SHORT COMMUNICATIONS

ON ONE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS DEGENERATED AT ONE POINT

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Abstract. The system of differential equations

$$z^{\nu}\,\zeta^{\,\mu}\,\frac{\partial^{n+q}w(z,\zeta)}{\partial\zeta^{\,n}\,\partial z^{q}} = B(z,\zeta)\,w^{*}(z,\zeta),$$

is investigated. The sufficient condition when the system has only trivial solution is derived. The interesting example, when this condition is not fulfilled is given.

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The basic concept of classical complex analysis is the Cauchy-Riemann system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

It is well-known that the previous system can be rewritten in a complex form

$$\frac{\partial w}{\partial \overline{z}} = 0,$$

where $w(z) = u(x,y) + iv(x,y)$, $z = x + iy$. The generalization of this system is a high-order differential system

$$\frac{\partial^{n} w}{\partial \overline{z^{n}}} = 0,$$

The solutions of which are called n-analytic functions. It has various applications in the theory of elasticity and was studied by numerous of authors.

In this paper more general system of differential equations is investigated by means of complex analysis method. This method is widely used by numerous of authors and has a long history (see for example [1–13]).

In a complex z plane ($z = x + iy$) consider the area G which contains the point $z = 0$. Let us denote by G^{*} the area for which $\zeta \in G$ implies $\overline{\zeta} \in G^{*}$. Let as consider the area G_4 in a 4D space which is defined by $G_4 = G \times G^{*}$.

In the area G_4 let us consider the differential equation given by

$$z^{\nu}\,\zeta^{\,\mu}\,\frac{\partial^{n+q}w(z,\zeta)}{\partial\zeta^{\,n}\,\partial z^{q}} = B(z,\zeta)\,w^{*}(z,\zeta), \tag{1}$$

where ν, μ, n, q are given non-negative integers, B is given analytic function with respect to z and ζ in the area G_4.

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big),$$

w is an unknown function and the function $w^*(z,\zeta)$ is defined in the area G_4 by the formula

$$w^*(z,\zeta) = \overline{w(\overline{\zeta}, \overline{z})}.$$

We will investigate solutions of the equation (1) in the class of analytic functions (with respect to x, y).

The following theorem is true.

Theorem. If $B(0,0) \neq 0$ and $\nu + \mu > n + q$, then the equation (1) has only the trivial solution.

Proof. As $B(z,\zeta)$ and $w(z,\zeta)$ are analytic functions in the area G_4 then they are representable by the double-series in the neighborhood of $(0,0)$

$$B(z,\zeta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} b_{km}\, z^k\, \zeta^m,$$

$$w(z,\zeta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w_{km}\, z^k\, \zeta^m.$$

It is obvious that in the neighborhood of $(0,0)$ we have

$$w^*(z,\zeta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \overline{w_{mk}}\, z^k\, \zeta^m.$$

Multiplying the doubly series we obtain

$$\begin{aligned} z^\nu\, \zeta^\mu\, \frac{\partial^{n+q} w}{\partial \zeta^n\, \partial z^q} &= \\ &= \sum_{k=\nu}^{\infty}\sum_{m=\mu}^{\infty} \frac{(m+n-\mu)!\,(k+q-\nu)!}{(m-\mu)!\,(k-\nu)!}\, w_{k+q-\nu,m+n-\mu}\, z^k\, \zeta^m, \quad (2) \\ B w^* &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Big(\sum_{l=0}^{k}\sum_{p=0}^{m} b_{k-l,m-p}\, \overline{w}_{pl}\Big) z^k\, \zeta^m. \quad (3) \end{aligned}$$

From (2) and (3) we obtain

$$\sum_{k=\nu}^{\infty}\sum_{m=\mu}^{\infty} \frac{(m+n-\mu)!\,(k+q-\nu)!}{(m-\mu)!\,(k-\nu)!}\, w_{k+q-\nu,m+n-\mu}\, z^k\, \zeta^m =$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \Big(\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w}_{pl} \Big) z^k \zeta^m.$$

Equalize the coefficients at the same degrees of z and ζ, we obtain the system of algebraic equations with respect to w_{mk}

$$\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w_{pl}} = 0, \quad \text{where} \;\; 0 \le \mathrm{m} \le \mu - 1, \;\; \mathrm{k} = 0, 1, 2, \dots,$$
$$\text{or} \quad \mathrm{m} = 0, 1, 2, \dots, \quad 0 \le \mathrm{k} \le \nu - 1, \tag{4}$$

$$\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w_{pl}} = \frac{(m+n-\mu)! \, (k+q-\nu)!}{(m-\mu)! \, (k-\nu)!} \, w_{k+q-\nu, m+n-\mu},$$
$$\text{where} \;\; \mathrm{m} \ge \mu, \;\; \mathrm{k} \ge \nu. \tag{5}$$

Let us prove, $w_{mk} = 0$, $m, k = 0, 1, 2, \dots$.

The integer $h = m + k$ we call the height of w_{mk}. We will prove $w_{mk} = 0$ by the mathematical induction method with respect to h.

If $h = 0$, then $m = k = 0$ and from (5) we obtain $b_{00} \overline{w}_{00} = 0$. $b_{00} = B(0) \neq 0$ implies $w_{00} = 0$.

Now, suppose for $h > 0$, $w_{mk} = 0$. Consider the coefficient w_{mk} for $h + 1$, i.e. $m + k = h + 1$.

We have

$$\begin{aligned} \sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w_{pl}} &= \sum_{l=0}^{k-1} \sum_{p=0}^{m-1} b_{k-l,m-p} \, \overline{w_{pl}} + \sum_{l=0}^{k-1} b_{k-l,0} \, \overline{w_{ml}} + \\ &+ \sum_{p=0}^{m-1} b_{0,m-p} \, \overline{w_{pk}} + b_{00} \, \overline{w_{mk}}. \end{aligned} \tag{6}$$

As $\nu + \mu > n + q$, then $(k + q - \nu) + (m + n - \mu) < k + m$, and as $b_{00} \neq 0$, from (4), (5), (6) follows that w_{mk} is the linear combination of the coefficients w_{pl} for the height h.

And as we suppose $w_{pl} = 0$ for $h = 0$, then $w_{mk} = 0$.

Hence $w(z, \zeta) = 0$ in G_4.

Note. In this theorem the condition $\nu + \mu > n + q$ is essential.

Let us consider the example:

1. Let $\nu, \mu, n, q, m, k > 0$, $\nu + \mu = n + q$ and $m - k = \nu - q = n - \mu$, $m \ge n$, $k \ge q$.

Let

$$B(z, \zeta) = \frac{m!}{(m-n)!} \frac{k!}{(k-q)!},$$

then we obtain

$$z^\nu \zeta^\mu \, \frac{\partial^{n+q} w(z, \zeta)}{\partial \overline{z}^{\,n} \, \partial z^q} = \frac{m!}{(m-n)!} \, \frac{k!}{(k-q)!} \, \overline{w^*(z, \zeta)}.$$

This equation has non-zero analytic solution $w(z, \zeta) = z^k \zeta^m$.

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