if the following conditions are fulfilled

$$\tau(x_2, t) := \frac{1}{\sqrt{g(x_2)}} \left(D(x_2) \left(\frac{q(x_2, t)}{g(x_2)} \right)_{,x_2 x_2} \right)_{,x_2 x_2} \in C[0, l],$$

$$\frac{\tau}{\sqrt{g}}(0, t) = -D(x_2) \left(\frac{\tau(x_2, t)}{\sqrt{g(x_2)}} \right)_{,x_2 x_2} \bigg|_{x_2=0_+} = \left(\frac{\tau(x_2, t)}{\sqrt{g(x_2)}} \right)_{,x_2} \bigg|_{x_2=l} \tag{2.73}$$

$$= \left(-D(x_2) \left(\frac{\tau(x_2, t)}{\sqrt{g(x_2)}} \right)_{,x_2 x_2} \right)_{,x_2} \Bigg|_{x_2 = l_-} = 0,$$

(for this, e.g., it is sufficient that $\frac{\partial^j}{\partial x_2^j} q(x_2, t) = O(x_2^{\gamma_j})$ $x_2 \to 0_+$, $\gamma_j > 7 - j - \frac{2\alpha}{3}$, $\frac{\partial^j}{\partial x_2^j} q(x_2, t) = O((l - x_2)^{\delta_j})$ $x_2 \to l_-$, $\gamma_j > 7 - j - \frac{2\beta}{3}$, $j = \overline{0, 8}$) we have the absolute and uniform convergence of the series (2.72) and

$$\frac{\partial^i}{\partial x_2^i} (D(x_2) w_{,x_2 x_2} (x_2, t)) = \sum_{n=1}^{\infty} \frac{d^i}{dx_2^i} (D(x_2) X_n'')(x_2) T_{1n}(t), \quad i = 0, 1,$$

on $[0, l]$, and the absolute and uniform convergence of the series

$$\frac{\partial^i}{\partial x_2^i} w(x_2, t) = \sum_{n=1}^{\infty} \frac{d^i}{dx_2^i} X_n(x_2) T_{1n}(t), \quad i = 1, ..., 4,$$

$$\frac{\partial^i}{\partial t^i} w(x_2, t) = \sum_{n=1}^{\infty} X_n(x_2) \frac{d^i}{dt^i} T_{1n}(t), \quad i = 1, 2,$$

on any $[a, b] \in]0, l[$.

Remark 2.7 Let $q(x_2, t)$, $\varphi_i(x_2) \not\equiv 0$. If conditions (2.60), (2.61) and (2.73) are satisfying then the solution of the Problems 1-10 can be expressed as follows

$$w(x_2, t) = \sum_{n=1}^{\infty} w_n(x_2, t),$$

where

$$w_n(x_2, t) = X_n(x_2)(T_{1n}(t) + T_n(t)),$$

$X_n(x_2) T_{1n}(t)$ is given by the formula (2.72) and $X_n(x_2) T_n(t)$ is given by the formula (2.54).

Remark 2.8 Similarly, we can solve the following initial boundary value problems which correspond to the Problems 1-7, 9, 10.

Problem 12 Let $0 \leq \alpha$, $\beta < 1$. Find a function $w(x_2, t)$, which satisfies following smoothness conditions

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (2.35), the boundary conditions (BCs)

$$w(0, t) = w_{,2}(0, t) = w(l, t) = w_{,2}(l, t) = 0, \quad t > 0,$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = \varphi_i'(l) = 0, \quad i = 1, 2.$$

Problem 13 *Let* $0 \leq \alpha$, $\beta < 1$. *Find a function* $w(x_2, t)$, *which satisfies following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]), \quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (2.35), the BCs

$$w(0, t) = w_{,2}(0, t) = w_{,2}(l, t) = Q_2(l, t) = 0, \quad t > 0,$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$\varphi_i(0) = \varphi'_i(0) = \varphi'_i(l) = (-D(x_2)\varphi''_i(x_2))' \big|_{x_2 = l_-} = 0, \ i = 1, 2.$$

Problem 14 *Let* $0 \leq \alpha, < 1$, $0 \leq \beta < 3$. *Find a function* $w(x_2, t)$, *which satisfies following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]) \cap C([0, l]), \quad M_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (2.35), the BCs

$$w(0, t) = w_{,2}(0, t) = w(l, t) = M_2(l, t) = 0, \quad t > 0,$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]) \cap C([0, l]),$$
$$\varphi_i(0) = \varphi'_i(0) = \varphi_i(l) = (-D(x_2)\varphi''_i(x_2)) \big|_{x_2 = l_-} = 0, \ i = 1, 2.$$

Problem 15 *Let* $0 \leq \alpha < 1$, $\beta \geq 0$. *Find a function* $w(x_2, t)$, *which satisfies following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]), \quad M_2(\cdot, t) \in C([0, l]), \quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 < l, \ t \geq 0),$$

equation (2.35), the BCs

$$w(0, t) = w_{,2}(0, t) = M_2(l, t) = Q_2(l, t) = 0, \quad t > 0,$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$

$$\begin{aligned} \varphi_i(0) &= \varphi'_i(0) = (-D(x_2)\varphi''_i(x_2)) \big|_{x_2 = l_-} \\ &= (-D(x_2)\varphi''_i(x_2))' \big|_{x_2 = l_-} = 0, \ i = 1, 2. \end{aligned}$$

Problem 16 *Let $0 \leq \alpha,\ \beta < 1$. Find a function $w(x_2,t)$, which satisfies following smoothness conditions*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]), \quad Q_2(\cdot,t) \in C([0,l]),$$
$$w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0),$$

equation (2.35), the BCs

$$w_{,2}(0,t) = Q_2(0,t) = w(l,t) = w_{,2}(l,t) = 0, \quad t > 0,$$

and ICs (2.36), where
$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l]),$$
$$\varphi_i'(0) = (-D(x_2)\varphi_i''(x_2))' \,|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i = 1,2.$$

Problem 17 *Let $0 \leq \alpha < 1,\ 0 \leq \beta < 3$. Find a function $w(x_2,t)$, which satisfies following smoothness conditions*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]), \quad M_2(\cdot,t) \in C([0,l]), \quad Q_2(\cdot,t) \in C([0,l]),$$
$$w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0),$$

equation (2.35), the BCs

$$w_{,2}(0,t) = Q_2(0,t) = w(l,t) = M_2(l,t) = 0, \quad t > 0,$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

$$\begin{aligned}
\varphi_i'(0) &= (-D(x_2)\varphi_i''(x_2))' \,|_{x_2=0_+} = \varphi_i(l) \\
&= (-D(x_2)\varphi_i''(x_2)) \,|_{x_2=l_-} = 0,\ i = 1,2.
\end{aligned}$$

Problem 18 *Let $0 \leq \alpha < 3,\ 0 \leq \beta < 1$. Find a function $w(x_2,t)$, which satisfies following smoothness conditions*

$$w(\cdot,t) \in C^4(]0,l[) \cap C^1(]0,l[) \cap C([0,l]), \quad M_2(\cdot,t) \in C([0,l]), \quad Q_2(\cdot,t) \in C([0,l]),$$
$$w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0),$$

equation (2.35), the BCs

$$w(0,t) = M_2(0,t) = w(l,t) = w_{,2}(l,t) = 0, \quad t > 0,$$

and ICs (2.36), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1(]0,l[) \cap C([0,l]),$$

$$\varphi_i(0) = (-D(x_2)\varphi_i''(x_2)) \,|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i = 1,2.$$

Problem 19 *Let $0 \leq \alpha$, $\beta < 3$. Find a function $w(x_2, t)$, which satisfies following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C([0, l]), \quad M_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 \leq x_2 \leq l, \ t \geq 0),$$

equation (2.35), the BCs

$$w(0, t) = M_2(0, t) = w(l, t) = M_2(l, t) = 0, \quad t > 0,$$

and ICs (2.36), where
$$\varphi_i(x_2) \in C^4(]0, l[) \cap C([0, l]),$$

$$\begin{aligned}
\varphi_i(0) &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2 = 0_+} = \varphi_i(l) \\
&= (-D(x_2)\varphi_i''(x_2))\,|_{x_2 = l_-} = 0, \ i = 1, 2.
\end{aligned}$$

Problem 20 *Let $\alpha \geq 0$, $0 < \beta < 1$. Find a function $w(x_2, t)$, which satisfies following smoothness conditions*

$$w(\cdot, t) \in C^4(]0, l[) \cap C^1(]0, l]), \quad M_2(\cdot, t) \in C([0, l]), \quad Q_2(\cdot, t) \in C([0, l]),$$
$$w(x_2, \cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \quad w(x_2, t) \in C(0 < x_2 \leq l, \ t \geq 0),$$

equation (2.35), the BCs

$$M_2(0, t) = Q_2(0, t) = w(l, t) = w_{,2}(l, t) = 0, \quad t > 0,$$

and ICs (2.36), where
$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l]),$$

$$\begin{aligned}
(-D(x_2)\varphi_i''(x_2)) &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2 = 0_+} \\
&= \varphi_i(l) = \varphi_i'(l) = 0, \ i = 1, 2.
\end{aligned}$$

In all these cases we get integral equations with symmetric kernels.

2.3.   Harmonic Vibration

We can avoid the restrictions (2.73) if we consider the problem of harmonic vibration. In this case
$$w(x_2, t) = e^{i\omega t} w_0(x_2), \quad q(x_2, t) = e^{i\omega t} q_0(x_2),$$

where $\omega = const$ is an oscillation frequency, $q_0(x_2) \in C([0, l])$ is a given function. Now, for $w_0(x_2)$ from (2.35) we get the following equation

$$(D(x_2)w_0''(x_2))'' = q_0(x_2) + 2\omega^2 \rho h(x_2) w_0(x_2), \tag{2.74}$$

which we solve under the above BVPs (see problems 1-10), where we replace $w(x_2)$ and $w'(x_2)$ by $w_0(x_2)$ and $w_0'(x_2)$. For bending moment and intersecting force we obtain

$$M_2(x_2) = -D(x_2)w_{0,22}, \quad Q_2(x_2) = M_{2,2}(x_2).$$

All these problems are equivalent to the following integral equation (which we get from (2.21) after replacing $w(x_2)$ and $q(x_2)$ by $w_0(x_2)$ and $q_0(x_2) + 2\omega^2 \rho h(x_2)w_0(x_2)$ respectively),

$$w_0(x_2) - \omega^2 \int_0^l K(x_2, \xi)\, g(\xi)\, w_0(\xi) d\xi = \Phi(x_2), \tag{2.75}$$

where

$$\Phi(x_2) := \int_0^l K(x_2, \xi)\, q_0(\xi) d\xi. \tag{2.76}$$

Introducing a new unknown function

$$w_1(x_2) = w_0(x_2)\sqrt{g(x_2)} \tag{2.77}$$

we can reduce (2.75) to the following integral equation

$$w_1(x_2) - \omega^2 \int_0^l R(x_2, \xi)\, w_1(\xi) d\xi = \Phi(x_2)\sqrt{g(x_2)} \tag{2.78}$$

where $R(x_2, \xi)$ is given by (2.49)

Further, in view of (2.52) we have

$$X_n(x_2) = \lambda_n \int_0^l g(\xi) K(x_2, \xi) X_n(\xi) d\xi. \tag{2.79}$$

If $\omega^2 \neq \lambda_n$, the unique solution of (2.78) can be written as follows (see, e.g., [66], Theorem XVIII, p.157)

$$\begin{aligned} w_1(x_2) &= \Phi(x_2)\sqrt{g(x_2)} \\ &+ \omega^2 \sum_{n=1}^\infty \left[\frac{1}{\lambda_n - \omega^2} \int_0^l \Phi(\xi)\, \sqrt{g(\xi)}\, Y_n(\xi) d\xi \right] Y_n(x_2), \end{aligned} \tag{2.80}$$

where the series in the right hand side of (2.80) is absolutely and uniformly convergent on $[0, l]$.

After substituting (2.80) into (2.77) we formally have

$$
\begin{aligned}
w_0(x_2) \;=\;& \Phi(x_2) \\
&+\; \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l \Phi(\xi)\, \sqrt{g(\xi)}\, Y_n(\xi)d\xi \right] X_n(x_2).
\end{aligned}
\qquad (2.81)
$$

We have to prove that (2.81) is a solution of (2.74) under homogeneous BCs.

Let differentiate (2.81) formally i-times with respect to x_2 and consider the following expressions

$$
\begin{aligned}
\frac{d^i}{dx_2^i} w_0(x_2) =\;& \frac{d^i}{dx_2^i} \Phi(x_2) \\
&+ \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l \Phi(\xi)\, \sqrt{g(\xi)}\, Y_n(\xi)d\xi \right] \frac{d^i}{dx_2^i} X_n(x_2), \\
& i = 1, ..., 4
\end{aligned}
\qquad (2.82)
$$

Proposition 2.9 *The series on the right hand side of (2.81) and (2.82) are absolutely and uniformly convergent on $]0, l[$.*

Proof Let denote by

$$
S_n(x_2) := \frac{1}{\lambda_n - \omega^2} \int_0^l \Phi(\xi)\, \sqrt{g(\xi)}\, Y_n(\xi)d\xi X_n(x_2).
$$

Taking into account (2.79), (2.49) we have

$$
\begin{aligned}
S_n(x_2) \;=\;& \frac{1}{\sqrt{g(x_2)}} \frac{1}{\lambda_n - \omega^2} \int_0^l \Phi(\xi)\, \sqrt{g(\xi)}\, Y_n(\xi)d\xi \cdot Y_n(x_2) \\
=\;& \frac{1}{\sqrt{g(x_2)}} \frac{\lambda_n}{\lambda_n - \omega^2} \frac{\int_0^l \Phi(\xi)\, \sqrt{g(\xi)}\, Y_n(\xi)d\xi \cdot Y_n(x_2)}{\lambda_n}
\end{aligned}
\qquad (2.83)
$$

According to Proposition 2.3, the number of eigenvalues is not finite, that means $\lambda_n \to \infty$ when $n \to \infty$, and further

$$
\frac{\lambda_n}{\lambda_n - \omega^2} = \frac{1}{1 - \frac{\omega^2}{\lambda_n}} \to 1.
\qquad (2.84)
$$

In view of $\Phi(x_2) := \int_0^l K(x_2, \xi) \, q_0(\xi) d\xi$ and (2.33) we obtain, that the following series

$$\sum_{n=1}^{\infty} \frac{\int_0^l \Phi(\xi) \sqrt{g(\xi)} \, Y_n(\xi) d\xi \cdot Y_n(x_2)}{\lambda_n}; \tag{2.85}$$

is absolutely and uniformly convergent on $]0, l[$.

Further, in view of (2.83)-(2.85) and (2.33) we get, that

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_n(x_2) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \int_0^l \sqrt{g(\eta)} K(x_2, \eta) \\
&\quad \times \left(\int_0^l \Phi(\xi) \sqrt{g(\xi)} \, Y_n(\xi) d\xi \right) Y_n(\eta) d\eta \\
&= \int_0^l \sqrt{g(\eta)} K(x_2, \eta) \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \\
&\quad \times \left(\int_0^l \Phi(\xi) \sqrt{g(\xi)} \, Y_n(\xi) d\xi \right) Y_n(\eta) d\eta
\end{aligned}
\tag{2.86}
$$

is also absolutely and uniformly convergent on $]0, l[$, i.e., $w_0(x_2) \in C(]0, l[)$.

Analogously, we obtain

$$
\begin{aligned}
\frac{d^i}{dx_2^i} w_0(x_2) &= \frac{d^i}{dx_2^i} \Phi + \int_0^l \sqrt{g(\eta)} \frac{\partial^i}{\partial x_2^i} K(x_2, \eta) \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \\
&\quad \times \left(\int_0^l \Phi(\xi) \sqrt{g(\xi)} \, Y_n(\xi) d\xi \right) Y_n(\eta) d\eta, \quad i = 1, ..., 4.
\end{aligned}
\tag{2.87}
$$

Because of

$$\sqrt{g(\eta)} \frac{\partial^i}{\partial x_2^i} K(x_2, \eta) \in C(]0, l[\times]0, l[),$$

we get $\Phi^{(i)} \in C(]0, l[)$ and $w^{(i)} \in C(]0, l[)$, $i = 1, ..., 4$. $\square$

Proposition 2.10

$$
w_0(x_2) \in \begin{cases}
C^1([0,l]), & \text{in case of Problems 1, 2, 5;} \\
C^1([0,l[) \cap C([0,l]), & \text{in case of Problems 3, 6;} \\
C^1(]0,l[) \cap C([0,l]), & \text{in case of Problems 7, 8;} \\
C([0,l]), & \text{in case of Problem 9;} \\
C(]0,l]), & \text{in case of Problem 10;} \\
C([0,l[), & \text{in case of Problem 4,}
\end{cases}
\tag{2.88}
$$

because of $q_0(x_2) \in C([0,l])$.

Proof. (2.81) can be rewritten as follows

$$
w_0(x_2) = \Phi(x_2) + \frac{\omega^2}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \frac{\int_0^l \Phi(\xi)\sqrt{g(\xi)}Y_n(\xi)d\xi}{\lambda_n} Y_n(x_2). \tag{2.89}
$$

Taking into account of (2.84), and Theorem 1.3 we have

$$
\frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} \frac{\int_0^l \Phi(\xi)\sqrt{g(\xi)}Y_n(\xi)d\xi \cdot Y_n(x_2)}{\lambda_n}
$$

$$
= \quad (\text{because of } \Phi\sqrt{g} \text{ is a continious function on } [0,l])
$$

$$
= \quad \frac{1}{\sqrt{g(x_2)}} \int_0^l R(x_2,\xi)\Phi(\xi)\sqrt{g(\xi)}d\xi = \int_0^l K(x_2,\xi)\Phi(\xi)g(\xi)d\xi
$$

$$
\in \begin{cases}
C^1([0,l]), & \text{in case of Problems 1, 2, 5;} \\
C^1([0,l[) \cap C([0,l]), & \text{in case of Problems 3, 6;} \\
C^1(]0,l[) \cap C([0,l]), & \text{in case of Problems 7, 8;} \\
C([0,l[), & \text{in case of Problem 10;} \\
C(]0,l]), & \text{in case of Problem 4;} \\
C([0,l]), & \text{in case of Problem 9.}
\end{cases}
\tag{2.90}
$$

According to (2.90), (2.89) we have (2.88). □

Similarly, it can be proved that the following series

$$
\frac{d^i}{dx_2^i}(D(x_2)w_0''(x_2)) = \frac{d^i}{dx_2^i}(D(x_2)F''(x_2))
$$

$$
+ \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l \Phi(\xi)\sqrt{g(\xi)}Y_n(\xi)d\xi \right]
$$

$$
\times \frac{d^i}{dx_2^i}(D(x_2)X_n''(x_2)), \quad i = 1, 2
$$

are absolutely and uniformly convergent in $[0, l]$.

So, we obtaine that the formal solution (2.81) is a solution of (2.74) under BCs 1-10.

Using the difference equation corresponding to (2.74) by means of MATLAB we get numerical and graphic (Figures 18-21) results for harmonic vibration problems.

Problem 10, Aluminium,
$\alpha = 0.3$, $\beta = 0.3$
$l = 2\pi$, $q(x_2, t) = cos(x_2) * cos(t)$, $t \in [0, 2\pi]$
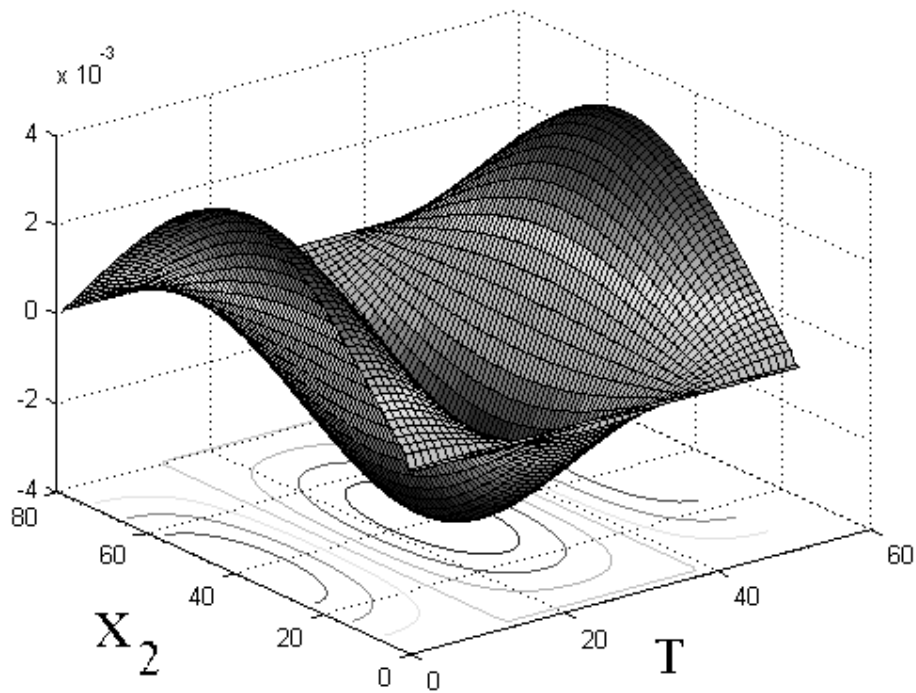
$$w_0(x_2)cos(t)$$
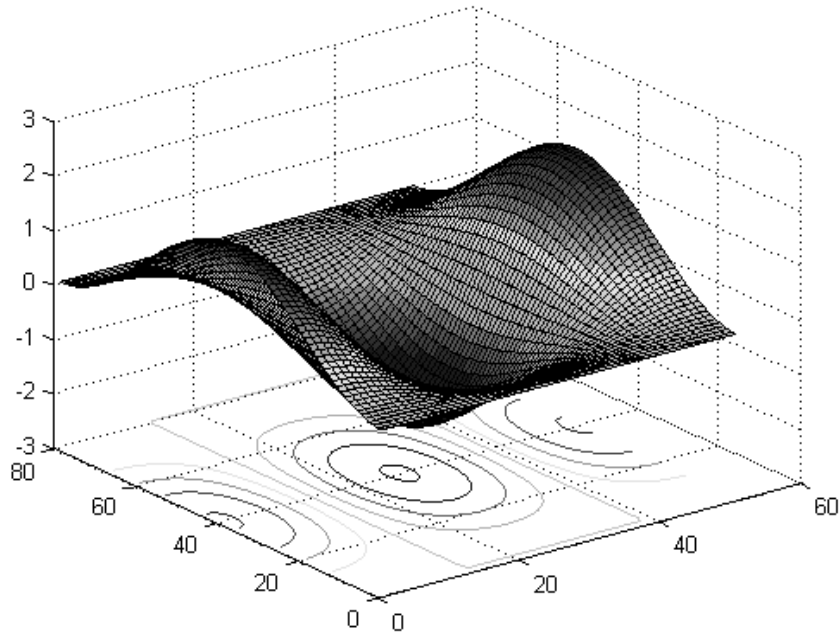

Fig. 18

$$M_2(x_2)cos(t)$$

Fig. 19

Problem 8, Iron,
$\alpha = 1,\ \beta = 0.3$
$l = 2,\ q(x_2, t) = \text{const} * cos(t),\ \ t \in [0, 2\pi]$
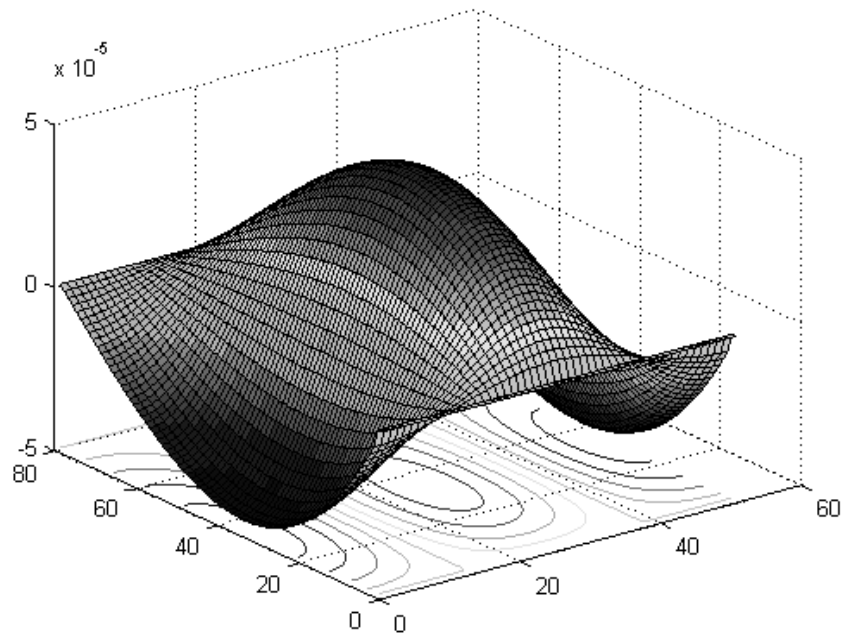
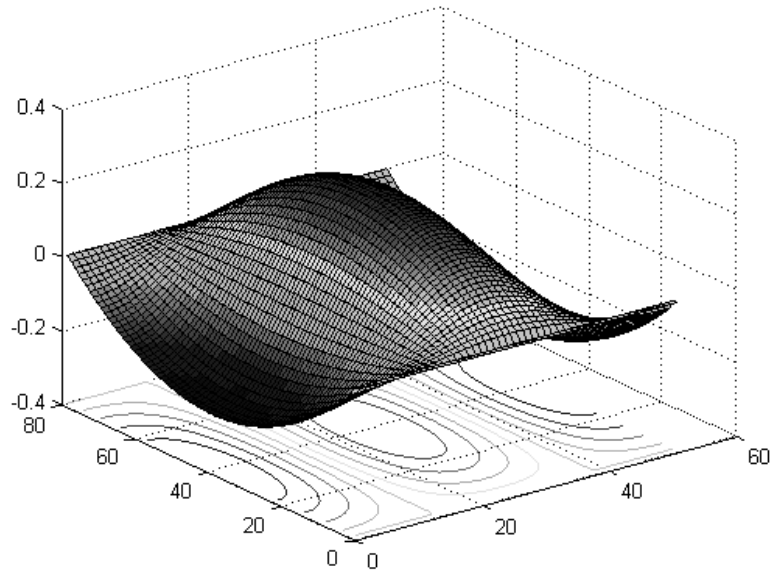$w_0(x_2)cos(t)$

Fig. 20

$$M_2(x_2)cos(t)$$

Fig. 21

Chapter 3

A Cusped Elastic Plate-Fluid Interaction Problem

Let us consider the interface problem of the interaction of a plate, whose variable flexural rigidity is given by the equation (2.3), and of a flow of fluid. Let the flow be independent of x_1, parallel to the plane $0x_2x_3$, i.e. $v_1 \equiv 0$, and generates a bending of the plate. Let at infinity, for pressure we have

$$p(x_2, x_3, t) \to p_\infty(t), \quad \text{when} \ |x| \to \infty, \tag{3.1}$$

and let for the velocity components conditions at infinity be either

$$v_2(x_2, x_3, t) = O(1), \quad v_3(x_2, x_3, t) \to v_{3\infty}(t), \quad \text{when} \ |x| \to \infty \tag{3.2}$$

or

$$v_j(x_2, x_3, t) = O(1), \quad j = 2, 3, \quad \text{when} \ |x| \to \infty \tag{3.3}$$

where $v := (v_2, v_3)$ is a velocity vector of the fluid, $p(x_2, x_3, t)$ is a pressure, and $v_{3\infty}(t)$, $p_\infty(t)$ are given functions.

In what follows we suppose that the plate is so thin that, we can assume: the fluid occupies the whole space $\mathbb{R}^3$ but the middle plane Ω of the plate.

Let,

$$I := \{[0, l] \times 0\},$$
$$\Omega^f := \left\{ x_1, x_2, x_3 : x_1 = 0, \ x := (x_2, x_3) \in \mathbb{R}^2 \backslash I \right\}.$$

If the middle plane of the plate lies in the plane $0x_1x_2$ and the flow of moving fluid involves bending of the plate then transmission conditions have the form:

$$\sigma_{N3}^f \left(x_1, x_2, \overset{(+)}{h}(x_1, x_2), t \right) - \sigma_{N3}^f \left(x_1, x_2, \overset{(-)}{h}(x_1, x_2), t \right) = q(x_1, x_2, t),$$

$$v_3 \left(x_1 - \overset{(+)}{h}(x_1, x_2)w_{,1}(x_1, x_2, t), x_2 - \overset{(+)}{h}(x_1, x_2)w_{,2}(x_1, x_2, t), \overset{(+)}{h}(x_1, x_2) \right.$$

$$\left. + w(x_1, x_2, t), t \right) = v_3 \left(x_1 - \overset{(-)}{h}(x_1, x_2)w_{,1}(x_1, x_2, t), x_2 \right.$$

$$\left. - \overset{(-)}{h}(x_1, x_2)w_{,2}(x_1, x_2, t), \overset{(-)}{h}(x_1, x_2) + w(x_1, x_2, t), t \right) = \frac{\partial w(x_1, x_2, t)}{\partial t}, \tag{3.4}$$

(the first of the last pair of equalities is valid since deflection of plate w is independent of x_3),

Because of incompressibility we have

$$\operatorname{div} v(x_2, x_3, t) = 0, \quad (x_2, x_3) \in \Omega^f, \quad t \geq 0, \tag{3.5}$$

and (see e.g., [23], p.5)

$$\sigma_{jk}^f = -p\delta_{jk} + \mu \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right), \quad j, k = \mathrm{const} = 2, 3, \tag{3.6}$$

where σ_{jk}^f is a stress tensor, μ is a coefficient of viscosity, δ_{jk} is Kroneker delta. In case of ideal fluid $\mu = 0$.

From (3.5) and (3.6) we obtain

$$\begin{aligned} \sigma_{33}^f(x_2, x_3, t) &= -p(x_2, x_3, t) + 2\mu \frac{\partial v_3(x_2, x_3, t)}{\partial x_3} \\ &= -p(x_2, x_3, t) - 2\mu \frac{\partial v_2(x_2, x_3, t)}{\partial x_2}. \end{aligned} \tag{3.7}$$

In case of ideal fluid in virtue of (3.7) we get

$$\sigma_{33}^f(x_2, \overset{(\pm)}{h}(x_2), t) = p(x_2, \overset{\pm}{h}(x_2), t).$$

Therefore, the transmission condition for p has the following form

$$\begin{aligned} &- p(x_2, \overset{(-)}{h}(x_2), t) \cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)), x_3) \\ &- p(x_2, \overset{(+)}{h}(x_2), t) \cos(\overrightarrow{n}(x_2, \overset{(+)}{h}(x_2)), x_3) = q(x_2, t), \quad x_2 \in]0, l[. \end{aligned} \tag{3.8}$$

Remark. *If the plate thickness is sufficiently small, we can assume that:*
1. the fluid occupies $\mathbb{R}^2 \backslash I$;
2. the plate occupies I (its geometry depending on the thickness is taken into account in the coefficient of the bending equation);
3. $\overset{(\pm)}{h}$ can be neglected. Since the normals of I are $(0, 0, 1)$ and $(0, 0, -1)$, (3.8) can be rewritten as follows

$$-p(x_2, 0_+, t) + p(x_2, 0_-, t) = q(x_2, t), \quad x_2 \in]0, l[. \tag{3.9}$$

4.Further, using (3.4), we can write transmission conditions for $v_3(x_2, x_3, t)$ in the following form (see [65], [98], [82])

$$v_3(x_2, 0, t) = \frac{\partial w(x_2, t)}{\partial t}, \quad x_2 \in]0, l[, \ t \geq 0. \tag{3.10}$$

In case of a viscous fluid we add to (3.10) the transmission condition for $v_2(x_2, x_3, t)$

$$v_2(x_2, 0, t) = 0, \quad x_2 \in]0, l[, \ t \geq 0. \tag{3.11}$$

In virtue of (3.7) and (3.11)

$$\sigma_{33}^{f \, (\pm)}(x_2, 0, t) = p^{\pm}(x_2, 0, t).$$

Further, taking into account of smallness of the thickness, in case of viscous fluid we rewrite transmission conditions for p can be (3.9)

3.1. Case of an Ideal Fluid

For the potential motion of the flow there exists a complex function $\Phi = -\psi + i\varphi$ such that

$$\frac{\partial \varphi(x_2, x_3, t)}{\partial x_2} = \frac{\partial \psi(x_2, x_3, t)}{\partial x_3} = v_2(x_2, x_3, t),$$

$$\frac{\partial \varphi(x_2, x_3, t)}{\partial x_3} = -\frac{\partial \psi(x_2, x_3, t)}{\partial x_2} = v_3(x_2, x_3, t). \tag{3.12}$$

The pressure is given by the formula

$$p(x_2, x_3, t) = \rho^f \left[\frac{v_\infty^2}{2} + \frac{p_\infty}{\rho^f} + \frac{\partial \varphi_\infty}{\partial t} - \frac{\partial \varphi}{\partial t} - \frac{1}{2}(v_2^2 + v_3^2) \right]. \tag{3.13}$$

In case under consideration $w(x_2, t)$ is given by the equation (2.35). Taking into account transmission condition (3.8), we have

$$\left(x_2^\alpha (l - x_2)^\beta w_{,22}(x_2, t) \right)_{,22} = -\frac{2h(x_2)\rho^s}{D_0} w_{,tt}(x_2, t) + \tag{3.14}$$

$$-\frac{p(x_2, \overset{(-)}{h}(x_2), t) \cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)), x_3) + p(x_2, \overset{(+)}{h}(x_2), t) \cos(\overrightarrow{n}(x_2, \overset{(+)}{h}(x_2)), x_3)}{D_0}.$$

Problem 21 *Find a function $w(\cdot, t) \in C^4(]0, l[)$ (satisfying additional smoothness conditions indicated in Problems 11-20), also the functions $v_2(x_2, x_3, t) \in C^2(\Omega^f) \cup C^1(t > 0)$, $v_3(x_2, x_3, t) \in C^2(\Omega^f) \cup C^1(t > 0)$ and $p(x_2, x_3, t) \in C(\Omega^f) \cup C(t > 0)$ which satisfy the system of equations (3.12), (3.13), (3.14), transmission conditions (3.10), (3.8), conditions at infinity (3.1), (3.2) and one of the BCs given in Problems 11-20.*

For $\Phi_{,2}(x_2, x_3, t) = v_3 + iv_2$, in view of (3.10) and (3.2), we get the following expression (see [72])

$$\Phi_{,2} = -\frac{1}{\pi i \sqrt{(x_2 + ix_3)(x_2 + ix_3 - l)}} \int_0^l \frac{\sqrt{(\xi_2 + ix_3)(\xi_2 + ix_3 - l)}}{(\xi_2 - x_2) - ix_3} w_{,t}(\xi_2, t) d\xi_2$$

$$+ v_{3\infty}(t) \frac{x_2 + ix_3 - l/2}{\sqrt{(x_2 + ix_3)(x_2 + ix_3 - l)}}. \tag{3.15}$$

Let

$$w(x_2, t) = e^{i\omega t} w_0(x_2), \quad q(x_2, t) = e^{i\omega t} q_0(x_2), \tag{3.16}$$

$$p(x_2, x_3, t) = e^{i\omega t} p_0(x_2, x_3),$$

$$u_2(x_2, x_3, t) = e^{i\omega t} u_2^0(x_2, x_3), \quad u_3(x_2, x_3, t) = e^{i\omega t} u_3^0(x_2, x_3), \tag{3.17}$$

where $\omega = \text{const} > 0$, $v_2 = u_{2,t}$ $(v_3 = u_{3,t})$. Further,

$$\varphi(x_2, x_3, t) = ie^{i\omega t} \varphi_0(x_2, x_3), \quad \psi(x_2, x_3, t) = ie^{i\omega t} \psi_0(x_2, x_3),$$

$$v_2(x_2, x_3, t) = ie^{i\omega t} v_2^0(x_2, x_3), \quad v_3(x_2, x_3, t) = ie^{i\omega t} v_3^0(x_2, x_3),$$

$$p_\infty(t) = e^{i\omega t} p_\infty^0, \quad v_{3\infty}(t) = ie^{i\omega t} v_{3\infty}^0, \quad p_\infty^0, \ v_{3\infty}^0 = \text{const}.$$

From (3.15), we have expressions for v_2 and v_3 as follows

$$v_2(x_2, x_3, t) = -\frac{1}{\pi} \int_0^l R_1(\xi, x_2, x_3) w_{,t}(\xi, t) d\xi + v_{3\infty}(t) R_3(x_2, x_3)$$

$$v_3(x_2, x_3, t) = \frac{1}{\pi} \int_0^l R_2(\xi, x_2, x_3) w_{,t}(\xi, t) d\xi + v_{3\infty}(t) R_4(x_2, x_3),$$

where

$$R_1(\xi, x_2, x_3) = \frac{\sqrt{r(\xi, x_3)}}{\sqrt{r(x_2, x_3)}}$$
$$\times \frac{(x_2 - \xi)\cos[(\phi(\xi, x_3) - \phi(x_2, x_3))/2] + x_3 \sin[(\phi(\xi, x_3) - \phi(x_2, x_3))/2]}{(\xi - x_2)^2 + x_3^2},$$

$$R_2(\xi, x_2, x_3) = \frac{\sqrt{r(\xi, x_3)}}{\sqrt{r(x_2, x_3)}}$$
$$\times \frac{(x_2 - \xi)\sin[(\phi(\xi, x_3) - \phi(x_2, x_3))/2] + x_3 \cos[(\phi(\xi, x_3) - \phi(x_2, x_3))/2]}{(\xi - x_2)^2 + x_3^2},$$

$$R_3(x_2, x_3) = \left\{ (x_2 - l/2)\cos\frac{\phi(x_2, x_3)}{2} + x_3 \sin\frac{\phi(x_2, x_3)}{2} \right\} \frac{1}{\sqrt{r(x_2, x_3)}},$$

$$R_4(x_2, x_3) = \left\{ (x_2 - l/2)\sin\frac{\phi(x_2, x_3)}{2} + x_3 \cos\frac{\phi(x_2, x_3)}{2} \right\} \frac{1}{\sqrt{r(x_2, x_3)}},$$

here $\phi(x_2, x_3)$ is defined by either

$$cos\phi(x_2, x_3) = (x_2^2 - x_3^2 - lx_2)/r(x_2, x_3)$$

or

$$sin\phi(x_2, x_3) = (2x_2 - l)x_3/r(x_2, x_3)$$

and

$$r(x_2, x_3) = \sqrt{(x_2^2 - x_3^2 - lx_2)^2 + ((2x_2 - l)x_3)^2}.$$

By means of the latter, in view of (3.12), we can calculate φ which we have to substitute in (3.13)

$$
\begin{aligned}
p(x_2, x_3, t) &= \frac{\rho^f}{\pi} \int_0^l w_{,tt}(\xi, t) \int_0^{x_2} R_1(\xi, x_2, x_3) dx_3 d\xi + v_{3\infty}(t)\rho^f \int_0^{x_2} R_3(x_2, x_3) dx_3 \\
&+ \rho^f \left[\frac{v_\infty^2(t)}{2} + \frac{p_\infty(t)}{\rho^f} + \frac{\partial\varphi_\infty(t)}{\partial t} \right] \\
&- \frac{\rho^f}{2} \left\{ \left(\frac{1}{\pi} \int_0^l R_1(\xi, x_2, x_3) w_{,t}(\xi, t) d\xi + v_{3\infty}(t) R_3(x_2, x_3) \right)^2 \right. \\
&+ \left. \left(\frac{1}{\pi} \int_0^l R_2(\xi, x_2, x_3) w_{,t}(\xi, t) d\xi + v_{3\infty}(t) R_4(x_2, x_3) \right)^2 \right\}.
\end{aligned}
$$

Then substituting the obtained expression of $p(x_2, x_3, t)$ in (3.8), by virtue of (3.16), we get the following expression for $q_0(x_2)$

$$
\begin{aligned}
q_0(x_2) \;=\;& \frac{\omega^2 \rho^f}{\pi} \int_0^l w_0(\xi) \int_0^{\overset{(-)}{h}(x_2)} R_1(\xi, x_2, x_3) \cdot \cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)), x_3) dx_3 d\xi \\[2mm]
&+ \int_0^l w_0(\xi) \int_0^{\overset{(+)}{h}(x_2)} R_1(\xi, x_2, x_3) \cdot \cos(\overrightarrow{n}(x_2, \overset{(+)}{h}(x_2)), x_3) dx_3 d\xi \\[2mm]
&- v_{3\infty}^0 \omega^2 \rho^f \left\{ \int_0^{\overset{(-)}{h}(x_2)} R_3(x_2, x_3) \cdot \cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)), x_3) dx_3 \right. \\[2mm]
&\left. + \int_0^{\overset{(-)}{h}(x_2)} R_3(x_2, x_3) \cdot \cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)), x_3) dx_3 \right\}
\end{aligned}
\tag{3.18}
$$

Taking into account (3.16), (3.17), (3.18) from (3.14) after four times integration with respect to x_2 we get the following relation

$$
\begin{aligned}
w_0(x_2) \;-\;& 2\rho^s \omega^2 \int_{x_2^0}^{x_2} h(\xi) K(x_2, \xi) w_0(\xi) d\xi = \int_{x_2^0}^{x_2} (c_1 \xi + c_2)(x_2 - \xi) D^{-1}(\xi) d\xi \\[2mm]
&+ c_3 x_2 + c_4 + \int_{x_2^0}^{x_2} K(x_2, \xi) q_0(\xi) d\xi,
\end{aligned}
\tag{3.19}
$$

where

$$
x_2^0 \in]0, l[, \quad K(x_2, \xi) = -\int_{\xi}^{x_2} (x_2 - \eta)(\xi - \eta) D^{-1}(\eta) d\eta.
$$

Constants c_i $(i = 1, ..., 4)$ should be defined from the admissible boundary conditions (see Section 2 Problems 1-10).

Let us consider, e.g., Problem 8. Then for $w_0(x_2)$ we get the following equation

$$w_0(x_2) \quad - \quad \omega^2 \int_0^l K_1(x_2, \xi) w_0(\xi) d\xi$$

$$- \quad 2\rho^s \omega^2 \left\{ \int_{x_2^0}^{x_2} h(\xi) K(x_2, \xi) w_0(\xi) d\xi + \int_{x_2^0}^l h(\xi) K_l(x_2, \xi) w_0(\xi) d\xi \right.$$

$$\left. + \quad \int_0^{x_2^0} h(\xi) K_0(x_2, \xi) w_0(\xi) d\xi \right\} \qquad (3.20)$$

$$= \quad f(x_2),$$

where

$$K_0(x_2, \xi) = \xi \left\{ \int_l^{x_2} x_2 D^{-1}(\eta) d\eta - \int_0^{x_2} \eta D^{-1}(\eta) d\eta \right\} - K(0, \xi),$$

$$K_l(x_2, \xi) = x_2 \int_l^{x_2} \eta D^{-1}(\eta) d\eta - \int_0^{x_2} \eta^2 D^{-1}(\eta) d\eta + x_2 \int_\xi^l (\eta - \xi) D^{-1}(\eta) d\eta,$$

$$K_1(x_2, \xi) = \frac{\rho^f}{\pi} \left\{ \int_{x_2^0}^l K_l(x_2, \zeta) \int_0^{\overset{(-)}{h}(\zeta)} R_1(\xi, \zeta, x_3) \cdot \cos(\overrightarrow{n}(\zeta, \overset{(-)}{h}(\zeta)), x_3) dx_3 d\zeta \right.$$

$$+ \int_{x_2^0}^l K_l(x_2, \zeta) \int_0^{\overset{(+)}{h}(\zeta)} R_1(\xi, \zeta, x_3) \cdot \cos(\overrightarrow{n}(\zeta, \overset{(+)}{h}(\zeta)), x_3) dx_3 d\zeta$$

$$+ \int_{x_2^0}^0 K_0(x_2, \zeta) \int_0^{\overset{(-)}{h}(\zeta)} R_1(\xi, \zeta, x_3) \cdot \cos(\overrightarrow{n}(\zeta, \overset{(-)}{h}(\zeta)), x_3) dx_3 d\zeta$$

$$+ \int_{x_2^0}^0 K_0(x_2, \zeta) \int_0^{\overset{(+)}{h}(\zeta)} R_1(\xi, \zeta, x_3) \cdot \cos(\overrightarrow{n}(\zeta, \overset{(+)}{h}(\zeta)), x_3) dx_3 d\zeta$$

$$+ \int_{x_2^0}^{x_2} K(x_2, \zeta) \int_0^{\overset{(-)}{h}(\zeta)} R_1(\xi, \zeta, x_3) \cdot \cos(\overrightarrow{n}(\zeta, \overset{(-)}{h}(\zeta)), x_3) dx_3 d\zeta$$

$$+ \int_{x_2^0}^{x_2} K(x_2, \zeta) \int_0^{\overset{(+)}{h}(\zeta)} R_1(\xi, \zeta, x_3) \cdot \cos(\overrightarrow{n}(\zeta, \overset{(+)}{h}(\zeta)), x_3) dx_3 d\zeta \,,$$

$$f(x_2) = x_2 \left(g_{22} + h_{22} \int_{x_2^0}^{l} \xi D^{-1}(\xi) d\xi + h_{11} \int_{x_2^0}^{l} D^{-1}(\xi) d\xi \right) + g_{11} + h_{22} \int_0^{x_2^0} \xi^2 D^{-1}(\xi) d\xi$$

$$- h_{11} \int_0^{x_2^0} \xi D^{-1}(\xi) d\xi - \int_{x_2^0}^{x_2} (h_{22}\xi + h_{11})(x_2 - \xi) D^{-1}(\xi) d\xi$$

$$-\omega^2 \rho^f \left\{ \left[\int_{x_2^0}^{l} K_l(x_2, \xi) \left[\int_0^{\overset{(-)}{h}(\xi)} R_3(\xi, x_3) \cdot \cos(\overrightarrow{n}(\xi, \overset{(-)}{h}(\xi)), x_3) dx_0 \right. \right. \right.$$

$$\left. + \int_0^{\overset{(+)}{h}(\xi)} R_3(\xi, x_3) \cdot \cos(\overrightarrow{n}(\xi, \overset{(+)}{h}(\xi)), x_3) dx_3 \right] d\xi$$

$$- \int_{x_2^0}^{0} K_0(x_2, \xi) \left[\int_0^{\overset{(-)}{h}(\xi)} R_3(\xi, x_3) \cdot \cos(\overrightarrow{n}(\xi, \overset{(-)}{h}(\xi)), x_3) dx_3 \right.$$

$$\left. + \int_0^{\overset{(+)}{h}(\xi)} R_3(\xi, x_3) \cdot \cos(\overrightarrow{n}(\xi, \overset{(+)}{h}(\xi)), x_3) dx_3 \right] d\xi$$

$$- \int_{x_2^0}^{x_2} K_0(x_2, \xi) \left[\int_0^{\overset{(-)}{h}(\xi)} R_3(\xi, x_3) \cdot \cos(\overrightarrow{n}(\xi, \overset{(-)}{h}(\xi)), x_3) dx_3 \right.$$

$$\left. \left. + \int_0^{\overset{(+)}{h}(\xi)} R_3(\xi, x_3) \cdot \cos(\overrightarrow{n}(\xi, \overset{(+)}{h}(\xi)), x_3) dx_3 \right] d\xi \right\}.$$

It is easy to show that $2\rho^s h(\xi) K(x_2, \xi)$, $2\rho^s h(\xi) K_0(x_2, \xi)$, $2\rho^s h(\xi) K_l(x_2, \xi)$, $K_1(x_2, \xi) \in C([0, l])$ (in our case $0 \le \alpha < 2$, $0 \le \beta < 1$).

The integral equation (3.20) can be solved by method of successive approximations when

$$\omega^2 < \frac{1}{Ml},$$

where

$$M := \max_{x_2,\xi\in[0,l]} \left\{ |2\rho^s h(\xi)K(x_2,\xi)|,\ |2\rho^s h(\xi)K_0(x_2,\xi)|,\ |2\rho^s h(\xi)K_l(x_2,\xi)|,\ |K_1(x_2,\xi)| \right\}.$$

Remark 3.1 *In case of the other boundary conditions (see problems 1-7, 9, 10), the problem under consideration is solved analogously and in all cases we get integral equations of type (3.20).*
Below we give expressions for kernels K_0 and K_l under BCs of Problem 1-7, 9, 10.
Problem 1.

$$
K_0(x_2,\xi) = -K(0,\xi) + \left\{ \frac{\left(-K(0,\xi) + l\int_\xi^l(\xi-\eta)D^{-1}(\eta)d\eta\right)\int_0^l\eta^2 D^{-1}(\eta)d\eta}{\int_0^l\eta^2 D^{-1}(\eta)d\eta\int_0^l D^{-1}(\eta)d\eta - \left(\int_0^l\eta D^{-1}(\eta)d\eta\right)^2} \right.
$$

$$
\left. - \frac{\int_\xi^0(\xi-\eta)D^{-1}(\eta)d\eta\int_0^l\eta D^{-1}(\eta)d\eta}{\int_0^l\eta^2 D^{-1}(\eta)d\eta\int_0^l D^{-1}(\eta)d\eta - \left(\int_0^l\eta D^{-1}(\eta)d\eta\right)^2} \right\} \int_0^l \eta(x_2-\eta)D^{-1}(\eta)d\eta
$$

$$
+ \left\{ -\frac{\int_\xi^0(\xi-\eta)D^{-1}(\eta)d\eta\int_0^l\eta^2 D^{-1}(\eta)d\eta}{\int_0^l\eta^2 D^{-1}(\eta)d\eta\int_0^l D^{-1}(\eta)d\eta - \left(\int_0^l\eta D^{-1}(\eta)d\eta\right)^2} \right.
$$

$$
\left. + \frac{K(0,\xi)\int_0^l\eta D^{-1}(\eta)d\eta}{\int_0^l\eta^2 D^{-1}(\eta)d\eta\int_0^l D^{-1}(\eta)d\eta - \left(\int_0^l\eta D^{-1}(\eta)d\eta\right)^2} \right\} \int_0^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta,
$$

$$
K_l(x_2,\xi) = -\left\{ \frac{\left(K(l,\xi) + l\int_\xi^l(\xi-\eta)D^{-1}(\eta)d\eta\right)\int_0^l D^{-1}(\eta)d\eta}{\int_0^l\eta^2 D^{-1}(\eta)d\eta\int_0^l D^{-1}(\eta)d\eta - \left(\int_0^l\eta D^{-1}(\eta)d\eta\right)^2} \right.
$$

$$
-\ \left.\frac{\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^{2}D^{-1}(\eta)d\eta\int\limits_{0}^{l}D^{-1}(\eta)d\eta-\left(\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta\right)^{2}}\right\}
$$

$$
\times\ \int\limits_{0}^{l}\eta(x_2-\eta)D^{-1}(\eta)d\eta
$$

$$
+\ \left\{K(l,\xi)+l\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta\right\}\int\limits_{0}^{l}(x_2-\eta)D^{-1}(\eta)d\eta.
$$

Problem 2.

$$
K_0(x_2,\xi)\ =\ -K(0,\xi)+\int\limits_{\xi}^{0}x_2(\xi-\eta)D^{-1}(\eta)d\eta-\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta
$$

$$
\times\ \frac{\int\limits_{0}^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}D^{-1}(\eta)d\eta},
$$

$$
K_l(x_2,\xi)\ =\ \int\limits_{0}^{x_2}\eta(x_2-\eta)D^{-1}(\eta)d\eta-\frac{\int\limits_{0}^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}D^{-1}(\eta)d\eta}\int\limits_{\xi}^{l}(\xi-\eta)D^{-1}(\eta)d\eta
$$

$$
-\ \frac{\int\limits_{0}^{x_2}\eta D^{-1}(\eta)d\eta\int\limits_{0}^{l}\eta D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}D^{-1}(\eta)d\eta}.
$$

Problem 3.

$$
K_0(x_2,\xi)\ =\ x_2\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta-\frac{\int\limits_{x_2}^{0}\eta(x_2-\eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l}\eta^{2}D^{-1}(\eta)d\eta}
$$

$$
\times\ \left\{-K(0,\xi)+l\int\limits_{\xi}^{0}(\xi-\eta)D^{-1}(\eta)d\eta\right\},
$$

$$K_l(x_2, \xi) \;=\; -K(l, \xi) + \xi \int\limits_{x_2}^{0} \eta D^{-1}(\eta)d\eta - \frac{K(l, \xi)\int\limits_{x_2}^{0} \eta(x_2 - \eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} \eta^2 D^{-1}(\eta)d\eta}$$

$$-\; \frac{(l - \xi)\int\limits_{x_2}^{0} \eta D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} \eta^2 D^{-1}(\eta)d\eta} \int\limits_{0}^{l} \eta(x_2 - \eta)D^{-1}(\eta)d\eta.$$

Problem 4.

$$K_0(x_2, \xi) = -K(0, \xi) + x_2 \int\limits_{\xi}^{0} (\xi - \eta)D^{-1}(\eta)d\eta,$$

$$K_l(x_2, \xi) = \int\limits_{x_2}^{0} (x_2 - \eta)(\xi - \eta)D^{-1}(\eta)d\eta.$$

Problem 5.

$$K_0(x_2, \xi) \;=\; \int\limits_{x_2}^{l} \eta(x_2 - \eta)D^{-1}(\eta)d\eta - \frac{\int\limits_{x_2}^{l} (x_2 - \eta)D^{-1}(\eta)d\eta \int\limits_{x_2}^{l} D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} D^{-1}(\eta)d\eta}$$

$$-\; \frac{\int\limits_{x_2}^{l} (x_2 - \eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} D^{-1}(\eta)d\eta} \int\limits_{\xi}^{0} (\xi - \eta)D^{-1}(\eta)d\eta,$$

$$K_l(x_2, \xi) \;=\; K(l, \xi) - (x_2 - l)\int\limits_{\xi}^{l} (\xi - \eta)D^{-1}(\eta)d\eta$$

$$-\; \int\limits_{\xi}^{l} (\xi - \eta)D^{-1}(\eta)d\eta \times \frac{\int\limits_{x_2}^{l} (x_2 - \eta)D^{-1}(\eta)d\eta}{\int\limits_{0}^{l} D^{-1}(\eta)d\eta}.$$

Problem 6.

$$K_0(x_2, \xi) = (x_2 - l)\int\limits_{0}^{x_2} \eta D^{-1}(\eta)d\eta - \int\limits_{l}^{x_2} \eta^2 D^{-1}(\eta)d\eta + (x_2 - l)\int\limits_{\xi}^{0} (\eta - \xi)D^{-1}(\eta)d\eta,$$

$$K_l(x_2, \xi) = \xi \left\{ \int_0^{x_2} (x_2 - l) D^{-1}(\eta) d\eta - \int_l^{x_2} \eta D^{-1}(\eta) d\eta \right\} - K(l, \xi)$$

Problem 7.

$$K_0(x_2, \xi) = x_2 \int_\xi^0 (\xi - \eta) D^{-1}(\eta) d\eta - \frac{\int_{x_2}^0 \eta(x_2 - \eta) D^{-1}(\eta) d\eta}{\int_0^l \eta^2 D^{-1}(\eta) d\eta}$$

$$\times \left\{ -K(0, \xi) + l \int_\xi^0 (\xi - \eta) D^{-1}(\eta) d\eta \right\},$$

$$K_l(x_2, \xi) = -K(l, \xi) + \xi \int_{x_2}^0 \eta D^{-1}(\eta) d\eta - \frac{K(l, \xi) \int_{x_2}^0 \eta(x_2 - \eta) D^{-1}(\eta) d\eta}{\int_0^l \eta^2 D^{-1}(\eta) d\eta}$$

$$- \frac{(l - \xi) \int_{x_2}^0 \eta D^{-1}(\eta) d\eta}{\int_0^l \eta^2 D^{-1}(\eta) d\eta} \int_{x_2}^0 \eta(x_2 - \eta) D^{-1}(\eta) d\eta.$$

Problem 9.

$$K_0(x_2, \xi) = \frac{\xi}{l} \int_{x_2^0}^{x_2} (l - \eta)(x_2 - \eta) D^{-1} d\eta + \frac{\xi(l - x_2)}{l^2} \int_{x_2^0}^0 \eta(l - \eta) D^{-1}(\eta) d\eta$$

$$- \frac{x_2 \xi}{l} \int_{x_2^0}^l (l - \eta) D^{-1}(\eta) d\eta - \frac{x_2}{l} K(0, \xi),$$

$$K_l(x_2, \xi) = \frac{\xi - l}{l} \int_{x_2^0}^{x_2} \eta(x_2 - \eta) D^{-1} d\eta - \frac{(\xi - l)(l - x_2)}{l^2} \int_{x_2^0}^0 \eta^2 D^{-1}(\eta) d\eta$$

$$- \frac{x_2(\xi - l)}{l} \int_{x_2^0}^l \eta D^{-1}(\eta) d\eta + \frac{x_2}{l} K(l, \xi).$$

Problem 10.

$$K_0(x_2, \xi) = -\int\limits_{x_2}^{l} (x_2 - \eta)(\xi - \eta) D^{-1}(\eta) d\eta,$$

$$K_l(x_2, \xi) = K(l, \xi) - (x_2 - \xi) \int\limits_{\xi}^{l} (\xi - \eta) D^{-1}(\eta) d\eta.$$

Thus, the following Preposition holds true.

Proposition 3.2 *Problem of the harmonic vibration corresponding to Problem 21 has a unique solution when*

$$\omega^2 < \frac{1}{Ml},$$

where

$$M := \max_{x_2, \xi \in [0, l]} \left\{ |2\rho^s h(\xi) K(x_2, \xi)|, \ |2\rho^s h(\xi) K_0(x_2, \xi)|, \ |2\rho^s h(\xi) K_l(x_2, \xi)|, \ |K_1(x_2, \xi)| \right\}.$$

3.2. Case of a Viscous Fluid

Let the motion of the fluid is sufficiently slow, i.e., v_j and $v_{j,k}$ $(j, k = 2, 3)$ be so small that linearized Navier-Stokes equations (see [65], [82], [98]) can be applied. Hence,

$$\frac{\partial v_2}{\partial t} = -\frac{1}{\rho^f} \frac{\partial p}{\partial x_2} + \nu \Delta v_2,$$

$$\frac{\partial v_3}{\partial t} = -\frac{1}{\rho^f} \frac{\partial p}{\partial x_3} + \nu \Delta v_3,$$

(3.21)

where $\nu = \mu/\rho^f$, $\Delta = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. Let

$$v_i \in C^2(\Omega^f) \cap C(\mathbb{R}^2) \cap C(t > 0), \quad i = 2, 3;$$
$$p \in C^2(\Omega^f);$$
$$q_{,2}(\cdot, t) \in H([0, l]).$$

(3.22)

After differentiation of the first equation of (3.21) with respect to x_2, of the second equation of (3.21) with respect to x_3 and termwise summation, by virtue of (3.5), we obtain that $p(x_2, x_3, t)$ is satisfying the following equation

$$\Delta p(x_2, x_3, t) = 0.$$

(3.23)

In case of harmonic vibration in the fluid part, from (3.17), (3.21), (3.23) we obtain the following system

$$\Delta p_0(x_2, x_3) = 0, \tag{3.24}$$

$$-\omega^2 u_j^0 = -\frac{1}{\rho^f}\frac{\partial p_0}{\partial x_j} + \nu i \omega \Delta u_j^0, \quad j = 2, 3. \tag{3.25}$$

Problem 22 *Find a function $w_0(x_2)$ on I, which satisfies the equation*

$$\left(D(x_2)w,_{22}\,(x_2,t)\right),_{22} = q(x_2,t) + F_3(x_2) - 2\rho h(x_2)\frac{\partial^2 w(x_2,t)}{\partial t^2}, \quad 0 < x_2 < l \tag{3.26}$$

(where $F := (0, F_3)$ is a plane volume forces), one of the boundary conditions given in the Problems 1-10, and also find functions $u_i^0(x_2, x_3)$, $p_0(x_2, x_3)$, $q_0(x_2)$ on Ω^f, which satisfy system of equations (3.24)-(3.25), smoothness conditions (3.22), following conditions at infinity

$$p_0\big|_{|x|\to\infty} = O(1), \quad u_j^0\big|_{|x|\to\infty} = O(1), \quad j = 2, 3, \tag{3.27}$$

and transmissions conditions as follows

$$-p_0(x_2, 0_+) + p_0(x_2, 0_-) = q_0(x_2), \quad x_2 \in]0, l[, \tag{3.28}$$

$$u_3^0(x_2, 0) = w_0(x_2), \quad u_2^0 = 0, \quad x_2 \in]0, l[, \tag{3.29}$$

where $q(x_2, t) = e^{i\omega t}q_0(x_2)$, p_0^∞ is a given constant.

Solution. After separating real and imaginer parts from (3.25) we have

$$u_j^0 = \frac{1}{\omega^2\rho^f}\frac{\partial p_0}{\partial x_j}, \quad j = 2, 3, \tag{3.30}$$

$$\Delta u_j^0 = 0, \quad j = 2, 3. \tag{3.31}$$

From the last equality, taking into account $u_2^0 = 0$, $x_2 \in]0, l[$ we get

$$\frac{\partial p_0}{\partial x_2} = 0, \quad x_2 \in]0, l[. \tag{3.32}$$

The solution of the equation (3.24) under condition (3.27), (3.29), and (3.32), has the following form (see [72])

$$p_0(x_2, x_3) \quad = \quad -\frac{x_3}{2\pi}\int_0^l \frac{q_0(\xi_2)d\xi_2}{(\xi_2 - x_2)^2 + x_3^2}. \tag{3.33}$$

Substituting (3.33) into (3.30), for u_2^0 and u_3^0 we get

$$u_2^0(x_2, x_3) \quad = \quad \frac{x_3}{\pi\omega^2\rho^f}\int_0^l \frac{q_0(\xi_2)(\xi_2 - x_2)d\xi_2}{[(\xi_2 - x_2)^2 + x_3^2]^2}, \tag{3.34}$$

$$u_3^0 = \frac{1}{2\pi\omega^2\rho^f} \int_0^l \frac{q_0(\xi_2)[x_3^2 - (\xi_2 - x_2)^2]}{[(\xi_2 - x_2)^2 + x_3^2]^2} d\xi_2. \tag{3.35}$$

Let now consider the following limit, when $x_2 \in]0, l[$,

$$\lim_{x_3 \to 0} \int_0^l \frac{q_0(\xi_2)[x_3^2 - (\xi_2 - x_2)^2]}{[(\xi_2 - x_2)^2 + x_3^2]^2} d\xi_2 = \lim_{x_3 \to 0} \left\{ q_0(l) \frac{l - x_2}{(l - x_2)^2 + x_3^2} \right.$$

$$+ q_0(0) \frac{x_2}{x_2^2 + x_3^2} - \int_0^l \frac{q_0(\xi_2)(\xi_2 - x_2)}{(\xi_2 - x_2)^2 + x_3^2} d\xi_2 \left. \right\} = \lim_{x_3 \to 0} \left\{ q_0(l) \frac{l - x_2}{(l - x_2)^2 + x_3^2} \right.$$

$$+ q_0(0) \frac{x_2}{x_2^2 + x_3^2} - \int_0^l \frac{[q_0'(\xi_2) - q_0'(x_2)](\xi_2 - x_2)}{(\xi_2 - x_2)^2 + x_3^2} d\xi_2$$

$$- \frac{q_0'(x_2)}{2} \int_0^l \left\{ \ln\left[(\xi_2 - x_2)^2 + x_3^2\right] \right\}_{,\xi_2} d\xi_2 \left. \right\} = \lim_{x_3 \to 0} \left\{ q_0(l) \frac{l - x_2}{(l - x_2)^2 + x_3^2} \right.$$

$$+ q_0(0) \frac{x_2}{x_2^2 + x_3^2} - \frac{q_0'(x_2)}{2} \ln \frac{(l - x_2)^2 + x_3^2}{x_2^2 + x_3^2} - \int_0^l \frac{[q_0'(\xi_2) - q_0'(x_2)](\xi_2 - x_2)}{(\xi_2 - x_2)^2 + x_3^2} d\xi_2 \left. \right\}$$

(because of $q_0' \in H([0,l])$)

$$= \frac{q_0(l)}{l - x_2} + \frac{q_0(0)}{x_2} - q_0'(x_2) \ln \frac{l - x_2}{x_2} - \int_0^l \frac{q_0'(\xi_2) - q_0'(x_2)}{\xi_2 - x_2} d\xi_2.$$

On the other hand if we define the following supersingular integral in H'adamard's finite part sense, we analogously obtain

$$\int_0^l \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2 = \lim_{\varepsilon \to 0} \left(\int_0^{x_2 - \varepsilon} \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2 + \int_{x_2 + \varepsilon}^l \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2 + \frac{2q_0(x_2)}{\varepsilon} \right)$$

$$= \frac{q_0(l)}{l - x_2} + \frac{q_0(0)}{x_2} - q_0'(x_2) \ln \frac{l - x_2}{x_2} - \int_0^l \frac{q_0'(\xi_2) - q_0'(x_2)}{\xi_2 - x_2} d\xi_2.$$

Hence, using transmission condition (3.28) for u_3^0, we get the following expression

$$w_0(x_2) = -\frac{1}{2\pi\omega^2\rho^f} \int_0^l \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2, \quad x_2 \in]0, l[,$$

where the supersingular integral on the right hand side we define in H'adamard's finite part sense (see [8], [3]).

Taking into account of Proposition 2.4 and Theorem 1.5 $R(x_2, \xi)$ is a positive definite kernel. Further, by view of Proposition 1.7 and (2.76), we can rewrite (2.81) as follows (see, also formula (3.26))

$$
\begin{aligned}
w_0(x_2) &= \int_0^l K(x_2, \xi) F_3(\xi) d\xi + \int_0^l K(x_2, \xi) q_0(\xi) d\xi \\
&+ \omega^2 \int_0^l \left(\int_0^l \Gamma(x_2, \eta, \omega^2) g(\eta) K(\eta, \xi) d\eta \right) q_0(\xi) d\xi \\
&:= \int_0^l K(x_2, \xi) F_3(\xi) d\xi + \int_0^l K_1(x_2, \xi) q_0(\xi) d\xi,
\end{aligned}
\tag{3.36}
$$

where $\Gamma(x_2, \xi, \omega^2)$ is a resolvent of the symmetric kernel $K(x_2, \eta) \sqrt{g(x_2) g(\eta)}$. Substituting (3.36) into (3.36), for $q_0(x_2)$ we obtain the following supersingular integral equation

$$
\int_0^l \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2 + 2\pi \omega^2 \rho^f \int_0^l K_1(x_2, \xi_2) q_0(\xi_2) d\xi_2
$$

$$
= \int_0^l K(x_2, \xi) F_3(\xi) d\xi =: f(x_2).
\tag{3.37}
$$

We will find approximate solution of (3.37) using the method of solving given in Section 1.3 (see equation (1.8), where interval [-1,1] should be replaced by $[0, l]$) for $q_0'(x_2) := (dq_0(x_2)/dx_2) \in H([0, l])$.

Let divide interval $[0, l]$ into N parts as follows

$$
y_k' := \frac{lk}{N}, \quad k = \overline{0, N}, \quad y_k := \frac{lk}{N} + \frac{l}{2N}, \quad k = \overline{0, N-1},
$$

$$
q_{0N} := (q_0(y_0), ..., q_0(y_{N_1})),
$$

we will call q_{0N} approximate solution of (3.37).

For q_{0N} we get the following system of linear equations [see Chapter 1, system (1.12)]

$$
a_{ii} q_0(y_i) - \sum_{j=0}^{N-1}{}' q_0(y_j) \left[\frac{1}{y_{j+i}' - y_i} - \frac{1}{y_j' - y_i} \right]
$$

$$
+ \frac{2\pi \omega^2 \rho^f l}{N} \sum_{j=0}^{N-1} K_1(y_i, y_j) q_0(y_j) = f(y_i), \quad i = \overline{0, N-1}.
\tag{3.38}
$$

where

$$a_{ii} := -\frac{4N}{l} \int\limits_{\Delta_{ii}} \frac{d\xi_2}{(\xi_2 - y_i)^2}, \quad \Delta_{ii} := [0, l] \cap \left[y_i' - \frac{n}{N}, y_{i+1}' + \frac{n}{N} \right],$$

$$n := \sqrt{N} \quad \sum{}' := \sum_{\substack{j=0 \\ j \neq i-1, \ i, \ i+1}}^{N-1} \quad .$$

After repeating the calculation given in Section 1.3, we get

$$|q_0^* - q_{0N}^*| \leq A \left(\frac{2n}{l} \right)^{-\alpha_1},$$

where q_0^* and q_{0N}^* are the solutions of the equations (3.37) and (3.38) respectively. After calculating q_{0N}, from (3.33) and (3.36) we get approximate expressions for $p_0(x_2, x_3)$ and $w_0(x_2)$, as follows

$$p_0(x_2, x_3) = -\frac{x_3 l}{2\pi N} \sum_{j=0}^{N-1} \frac{q_0(y_j)}{(y_j - x_2)^2 + x_3^2}, \quad (x_2, x_3) \in \Omega^f;$$

$$w_0(y_i) = -\frac{1}{2\pi\omega^2\rho^f} \left\{ a_{ii} q_0(y_i) - \sum_{j=0}^{N-1}{}' q_0(y_j) \left[\frac{1}{y_{j+i}' - y_i} - \frac{1}{y_j' - y_i} \right] \right\}, \quad x_2 \in]0, l[,$$

Let us denote by $\bar{w}_0(y_i)$ the projection of w_0 on y_i and let estimate the error of the approximate solution of deflection. If we repeat the above calculation we get

$$|\bar{w}_0(y_i) - w_0(y_i)| \leq \frac{A}{2\rho^f\pi\omega^2} \left(\frac{2n}{l} \right)^{-\alpha_1}.$$

Further, after substituting $p_0(x_2, x_3)$ in (3.28) we obtain $u_j^0(x_2, x_3)$.

$$u_2^0(x_2, x_3) = \frac{x_3 l}{\pi N \omega^2 \rho^f} \sum_{j=0}^{N-1} \frac{q_0(y_j)(y_j - x_2)}{[(y_j - x_2)^2 + x_3^2]^2},$$

$$u_3^0(x_2, x_3) = -\frac{1}{2\pi N \omega^2 \rho^f} \sum_{j=0}^{N-1} \frac{q_0(y_j)(x_3^2 - (y_j - x_2)^2)}{[(y_j - x_2)^2 + x_3^2]^2}, \quad (x_2, x_3) \in \Omega^f.$$

Proposition 3.3 *In case of the harmonic vibration of the plate with two cusped edges under action of the incompressible viscous fluid all quantities can be expressed by lateral load ($q_0(x_2)$) (see formulas (3.33)-(3.34)) and for the calculating of $q_0(x_2)$ we get (3.37) type supersingular integral equation, where supersingular integral is defined in H'adamard's finite part sense. This equation has solution in class $q_0' \in H([0, l])$.*

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